UNIT 15 LARGE SCALE STRUCTURE AND THE EXPANDING UNIVERSE

Structure

15.1 INTRODUCTION

In previous units you have learnt about stars and galaxies and their properties. You have learnt that there are billions of galaxies in the universe. It is now time to estimate the size of the universe and discuss the current ideas about its major components and its origin. The science which deals with the origin of the universe is called **cosmology**.

In this unit, we discuss some aspects of cosmology. First we learn how distances of distant objects can be estimated so that we can get an idea of the size of the universe. Estimation of distances is a very tricky problem because there is no way of verifying these distances directly. So, we look for internal consistency in various methods, which means that the distance estimates given by them agree with one another.

In this unit we discuss the distribution of matter on very large scales also called the large scale structure of the universe. We also look at the kind of matter that may be forming the bulk of the universe. This matter is not visible and is called the '**dark matter**'. It shows itself up only through its gravitational effect.

Objectives

After studying this unit, you should be able to:

- explain how astronomical distances are measured;
- state Hubble's law and explain how Hubble's constant indicates the age of the universe;
- explain the need to postulate the existence of dark matter in the universe;
- derive Friedmann equation and solve it in simple cases; and
- explain why a hot and dense phase in the early universe is needed to explain the existence of cosmic background radiation and to synthesise light elements.

15.2 COSMIC DISTANCE LADDER

An important step in this direction is to estimate the physical size of distant objects and their distances from us. You know that to specify the position of an object in three-dimensional space, we need 3 coordinates. Since we are observing from the Earth (or its vicinity from satellites), it is best to use spherical polar coordinate system with us as the origin. Two of the coordinates, namely, θ and ϕ (which are related to the declination and right ascension, respectively) are easily fixed by pointing a telescope in the direction of the object. Fixing the distance to an astronomical object is relatively trickier.

In this section, we give an introduction to the basic principles that are used to measure distances. The basic principle involved is to use the properties of the nearby objects and deduce distances of similar objects farther off using these properties. Then we use the latter objects to deduce the distances to objects still farther, and so on. This series of steps which takes us from one step (in terms of distances) to the next step (in terms of distances even farther) is termed the **Cosmic Distance Ladder**.

All this is best understood through an example of estimating distances on the Earth.

15.2.1 An Example from Terrestrial Physics

Consider the following situation. You are in a house and there are small plants around the lawn in your house. Outside the house also there are similar plants and in addition there are also some trees. We do not know the height of these trees. Very far away there is the sea and on the beach also there are a number of such trees. The problem is to estimate the distance of the sea from your house. To begin with we do not know about the height of the trees.

The obvious thing would be to walk down to the sea (keeping in mind that all the steps should approximately be of the same size) and count the number of steps. The length of the step multiplied by the number of steps gives us the distance to the beach. But you cannot use this method if you are not allowed to, or are unable to come out of the house and go to the beach. The situation is more like this in the context of distance measurement in astronomy. We can make direct measurement only on the Earth's surface. Since it will take about 1,00,000 years, even to reach the other end of the galaxy by moving at the speed of light, we need to devise a better strategy.

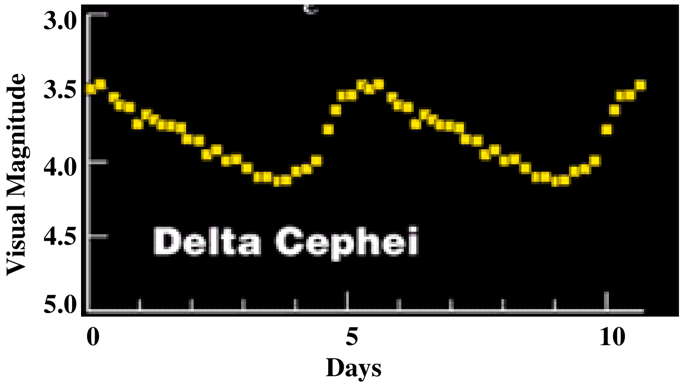
Let us come back to our problem of measuring the distance to the sea beach. We will put an extra condition that we are not allowed to come out of the gates of the house. With this constraint we can proceed in the following way.

