UNIT 12 CHI-SQUARE AND F-TESTS

Structure

12.1 Introduction
Objectives
12.2 Testing of Hypothesis for Population Variance Using χ^2-Test
12.3 Testing of Hypothesis for Two Population Variances Using F-Test
12.4 Summary
12.5 Solutions / Answers

12.1 INTRODUCTION

Recall from the previous unit when we test the hypothesis about the difference of means of two populations, t-test needs an assumption of equality of variances of two populations under study. Other than this there are situations where we want to test the hypothesis about the variances of two populations. For example, an economist may want to test whether the variability in incomes differ in two population. In such situations, we use F-test when the populations under study follow the normal distributions.

Similarly, there are many other situations where we need to test the hypothesis about the hypothetical or specified value of the variance of the population under study. For example, the manager of the electric bulbs company would probably be interested whether or not the variability in the life of bulbs is within acceptable limits, the product controller of a milk company may be interested in the variance of the amount of fat in the whole milk processed by the company is no more than the specified level. In such situations, we use χ^2-test when the population under study follows the normal distribution.

This unit is divided into five sections. Section 12.1 is described the need of χ^2 and F-tests. χ^2-test for testing the hypothesis about the population variance is discussed in Section 12.2. And F-test for equality of variances of two populations is discussed in Section 12.3. Unit ends by providing summary of what we have discussed in this unit in Section 12.4 and solution of exercises in Section 12.5.

Objectives

After studying this unit, you should be able to:

- describe the testing of hypothesis for population variance; and
- explain the testing of hypothesis for two population variances.

12.2 TESTING OF HYPOTHESIS FOR POPULATION VARIANCE USING χ^2-TEST

In Section 11.3 of previous unit, we have discussed testing of hypothesis for population mean when the characteristic under study follows the normal distribution but when analysing quantitative data, it is often important to draw conclusion about the average as well as the variability of a characteristic under study. For example, if a company manufactured the electric bulbs then the manager of the company would probably be interested in determining the average life of the bulbs and also determining whether or not the variability in the life of bulbs is within acceptable limits, the product controller of a milk company may be interested in the variance of the amount of fat in the whole milk processed by the company is no more than the specified level, etc.

The procedure of testing a hypothesis for population variance or standard deviation is similar to the testing of population mean. The basic difference is that here we use chi-square test instead of t-test because here the sampling distribution of test statistic follows the chi-square distribution.

Assumptions

This test works under the following assumptions:

(i) The characteristic under study follows normal distribution. In other words, populations from which random sample is drawn should be normal with respect to the characteristic of interest.

(ii) Sample observations are random and independent.

Let $X_1, X_2, ..., X_n$ be a random sample of size n drawn from a normal population with mean μ and variance σ^2, where μ and σ^2 are unknown.

The general procedure of this test is explained below in detail:

As we are doing so far in all tests, first step in hypothesis testing problems is to setup null and alternative hypotheses. Here, we want to test the claim about the hypothesized or specified value σ_0^2 of population variance σ^2 so we can take our null and alternative hypotheses as

$$H_0 : \sigma^2 = \sigma_0^2 \text{ and } H_1 : \sigma^2 \neq \sigma_0^2 \qquad [\text{for two-tailed test}]$$

or

$$\left. \begin{array}{l} H_0 : \sigma^2 \leq \sigma_0^2 \text{ and } H_1 : \sigma^2 > \sigma_0^2 \\ H_0 : \sigma^2 \geq \sigma_0^2 \text{ and } H_1 : \sigma^2 < \sigma_0^2 \end{array} \right\} \qquad [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic χ^2 is given by

$$\chi^2 = \frac{\sum \left(X_i - \bar{X} \right)^2}{\sigma_0^2} = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{(n-1)} \text{ under } H_0 \qquad \text{... (1)}$$

where, $S^2 = \dfrac{1}{n-1} \sum \left(X - \bar{X} \right)^2$

Here, the test statistic χ^2 follows chi square distribution with $(n-1)$ degrees of freedom as we have discussed in Unit 3 of this course.

After substituting values of n, S and σ_0^2, we get calculated value of test statistic. Let χ^2_{cal} be the calculated value of test statistic χ^2.

Obtain the critical value(s) or cut-off value(s) in the sampling distribution of the test statistic χ^2 and construct rejection (critical) region of size α. The critical value of the test statistic χ^2 for various df and different level of significance α are given in **Table III** of the Appendix given at the end of the Block 1of this course.

After doing all the calculation discussed above, we have to take the decision about rejection or non-rejection of the null hypothesis. The procedure of taking the decision about the null hypothesis is explained is the next page:

For one-tailed test:

Case I: When $H_0: \sigma^2 \le \sigma_0^2$ and $H_1: \sigma^2 > \sigma_0^2$ (right-tailed test)

In this case, the rejection (critical) region falls under the right tail of the probability curve of the sampling distribution of test statistic χ^2. Suppose $\chi^2_{(\nu),\alpha}$ is the critical value at α level of significance where, $\nu = n - 1$, so entire region greater than or equal to $\chi^2_{(\nu),\alpha}$ is the rejection region and less than $\chi^2_{(\nu),\alpha}$ is the non-rejection region as shown in Fig. 12.1.

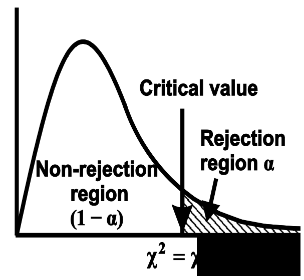

Fig. 12.1

If $\chi^2_{cal} \ge \chi^2_{(\nu),\alpha}$, that means calculated value of test statistic lies in the rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that sample data provides us sufficient evidence against the null hypothesis and there is a significant difference between hypothesized value and observed value of the population variance σ^2.

