UNIT 9 THE OVERLAPPING GENERATIONS MODEL

Structure

9.0 OBJECTIVES

After going through this Unit you should be able to exlain:

- the standard two-period overlapping generations model with production;
- the concept of dynamic efficiency and examine whether the dynamic efficiency property holds for an overlapping generations economy; and
- the role of social security system in the overlapping generations framework in eliminating dynamic inefficiency.

9.1 INTRODUCTION

In the context of intertemporal decision making on the part of the households, you have come across the optimal growth model in the previous unit. There is another important framework that also considers households' intertemporal decision making – although over finite time horizon. This framework is called the overlapping generations framework. The framework was first developed by Samuelson (1954) which has subsequently been widely used in macro dynamics.

In the overlapping generations framework, individuals have a finite time horizon (in the standard case, a two-period time horizon), but the society lives forever. The term 'overlapping generations' implies that at each point of time the life-time of two generations overlaps. To clarify, let us take the standard case where each generation lives exactly for two periods. Thus for the cohort of individuals who are born at the beginning of period 't', they are alive in period 't' (when they are young) and in period '$t+1$' (when they are old). On the other hand, a new set of individuals are born at the beginning'of period $t+1$, who would be alive in period '$t+1$' and period '$t+2$' respectively. Thus between the lifetimes of these two successive generations, there is an exact overlap of one period. In the following discussion we shall concentrate on the two-period overlapping generations framework only.

9.2 STRUCTURE OF THE MODEL

The set of people who are born at the beginning of period t will be called 'generation t'.

Let us now look at the activities of a representative member of generation t. Each person is born with an endowment of one unit of labor. In the first period of his life, when he is young, he works and earns a wage income of which he consumes a part and saves (and invests) the rest. In the next period of his life, when he is old, he does not work anymore. He nonetheless earns an interest income on his previous period's savings (investment). He also gets back the principal amount that he invested, which he consumes in the second period along with the interest earning. Thus his first and second period budget constraints are respectively given by:

i) $c_{1t} + s_t = w_t$;

ii) $c_{2t} = (1 + r_{t+1})s_t$,

where c_{1t} and c_{2t} are the first and second period consumption of the representative member of generation t, and w_t and r_{t+1} are the wage rate at period t and the **rate** of interest at period $t+1$ respectively.[1] The representative member maximizes his two-period utility function $U(c_{1t}, c_{2t})$ subject to these two budget constraints. Noting that $s_t = \frac{c_{2t}}{(1 + r_{t+1})}$, we can combine the two budget constraints by eliminating s_t, which gives us the following single equation that represents the ***life-time budget constraint*** of the agent: $c_{1t} + \frac{c_{2t}}{(1 + r_{t+1})} = w_t$.

Maximizing $U(c_{1t}, c_{2t})$ subject to the life-time budget constraint generates the following two first order conditions:

1) $$\frac{U_1}{U_2} = (1 + r_{t+1}),$$

2) $$c_{1t} + \frac{c_{2t}}{(1 + r_{t+1})} = w_t.$$

From these two equations we can write the optimal c_{1t} and c_{2t} as functions of w_t and r_{t+1}. Since $s_t = \frac{c_{2t}}{(1 + r_{t+1})}$, the optimal value of s_t also becomes a function of w_t and r_{t+1}.

We shall assume that all members of all generations are identical in terms of tastes and preferences, i.e., they have similar utility functions.[2]

Let us now look at the overall macroeconomic picture. As we have mentioned before, at each period there are two generations who are simultaneously alive. Thus at period t, there is a set of people who belong to generation *t-1* (these are the people who are currently old) and a set of people who belong to generation t (these are the people who are currently young). The generation t people are the workers in period t, each of

---

[1] **Note that though the member of generation *t* made his saving and investment decision at period *t*, he earns the interest on that savings only in the next period. Hence r_{t+1} is the relevant rate of interest.**

[2] **Similar, but not identical. To be more precise, the time subscripts in the utility function will be different for different generations.**

whom earn a wage income w_t. The generation $t-1$ people are the interest earners who earn an interest income on their previous period's savings (i.e., savings made in period $t-1$) at the rate r_t.

