UNIT 12 FACTORIAL EXPERIMENTS

Structure

12.1 INTRODUCTION

Factorial experiments are the experiments that investigate the effects of two or more factors or input parameters on the output response of a process. Factorial experiment design or simply factorial design is a systematic method for formulating the steps needed to successfully implement a factorial experiment. Estimating the effects of various factors on the output of a process with minimum number of observations is crucial to being able to optimize the output of the process.

In a factorial experiment, the effects of varying the levels of the various factors affecting the process output are investigated. Each complete trial or replication of the experiment takes into account all the possible combinations of the varying levels of these factors. Effective factorial design ensures that least number of experimental runs are conducted to generate the maximum amount of information about how input variables affect the output of a process.

The basic definitions and different types of factorial experiments are described in Section 12.2. In Section 12.3, the statistical analysis of 2^2 factorial experiment is explained whereas the statistical analysis of 2^3 factorial experiment is given in Section 12.4.

Objectives

After studying this unit, you would be able to

- define the factorial experiments;
- describe the 2^2 factorial experiments;
- explain the statistical analysis of 2^2 factorial experiments;
- describe the 2^3 factorial experiments; and
- explain the statistical analysis of 2^3 factorial experiments.

12.2 FACTORIAL EXPERIMENTS

In factorial experiments, if the effet of two factors with levels r and s on the output of a process are investigated, then one would need to run $r \times s$ treatment combinations to complete the experiment. For instance if the effect of two factors A (Fertilizer) and B (Irrigation) on the output of a process are investigated, and A has 3 levels (0 kg N/ha, 20 kg N/ha and 40 kg N/ha) while B has two levels (5 cm and 10 cm), then one would need to run 6 treatment combinations to complete the experiment, observing the process output for each of the following combinations:

0 kg N/ha-5 cm I,

0 kg N/ha-10 cm I,

20 kg N/ha-5 cm I,

20 kg N/ha-10 cm I,

40 kg N/ha-5 cm I,

40 kg N/ha-10 cm I,

The amount of change produced in the process output for the change in the level of the given factor is referred as the main effect of that factor. Table 1 shows an example of simple factorial experiment involving two factors with two levels each. The two levels of each factor may be denoted by ‘low’ and ‘high’ which are usually symbolized by ‘–’ and ‘+’ in factorial designs, respectively.

Table 1: A Simple 2^2 Factorial Experiment

	A(–)	A(+)
B(–)	20	40
B(+)	30	52

The main effect of a factor is basically the average change in the output response as that factor goes from ‘–’to ‘+’. Mathematically this is the average of two numbers: 1) the change in the output when the factor goes from low to high level as the other factor stays low, and 2) the change in the output when the factor goes from low to high level as the other factor stays high.

In the example given in Table 1, the output of the process is just 20 (lowest output) when both A and B are at their ‘–’level. While the output is maximum at 52 when both A and B are at their ‘+’ level. The main effect of A is the average of the change in the output response when B stays at ‘–’as A goes from ‘–’to ‘+’ or (40−20) = 20 and the change in the output response when B stays ‘+’ as A goes from ‘–’ to ‘+’ or (52– 30) = 22. The main effect of A, therefore, is equal to $\frac{20+22}{2} = 21$.

Similarly, the main effect of B is the average change in output as it goes from ‘–’ to ‘+’, i.e. the average of 10 and 12, 11. Thus, the main effect of B in this process is 11. Here one can see that the factor A exerts a greater influence on the output of the process, having a main effect of 21 versus factor B's main effect of only 11.

It must be noted that aside from main effects factors can likewise result in interaction effects. Interaction effects are changes in the process output caused by two or more factors that are interacting with each other. Large interactive effects can make the main effects insignificant, such that it becomes more important to pay attention to the interaction of the involved factors than to investigate them individually. In above table as effects of A (B) is not same at all the levels of B (A) hence, A and B are interacting, thus interaction is the failure of the differences in response to changes in levels of one factor, to retain the same order and magnitude of performance throughout all the levels of other factors or the factors are said to interact if the effect of one factor changes as the levels of other factor(s) changes.

If interaction exists which is fairly common, we should plan our experiment in such a way that they can be estimated and tested. It is clear that we cannot do this if we vary only one factor at a time. For this purpose, we must use multilevel, multifactor experiments.

The running of factorial combinations and the mathematical interpretation of the output responses of the process to such combinations is the essence of factorial experiments. It allows us to understand which factors affect the process most so that improvements may be graded towards these.