We first measure the heights h of the plants inside the garden of the house. Next we measure the angle θ subtended by similar plants outside which are in the vicinity of the trees. The distance of these plants from us equals $d = \frac{h}{\theta}$. Since the trees are in the vicinity of these plants we know that their distance from us is also d. Further, we can measure the angle θ_1 subtended by the heights of these trees. The height h_1 of these trees is now given by $h_1 = d\,\theta_1$. In this way we have achieved the first step in the ladder of finding distances. We next use this information to estimate the distance to the trees near the seaside. Since the trees near the beach are similar to the ones nearby, we assume that their height is also h_1. By measuring the angle subtended by these trees we can compute the distance of these trees from us. Because of the physical association of these trees with the sea side we know the distance of the sea from your house. This is the second step of the ladder. Like this we can add more steps and go on to estimate the distances of far off objects.

15.2.2 Distance Measurement using Cepheid Variables

We will now give an example of using this principle to measure distances in astronomy. In Unit 1 you have learnt the technique of measuring distances by parallax method. In this section we will discuss how this method can be used to calibrate another technique for distance measurements using Cepheid variable stars, which in turn are used to measure even further distances. The brightness of Cepheid variable stars is a periodic function of time (Fig. 15.1b).

(a)



(b)

Fig.15.1: a) Cepheid variable stars in NGC 300; b) the brightness of Cepheid stars as a function of time

We have a large number of such stars in our neighbourhood. Their distances can be measured by the parallax method. From these distances and from their observed apparent magnitudes, their absolute magnitudes and luminosities can be calculated. It turns out that their absolute magnitudes are directly proportional to their periods (Fig. 15.2). This is called the **period-luminosity relation** for the Cepheids.

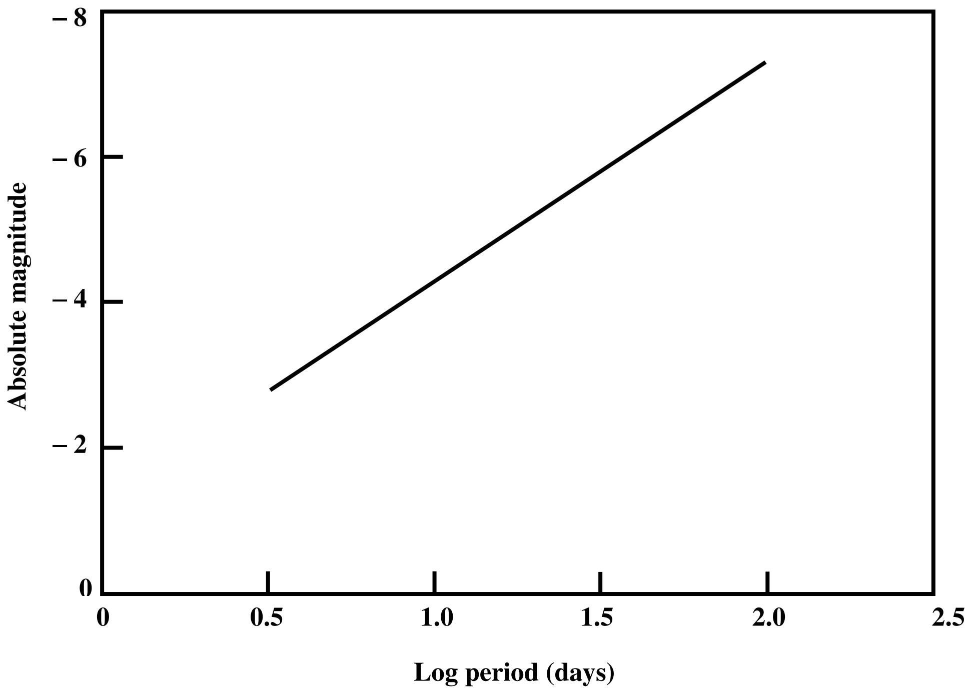


Fig.15.2: Period-luminosity relation for Cepheids

This relation is firmly established for the local sample. Now **we assume that the distant sample of these stars** also obeys this relation. So, the observed period of a member of the distant sample is used to find its absolute magnitude. The apparent magnitude can be observed directly. Using the relation between the absolute magnitude and the apparent magnitude:

$$M = m + 5 - 5 \log r, \qquad (15.1)$$

we can find the distance of this star. In this way, the Cepheid variable stars have been used to find distances of nearby galaxies. Used in this manner, Cepheids are called standard **candles**.