Suppose the population in successive generations grows at a constant rate n. Thus if L_t denotes the number of people belonging to generation t, and L_{t-1} denotes the number of people belonging to generation $t-1$, then $L_t = (1+n)L_{t-1}$. Total population in the economy at period t is given by $L_t + L_{t-1}$.

The production side of the economy is like any standard neoclassical growth model that you have seen before. A single final commodity is produced which is used both as a consumption good as well as an investment good. Technology for final commodity production is given by a neoclassical production function: $Y_t = F(K_t, L_t)$, where F is continuous, concave, exhibits constant returns to scale (CRS) with respect to its two factors – capital (K) and labour (L). As in the Ramsey model, the CRS property of the production function implies that per capita output (y) can be written as the function of the capital-labour ratio (k) in the following way:

$$y \equiv \frac{Y}{L} = \frac{F(K,L)}{L} = \frac{LF(K/L,1)}{L} = F\left(\frac{K}{L},1\right) \equiv f(k).$$

Moreover the marginal products of capital and labour can also written as the following functions of the capital-labour ratio:

$$\frac{\partial F}{\partial K} = f'(k) \; ; \; \frac{\partial F}{\partial L} = f(k) - kf'(k).$$

The continuity and concavity properties of $F(L, K)$ ensure that f (k) is also continuous and concave. Additionally we assume that the Inada conditions hold: $f(0) = 0$; $f'(0) = \infty$; $f'(\infty) = 0$.

In a market economy with perfect competition, the wage rate and the rate of interest are equated with the respective marginal products of labour and capital. Thus $w_t = f(k_t) - k_t f'(k_t)$ and $r_t = f'(k_t)$.

For the economy as a whole, savings-investment equality (which generates goods market equilibrium for the aggregate economy) tells us that $C_t + I_t = Y_t = F(K_t, L_t)$. Noting that investment is nothing but the augmentation of the capital stock (assuming no depreciation), i.e., $I_t = K_{t+1} - K_t$, and also noting that due to the property of CRS $F(L_t, K_t) = w_t L_t + r_t K_t$, we can write the above savings-investment equality condition as: $C_t + K_{t+1} - K_t = w_t L_t + r_t K_t$. Notice that C_t denotes the aggregate consumption at period t, which includes consumption by the old group and consumption by the young group. There are L_{t-1} number of old people who are alive at period t and each of them consume an amount c_{2t-1}.[3] On the other hand, there are L_t number of people

[3] c_{2t-1} is the 2nd period consumption of the representative member of generation $t-1$.

at period t belonging to the young generation, each of them consuming an amount c_{1t}. Hence $C_t = L_{t-1}c_{2t-1} + L_t c_{1t}$.

At this p i n t it is important to recall that each young person is born with an endowment of one unit of labour. There is no bequest; therefore the young people do not own any capital stock at period t (capital stock being the only asset in this economy). Thus the entire capital stock is owned by the older generation. And since the older generation is going to die at the end of this period, they consume their entire interest earning plus the capital stock (which, in the one good world, is directly consumable). Hence $L_{t-1}c_{2t-1} = K_t + r_t K_t$. Again, each young person earns a wage w_t, of which he consumes c_{1t} and saves s_t. Thus $L_t c_{1t} = L_t(w_t - s_t)$. Using all these information, we can write the savings-investment equality condition as:

$$K_t + r_t K_t + L_t(w_t - s_t) + K_{t+1} - K_t = w_t L_t + r_t K_t .$$

Simplifying, we get the goods market equilibrium condition for the aggregate economy as: $K_{t+1} = L_t s_t$. Recall that s_t is a function of w_t and r_{t+1}. w_t and r_{t+1} are in turn functions of k_t and k_{t+1} respectively. Thus s_t can be written as a function of k_t and k_{t+1}: $s(w(k_t), r(k_{t+1}))$. This allows us to write the goods market equilibrium condition in terms of the capital labour ratio as follows:

$$K_{t+1} = L_t s(w(k_t), r(k_{t+1}))$$
$$\Rightarrow \frac{K_{t+1}}{L_{t+1}} = \frac{L_t s(k_t, k_{t+1})}{L_{t+1}} = \frac{L_t s(w(k_t), r(k_{t+1}))}{(1+n)L_t}$$
$$\Rightarrow k_{t+1} = \frac{s(w(k_t), r(k_{t+1}))}{(1+n)}.$$