We may define factorial experiments as experiments in which the effects (main effects and interactions) of more than one factor are studied together. In general if there are n factors say F_1, F_2, …, F_n and i^{th} factor has s_i levels, i=1, 2, …, n, then total number of treatment combinations is $\prod s_i$. Factorial experiments are of two types.

Factorial experiments in which the number of levels of all the factors are same i.e. all s_i are equal, are called symmetrical factorial experiments and the experiments in which at least two of the s_i's are different are called as asymmetrical factorial experiments.

If there are p different varieties, then we shall say that there are p levels of the factor variety. Similarly, the second factor manure may have q levels i.e. there may be q different manures or different doses of the same manure. Then this factorial experiment is called $p \times q$ experiment and this example of variety (at p-levels) and manure (at q-levels) is related to asymmetrical factorial experiment.

12.2.1 2^2 Factorial Experiments

The simplest of the symmetrical factorial experiments are the experiments with each of two factors at 2 levels. If there are n factors each at 2 levels it is called as a 2^n factorial experiments, where power stands for number of factors and the base the level of each factor. Simplest of the symmetrical factorial experiments is the 2^2 factorial experiment i.e. 2 factors say A and B each at two levels 0 (low) and 1(high). There will be 4 treatment combinations which can be written as

$00 = a_0b_0 = 1$: A and B both at first (low) levels.

$10 = a_1b_0 = a$: A at second (high) level and B at first (low) level.

$01 = a_0b_1 = b$: A at first (low) level and B at second (high) level.

$11 = a_1b_1 = ab$: A and B both at second (high) level.

In a 2^2 experiment wherein r replicates were run for each treatment combination, the main and interaction effects of A and B on the output may be mathematically expressed as follows:

$A = [ab + a - b - (1)] / 2r$; (main effect of factor A)

$B = [ab + b - a - (1)] /2r$; (main effect of factor B)

$AB = [ab + (1) - a - b]/ 2r$; (interaction effect of factor A and B)

12.3 STATISTICAL ANALYSIS OF 2^2 FACTORIAL EXPERIMENTS

If the two factors remain independent, then $[ab+ (1) - a- b]/ 2r$ will be of the order of zero. If not then this will give an estimate of interdependence of the two factors and it is called the interaction between A and B. It is easy to verify that the interaction of the factor B with the factor A is BA which will be same as the interaction AB and hence the interaction does not depend on the order of the factors. It is also easy to verify that main effect of factor B, a contrast of the treatments totals is orthogonal to each of A and AB.

2^2 Factorial Experiment Pattern

RUN	Comb.	M	A	B	AB
1	(1)	+	–	–	+
2	a	+	+	–	–
3	b	+	–	+	–
$4 = 2^2$	ab	+	+	+	+

The signs for the main effects can be written according to the rule “Give a plus sign to each of the treatment means when the corresponding factor is at the high level and a minus sign where it is of the low level. Or give a plus sign to the treatment combinations containing the corresponding small letter and give a minus sign where the corresponding small letter is absent. For a two factor interaction, the signs are obtained by combining the corresponding signs of two main effects i.e. two opposite signs will give a minus sign and two identical signs will give a plus sign to the interaction.”

12.3.1 Steps for Analysis

1. The sum of squares (SS) due to treatments, replications (in case of RBD is used) due to rows and columns (in case a row-column design has been used), total sum of squares and error sum of squares is obtained as per established procedures. No replication sum of squares is required in case of a CRD. The treatment sum of squares is divided into different components i.e. main effects and interactions each with single df. The sum of squares due to these factorial effects is obtained by factorial effect

totals rather than treatment means. Factorial effect totals are may be defined as

$$[A] = [ab] + [a] – [b] – [1]$$

$$[B] = [ab] – [a] + [b] – [1]$$

$$[AB] = [ab] – [a] – [b] + [1]$$

where, each [ab], [a], [b], [1] is the total of outputs of each of the replicates of the treatment combinations ab, a, b,1 respectively.

2. For a 2^2 factorial experiment, the sum of squares due to a main effect or the interaction effect is obtained by dividing the square of the effect total by 4r. Thus

   Sum of squares due to main effect of A = $[A]^2/4r$, with 1 df

   Sum of squares due to main effect of B = $[B]^2/4r$, with 1 df

   Sum of squares due to interacton effect AB = $[AB]^2/4r$, with 1 df

   Mean squares (MS) is obtained by dividing each SS by corresponding

   degrees of freedom.

3. After obtaining the different SS's the usual ANOVA table is prepared and the different effects are tested against error mean square and conclusions drawn.