We can now use Cepheids to define some other objects, such as supernovae, to act as standard candles to estimate even larger distances.

What is the result of these investigations? We find that distant galaxies are rushing away from us with velocities which are proportional to their distances. This is called **Hubble's law**.

Spend 5 min.

SAQ 1

Explain the concept of a distance ladder.

15.3 HUBBLE'S LAW

Hubble's law is probably the single most important step in our attempt to understand the Universe. This law was discovered by Edwin Hubble and it relates the distances of galaxies with the velocities with which they are receding away from us (Fig. 15.3).

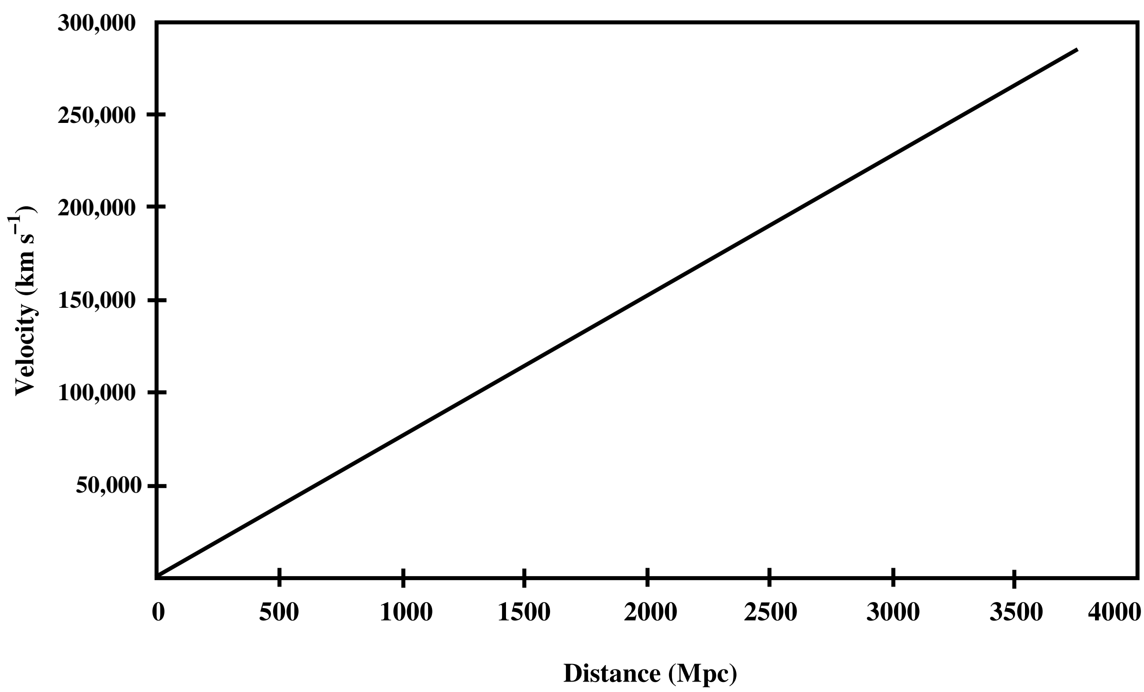


Fig.15.3: Hubble's law

The velocity of any object can be split into a component that is along the line-of-sight and another component that is transverse (perpendicular to) the line-of-sight. The line-of-sight component of the velocity can be determined very accurately by Doppler shift of the light that we received from the object. Hence this gives the velocity with which the object is coming towards us or receding away from us. Let us look at this process.

15.3.1 Distance-Velocity Relation

Using the methods similar to those mentioned above, Hubble estimated the distances and velocities of a set of galaxies and plotted them. He found that the galaxies in general seem to be receding away from us. This is popularly known as the expanding universe. Further, the velocities with which they are receding away, are directly proportional to their distances from us (Fig. 15.3). This prompted him to propose a law, now known by his name.

Hubble's law

$$v = Hr, \tag{15.2}$$

where v is the line-of-sight velocity of an object, r its distance from us and the proportionality constant H is called the **Hubble constant**.

The importance of this relation is that, once we know the velocity of a galaxy (by red shift measurement), we can calculate the distance at which it is located. It is important to point out here that Hubble's law holds even if we were on some other galaxy. Our location in the universe does not have any special importance.