This last line above represents the basic dynamic equation of this overlapping generations model which specifies the relationship between the capital-labour ratio of today and the capital-labour ratio of tomorrow. Tracing this dynamic equation will tell us how the capital-labour ratio of the economy changes over time. It is easy to see that this equation is a first order non-linear difference equation in k. To characterize the solution path we shall use the phase diagram technique. The phase diagram plots k_{t+1} on the vertical axis and k_t on the horizontal axis. The intersection of the k_{t+1} line with the 45° line denotes the steady state. Let us now determine the slope of this line. The slope is given by the derivative $\frac{dk_{t+1}}{dk_t}$. As you can see, the LHS of the dynamic equation is also a function of k_{t+1}. Hence total differentiating both sides,

$$dk_{t+1} = \frac{s_w \frac{dw_t}{dk_t} dk_t + s_r \frac{dr_{t+1}}{dk_{t+1}} dk_{t+1}}{(1+n)} .$$

Noting that $\frac{dw}{dk} = -kf''(k)$ and $\frac{dr}{dk} = f''(k)$ we can write the above equation as:

$$\frac{dk_{t+1}}{dk_t} = \frac{s_w[-k_t f''(k_t)]}{(1+n) - s_r f''(k_{t+1})}.$$

Notice that **as** yet we do not know the signs of s_w and s_r; hence we cannot say whether the k_{t+1} **line** is positively sloping or negatively sloping. It is easy to see that **under** the assumption that consumption in both periods are **normal** goods (i.e., both c_1 **and** c_2 increases with an increase in w), $0 < s_w < 1$. The sign of s_r however is ambiguous. Since an **increase** in r implies that the relative price of future consumption in terms of current consumption falls. Hence due to substitution effect current consumption should fall, which means that with unchanged wage rate, savings would rise. However, a fall in relative price will also be associated with a positive income *e f f i* ton **current** consumption. Thus whether **current** consumption increases due to an **increase** in r depends **crucially** on whether the income effect dominates the substitution effect. If the income effect dominates the substitution effect, thenc, rises and consequently $s_r < 0$. **On the** other hand, **if the** substitution effect dominates the income effect, then c, **falls** and **consequently** $s_r > 0$. We shall assume here that the latter holds, i.e., $s_r > 0$. Under this assumption the k_{t+1} line is positively sloping. We still do not know the curvature of this line. Depending on the **curvature** multiple equilibria (i.e., multiple steady states) are possible, as shown in Fig. 9.1.