4. If we consider n factors then standard error (SE's) for main effects and two factors interaction:

   SE of difference between main effect means = $\sqrt{\dfrac{2MSSE}{r.2^{n-1}}}$,

Therefore, for n = 2, SE of difference between main effect means

$$= \sqrt{\frac{2MSSE}{r.2^{2-1}}}$$

SE of difference between A means at same level of B

= SE of difference between B means at same level of A

$$= \sqrt{\frac{2MSSE}{r.2^{n-2}}},$$

Therefore, for n = 2, SE of difference between A means at same level of B

$$= \sqrt{\frac{2MSSE}{r.2^{2-2}}}$$

In general, SE for testing the difference between means in case of r-factor interaction

$$= \sqrt{\frac{2MSSE}{r.2^{n-r}}}$$

The critical differences are obtained by multiplying the SE by the student's t value at α level of significance at error degree of freedom.

The ANOVA for 2^2 factorial experiments with r replications conducted using RBD is as follows:

ANOVA Table

Source of Variation	DF	SS	MSS	F
Between Replications	r – 1	SSR	MSSR= SSR/(r – 1)	MSSR/MSSE
Between Treatments	$2^2 - 1 = 3$	SST	MSST= SST/3	MSST/MSSE
A	1	SSA=$[A]^2$ /4r	MSSA = SSA/1	MSSA/MSSE
B	1	SSB=$[B]^2$ /4r	MSSB = SSB/1	MSSB/MSSE
AB	1	SSAB=$[AB]^2$ /4r	MSSAB = SSAB/1	MSSAB/MSSE
Error	3(r – 1)	SSE	MSSE = SSE / 3(r– 1)	
Total	r. 2^2 – 1 = 4r– 1	TSS		

12.4 STATISTICAL ANALYSIS OF 2^3 FACTORIAL EXPERIMANT

Consider the case of 3 factors A, B and C each at two levels (0 and 1) i.e. 2^3 factorial experiment. There will be 8 treatment combinations which are written as

000 = $a_0b_0c_0$ = (1); A, B and C (all three) at first level.

100 = $a_1b_0c_0$ = a; A at second level and B and C are at first level.

010 = $a_0b_1c_0$ = b; A and C are both at first level and B at second level.

001 = $a_0b_0c_1$ = c; A and B both at first level and C at second level.

110 = $a_1b_1c_0$ = ab; A and B both at second level and C at first level.

101 = $a_1b_0c_1$ = ac; A and C both at second level and B at first level.

011 = $a_0b_1c_1$ = bc; A at first level and B and C at second level.

111 = $a_1b_1c_1$ = abc; A, B and C (all the three) at second level.

In a three factor experiment there are three main effects A,B and C; 3 first order or two factor interaction AB, AC, BC and one second order or three factor interaction ABC.

2^3 Factorial Experiment Pattern

RUN	Comb.	M	A	B	AB	C	AC	BC	ABC
1	(1)	+	–	–	+	–	+	+	–
2	a	+	+	–	–	–	–	+	+
3	b	+	–	+	–	–	+	–	+
4	ab	+	+	+	+	–	–	–	–
5	c	+	–	–	+	+	–	–	+
6	ac	+	+	–	–	+	+	–	–
7	bc	+	–	+	–	+	–	+	–
8 = 2^3	abc	+	+	+	+	+	+	+	+

$$\text{Main effect A} = \frac{1}{4}[(abc)-(bc)+(ac)-(c)+(ab)-(b)+(a)-(1)]$$

$$=\frac{1}{4}(a-1)(b+1)(c+1)$$

$$AB = \frac{1}{4}[(abc)-(bc)-(ac)+(c)+(ab)-(b)-(a)+(1)]$$

$$ABC = \frac{1}{4}[(abc)-(bc)-(ac)+(c)-(ab)+(b)+(a)-(1)]$$

Or equivalently,

$$AB = \frac{1}{4}(a-1)(b-1)(c+1)$$

$$ABC = \frac{1}{4}(a-1)(b-1)(c-1)$$

The method of representing the main effect or interaction as above is due to Yates and is very useful and quite straightforward. For example if the design is 2^4 then

$$A = (1/2^{3)}\,[(a-1)(b+1)(c+1)(d+1)]$$

$$ABC = (1/2^{3)}\,[(a-1)(b-1)(c-1)(d+1)]$$

In case of 2^n factorial experiment there will be $2^n = \nu$ treatment combinations with n main effects, ‘$\binom{n}{2}$’ first order or two factor interactions, ‘$\binom{n}{3}$’ second order or three factor interactions, ‘$\binom{n}{4}$’ third order or four factor interactions and so on, ‘$\binom{n}{r}$’ $(r-1)^{th}$ order or r factor interactions and ‘$\binom{n}{n}$’ $(n-1)^{th}$ order or n factor interaction. Using these ν treatments combinations, the experiment may be laid out using any of the suitable experimental designs for example CRD or RBD, etc.