Notice that in the above relation H is the slope of the curve shown in Fig. 15.3. Notice also that $1/H$ has the dimensions of time. In a very simple picture, we can imagine that all the galaxies which are today moving away from one another were at some time in the past together at one point. Some event occurred at that time which triggered the expansion of the universe. This event is usually called the **Big Bang**. The quantity $1/H$ measures the time since that even, or the age of the universe.

SAQ 2

Spend 5 min.

In astronomy, the velocity is measured in km s^{-1}. The distance of galaxies is measured in Mpc or million parsec. Find the dimensions of H.

Unfortunately the measured value of H has lots of errors. But we know today that it is roughly 70 km s^{-1}/Mpc^{-1}. Estimate the age of the universe.

15.4 CLUSTERS OF GALAXIES

Galaxies mostly exist as members of large groups called **galaxy clusters** (see Fig. 15.4). A cluster of galaxies contains about a thousand galaxies. A galaxy cluster consists of a variety of galaxies. It is an observed fact that the gross features of these clusters are very similar. This fact alone has a far reaching implication. We expect that roughly the same kind of physical processes are responsible for their evolution. Further, it is natural to assume that they were created at different times and began to evolve. This is because, in order to initiate the process of creation and subsequent evolution of these clusters one needs a certain combination of astrophysical conditions.

These conditions need not be the same at all points at a given time. Let us illustrate this situation in the following manner:

Consider a set of systems, A, B, C, … Let the evolution in each of these systems be governed by the same physical processes. If they started to evolve at different times, we would expect that at any time, in particular today, they should be in different stages of evolution. Hence, we would expect they should not show great similarity.

Fig. 15.4: Cluster of galaxies

Alternatively, if we find that they look similar today, we may naively think that they must have got created at the same time, so that they have had the same time for evolution. For the systems under consideration, namely, the clusters of galaxies, we know that they look similar but at the same time we have reasons to believe that they started their evolution at different times.

Our simple-minded reasoning leads to the suggestion that they may have been created at different times but they have reached some kind of steady state today. Such systems have a simple, but important relationship for their kinematic parameters such as their mass and velocity. This relationship is called the virial theorem which we now describe.

15.4.1 The Virial Theorem and Dark Matter

The virial theorem says that if a system is bounded and is in equilibrium, then its moment of inertia does not change with time. Such a system will have the following relation between its total kinetic and potential energies:

$$2T + V = 0, \qquad (15.3)$$

where T is the total kinetic and V is the total potential energy of the system.

Suppose we have a spherical system consisting of N particles that are interacting gravitationally. Let the position of the i^{th} particle be $\mathbf{r}_i$ and its velocity $\mathbf{v}_i$. The total kinetic energy is then

$$T = \frac{1}{2} m \sum_{i=1}^{N} v_i^2 . \qquad (15.4)$$

The total potential energy is obtained by summing over the potential energy of all the pairs. We have

$$V = \sum_{i>j} \frac{G m_i m_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|} \qquad (15.5)$$

For a spherically symmetric distribution, V will come out to be proportional to $M(R)\,R^{-1}$, where R is the radius of the system and M the total mass enclosed in a sphere of radius R. The total kinetic energy can be estimated in the following manner. If the velocities are in random directions, some of the particles of the system will contribute a blue shift and some red shift. This will result in broadening of the spectral lines. Hence, from the width of the spectral lines we can estimate the *root mean square* (*rms*) velocity. This gives the total kinetic energy. If the system is in equilibrium, we should have

$$T = -V/2 \tag{15.6}$$

or

$$v_{rms}^2 \propto R^{-1}. \tag{15.7}$$

When one tries to estimate the mass of the system in the above manner, one finds that there is much more mass in the system as compared to that suggested by the luminous mass alone. Hence, it is postulated that there should be a significant fraction of mass in the form of **dark matter**. What form this dark matter takes, we do not yet know. One possibility is that it consists of cold, burnt out stars which emit very little radiation. Another possibility is that it is in the form of particles which interact very weakly with normal matter. It is a very active area of research at present.

15.5 FRIEDMANN EQUATION AND ITS SOLUTIONS

Cosmology is the study of the overall features of the universe. In this framework, one does not bother about local features of the universe like planets or stars. In cosmology, we set up equations and solve them to obtain the very large scale features of the universe.