If $\chi^2_{cal} < \chi^2_{(\nu),\alpha}$, that means calculated value of test statistic lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that the sample data fails to provide us sufficient evidence against the null hypothesis and the difference between hypothesized value and observed value of the population variance σ^2 due to fluctuation of sample.

Case II: When $H_0: \sigma^2 \ge \sigma_0^2$ and $H_1: \sigma^2 < \sigma_0^2$ (left-tailed test)

In this case, the rejection (critical) region falls under the left tail of the probability curve of the sampling distribution of test statistic χ^2. Suppose $\chi^2_{(\nu),(1-\alpha)}$ is the critical value at α level of significance then entire region less than or equal to $\chi^2_{(\nu),(1-\alpha)}$ is the rejection (critical) region and greater than $\chi^2_{(\nu),(1-\alpha)}$ is the non-rejection region as shown in Fig. 12.2.

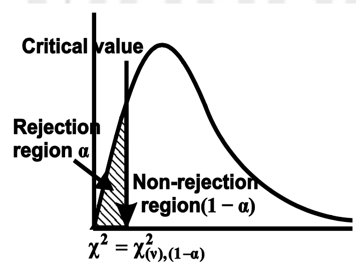

Fig. 12.2

If $\chi^2_{cal} \le \chi^2_{(\nu),(1-\alpha)}$, that means calculated value of test statistic lies in rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance.

For $\chi^2_{cal} > \chi^2_{(\nu),(1-\alpha)}$, that means calculated value of test statistic lies in the non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

For two-tailed test:

When $H_0: \sigma^2 = \sigma_0^2$ and $H_1: \sigma^2 \ne \sigma_0^2$

In this case, the rejection region falls under both tails of the probability curve of sampling distribution of the test statistic χ^2 and half the area (α) i.e. α/2 of rejection (critical) region lies at left tail and other half on the right tail. Suppose $\chi^2_{(\nu),(1-\alpha/2)}$ and $\chi^2_{(\nu),\alpha/2}$ are the two critical values at the left-tailed and right-tailed respectively on pre-fixed α-level of significance. Therefore, entire region less than or equal to $\chi^2_{(\nu),(1-\alpha/2)}$ and greater than or equal to $\chi^2_{(\nu),\alpha/2}$ are the rejection

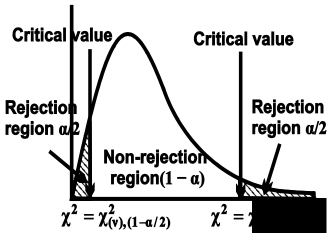

Fig. 12.3

(critical) regions and between $\chi^2_{(\nu),(1-\alpha/2)}$ and $\chi^2_{(\nu),\alpha/2}$ is the non-rejection region as shown in Fig. 12.3.

If $\chi^2_{cal} \geq \chi^2_{(\nu),\alpha/2}$ or $\chi^2_{cal} \leq \chi^2_{(\nu),(1-\alpha/2)}$, that means calculated value of test statistic χ^2 lies in rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance.

If $\chi^2_{(\nu),(1-\alpha/2)} < \chi^2_{cal} < \chi^2_{(\nu),\alpha/2}$, that means calculated value of test statistic χ^2 lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

Procedure of taking the decision about the null hypothesis on the basis of p-value:

You have done it in so many test, it has become a routine thing for you. You know that to take the decision about the null hypothesis on the basis of p-value, the p-value is compared with level of significance (α) and if p-value is less than or equal to α then we reject the null hypothesis and if the p-value is greater than α we do not reject the null hypothesis.

For χ^2-test, p-value is defined as:

For one-tailed test:

For $H_1: \sigma^2 > \sigma_0^2$ (right-tailed test)

$$\text{p-value} = P\left[\chi^2 \geq \chi^2_{cal}\right]$$

For $H_1: \sigma^2 < \sigma_0^2$ (left-tailed test)

$$\text{p-value} = P\left[\chi^2 \leq \chi^2_{cal}\right]$$

For two-tailed test: $H_1: \sigma^2 \neq \sigma_0^2$

For two-tailed test the p-value is approximated as

$$\text{p-value} = 2P\left[\chi^2 \geq \chi^2_{cal}\right]$$

The p-value for χ^2-test can be obtained with the help of the **Table-III (χ^2-table)** given in the Appendix at the end of Block 1 of this course. Similar to t-test, this table gives the χ^2 values corresponding to the standard values of α such as 0.995, 0.99, 0.10, 0.05, 0.025, 0.01, etc only. therefore, the exact p-value is not obtained with the help of this table and we can approximate the p-value for this test.

For example, if test is right-tailed and calculated (observed) value of test statistic χ^2 is 25.10 with 12 df then p-value is calculated as:

Since test statistic is based on the 12 df therefore, we use row for 12 df in the χ^2-table and move across this row to find the values in which calculated χ^2-value falls. Since calculated χ^2-value falls between 23.24 and 26.22, corresponding to one-tailed area $\alpha = 0.025$ and 0.01 respectively, therefore p-value lies between 0.01 and 0.025, that is,

$$0.01 < \text{p-value} < 0.025$$

If in the above example the test is two-tailed then the two values 0.01 and 0.005 would be doubled for p-value, that is,

$$2 \times 0.01 = 0.02 < \text{p-value} < 0.05 = 2 \times 0.025$$

Note 1: With the help of computer packages and softwares such as SPSS, SAS, MINITAB, EXCEL, etc. we can find the exact p-value for χ^2-test.

Let us do some examples to become more user friendly with the test explained above.

Example 1: The variance of a certain dimension article produced by a machine is 7.2 over a long period. A random sample of 20 articles gave a variance 8. Is it justifiable to conclude that variability has increased at 5% level of significance assuming that the measurement of dimension article is normally distributed?