Steps for the analysis are same as explained for the 2^2 factorial experiments. ANOVA for 2^3 factorial experiment conducted in RBD with r replications is given by

ANOVA Table

Source of Variation	**DF**	**SS**	**MSS**	**Variance Ratio (F)**
Between Replications	(r – 1)	SSR	MSSR = SSR/ (r – 1)	MSSR/MSSE
Between Treatments	$2^3 - 1 = 7$	SST	MSST =SST/7	MSST/MSSE
A	1	SSA	MSSA = SSA/1	MSSA/MSSE
B	1	SSB	MSSB = SSB/1	MSSB/MSSE
AB	1	SSAB	MSSAB=SSAB/1	MSSAB/MSSE
C	1	SSC	MSSC=SSC/1	MSSC/MSSE
AC	1	SSAC	MSSAC=SSAC/1	MSSAC/MSSE
BC	1	SSBC	MSSBC=SSBC/1	MSSBC/MSSE
ABC	1	SSABC	MSSABC=SSABC/1	MSSABC/MSSE
Error	7(r – 1)	SSE	MSSE=SSE/7(r −1)	
Total	$r.\,2^3 - 1 = 8r - 1$	TSS		

Similarly, ANOVA table for 2^n factorial experiment can be made.

Example 1: Analyse the data of a 2^3 factorial experiment conducted using a RBD with three replications. The three factors were fertilizers viz. Nitrogen (N), Phosphorus (P) and Potassium (K). The purpose of the experiment is to determine the effect of different kinds of fertilizers on potato crop yield. The yields under 8 treatment combinations for each of the three randomised blocks are given below:

Block I

npk	(1)	k	np	p	n	nk	pk
450	101	265	373	312	106	291	391

Block II

p	nk	k	np	(1)	npk	pk	n
324	306	272	338	106	449	407	89

Block III

p	npk	nk	(1)	n	k	pk	np
323	471	334	87	128	279	423	324

Solution: The data arranged in following table

Blocks ↓	**Treatment Combinations**								**Total**
	(1)	**n**	**p**	**np**	**k**	**nk**	**pk**	**npk**	
B_1	101	106	312	373	265	291	391	450	2289
B_2	106	89	324	338	272	306	407	449	2291
B_3	87	128	323	324	279	334	423	471	2369
Total	294 (T_1)	323 (T_2)	959 (T_3)	1035 (T_4)	816 (T_5)	931 (T_6)	1221 (T_7)	1370 (T_8)	6949 (G)

Grand Total = 6949

Number of observations (N) = $r \times 2^n = 3 \times 2^3 = 3 \times 8 = 24$

Correction Factor (CF) = $\frac{G^2}{N} = \frac{(6949)^2}{24} = 2012025.042$

Total Sum of Squares (TSS) = Raw Sum of Squares – CF

$$= (101^2 + 106^2 + \ldots.. + 449^2 + 471^2) - CF$$

$$= 352843.958$$

Block (Replication) Sum of Squares (SSB) = $\sum_{j=1}^{r} \frac{B_j^2}{2^3} - CF$

$$= \frac{[(2289)^2 + (2291)^2 + (2369)^2]}{8} - CF$$

$$= 520.333$$

Treatment Sum of Squares (SST) = $\sum_{i=1}^{v} \frac{T_i^2}{r} - CF$

$$= \frac{[(294)^2 + (323)^2 + (959)^2 + (1035)^2 + (816)^2 + (931)^2 + (1221)^2 + (1370)^2]}{3} - CF$$

$$= \frac{7082029}{3} - 2012025.042$$

$$= 348651.291$$

Error Sum of Squares (SSE) = TSS – SSB – SST

= 352843.958 - 520.333 -348651.291

= 3672.334

Main effects totals and interactions totals are obtained as follows:

[N] = [npk] - [pk] + [nk] – [k] + [np] – [p] + [n] – [1] = 369

[P] = [npk] + [pk] – [nk] – [k] + [np] + [p] – [n] – [1] = 2221

[K] = [npk] + [pk] + [nk] + [k] - [np] – [p] – [n] – [1] =1727

[NP] = [npk] – [pk] – [nk] + [k] + [np]-[p] – [n] + [1] = 81

[NK] = [npk] – [pk] + [nk] – [k] – [np] + [p] – [n] + [1] = 159

[PK] = [npk] + [pk] – [nk] – [k]-[np] + [p] + [n] + [1] = -533

[NPK] = [npk] – [pk] – [nk] + [k]-[np] + [p] + [n] – [1] = -13

Factorial effects = $\dfrac{\text{Fcatorial effect total}}{r.2^{n-1}(=12)}$

Factorial effect Sum of Squares = $\dfrac{(\text{Fcatorial effect total})^2}{r.2^{n}(=24)}$

Here factorial effects,

N = 30.750, P = 185.083, K = 143.917, NP = 6.750, NK = 13.250,

PK = − 44.417, NPK = − 1.083

Sum of Squares due to N = 5673.375

Sum of Squares due to P = 202235.042

Sum of Squares due to K = 124272.042

Sum of Squares due to NP = 273.375

Sum of Squares due to NK = 1053.375

Sum of Squares due to PK = 11837.042

Sum of Squares due to NPK = 7.042

Mean sum of squares is obtained by dividing the sum of squares by their respective df.

ANOVA Table

Source of Variation	DF	SS	MSS	Variance Ratio (F)
Between Replications	$(r-1) = 2$	520.333	260.167	0.9918
Between Treatments	$2^3 - 1 = 7$	348651.291	49807.327	189.8797
N	$(2-1) = 1$	5673.375	5673.375	21.6285
P	1	205535.042	205535.042	783.5582
K	1	124272.042	124272.042	473.7606
NP	1	273.375	273.375	1.0422
NK	1	1053.375	1053.375	4.0158
PK	1	11837.041	11837.041	45.1262
NPK	1	7.041	7.041	0.0268
Error	$7(r-1) = 14$	3672.337	262.310	
Total	$r.\,2^3 - 1 = 23$	352843.958		

Standard Error of difference between main effect means

$$= \sqrt{\frac{\text{MSSE}}{r.2^{n-2}}} = 8.098$$

SE of difference between N means at same level of P or K

= SE of difference between P (or K) means at same level of N

= SE of difference between P means at same level of K

= SE of difference between K means at same level of P

$$= \sqrt{\frac{\text{MSSE}}{r.2^{n-3}}} = 11.4523,$$

And $t_{0.05}$ at 14 df = 2.145. Accordingly Critical difference (CD) can be calculated.

E1) An experiment was planned to study the effect of Sulphate, Potash and Super Phosphate on the yield of potatoes. All the combinations of 2 levels of Super Phosphate [0 cent (p_0) and 5 cent (p_1)/ acre] and two levels of Sulphate and Potash [0 cent (k_0) and 5 cent (k_1)/acre] were studied in a randomised block design with 4 replications each. The (1/70) yields [lb. per plot = (1/70) acre] obtained are given in table below:

Blocks	Yields (lbs per plot)			
I	(1) 23	k 25	p 22	kp 38
II	p 40	(1) 26	k 36	kp 38
III	(1) 29	k 20	pk 30	p 20
IV	kp 34	k 31	p 24	(1) 28

Analyse the data and give your conclusions.

12.4.1 Advantages and Disadvantages of Factorial Experiments

Advantages

1. To investigate the interactions of factors. Single factor experiments provide a disorderly and incomplete picture.
2. In exploratory work for quick determination of which factors are independent and can therefore be more fully analyzed in separate experiments.
3. To lead to recommendations that must extend over a wide range of conditions.

Disadvantages

1. Large numbers of combination are required to study several factors at several levels and need a large sized experiment: 7 factors at 3 levels requires 2187 combinations.
2. Large numbers of factor, complicate the interpretation of high order interactions.

12.5 SUMMARY

In this unit, we have discussed:

1. The factorial experiments;
2. The layout of 2^2 factorial experiments;
3. The statistical analysis of 2^2 factorial experiments;
4. The layout of 2^3 factorial experiments, and
5. The statistical analysis of 2^3 factorial experiments.