To set up the basic equations governing the evolution of the universe as a whole, Newtonian theory of gravity is inadequate and is, rigorously speaking, inapplicable. The correct theory to use is Einstein's General Theory of Relativity. This, however, is outside the scope of this course. All is, however, not lost. It so happens that if we go ahead and apply Newton's laws (which strictly speaking is not the correct thing to do) the final equations which we get are the same equations that result from the correct theory, namely, the General Theory of Relativity.

The aim of this Section is to study the evolution of the Universe using these equations. Hence, we will not be disturbed by the fact that the derivation of these is not rigorous. We will happily go ahead and use these equations since we are aware that the final equations are the correct ones.

We know that over a variety of scales the universe is not homogeneous. We have planets, stars, galaxies and clusters of galaxies, which indicate that the universe is far from being homogeneous. However, if we consider the universe as a whole, then at large enough scales the universe seems to be homogeneous and isotropic. This last point needs some explanation.

Consider a well maintained lawn. From a distance, the lawn appears uniformly green. But as we start analysing the lawn on smaller scales, it begins to lose its homogeneity. Consider a grass hopper sitting in the lawn. It observes the lawn at a scale which is about the blade of grass. Clearly it will notice that the lawn is not at all homogeneous. This means that the lawn is homogeneously green over distance scales which are much bigger than the size of a blade of grass. In a similar way, we say that over scales of sizes, much bigger than the size of galaxies, the universe is homogeneous.

The equations that we now derive are the equations that govern the overall evolution of a homogeneous universe. Consider two points *A* and *O*. With *O* as centre, and *AO* as radius, draw a sphere. The force with which a test particle at *A* is gravitationally pulled towards *O* can be calculated by just using the mass enclosed by the sphere. As we know, the mass outside the sphere will not exert any net force on *A*.

Let the density of matter be ρ. Notice that density does not depend upon space. This is because of our condition of homogeneity. However, there is no such restriction on time dependence. So we will include time dependence for generality. The mass of the sphere of radius $R = OA$ is

$$M = \frac{4\pi}{3}\rho(t)R^3(t), \tag{15.8}$$

The gravitational potential at *A* due to this sphere is $-GM/R$. Further, let $\mathbf{v} = \dot{\mathbf{R}}$ be the velocity with which *A* is moving with respect to *O*. Depending on the magnitudes of the kinetic energy and the potential energy, the particle at point *A* may keep moving away from *O* or may turn back and fall towards *O*. This is the well known condition for escape velocity.

The total energy per unit mass of a test particle at *A* is

$$E = -\frac{GM}{R} + \frac{v^2}{2} \tag{15.9}$$

or

$$\dot{R}^2 - 2E = \frac{2GM}{R} \tag{15.10}$$

If the energy *E* is positive, then the distance between *A* and *O* will keep increasing and if *E* is negative, *A* will attain a maximum distance from *O* and then begin to fall towards *O*.

This condition on *E* can alternatively be expressed by replacing $-2E$ by $k.2|E|$; $k = +1$ corresponds to the case $E < 0$.

An alternative way of writing the last condition is then

$$\dot{R}^2 + k.2|E| = \frac{2GM}{R} \tag{15.11}$$

Dividing both sides by R^2 and expressing the mass *M* in terms of the density ρ as

$$M = \frac{4}{3}\pi R^3 \rho$$

we get,

$$\frac{\dot{R}^2}{R^2} + \frac{k.2|E|}{R^2} = \frac{8\pi G}{3}\rho \tag{15.12}$$

Defining a new variable $a = R\big/\sqrt{2|E|}$, we can now write the **Friedmann equation** governing the evolution of the distance between two particles.

Friedmann equation

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3} \tag{15.13}$$

We get the same equation from the general theory of relativity. The parameter k then signifies the curvature of space. In the general theory of relativity, the effect of the gravitational field is to make the space curved. The curvature of space is denoted by k, which can take values +1, 0 or −1, depending on the overall density of the universe. The quantity a is called the **scale factor** and the nature of a as function of t indicates the nature of the expansion of the universe.