Solution: Here, we are given that

Sample size = n = 20

Sample variance = S^2 = 8

Specified value of population variance under test = $\sigma_0^2 = 7.2$

Here, we want to test that variability of dimension article produced by a machine has increased. Since variability is measured in terms of variance (σ^2) so our claim is $\sigma^2 > 7.2$ and its complement is $\sigma^2 \le 7.2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \sigma^2 \le \sigma_0^2 = 7.2$$

$$H_1 : \sigma^2 > 7.2 \qquad [\text{variability of dimension article has increased}]$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Here, we want to test the hypothesis about the population variance and sample size is small n = 20(< 30). Also we are given that measurement of dimension article follows normal distribution so we can go for χ^2 test for population variance.

So, test statistic is given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)} \text{ under } H_0$$

$$= \frac{19 \times 8}{7.2} = 21.11$$

The critical (tabulated) value of test statistic χ^2 for right-tailed test corresponding (n-1) = 19 df at 5% level of significance is $\chi^2_{(n-1),\alpha} = \chi^2_{(19),0.05} = 30.14$.

Since calculated value of test statistic (= 21.11) is less than the critical (tabulated) value (= 30.14), that means calculated value of test statistic lies in non-rejection region as shown in Fig. 12.4, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject our claim at 5% level of significance.

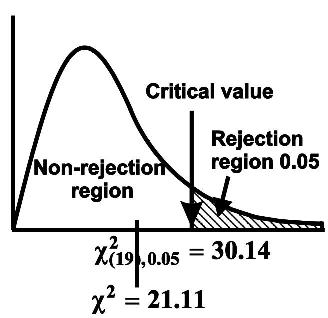

Fig. 12.4

Decision according to p-value:

Since test statistic is based on 19 df therefore, we use row for 19 df in the χ^2-table and move across this row to find the values in which calculated χ^2-value falls. Since calculated χ^2-value falls between 11.65 and 27.20, corresponding to one-tailed area α = 0.90 and 0.10 respectively therefore p-value lies between 0.10 and 0.90, that is,

$$0.10 < \text{p-value} < 0.90$$

Since p-value is greater than $\alpha (= 0.05)$ so we do not reject the null hypothesis at 5% level of significance.

Thus, we conclude that sample provides us sufficient evidence against the claim so we may assume that the variability of dimension article produced by a machine is not increased.

Example 2: The12 measurements of the same object on an instrument are given below:

1.6, 1.5, 1.3, 1.5, 1.7, 1.6, 1.5, 1.4, 1.6, 1.3, 1.5, 1.5

If the measurement of the instrument follows normal distribution then carry out the test at 1% level of significance that variance in the measurement of the instrument is less than 0.016.

Solution: Here, we are given

Sample size = n = 12

Specified value of population variance under test $\sigma_0^2 = 0.016$

Here, we want to test that the variance (σ^2) in the measurements of the instrument is less than 0.016. So our claim is $\sigma^2 < 0.016$ and its complement is $\sigma^2 \geq 0.016$. Since complement contains the equality sign so we can take complement as null hypothesis and claim as the alternative hypothesis. Thus,

$$H_0 : \sigma^2 \geq \sigma_0^2 = 0.016 \text{ and } H_1 : \sigma^2 < 0.016$$

Since the alternative hypothesis is left-tailed so the test is left-tailed test.

Here, we want to test the hypothesis about the population variance and sample size is small n = 12(< 30). Also we are given that the measurement of the instrument follows normal distribution so we can go for χ^2-test for population variance.

For testing the null hypothesis, the test statistic χ^2 is given by

$$\chi^2 = \frac{\sum \left(X_i - \bar{X} \right)^2}{\sigma_0^2} \sim \chi^2_{(n-1)} \text{ under } H_0 \qquad \text{... (2)}$$

Calculation for $\sum \left(X_i - \bar{X} \right)^2$:

X	$(X-\bar{X})$	$(X-\bar{X})^2$
1.6	0.1	0.01
1.5	0	0
1.3	-0.2	0.04
1.5	0	0
1.7	0.2	0.04
1.6	0.1	0.01
1.5	0	0
1.4	-0.1	0.01
1.6	0.1	0.01
1.3	-0.2	0.04
1.5	0	0
1.5	0	0
$\sum X = 18$	0	$\sum (X-\bar{X})^2 = 0.16$

From above calculation, we have

$$\bar{X} = \frac{1}{n}\sum X = \frac{1}{12}\times 18 = 1.5$$

Putting the values of $\sum(X-\bar{X})^2$ and σ_0^2 in equation (2), we have

$$\chi^2 = \frac{\sum(X_i-\bar{X})^2}{\sigma_0^2} = \frac{0.16}{0.016} = 10$$

The critical value of test statistic χ^2 for left-tailed test corresponding (n - 1) = 11 df at 5% level of significance is $\chi^2_{(n-1),(1-\alpha)} = \chi^2_{(11),0.95} = 4.57$.

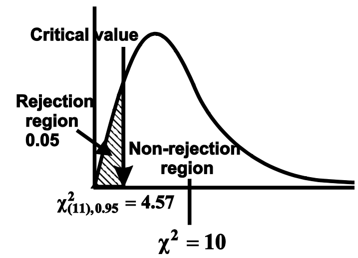

Fig. 12.5

Since calculated value of test statistic (= 10) is greater than the critical value (= 4.57), that means calculated value of test statistic lies in non-rejection region as shown in Fig. 12.5 so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that sample provide sufficient evidence against the claim so the variance in the measurement of the instrument is not less than 0.016.

In the same way, you can try the following exercises.

E1) An ambulance agency claims that the standard deviation in the length of serving times is less than 15 minutes. Investigator suspects that this claim is wrong and takes a random sample of 20 serving times which has a standard deviation of 17 minutes. Assume that the service time of the ambulance follows normal distribution. Test at α = 0.01, is there enough evidence to reject the agency's claim?