12.6 SOLUTIONS/ANSWERS

E1) Correction Factor (CF) = 0

Raw Sum of Squares = 660

Total Sum of Squares = 660

Block Sum of Squares = 232.5

Treatment Sum of Squares = 198

Error Sum of Squares = 229.5

Sum of Squares due to k = 100

Sum of Squares due to p = 49

Sum of Squares due to kp = 49

ANOVA Table

Source of Variation	DF	SS	MSS	Variance Ratio	
				Calculated	Tabulated
Treatments	3	198	77.5	3.04	3.86
Blocks	3	232.5	66	2.59	3.86
k	1	100	100	3.92	5.12
p	1	49	49	1.92	5.12
kp	1	49	49	1.92	5.12
Error	6	229.5	25.5		
Total	11	660			

Calculated value of F is less than the corresponding tabulated value, so there are no significant main or interaction effects present in the experiment. The blocks as well as treatments do not differ considerably.

TABLE: The F Table

Value of F Corresponding to 5% (Normal Type) and 1% (**Bold Type**) of the Area in the Upper Tail

Degrees of Freedom: (Denominator)	1	2	3	4	5	6	7	8	9	10	11	12	14	16	20	24	30	∞
	Degrees of Freedom (Numerator)																	
1	161	200	216	225	230	234	237	239	241	242	243	244	245	246	248	249	250	254
	4,052	**4,999**	**5,403**	**5,625**	**5,764**	**5,859**	**5,928**	**5,981**	**6,022**	**6,056**	**6,082**	**6,106**	**6,142**	**6,169**	**6,208**	**6,234**	**6,258**	**6,366**
2	18.51	19.00	19.16	19.25	19.30	19.33	19.36	19.37	19.38	19.39	19.40	19.41	19.42	19.43	19.44	19.45	19.46	19.50
	98.49	**99.00**	**99.17**	**99.25**	**99.30**	**99.33**	**99.34**	**99.36**	**99.38**	**99.40**	**99.41**	**99.42**	**99.43**	**99.44**	**99.45**	**99.46**	**99.47**	**99.50**
3	10.13	9.55	9.28	9.12	9.01	8.94	8.88	8.84	8.81	8.78	8.76	8.74	8.71	8.69	8.66	8.64	8.62	8.5
	34.12	**30.82**	**29.46**	**28.71**	**28.24**	**27.91**	**27.67**	**27.49**	**27.34**	**27.23**	**27.13**	**27.05**	**26.92**	**26.83**	**26.69**	**26.60**	**26.50**	**26.12**
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.93	5.91	5.87	5.84	5.80	5.77	5.74	5.63
	22.20	**18.00**	**16.69**	**15.98**	**15.52**	**15.21**	**14.98**	**14.80**	**14.66**	**14.54**	**14.45**	**14.37**	**14.24**	**14.15**	**14.02**	**13.93**	**13.83**	**13.46**
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.78	4.74	4.70	4.68	4.64	4.60	4.56	4.53	4.40	4.36
	16.26	**13.27**	**12.06**	**11.39**	**10.97**	**10.67**	**10.45**	**10.27**	**10.15**	**10.05**	**9.96**	**9.89**	**9.77**	**9.68**	**9.55**	**9.47**	**9.38**	**9.02**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.96	3.92	3.87	3.84	3.81	3.67
	13.74	**10.92**	**9.78**	**9.15**	**8.75**	**8.47**	**8.26**	**8.10**	**7.98**	**7.87**	**7.79**	**7.72**	**7.60**	**7.52**	**7.39**	**7.31**	**7.23**	**6.88**
7	5.59	4.47	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.63	3.60	3.57	3.52	3.49	3.44	3.41	3.38	3.23
	12.25	**9.55**	**8.45**	**7.85**	**7.46**	**7.19**	**7.00**	**6.84**	**6.71**	**6.62**	**6.54**	**6.47**	**6.35**	**6.27**	**6.15**	**6.07**	**5.98**	**5.65**
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.34	3.31	3.28	3.23	3.20	3.15	3.12	3.08	2.93
	11.26	**8.65**	**7.59**	**7.01**	**6.63**	**6.37**	**6.19**	**6.03**	**5.91**	**5.82**	**5.74**	**5.67**	**5.56**	**5.48**	**5.36**	**5.28**	**5.20**	**4.86**
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.13	3.10	3.07	3.02	2.98	2.93	2.90	2.86	2.71