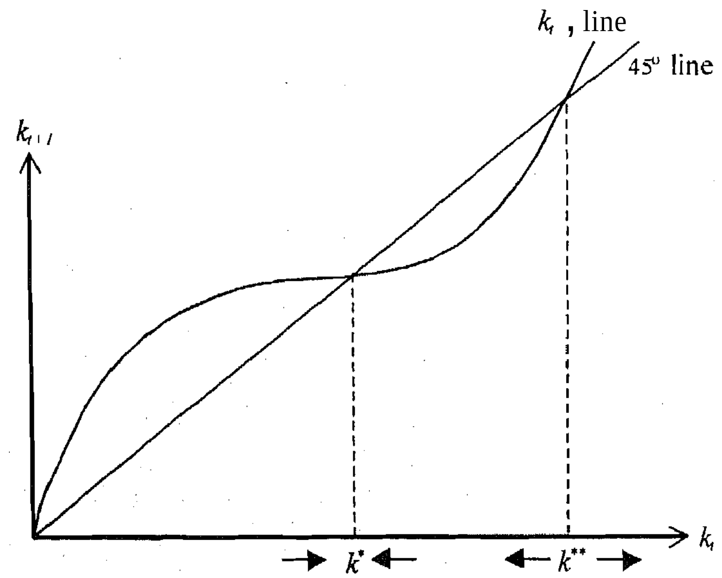

Fig. 9.1 : Multiple Steady State

Local stability of a steady state depends on whether the k_{t+1} line intersects the 45° line from above or fiom below. In Fig. 9.1 we find that k^*, where the k_{t+1} line intersects the 45° line from above, is a locally stable equilibrium. On the other hand, k*', where the k_{t+1} line intersects the 45° line fiom below, is a locally unstable equilibrium.

9.3 DYNAMIC INEFFICIENCY IN OVER LAPPING GENERATIONS MODEL

In the following discussion we shall assume that the parametric conditions are such that there exists a unique equilibrium which is locally stable.

Even when the steady state is unique and stable, in an overlapping generation **framework** the equilibrium can still be characterized by dynamic inefficiency. This is a serious shortcoming of the overlapping generations framework in comparison to the optimal growth framework. Before we elaborate this point further it is important to know what we mean by dynamic efficiency or inefficiency.

The concept of dynamic efficiency is closely related to the concept of Pareto efficiency. To understand the concept consider a situation where we are comparing between various possible steady states or equilibrium points. Each of these steady states is characterized by a different capital-labour ratio and corresponding steady state levels of per capita consumption for the old and the young. Since utility depends on the consumption during youth and during old-age, each of these steady states is therefore associated with a different level of steady state utility. Now among all these steady states, the one which provides maximum steady state value of utility is called the 'golden rule' point and the corresponding capital-labour ratio is called the **'golden rule'** capital labour ratio. This point is the best possible steady state point which provides maximum life-time utility to each person. One can show that steady state utility is maximized at the point where $f'(k) = n$. Thus golden rule capital-labour ratio is defined by k_g such that $f'(k_g) = n$. Fig. 9.2 depicts the golden rule capital labour ratio as the point which maximizes steady state utility.