E2) A cigarette manufacturer claims that the variance of nicotine content of its cigarettes is 0.62. Nicotine content is measured in milligrams and is normally distributed. A sample of 25 cigarettes has a variance of 0.65. Test the manufacturer's claim at 5% level of significance.

12.5 TESTING OF HYPOTHESIS FOR TWO POPULATION VARIANCES USING F-TEST

In Section 12.1 we have already mentioned that before applying t-test for difference of two population means, one of the requirements is to check the equality of variances of two populations. This assumption can be checked with the help of F-test for two population variances. This F-test is also important in a number of contexts. For example, an economist may want to test whether the variability in incomes differ in two populations, a quality controller may want to test whether the quality of the product is changing over time, etc.

Some author uses this test as the name "homoscedastic test". Because two populations with equal variances are called homoscedastic. This word is derived from two wards, homo means "the same" and scedastic means "variability".

Assumptions

The assumptions for F-test for testing the variances of two populations are:

1. The populations from which the samples are drawn must be normally distributed.
2. The samples must be independent of each other.

Now, we come to the general procedure of this test.

Let $X_1, X_2, ..., X_{n_1}$ be a random sample of size n_1 from a normal population with mean μ_1 and variance σ_1^2. Similarly, $Y_1, Y_2, ..., Y_{n_2}$ be a random sample of size n_2 from another normal population with mean μ_2 and variance σ_2^2. Here, we want to test the hypothesis about the two population variances so we can take our alternative null and hypotheses as

$$H_0 : \sigma_1^2 = \sigma_2^2 = \sigma^2 \text{ and } H_1 : \sigma_1^2 \neq \sigma_2^2 \qquad [\text{for two-tailed test}]$$

or

$$\left.\begin{array}{l} H_0 : \sigma_1^2 \leq \sigma_2^2 \text{ and } H_1 : \sigma_1^2 > \sigma_2^2 \\ H_0 : \sigma_1^2 \geq \sigma_2^2 \text{ and } H_1 : \sigma_1^2 < \sigma_2^2 \end{array}\right\} \qquad [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic F is given by

$$F = \frac{S_1^2}{S_2^2} \sim F_{(n_1-1,\, n_2-1)} \quad \text{under } H_0 \qquad \ldots (3)$$

where, $S_1^2 = \frac{1}{n_1-1}\sum\left(X-\bar{X}\right)^2$ and $S_2^2 = \frac{1}{n_2-1}\sum\left(Y-\bar{Y}\right)^2$.

For computational simplicity, we can also write

$$S_1^2 = \frac{1}{n_1-1}\left(\sum X^2 - \frac{\left(\sum X\right)^2}{n_1}\right) \text{ and } S_2^2 = \frac{1}{n_2-1}\left(\sum Y^2 - \frac{\left(\sum Y\right)^2}{n_2}\right)$$

The test statistic F follows F-distribution with $\nu_1 = (n_1 - 1)$ and $\nu_2 = (n_2 - 1)$ degrees of freedom as discussed in Unit 4 of this course.

After substituting the values of S_1^2 and S_2^2, we get calculated value of test statistic. Let F_{cal} be calculated value of test statistic F.

Obtain the critical value(s) or cut-off value(s) in the sampling distribution of the test statistic F and construct rejection (critical) region of size α. The critical values of the test statistic F for various df and different level of significance α are given in **Table IV (F-table)** of the Appendix at the end of Block 1of this course.

After doing all calculations discussed above, we have to take the decision about rejection or non rejection of the null hypothesis. This is explained below:

In case of one-tailed test:

Case I: When $H_0 : \sigma_1^2 \leq \sigma_2^2$ and $H_1 : \sigma_1^2 > \sigma_2^2$ (right-tailed test)

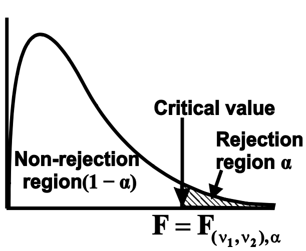

Fig. 12.6

In this case, the rejection (critical) region falls at the right side of the probability curve of the sampling distribution of test statistic F. Suppose $F_{(\nu_1,\nu_2),\alpha}$ is the critical value of test statistic F with ($\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$) df at α level of significance so entire region greater than or equal to $F_{(\nu_1,\nu_2),\alpha}$ is the rejection (critical) region and less than $F_{(\nu_1,\nu_2),\alpha}$ is the non-rejection region as shown in Fig. 12.6.

If $F_{cal} \geq F_{(\nu_1,\nu_2),\alpha}$, that means calculated value of test statistic lies in rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that samples data provide us sufficient evidence against the null hypothesis and there is a significant difference between population variances.

If $F_{cal} < F_{(\nu_1, \nu_2), \alpha}$, that means calculated value of test statistic lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance. Therefore, we conclude that the samples data fail to provide us sufficient evidence against the null hypothesis and the difference between population variances due to fluctuation of sample.

Case II: When $H_0 : \sigma_1^2 \geq \sigma_2^2$ and $H_1 : \sigma_1^2 < \sigma_2^2$ (left-tailed test)

In this case, the rejection (critical) region falls at the left side of the probability curve of the sampling distribution of test statistic F. Suppose $F_{(\nu_1, \nu_2), (1-\alpha)}$ is the critical value at α level of significance then entire region less than or equal to $F_{(\nu_1, \nu_2), (1-\alpha)}$ is the rejection(critical) region and greater than $F_{(\nu_1, \nu_2), (1-\alpha)}$ is the non-rejection region as shown in Fig. 12.7.