	10.56	**8.02**	**6.99**	**6.42**	**6.06**	**5.80**	**5.62**	**5.47**	**5.35**	**5.26**	**5.18**	**5.11**	**5.00**	**4.92**	**4.80**	**4.73**	**4.64**	**4.31**
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.97	2.94	2.91	2.86	2.82	2.77	2.74	2.70	2.54
	10.04	**7.56**	**6.55**	**5.99**	**5.64**	**5.39**	**5.21**	**5.06**	**4.95**	**4.85**	**4.78**	**4.71**	**4.60**	**4.52**	**4.41**	**4.33**	**4.25**	**3.91**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.86	2.82	2.79	2.74	2.70	2.65	2.61	2.57	2.40
	9.65	**7.20**	**6.22**	**5.67**	**5.32**	**5.07**	**4.88**	**4.74**	**4.63**	**4.54**	**4.46**	**4.40**	**4.29**	**4.21**	**4.10**	**4.02**	**3.94**	**3.60**
12	4.75	3.88	3.49	3.26	3.11	3.00	2.92	2.85	2.80	2.76	2.72	2.69	2.64	2.60	2.54	2.50	2.46	2.30
	9.33	**6.93**	**5.95**	**5.41**	**5.06**	**4.82**	**4.65**	**4.50**	**4.39**	**4.30**	**4.22**	**4.16**	**4.05**	**3.98**	**3.86**	**3.78**	**3.70**	**3.36**
13	4.67	3.80	3.41	3.18	3.02	2.92	2.84	2.77	2.72	2.67	2.63	2.60	2.55	2.51	2.46	2.42	2.38	2.21
	9.07	**6.70**	**5.74**	**5.20**	**4.86**	**4.62**	**4.44**	**4.30**	**4.19**	**4.10**	**4.02**	**3.96**	**3.85**	**3.78**	**3.67**	**3.59**	**3.51**	**3.16**
14	4.60	3.74	3.34	3.11	2.96	2.85	2.77	2.70	2.65	2.60	2.56	2.53	2.48	2.44	2.39	2.35	2.31	2.13
	8.86	**6.51**	**5.56**	**5.03**	**4.69**	**4.46**	**4.28**	**4.14**	**4.03**	**3.94**	**3.86**	**3.80**	**3.70**	**3.62**	**3.51**	**3.43**	**3.34**	**3.00**
15	4.54	3.68	3.29	3.06	2.90	2.79	2.70	2.64	2.59	2.55	2.51	2.48	2.43	2.39	2.33	2.29	2.25	2.07
	8.68	**6.36**	**5.42**	**4.89**	**4.56**	**4.32**	**4.14**	**4.00**	**3.89**	**3.80**	**3.73**	**3.67**	**3.56**	**3.48**	**3.36**	**3.29**	**3.20**	**2.87**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.45	2.42	2.37	2.33	2.28	2.24	2.20	2.01
	8.53	**6.23**	**5.29**	**4.77**	**4.44**	**4.20**	**4.03**	**3.89**	**3.78**	**3.69**	**3.61**	**3.55**	**3.45**	**3.37**	**3.25**	**3.18**	**3.10**	**2.75**
17	4.45	3.59	3.20	2.96	2.81	2.70	2.62	2.55	2.50	2.45	2.41	2.38	2.33	2.29	2.23	2.19	2.15	1.96
	8.40	**6.11**	**5.18**	**4.67**	**4.34**	**4.10**	**3.93**	**3.79**	**3.68**	**3.95**	**3.52**	**3.45**	**3.35**	**3.27**	**3.16**	**3.08**	**3.00**	**2.65**
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.29	2.25	2.19	2.15	2.11	1.92
	8.28	**6.01**	**5.09**	**4.58**	**4.25**	**4.01**	**3.85**	**3.71**	**3.60**	**3.51**	**3.44**	**3.37**	**3.27**	**3.19**	**3.07**	**3.00**	**2.91**	**2.57**
19	4.38	3.52	3.13	2.90	2.74	2.63	2.55	2.48	2.43	2.38	2.34	2.31	2.26	2.21	2.15	2.11	2.07	1.88
	8.18	**5.93**	**5.01**	**4.50**	**4.17**	**3.94**	**3.77**	**3.63**	**3.52**	**3.43**	**3.36**	**3.30**	**3.19**	**3.12**	**3.00**	**2.92**	**2.84**	**2.49**
20	4.35	3.49	3.10	2.87	2.71	2.60	2.52	2.45	2.40	2.35	2.31	2.28	2.23	2.18	2.12	2.08	2.04	1.84
	8.10	**5.85**	**4.94**	**4.43**	**4.10**	**3.87**	**3.71**	**3.56**	**3.45**	**3.37**	**3.30**	**3.23**	**3.13**	**3.05**	**2.94**	**2.86**	**2.77**	**2.42**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.20	2.15	2.09	2.05	2.00	1.81
	8.02	**5.78**	**4.87**	**4.37**	**4.04**	**3.81**	**3.65**	**3.51**	**3.40**	**3.31**	**3.24**	**3.17**	**3.07**	**2.99**	**2.88**	**2.80**	**2.72**	**2.36**