Fig. 15.5 shows the behaviour of a as a function of t for the three values of k, i.e., +1, 0 and −1. These curves are the **solutions of the Friedmann equation**. We see that when $k = -1$ (which according to general theory of relativity implies that the overall density of the universe is less than a certain critical density), or $k = 0$ (the overall density of the universe equals the critical density), the universe keeps expanding. When $k = +1$ (the overall density of the universe is greater than the critical density), the universe expands up to a point and then starts contracting. Present observations indicate that the universe will keep expanding, and its expansion will not be followed by contraction.

In the early phase of the universe, the curvature must have been small, so it is sufficient to consider the case of $k = 0$. The solution of the Friedmann equations, of course, depends on the nature of energy density. If $\rho \propto a^{-n}$, the equation can be solved for $k = 0$ and the result comes out to be

$$a \propto (n/2)^{2/n} t^{2/n} \tag{15.14}$$

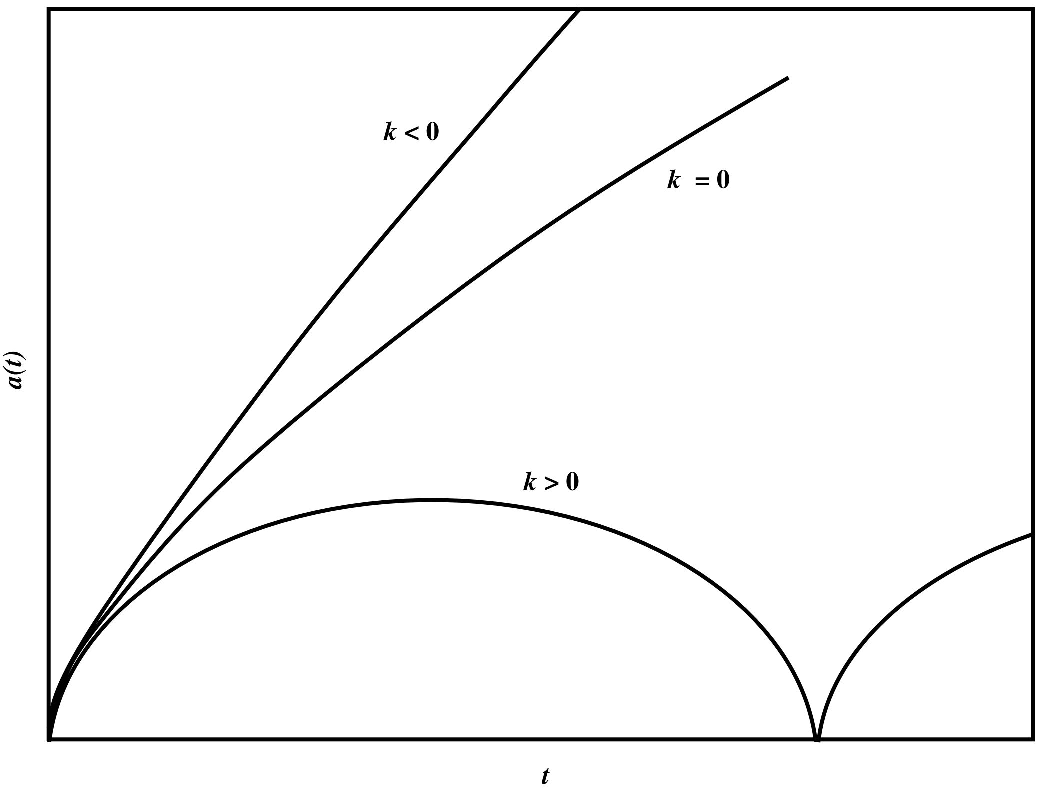


Fig.15.5: The variation of *a* with time

Spend 5 min.

SAQ 3

Verify that Eq. (15.14) is a solution of Eq. (15.13) when $\rho \propto a^{-n}$.

Time-temperature Relationship

We know from the thermodynamics of radiation that it has pressure and energy density which we denote by p_{rad} and ρ_{rad}, respectively. They are related to each other through the relation,

$$p_{rad} = \rho_{rad}\, c^2/3 \tag{15.15}$$

where c is the speed of light. The distance between points increases as R which, in turn, is proportional to a. Hence, the volume increases as a^3 and the energy contained in it as $\rho\, a^3$. Now, using the first law of thermodynamics, $dU + pdV = 0$, we have

$$d[\rho c^2 a^3] + pd[a^3] = 0. \tag{15.16}$$

Applying it to radiation, we get

$$\rho_{rad} \propto a^{-4} \tag{15.17}$$

Using this relation in the expression for ρ derived in the last section (with $n = 4$) we get,

$$a \propto \sqrt{t} \tag{15.18}$$

At the same time we know that the temperature of radiation is related to its energy-density by

$$\rho_{rad} \propto T^4 \tag{15.19}$$

for isotropic and homogeneous radiation field.

