UNIT 8 STELLAR STRUCTURE

Structure

8.1 INTRODUCTION

In Unit 7, you have learnt about the classification of stars on the basis of their spectra. You know that the H-R diagram is obtained on the basis of the luminosity, and effective temperature of stars and enables us to classify them in the most comprehensive manner. A careful look at this diagram reveals that there are gaps between families of stars, e.g., between the main sequence and giants. You may wonder why such gaps should exist when we have such a large number of observable stars! Further, you may like to know: **What causes some ordinary stars to become giants and others to become dwarfs?** These and similar other questions cannot be answered on the basis of observations alone. We require the knowledge of the physical conditions in the interior of the stars. In other words, we need to know: How the temperature, pressure and density of a star vary in its interior? In the present unit, you will study the physical principles which form the basis for understanding the internal structure of stars.

You know that the Sun is emitting radiation at a constant rate and its diameter shows no significant variation with time. This implies that the Sun as well as other stars are in mechanical and thermal equilibria. In Sec. 8.2, you will study about hydrostatic equilibrium and its consequences for the variation of density and pressure inside a star. In Sec. 8.3, you will learn how to estimate the internal temperature of a star on the basis of the virial theorem: statement of relation between the kinetic and potential energies of a system in equilibrium. In addition to the considerations of hydrostatic and thermal equilibria, the mechanism of energy generation and transport play an important role in deciding stellar structures. In Sec. 8.4, you will learn why only the energy generated due to nuclear reaction needs to be considered as the source of stellar energy. You will also learn some important mechanisms of nuclear energy generation in stars. In Sec. 8.5, various modes of transportation of energy from the interior to the surface of stars have been discussed. You will discover the conditions for the formations of convective and radiative zones in the stellar interior.
Finally, in Sec. 8.6, we discuss the computation of a simple stellar model. We also discuss how the results of this model compare with the observations.

Objectives

After studying this unit, you should be able to:

- list the basic assumptions for the theoretical study of stellar structure;

- show that the values of interior pressure and temperature of a star are higher than the values at the surface by several order of magnitudes;
- explain that the nuclear energy generation is the only important energy generation process in stars;
- predict when a radiative or a convective zone will be formed in the stellar interior;
- compute polytropic stellar model and compare the theoretical results with observations; and
- solve numerical problems based on these concepts.

8.2 HYDROSTATIC EQUILIBRIUM OF A STAR

You know that stars, including the Sun, are made of hot gas. We cannot probe the interior of stars to determine their physical parameters and their variation with time and distance because of their high temperatures and enormous distances from the Earth. The question, therefore, is: **How do we determine the internal structure of a star?** Astrophysicists construct theoretical models of stars and compare their predictions with observations. To keep the theoretical analysis simple, the following assumptions are made:

i) ***The star is spherically symmetric:*** You know that stars have rotational motion which alters their spherical shape. Since the rotational motion is slow in most cases, it does not have appreciable effect on the shape of the stars. Spherical symmetry is, therefore, a valid assumption.

ii) ***The star is in dynamic equilibrium:*** Dynamic equilibrium means that the energy radiated by a star is equal to the energy supplied from its core. This assumption seems valid because luminosities of stars have been observed to be constant over a considerable period of time.

iii) ***The star is in thermally steady state:*** This implies that the temperature at *each point* within a star is constant over a considerable period of time. Note that this assumption does not mean that the entire interior of a star is at the same temperature.

Under these assumptions, the theoretical understanding of stellar structure is based on four equations based on certain fundamental principles of physics. First of all, let us consider the **principle of hydrostatic equilibrium**.