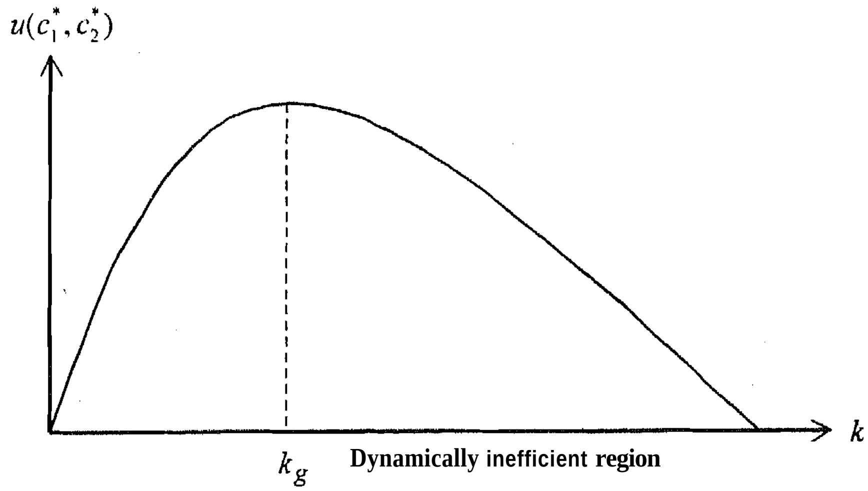

Fig. 9.2: Golden Rule Capital Labour Ratio

Now consider all the points which lie to the right of k_g. Here the capital-labour ratio is more than optimal. In other words people are over-saving and over-investing. If instead of saving, people consume a part of their savings in the first period of their life, then the capital-labour ratio will fall to k_g and at the same time their life time utility will increase. Thus all the points lying to the right ofk, are Pareto inferior to k_g: one can improve the current consumption without reducing future consumption and thus improve total life-time utility. These points are called dynamically inefficient points. Now consider all the points to the left of k_g. Here also the utility level is less than k_g; hence by moving to k_g one can improve the steady state value of utility. However such a move is not costless anymore. If one wants to move from a point to the left of k_g to k_g, then he has to save more. In other words he has to forgo some amount of current consumption. Thus his current utility would *fall*, even though he would be better off in the future (once he reaches k_g). Since such a move from the left to right involves a current utility loss, we cannot say for sure whether these points are better or worse then k_g. All these points are Pareto efficient or dynamically efficient.

We have seen that in the optimal growth framework the steady state is defined by $f'(k) = n + \rho$. Thus the steady state point in that framework is always to the left of the golden rule point and is therefore dynamically efficient. In the case of the overlapping generations framework however dynamic efficiency of the equilibrium point cannot be guaranteed. In fact under very reasonable parametric values the steady state could be dynamically inefficient.

To see how, consider the following example where we assume a specific utility function and a specific production function. Let the utility function of the representative member of generation t be $U(c_{1t}, c_{2t}) \equiv \log c_{1t} + \log c_{2t}$. Also let the per worker production function be $f(k_t) \equiv Ak_t^{\alpha}$, $0 < \alpha < 1$. Both the log utility function and the Cobb-Douglas production function are known to be well-behaved which satisfy all the standard neoclassical properties.

With the log utility function as specified above, one can easily verify that the first order conditions for individual's utility maximization exercise are given by:

i) $$\frac{c_{2t}}{c_{1t}} = (1 + r_{t+1})$$

ii) $$c_{1t} + \frac{c_{2t}}{(1 + r_{t+1})} = w_t.$$

These two first order conditions generate a savings function which is given by: $s_t = \frac{1}{2} w_t$, Also, with Cobb-Douglas production function $w_t = (1 - \alpha)Ak_t^{\alpha}$. Thus the basic dynamic equation in this example is given by: $k_{t+1} = \frac{1}{(1+n)}(1 - \alpha)Ak_t^{\alpha}$. At steady state $k_t = k_{t+1} = k^*$. Hence putting k^* at the LHS and RHS of the above equation, we can solve for the steady state value as: $k^* = \left[\frac{A(1-\alpha)}{(1+n)}\right]^{1/(1-\alpha)}$

Let us now compare this steady state value of capital-labour ratio with the corresponding golden rule capital labour ratio. Note that with the Cobb-Douglas production function, the golden rule capital labour ratio in this example is defined by: $A\alpha(k)^{\alpha-1} = n$. Solving, we get k_g as: $k_g = \left[\frac{A\alpha}{n}\right]^{1/(1-\alpha)}$. Thus whenever $\frac{(1-\alpha)}{\alpha} > \frac{(1+n)}{n}$, $k^* > k_g$, i.e., the steady state will be dynamically inefficient. For example, the equilibrium will be dynamically inefficient if $\alpha = 1/4$ and $n = 1$.

An obvious question that arises here is: why is it that the steady state could be dynamically inefficient in the overlapping generations framework, while such possibility is ruled out in the optimal growth framework? The answer lies in the fact that in the overlapping generations individuals are selfish (no bequest). Since they do not have to share the benefits of their investment with their successive generations (who are growing at the rate n), when they consider the possible future return to their investment, the return is **not** net of the population growth. To put it differently, the relevant return for them is not $(f'(k) - n)$ but just $f'(k)$. Hence they would be interested in investing as long as this return is positive, even if it falls short of n. This is the reason for their possible over-saving.

Check Your Progress 1

1) Derive the basic dynamic equation of the standard two period overlapping generations model with production. Explain the intuition behind this equation.

----

2) What is dynamic efficiency? Is the steady state under the overlapping generations structure necessarily dynamically efficient? Give an example to elaborate your answer.