From the last three equations, we get the relationship of temperature with time as

$$T \propto 1/\sqrt{t} \tag{15.20}$$

We see that $t = 0$ is both interesting as well as disturbing. This is because at that point of time, the temperature shoots to infinity and so does the energy density. It is this epoch which is termed as the **Big Bang**.

15.6 EARLY UNIVERSE AND NUCLEOSYNTHESIS

We saw that the temperature of radiation varies as inverse of the square root of time. The immediate consequence of this is that the radiation temperature in early times should have been very high. If matter and radiation were in thermal equilibrium, the above statements imply that the temperature of matter was also high in early times.

We know from thermodynamics that temperature is a measure of the mean kinetic energy which in turn implies high energy collisions between particles. The earlier the epoch, the higher the energy with which these particles collide with each other. Today we know about the physical phenomena at high energies from experimental investigations using high energy particle accelerators and colliders. We should expect that the same phenomenon must have taken place in the early universe. In fact, for this reason, early universe is often called the poor man's laboratory.

An important class of reactions at high energy is those which lead to the synthesis of nuclei of elements. Very early on, there were no complex nuclei. The only ones were the hydrogen nuclei, i.e., protons. At those energies, even if the protons and neutrons combined to form higher atomic number nuclei (e.g., the Helium nucleus), the kinetic energies of the particles were so high that the collisions would have immediately disintegrated them.

As the universe cools, a certain temperature is reached when the energies are low enough that this backward reaction (namely disintegration) begins to get suppressed. Hence, stable helium nucleus begins to get formed. As the temperature lowers further, we expect that higher atomic number nuclei will begin to get synthesized. So the question is, "Can we proceed in this way and synthesize all the naturally occurring nuclei"? The answer unfortunately is no!

This line of reasoning works for only the elements with first few atomic numbers. As the temperature decreases, we can form Lithium and some Boron. The problem comes up when we need to form Beryllium. In the process of the formation of the stable Beryllium nucleus, one passes through an intermediate stage where, spontaneous disintegration is faster than the fusion. So even before there can be fusion the nucleus which is supposed to participate in the fusion, disintegrates. Hence one cannot form Beryllium by this procedure. Only after the formation of Beryllium, can the nuclei of higher atomic numbers be formed. Hence this is called the "Beryllium Bottleneck". The universe has to wait for a very long time, namely, till stars form, in order to synthesize elements of atomic number 5 and higher. (Refer to Unit 10 for details of nucleo-synthesis inside the stars).

15.6.1 Cosmic Background Radiation

Do we have any signature of the early hot phase of the universe? The answer is yes. In 1965, two scientists at the Bell Telephone Laboratories in America discovered accidentally a radiation at a very low temperature of only 3 K which seemed to come from all directions. It was highly isotropic. It was suggested that the radiation fills the whole universe. Since the wavelength of the peak radiation, ~ 1 mm, falls in the microwave region, it was called the **cosmic microwave background radiation** (CMBR). This is the relic of the era when the universe was very hot and dense. It is argued that the radiation was once very hot and has been cooled to its present temperature due to the expansion of the universe over billions of years (recall from Eqs. (15.20) and (15.19) that $T \propto 1/\sqrt{t}$ and $\rho_{rad} \propto T^4$). Put in another way, the same energy fills an every increasing volume, so its energy density decreases and so does its temperature.

The discovery of the CMBR has great significance for cosmology. (The discoverers of the radiation were awarded the Nobel Prize for their work.) It shows that the universe was once very hot and dense. This lends great support to theories of the universe which maintain that the universe is changing with time, that is, it is evolving. Its existence is a very powerful argument against theories which propose that the universe is steady, that is, it is unchanging. In the theories of the latter type, it is extremely hard to produce such a radiation.

CMBR is a topic of intense research today.