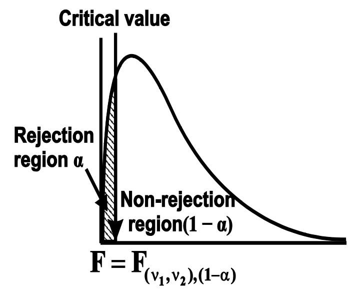

Fig. 12.7

If $F_{cal} \leq F_{(\nu_1, \nu_2), (1-\alpha)}$, that means calculated value of test statistic lies in rejection (critical) region, then we reject the null hypothesis H_0 at α level of significance.

If $F_{cal} > F_{(\nu_1, \nu_2), (1-\alpha)}$, that means calculated value of test statistic lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

Note 2: F-table mentioned above gives only the right tailed critical values with different degrees of freedom at different level of significance. The left tailed critical value of F-test can always be obtained by the formula given below (as described in Unit 4 of this course):

$$F_{(\nu_1, \nu_2), (1-\alpha)} = \frac{1}{F_{(\nu_2, \nu_1), \alpha}}$$

In case of two-tailed test:

When $H_0 : \sigma_1^2 = \sigma_2^2$ and $H_1 : \sigma_1^2 \neq \sigma_2^2$

In this case, the rejection (critical) region falls at both sides of the probability curve of the sampling distribution of test statistic F and half the area(α) i.e. α/2 of rejection (critical) region lies at left tail and other half on the right tail.

Suppose $F_{(\nu_1, \nu_2), (1-\alpha/2)}$ and $F_{(\nu_1, \nu_2), \alpha/2}$ are the two critical values at the left-tailed and right-tailed respectively on pre-fixed α level of significance. Therefore, entire region less than or equal to $F_{(\nu_1, \nu_2), (1-\alpha/2)}$ and greater than or equal to $F_{(\nu_1, \nu_2), \alpha/2}$ are the rejection (critical) regions and between $F_{(\nu_1, \nu_2), (1-\alpha/2)}$ and $F_{(\nu_1, \nu_2), \alpha/2}$ is the non-rejection region as shown in Fig. 12.8.

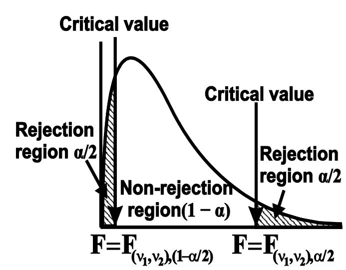

Fig. 12.8

If $F_{cal} \geq F_{(\nu_1, \nu_2), \alpha/2}$ or $F_{cal} \leq F_{(\nu_1, \nu_2), (1-\alpha/2)}$, that means calculated value of test statistic lies in rejection(critical) region, then we reject the null hypothesis H_0 at α level of significance.

If $F_{(\nu_1,\nu_2),(1-\alpha/2)} < F_{cal} < F_{(\nu_1,\nu_2),\alpha/2}$, that means calculated value of test statistic F lies in non-rejection region, then we do not reject the null hypothesis H_0 at α level of significance.

Procedure of taking the decision about the null hypothesis on the basis of p-value:

To take the decision about the null hypothesis on the basis of p-value, the p-value is compared with level of significance (α) and if p-value is less than or equal to α then we reject the null hypothesis and if the p-value is greater than α we do not reject the null hypothesis.

For F-test, p-value can be defined as:

For one-tailed test:

For $H_1 : \sigma_1^2 > \sigma_2^2$ (right-tailed test)

$$\text{p-value} = P\left[F \geq F_{cal}\right]$$

For $H_1 : \sigma_1^2 < \sigma_2^2$ (left-tailed test)

$$\text{p-value} = P\left[F \leq F_{cal}\right]$$

For two-tailed test: $H_1 : \sigma_1^2 \neq \sigma_2^2$

For two-tailed test the p-value is approximated as

$$\text{p-value} = 2P\left[F \geq F_{cal}\right]$$

The p-value for F-test can be obtained with the help of **Table-IV (F-table)** given in the Appendix at the end of Block 1 of this course. Similar to t-test or χ^2-test, this table gives F values corresponding to the standard values of α such as 0.10, 0.05, 0.025 and 0.01 only. Therefore with the help of F-table, we cannot find the exact p-value. So we can approximate p-value for this test.

For example, if test is right-tailed and calculated (observed) value of test statistic F is 2.65 with 24 degrees of freedom of numerator and 14 degrees of freedom of the denominator then we can find the p-value as:

Since test statistic is based on (24, 14) df therefore, we move across all the values of tabulated F corresponding to (24, 14) df at α such as 0.10, 0.05, 0.025 and 0.01 and find the values in which calculated F-value falls. Since we have F tabulated with (24, 14) df at α = 0.10, 0.05, 0.025 and 0.01 as

α =	0.10	0.05	0.025	0.01
$F_{(24, 14), \alpha}$	1.94	2.35	2.79	3.43

Since calculated F-value (= 2.65) falls between 2.35 and 2.79, corresponding to one-tailed area α = 0.05 and 0.025 respectively therefore p-value lies between 0.0025 and 0.05, that is,

$$0.025 < \text{p-value} < 0.05$$

If in the above example the test is two-tailed then the two values 0.025 and 0.05 would be doubled for p-value, that is,

$$0.05 < \text{p-value} < 0.10$$

Note 3: With the help of computer packages and software such as SPSS, SAS, MINITAB, EXCEL, etc. we can find the exact p-value for F-test.

Let us do some examples based on this test.

Example 3: The following data relate to the number of items produced in a shift by two workers A and B for some days:

A	26	37	40	35	30	30	40	26	30	35	45
B	19	22	24	27	24	18	20	19	25		

Assuming that the parent populations are normal, can it be inferred that B is more stable (or consistent) worker compared to A?