You know that the observable stellar parameters such as luminosity change very slowly. We can, therefore, safely assume that a star is in hydrostatic equilibrium, that is, it is neither expanding nor contracting (at least not very rapidly). This equilibrium is maintained by a balance between the force of gravity acting inwards and that due to the gradient of pressure of the gas acting outwards. To find out the consequences of the hydrostatic equilibrium, let us consider an element of volume dV at a point A inside a star at a distance r from its centre (Fig. 8.1). If $\rho(r)$ is the density of matter inside dV, that is, at a distance r from the centre, the mass enclosed in the volume element dV is $\rho(r)\,dV$. Further, if $M(r)$ is the mass inside the sphere of radius r, the gravitational force acting on the mass inside dV is given by:

$$\frac{GM(r)}{r^2}\rho(r)dV \tag{8.1}$$

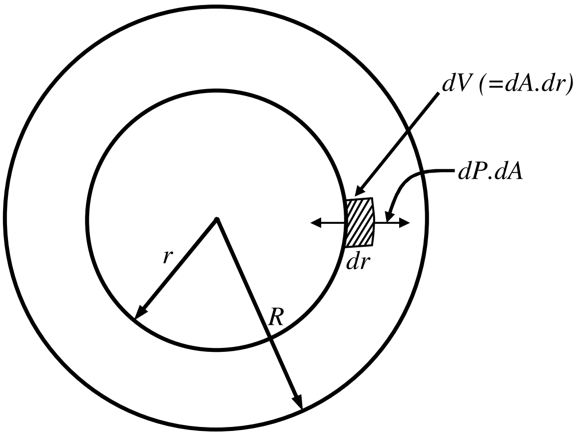

Fig.8.1: An element of volume *dV* at a distance *r* inside a star in hydrostatic equilibrium

Now, in view of the spherical symmetry of the star, the pressure, density and temperature may be taken as identical at all points over the spherical surface of radius *r*. Therefore, the net hydrostatic force acting on the volume element *dV* and pushing it outward can be written as:

$$dP \, . \, dA, \tag{8.2}$$

where *dA* is the area of the volume element *dV* perpendicular to *r* and *dP* is the pressure difference between two sides of the volume element along the radius. For hydrostatic equilibrium, the gravitational force must be equal and opposite to the hydrostatic force. Thus, on the basis of Eqs. (8.1) and (8.2), we can write:

$$\frac{GM(r)}{r^2}\rho(r)dV = -dP.dA$$

$$= -\frac{dP}{dr}.dV$$

This yields the equation of **hydrostatic equilibrium.**

Equation of Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r) \tag{8.3}$$

We can also write Eq. (8.3) as

$$\frac{dP}{dr} = -\rho(r)g(r) \tag{8.4}$$

where $g(r)$ is the acceleration due to gravity given by $g(r) = \frac{GM(r)}{r^2}$. From Eq. (8.3), it is obvious that *it is the pressure gradient that supports the star and not the pressure.* If we denote the mass of the whole star by *M* and its radius by *R*, the mean density, $<\rho>$ of the whole star can be expressed as:

$$<\rho> = \frac{M}{\left(\frac{4}{3}\pi R^3\right)} \tag{8.5}$$

Let us now consider a spherical shell of the star between radii r and $r + dr$. The volume of the matter enclosed in this shell is $4\pi r^2 dr$. Since $\rho(r)$ is the density of stellar matter at distance r, the mass of this spherical shell is:

$$dM(r) = 4\pi r^2 \rho(r)\, dr$$

Thus, we can express the total mass inside the sphere of radius r as:

$$M(r) = \int_0^r \rho(r)\, 4\pi r^2\, dr$$

Differentiating both sides of the above equation with respect to r, we get the **mass continuity equation** for the star:

Mass continuity equation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \qquad (8.6)$$

Eqs. (8.3) and (8.6) constitute two basic equations of stellar structure. Further, the state of hydrostatic equilibrium in a star enables us to obtain a relation between its gravitational potential energy and the kinetic energy of its constituent particles. This relation is known as the ***virial theorem***. You will learn about it now.

8.3 SOME INSIGHT INTO A STAR: VIRIAL THEOREM

You may be aware that the virial theorem is applicable for a system of perfect gas particles. However, this theorem can also be applied to a star because it (star) can be considered as a system of free particles. To obtain the relation between the potential and kinetic energies of a star, let us consider the equation of hydrostatic equilibrium (Eq. (8.3)). On multiplying Eq. (8.3) by $4\pi r^3$ and integrating over the radius of the star, we get:

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R \rho(r) \frac{GM(r)}{r^2} 4\pi r^3 dr \qquad (8.7)$$

On integrating by parts, the left hand side of Eq. (8.7) gives:

Actually, the pressure on the surface of a star is not zero. Its value is, however, much smaller than the pressure in the interior. Therefore, it can be assumed to be zero. This is also the case with density.