TABLE (Continued)

Degrees of Freedom: Denominator	1	2	3	4	5	6	7	8	9	10	11	12	14	16	20	24	30	∞
22	4.30	3.44	3.05	2.82	2.66	2.55	2.47	2.40	2.35	2.30	2.23	2.23	2.18	2.13	2.07	2.03	1.98	1.78
	7.94	**5.72**	**4.82**	**4.31**	**3.99**	**3.76**	**3.59**	**3.45**	**3.35**	**3.26**	**3.18**	**3.12**	**3.02**	**2.94**	**2.83**	**2.75**	**2.67**	**2.31**
23	4.28	3.42	3.03	2.80	2.64	2.53	2.45	2.38	3.32	2.28	2.24	2.20	2.14	2.10	2.04	2.00	1.96	1.76
	7.88	**5.66**	**4.76**	**4.26**	**3.94**	**3.71**	**3.54**	**3.41**	**3.30**	**3.21**	**3.14**	**3.07**	**2.97**	**2.89**	**2.78**	**2.70**	**2.62**	**2.26**
24	4.26	3.40	3.01	2.78	2.62	2.51	2.43	2.36	2.30	2.26	2.22	2.18	2.13	2.09	2.02	1.98	1.94	1.73
	7.82	**5.61**	**4.72**	**4.22**	**3.90**	**3.67**	**3.50**	**3.36**	**3.25**	**3.17**	**3.09**	**3.03**	**2.93**	**2.85**	**2.74**	**2.66**	**2.58**	**2.21**
25	4.24	3.38	2.99	2.76	2.60	2.49	2.41	2.34	2.28	2.24	2.20	2.16	2.11	2.06	2.00	1.96	1.92	1.71
	7.77	**5.57**	**4.68**	**4.18**	**3.86**	**3.63**	**3.46**	**3.32**	**3.21**	**3.13**	**3.05**	**2.99**	**2.89**	**2.81**	**2.70**	**2.62**	**2.54**	**2.17**
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.10	2.05	1.99	1.95	1.90	1.69
	7.72	**5.53**	**4.64**	**4.14**	**3.82**	**3.59**	**3.42**	**3.29**	**3.17**	**3.09**	**3.02**	**2.96**	**2.86**	**2.77**	**2.66**	**2.58**	**2.50**	**2.13**
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.30	2.25	2.20	2.16	2.13	2.08	2.03	1.97	1.93	1.88	1.67
	7.68	**5.49**	**4.60**	**4.11**	**3.79**	**3.56**	**3.39**	**3.26**	**3.14**	**3.06**	**2.98**	**2.93**	**2.83**	**2.74**	**2.63**	**2.55**	**2.47**	**2.10**
28	4.20	3.34	2.95	2.71	2.56	2.44	2.36	2.29	2.24	2.19	2.15	2.12	2.06	2.02	1.96	1.91	1.87	1.65
	7.64	**5.45**	**4.57**	**4.07**	**3.76**	**3.53**	**3.36**	**3.23**	**3.11**	**3.03**	**2.95**	**2.90**	**2.80**	**2.71**	**2.60**	**2.52**	**2.44**	**2.06**
29	4.18	3.33	2.93	2.70	2.54	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.05	2.00	1.94	1.90	1.85	1.64
	7.60	**5.42**	**4.54**	**4.04**	**3.73**	**3.50**	**3.33**	**3.20**	**3.08**	**3.00**	**2.92**	**2.87**	**2.77**	**2.68**	**2.57**	**2.49**	**2.41**	**2.03**
30	4.17	3.32	2.92	2.69	2.53	2.42	2.34	2.27	2.21	2.16	2.12	2.09	2.04	1.99	1.93	1.89	1.84	1.62
	7.56	**5.39**	**4.51**	**4.02**	**3.70**	**3.47**	**3.30**	**3.17**	**3.06**	**2.98**	**2.90**	**2.84**	**2.74**	**2.66**	**2.55**	**2.47**	**2.38**	**2.01**
∞	3.84	2.99	2.60	2.37	2.21	2.09	2.01	1.94	1.88	1.83	1.79	1.75	1.69	1.64	1.57	1.52	1.46	1.00
	6.64	**4.60**	**3.78**	**3.32**	**3.02**	**2.80**	**2.64**	**2.51**	**2.41**	**2.32**	**2.24**	**2.18**	**2.07**	**1.99**	**1.87**	**1.79**	**1.69**	**1.00**