Solution: Here, we want to test that worker B is more stable than worker A. As we know that stability of data is related to variance of the data. Smaller value of the variance implies data that it is more stable. Therefore, to compare stability of two workers, it is enough to compare their variances. If σ_1^2 and σ_2^2 denote the variances of worker A and worker B respectively then our claim is $\sigma_1^2 > \sigma_2^2$ and its complement is $\sigma_1^2 \le \sigma_2^2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \sigma_1^2 \le \sigma_2^2$$

$$H_1 : \sigma_1^2 > \sigma_2^2 \; [\text{worker B is more stable than worker A}]$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Here, we want to test the hypothesis about two population variances and sample sizes n_1 = 11(< 30) and n_2 = 9 (< 30) are small. Also populations under study are normal and both samples are independent so we can go for F-test for two population variances.

For testing the null hypothesis, test statistic is given by

$$F = \frac{S_1^2}{S_2^2} \sim F_{(n_1-1,\, n_2-1)} \quad \ldots (4)$$

where, $S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(X_i - \bar{X}\right)^2$ and $S_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(Y_i - \bar{Y}\right)^2$

Calculation for S_1^2 and S_2^2:

Items Produced by A (Variable X)	$(X - \bar{X}) = (X - 34)$	$(X - \bar{X})^2$	Items Produced by B (Variable Y)	$(Y - \bar{Y}) = (Y - 22)$	$(Y - \bar{Y})^2$
26	−8	64	19	−3	9
37	3	9	22	0	0
40	6	36	24	2	4
35	1	1	27	5	25
30	−4	16	24	2	4
30	−4	16	18	−4	16
40	6	36	20	−2	4
26	−8	64	19	−3	9
30	−4	16	25	3	9
35	1	1			
45	11	121			
Total = 374	0	380	198	0	80

Therefore, we have

$$\bar{X} = \frac{1}{n_1}\sum X = \frac{1}{11} \times 374 = 34$$

and

$$\bar{Y} = \frac{1}{n_2}\sum Y = \frac{1}{9} \times 198 = 22$$

Thus,

$$S_1^2 = \frac{1}{n_1 - 1}\sum\left(X - \bar{X}\right)^2 = \frac{1}{10} \times 380 = 38$$

$$S_2^2 = \frac{1}{n_2 - 1}\sum\left(Y - \bar{Y}\right)^2 = \frac{1}{8} \times 80 = 10$$

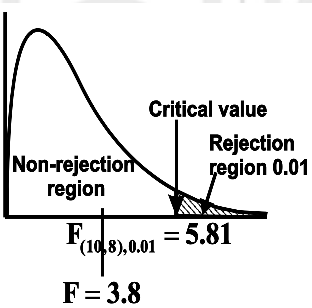

Fig. 12.9

Putting the value of S_1^2 and S_2^2 in equation (4), we have

$$F = \frac{38}{10} = 3.8$$

The critical (tabulated) value of test statistic F for right-tailed test corresponding $(n_1 - 1, n_2 - 1) = (10, 8)$ df at 1% level of significance is $F_{(n_1-1, n_2-1), \alpha} = F_{(10,8), 0.01} = 5.81$.

Since calculated value of test statistic (= 3.8) is less than the critical value (= 5.81), that means calculated value of test statistic lies in rejection region as shown in Fig. 12.9, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 1% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so worker B is not more stable (or consistent) worker compared to A.

Example 4: Two random samples drawn from two normal populations gave the following results:

Sample	Size	Mean	Sum of Squares of Deviation from the Mean
Sample I	9	59	26
Sample II	11	60	32

Test whether both samples are from the same normal populations?

Solution: Since we have to test whether both the samples are from same normal population, therefore, we will test two hypotheses separately:

(i) Two population means are equal, i.e. $H_0 : \mu_1 = \mu_2$

(ii) Two population variances are equal, i.e. $H_0 : \sigma_1^2 = \sigma_2^2$

Since sample sizes are small and populations under study are normal so two means will be tested using t-test whereas two variances will be tested using F-test. But t-test is based on the prior assumption that both population variances are same, therefore, first we apply F-test and later the t- test (when F- test accepts equality hypothesis).

Given that

$$n_1 = 9, \quad \bar{X} = 59, \quad \sum\left(X - \bar{X}\right)^2 = 26$$

$$n_2 = 11, \quad \bar{Y} = 60, \quad \sum\left(Y - \bar{Y}\right)^2 = 32$$

Therefore,

$$S_1^2 = \frac{1}{n_1 - 1}\sum\left(X - \bar{X}\right)^2 = \frac{1}{9-1} \times 26 = 3.25$$

$$S_2^2 = \frac{1}{n_2 - 1}\sum\left(Y - \bar{Y}\right)^2 = \frac{1}{11-1} \times 36 = 3.60$$

First we want to test that the variances of both normal populations are equal so our claim is $\sigma_1^2 = \sigma_2^2$ and its complement is $\sigma_1^2 \neq \sigma_2^2$. Thus, we can take the null and alternative hypotheses as

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ and } H_1 : \sigma_1^2 \neq \sigma_2^2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

For testing this, the test statistic F is given by

$$F = \frac{S_1^2}{S_2^2} \sim F_{(n_1 - 1, n_2 - 1)}$$

$$= \frac{3.20}{3.65} = 0.88$$

The critical (tabulated) value of test statistic F for two-tailed test corresponding $(n_1 - 1, n_2 - 1) = (8, 11)$ df at 5% level of significance are $F_{(n_1-1, n_2-1), \alpha/2} =$

$$F_{(8,11), 0.025} = 3.66 \text{ and } F_{(n_1-1, n_2-1),(1-\alpha/2)} = \frac{1}{F_{(n_2-1, n_1-1), \alpha/2} =} = \frac{1}{F_{(11,8), 0.025}} = \frac{1}{4.24} = 0.24.$$

Since calculated value of test statistic (= 0.88) is less than the critical value (= 3.66) and greater than the critical value (= 0.24), that means calculated value of test statistic F lies in non-rejection region so we do not reject the null hypothesis i.e. we support the claim.