9.4 SOCIAL SECURITY

Given that the overlapping generations equilibrium could be characterized by dynamic inefficiency, one can ask if there is any role for the government here. In other words, does there exist any government policy which could check the over-saving by the households? Typically the government could influence the savings decision through some kind of social security programmes, What is a social security programme? A social security programme is meant to provide some income to individuals after retirement. There are usually two types of social security programmes which *are* in vogue – a fully funded system and a pay-as-you-go system, In a fully funded system a stipulated amount is taken by the government from each young person at period t as social security contribution. The same amount is invested by the government and returned with interest to the old people at period $t+1$. Since the young at period tare the old at period $t+1$, in a fully funded system the set of people who made the contributions today and the set of people who will receive the corresponding benefit tomorrow are the same. If d_t is the contribution made by each young person at period t, and b_{t+1} is the benefit received by each old person at period $t+1$, then $b_{t+1} = (1 + r_{t+1})d_t$.

In an unfunded pay-as-you-go system, a stipulated amount is taken by the government from each young person at period t and the entire amount is immediately distributed to the old people at period t. In other words, the pay-as-you-go system involves a transfer of funds from the current young to the current youth. Note that population is growing at the rate n; therefore at each point of time there are n more people belonging to the currently young group compared to the currently old group. Thus if d_t is the contribution made by each young person at period t, and b_t is the benefit received by each old person at period t, then $b_t = (1 + n)d_t$.

The type of social security system has important implications for the savings decisions of the young. Note that in a filly funded system the government is doing part of the savings on behalf of the individuals. The representative individual is effectively saving an amount equal to $(s_t + d_t)$ and in the next period earning an interest income equal to $(1 + r_{t+1})(s_t + d_t)$. Since each person knows that this is the amount that he would receive in the next period, in his own savings decision, the individual will optimally adjust (cut back) his own savings so that his total effective savings remains the same as in the pre-social security economy. Thus a fully funded social security system has no impact on the total savings and capital accumulation. If the economy was in a dynamically inefficient steady state in the pre-social security economy, it will remain so even after introducing a fully funded social security system.

In contrast, in a pay-as-you-go social security system, the relevant rate of return on government securities for an individual is n. On the other hand, the rate of interest on capital accumulation is r. Thus as long as $r<n$, it pays to save less and contribute more in the form of social security. Hence if the economy was in a dynamically inefficient steady state in the pre-social security economy (which implies that r is indeed less than n), introducing a pay-as-you-go type of social security system will reduce private savings and capital accumulation. As a result the economy will move to the dynamically efficient region.

Check Your Progress 2

1) Does any type of social security system necessarily eliminate the dynamic inefficiency problem in the overlapping generations framework? Elaborate your answer with the two types of social security systems.

9.5 LET US SUM UP

In this unit, we discussed at the standard overlapping generations framework with production. We have derived the basic dynamic equation which basically states that the tomorrow's capital stock is equal to the savings by the young generations today. We have derived the steady state conditions. It has been shown that the steady state under the OLG framework may not be dynamically efficient. The government can play an important role here to ensure dynamic efficiency by introducing a social security system. However the type of the social security system is important: apay-as-you-go type of social security system is effective in eliminating inefficiency while a fully funded social security system is not effective.

9.6 KEY WORDS

Dynamic Efficiency : The situation where an increase in future consumption implies a fall in current consumption so that future utility cannot be increased without reducing current utility.

Overlapping Generations : A framework where agents have finite lifetime and at every point of time more than one generation of agents coexist.

Social Security System : A provision for retirement benefit such that an agent receives certain monetary benefit from the government when he is old.

9.7 SOME USEFUL BOOKS

Blanchard, Olivier and Stanley Fischer,1989, ***Lectures on Macroeconomics***, MIT Press, chapter 2.

9.8 ANSWER/HINTS TO CHECK YOUR PROGRESS EXERCISES

Check Your Progress 1

1) Read Section 9.2 and Mite your answer.

2) Read Section **9.3** and write your answer.

Check Your Progress 2

1) Read Section 9.4 **and** write your answer.