15.6.2 Evolving vs. Steady State Universe

You have just seen that CMBR points to a phase of the universe when it was very hot and dense. This phase is generally known as the **'hot Big Bang'**. The idea is that some violent event took place at that time which sent the universe expanding. You have also seen above that if the early universe had not been hot and dense, it would not have been possible to synthesise light elements, such as H^2, H_e^4 and Lithium. In fact, the prediction of the precise observed abundances of these light elements is a very powerful argument in favour of the universe that changes with time: **an evolutionary universe**.

Yet there is a set of scientists who believe in a **steady state universe**, a universe which has no beginning and no end, which appears the same at all points in space and at all times. Historically, the steady state theory emerged in the1940s and 1950s, when observational techniques were not much developed, and so the Hubble constant H could not be measured accurately. Recall that $1/H$ gives a rough time scale of the age of the universe. So, the age of the universe inferred from the value of H at that time turned out to be less than the age of some fossils on the Earth. This was quite embarrassing. To overcome this age problem, the scientists proposed the steady state universe. However, CMBR and synthesis of light elements in the early universe are very powerful arguments against this theory and in favour of the hot Big Bang theory.

With this we come to an end of this unit in which you have studied about the large scale structure of the universe. We now present its summary.

15.7 SUMMARY

- Distances of far-off objects inform us about the large scale structure of the universe. To find distances of other galaxies we employ the concept of the **distance ladder**. The first step of the ladder is the Cepheid variable stars found in nearly galaxies. Subsequent steps include objects such as supernovae which can be detected even in galaxies which are very far off.
- The outcome of the exploration of the universe is that the galaxies are rushing away from one another and **the universe is expanding**.
- The expansion of the universe is in accordance with **Hubble's law**, $v = Hr$, so that $1/H$ gives a rough estimate of the age of the universe.
- The behaviour of the universe is governed by the **Friedmann equation:**

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G\rho}{3}$$

- The need for a hot and dense early universe can be explained in connection with the **synthesis of light elements** and the existence of the **cosmic microwave background radiation**.
- The present understanding is that if the universe was once hot and dense, then it must be an evolving universe, and not a steady state universe.

15.8 TERMINAL QUESTIONS *Spend* 30 *min.*

1. Explain how Cepheid variables have been used to measure astronomical distances.
2. State Hubble's law. How can this be used to get an estimate of the age of the universe?
3. Explain why at one time, the steady state theory appeared necessary. What is its status now?

15.9 SOLUTIONS AND ANSWERS

Self Assessment Questions (SAQs)

1. See Section 15.2.

2. $H = 70 \text{ km s}^{-1}\text{Mpc}^{-1}$

$$= 70\times10^5 \text{ cm s}^{-1} / 3\times10^{18}\times10^6 \text{ cm}$$

$$= \frac{7\times10^6}{3\times10^{24}} \text{ s}^{-1}$$

$$\therefore \text{ Age of the universe } = \frac{1}{H} = \frac{3\times10^{24}}{7\times10^6} \text{ s}$$

$$= \frac{3\times10^{24}}{7\times10^6\times3\times10^7} \text{ yr}$$

$$= 1.4\times10^{10} \text{ yr} = 14 \text{ billion years.}$$

3. Eq. (15.13) $\Rightarrow \quad \dfrac{\dot{a}^2}{a^2} = \dfrac{8\pi G}{3}\rho$

Since $\rho = Ca^{-n} \quad \Rightarrow \quad \dfrac{\dot{a}^2}{a^2} = \dfrac{8\pi G}{3} C\, a^{-n}$ [C is a constant]

$$\therefore \quad \frac{\dot{a}}{a} = \sqrt{\frac{8\pi GC}{3}}\, a^{-n/2} \quad \Rightarrow \quad \dot{a} = \frac{da}{dt} = A\, a^{1-\frac{n}{2}} \quad \left[A = \sqrt{\frac{8\pi GC}{3}}\right]$$

$$\therefore \quad \int a^{\frac{n-2}{2}}\, da = \int A\, dt \quad \Rightarrow \quad \frac{a^{n/2}}{n/2} \propto t \quad \Rightarrow \quad a \propto \left(n/2\right)^{2/n} .\, t^{2/n} .$$

Hence, proved.

Terminal Questions

1. See Text.
2. See Text.
3. See Text.