Thus, we conclude that both samples may be taken from normal populations having equal variances.

Now, we test that the means of two normal populations are equal so our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Thus, we can take the null and alternative hypotheses as

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

The test statistic is given by

$$t = \frac{\bar{X} - \bar{Y}}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad \ldots (5)$$

where, $S_p^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum\left(X - \bar{X}\right)^2 + \sum\left(Y - \bar{Y}\right)^2\right]$

$$= \frac{1}{9 + 11 - 2}\left(26 + 32\right) = \frac{1}{18} \times 58 = 3.22$$

$$S_p = \sqrt{3.22} = 1.79$$

Putting the values of $\bar{X}, \bar{Y}, S_p, n_1$ and n_2 in equation (5), we have,

$$t = \frac{59-60}{1.79\sqrt{\frac{1}{9}+\frac{1}{11}}} = \frac{-1}{1.79 \times 0.45} = \frac{-1}{0.81} = -1.23$$

The critical values of test statistic t for $(n_1 + n_2 - 2) = 18$ df at 5% level of significance for two-tailed test are $\pm t_{(18),0.025} = \pm 2.101$.

Since calculated value of test statistic t (= −1.23) is less than the critical value (= 2.101) and greater than the critical value (= –2.101), that means calculated value of test statistic t lies in non-rejection region so we do not reject the null hypothesis i.e. we support the claim.

Thus, we conclude that both samples may be taken from normal populations having equal means.

Hence, overall we conclude that both samples may come from the same normal populations.

Now, you can try the following exercises.

E3) Two sources of raw materials are under consideration by a bulb manufacturing company. Both sources seem to have similar characteristics but the company is not sure about their respective uniformity. A sample of 12 lots from source A yields a variance of 125 and a sample of 10 lots from source B yields a variance of 112. Is it likely that the variance of source A significantly differs to the variance of source B at significance level $\alpha = 0.01$?

E4) A laptop computer maker uses battery packs of two brands, A and B. While both brands have the same average battery life between charges (LBC), the computer maker seems to receive more complaints about shorter LBC than expected for battery packs of brand A. The computer maker suspects that this could be caused by higher variance in LBC for brand A. To check that, ten new battery packs from each brand are selected, installed on the same models of laptops, and the laptops are allowed to run until the battery packs are completely discharged. The following are the observed LBCs in hours:

Brand A	3.2	3.7	3.1	3.3	2.5	2.2	3.2	3.1	3.2	4.3
Brand B	3.4	3.6	3.0	3.2	3.2	3.2	3.0	3.1	3.2	3.2

Assuming that the LBCs of both brands follows normal distribution, test the LBCs of brand A have a larger variance that those of brand B at 5% level of significance.

We now end this unit by giving a summary of what we have covered in it.

12.4 SUMMARY

In this unit, we have discussed the following points:

1. Testing of hypothesis for population variance using χ^2-test.
2. Testing of hypothesis for two population variances using F-test.

12.5 SOLUTIONS / ANSWERS

E1) Here, we are given that

$$\sigma_0 = 15, \quad n = 20, \quad S = 17$$

Here, we want to test the agency's claim that the standard deviation (σ) of the length of serving times is less than 15 minutes. So our claim is σ < 15 and its complement is σ ≥ 15. Since complement contains the equality sign so we can take complement as null hypothesis and claim as the alternative hypothesis. Thus,

$$H_0 : \sigma \geq \sigma_0 = 15$$

$$H_1 : \sigma < 15 \qquad \left[\begin{array}{l}\text{SD of the length of serving} \\ \text{times is less than 15 minutes}\end{array}\right]$$

Since the alternative hypothesis is left-tailed so the test is left-tailed test.

Here, we want to test the hypothesis about the population standard deviation and sample size is small n = 20(< 30). Also we are given that the service time of the ambulance follows normal so we can go for χ^2 test for population variance.

The test statistic is given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$= \frac{19 \times (17.0)^2}{(15.0)^2} = 24.40$$

The critical value of test statistic χ^2 for left-tailed test corresponding (n - 1) = 19 df at 1% level significance is $\chi^2_{(n-1),(1-\alpha)} = \chi^2_{(19),0.99} = 7.63$.

Since calculated value of test statistic (= 24.40) is greater than the critical value (= 7.63), that means calculated value of test statistic lies in non-rejection region so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject our claim at 1% level of significance.

Thus, we conclude that sample provides us sufficient evidence against the claim so agency's claim that the standard deviation (σ) of the length of serving times is less than 15 minutes is not true.

E2) Here, we are given that

$$\sigma_0^2 = 0.62, \quad n = 25, \quad S^2 = 0.65$$

Here, we want to test the cigarette manufacturer's claims that the variance (σ^2) of nicotine content of its cigarettes is 0.62 milligrams. So claim is $\sigma^2 = 0.62$ and its complement is σ ≠ 0.62. Since claim contains the equality sign so we can take claim as null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \sigma^2 = \sigma_0^2 = 0.62 \text{ and } H_1 : \sigma^2 \neq 0.62$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis about the population variance and sample size is small n = 25(< 30). Also we are given that the nicotine content of its cigarettes follows normal distribution so we can go for χ^2 test for population variance.

The test statistic is given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

$$= \frac{24 \times 0.65}{0.62} = 25.16$$

The critical (tabulated) values of test statistic χ^2 for two-tailed test corresponding (n - 1) = 24 df at 5% level of significance are

$\chi^2_{(n-1),\alpha/2} = \chi^2_{(24),0.025} = 39.36$ and $\chi^2_{(n-1),(1-\alpha/2)} = \chi^2_{(24),0.975}$ =12.40.

Since calculated value of test statistic (= 25.16) is less than the critical value (= 39.36) and greater than the critical value (= 12.40), that means calculated value of test statistic lies in non-rejection region, so we do not reject the null hypothesis i.e. we support the claim at 5% level of significance.

Thus, we conclude that sample fails to provide sufficient evidence against the claim so we may assume that manufacturer's claim that the variance of the nicotine content of the cigarettes is 0.62 milligram is true.

E3) Here, we are given that

$$n_1 = 12, \quad S_1^2 = 125, \quad n_2 = 10, \quad S_2^2 = 112$$

Here, we want to test that variance of source A significantly differs to the variances of source B. If σ_1^2 and σ_2^2 denote the variances in the raw materials of sources A and B respectively so our claim is $\sigma_1^2 \neq \sigma_2^2$ and its complement is $\sigma_1^2 = \sigma_2^2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ and } H_1 : \sigma_1^2 \neq \sigma_2^2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed test.

Here, we want to test the hypothesis about two population variances and sample sizes n_1 = 12(< 30) and n_2 = 10 (< 30) are small. Also populations under study are normal and both samples are independent so we can go for F-test for two population variances.

For testing this, the test statistic is given by

$$F = \frac{S_1^2}{S_2^2} \sim F_{(n_1-1, n_2-1)}$$

$$= \frac{125}{112} = 1.11$$

The critical (tabulated) value of test statistic F for two-tailed test corresponding (n_1 - 1, n_2 - 1) = (11, 9) df at 5% level of significance are

$F_{(n_1-1, n_2-1),\alpha/2} = F_{(11,9),0.025} = 3.91$ and

$$F_{(n_1-1, n_2-1),(1-\alpha/2)} = \frac{1}{F_{(n_2-1, n_1-1),\alpha/2}} = \frac{1}{F_{(9, 11), 0.025}} = \frac{1}{3.59} = 0.28.$$

Since calculated value of test statistic (= 1.11) is less than the critical value (= 3.91) and greater than the critical value (= 0.28), that means calculated value of test statistic lies in non-rejection region, so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so we may assume that the variances of source A and B is differ.

E4) Here, we want to test that the LBCs of brand A have a larger variance than those of brand B. If σ_1^2 and σ_2^2 denote the variances in the LBCs of brands A and B respectively so our claim is $\sigma_1^2 > \sigma_2^2$ and its complement is $\sigma_1^2 \le \sigma_2^2$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \sigma_1^2 \le \sigma_2^2 \text{ and } H_1 : \sigma_1^2 > \sigma_2^2$$

Since the alternative hypothesis is right-tailed so the test is right-tailed test.

Here, we want to test the hypothesis about two population variances and sample sizes n_1 = 10(< 30) and n_2 = 10 (< 30) are small. Also populations under study are normal and both samples are independent so we can go for F-test for two population variances.

For testing the null hypothesis, test statistic is given by

$$F = \frac{S_1^2}{S_2^2} \sim F_{(n_1-1,\, n_2-1)} \qquad \ldots (6)$$

where, $S_1^2 = \frac{1}{n_1-1}\sum\left(X-\bar{X}\right)^2 = \frac{1}{n_1-1}\left[\sum X^2 - \frac{\left(\sum X\right)^2}{n_1}\right]$ and

$$S_2^2 = \frac{1}{n_2-1}\sum\left(Y-\bar{Y}\right)^2 = \frac{1}{n_2-1}\left[\sum Y^2 - \frac{\left(\sum Y\right)^2}{n_2}\right].$$

Calculation for S_1^2 and S_2^2 :

LBCs of Brand A (X)	X^2	LBCs of Brand A (Y)	Y^2
3.7	13.69	3.6	12.96
3.2	10.24	3.2	10.24
3.3	10.89	3.2	10.24
3.1	9.61	3.0	9.00
2.5	6.25	3.0	9.00
2.2	4.84	3.2	10.24
3.1	9.61	3.2	10.24
3.2	10.24	3.1	9.61
4.3	18.49	3.2	10.24
3.2	10.24	3.1	9.61
Total = 3.18	104.1	31.8	101.38

From the calculation, we have

$$\bar{X} = \frac{1}{n_1}\sum X = \frac{1}{10}\times 31.8 = 3.18,$$

$$\bar{Y} = \frac{1}{n_2}\sum Y = \frac{1}{10}\times 31.8 = 3.18$$

Thus,

$$S_1^2 = \frac{1}{n_1 - 1}\left[\sum X^2 - \frac{\left(\sum X\right)^2}{n_1}\right] = \frac{1}{9}\left[104.10 - \frac{(31.8)^2}{10}\right]$$

$$= \frac{1}{9}\times 2.98 = 0.33$$

$$S_2^2 = \frac{1}{n_2 - 1}\left[\sum Y^2 - \frac{\left(\sum Y\right)^2}{n_2}\right] = \frac{1}{9}\left[101.38 - \frac{(31.8)^2}{10}\right]$$

$$= \frac{1}{9}\times 0.26 = 0.03$$

Putting the values of S_1^2 and S_2^2 in equation (6), we have

$$F = \frac{0.33}{0.03} = 1.1$$

The critical (tabulated) value of test statistic F for right-tailed test corresponding $(n_1 - 1, n_2 - 1) = (9, 9)$ df at 1% level of significance is $F_{(n_1-1, n_2-1), \alpha} = F_{(9,9), 0.01} = 5.35$.

Since calculated value of test statistic (= 1.1) is less than the critical value (= 5.35), that means calculated value of test statistic lies in non-rejection region so we do not reject the null hypothesis and reject the alternative hypothesis i.e. we reject the claim at 1% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so variance in LBCs of brand A is not greater than variance in LBCs of brand B.