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = P\, 4\pi r^3 \Big|_0^R - \int_0^R 3P 4\pi r^2 dr$$

Since $P = 0$ at $r = R$, the first integral on the right hand side vanishes and the above equation reduces to:

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -3\int_0^R P\, 4\pi r^2 dr$$

Thus, Eq. (8.7) can be written as:

$$-3\int_0^R P\, 4\pi r^2 dr = -\int_0^R \rho(r) \frac{GM(r)}{r} 4\pi r^2 dr \qquad (8.8)$$

Now, assuming that the star comprises of monoatomic gas, its total thermal (or internal) energy U can be written as:

$$2U = 3\int_0^R P\, 4\pi r^2 dr \qquad (8.9)$$

SAQ 1

Spend 3 min.

Derive Eq. (8.9).

Substituting Eq. (8.9) in Eq. (8.8), we obtain:

$$2U = \int_0^R \rho(r) \frac{GM(r)}{r}\; 4\pi\, r^2 dr \qquad (8.10)$$

The right hand side of Eq. (8.10) can be expressed in terms of the gravitational potential energy. The gravitational potential due to the mass $M(r)$ inside the sphere of radius r is $-\frac{GM(r)}{r}$. Therefore, total potential energy due to all mass elements dM (= $4\pi r^2 \rho(r)\, dr$) of the star is:

$$\Omega = -\int_0^M \frac{GM(r)}{r} dM = -\int_0^R \frac{GM(r)}{r} 4\pi r^2 \rho(r) dr \qquad (8.11)$$

Thus, on the basis of Eqs. (8.10) and (8.11) we get the **virial theorem**.

Virial theorem

$$2U + \Omega = 0 \qquad (8.12)$$

You can see from Eq. (8.11) that the gravitational potential energy Ω of the whole star can be determined only if we know the variation of density $\rho(r)$ inside. Assuming that $\rho(r) \approx <\rho>$, the mean density of stellar matter, we may write:

$$M(r) \approx \frac{4}{3}\pi r^3 <\rho> \qquad (8.13)$$

Substituting $<\rho>$ for $\rho(r)$ and $M(r)$ from Eq. (8.13) in Eq. (8.11) and integrating, we get:

$$\Omega \approx -\frac{3}{5}\frac{GM^2}{R} \qquad (8.14)$$

Now, to find an expression for the internal energy of a star, we make use of the equation of state of a gaseous system:

$$P = \frac{\rho}{\mu m} k_B T \qquad (8.15)$$

$$M(r) = \frac{4}{3}\pi r^3 <\rho>$$

$$dM = \frac{4}{3}\pi . 3r^2 <\rho> dr$$

$$\frac{dM}{<\rho>} = 4\pi r^2 dr$$

where k_B (= 1.38×10^{-23} J K^{-1}) is the Boltzman constant, T is the temperature and μm is the mean mass of a gas particle. For pure hydrogen gas, $\mu = 1$. Substituting Eq. (8.15) in Eq. (8.9), we get:

$$U = \frac{3}{2}\int_0^R P\, 4\pi r^2 dr = \frac{3}{2}\int_0^M \frac{k_B T}{m} dM$$

since $4\pi r^2 dr = dV$ and $\rho dV = dM$. If we define the mean temperature of the star, $< T > = \frac{1}{M}\int_0^M T dM$, then the above expression can be written as:

$$U = \frac{3}{2}\frac{k_B}{\mu m} M < T > \tag{8.16}$$

Substituting the expressions for potential energy (Eq. (8.14)) and internal energy (Eq. (8.16)) in the virial theorem (Eq. (8.12)), we get:

$$< T > = \frac{1}{5}\frac{\mu m G}{k_B}\frac{M}{R}$$

Thus

$$< T > \propto M^{2/3} < \rho >^{1/3} \tag{8.17}$$

where R has been expressed in terms of the mean density.

Spend 10 *min.*

SAQ 2

a) Verify the results contained in Eqs. (8.14) and (8.17).

b) Assume that the Sun is made of pure hydrogen ($\mu = 1$). Show that the mean temperature of the Sun is $< T > \cong 4 \times 10^6$K.

On solving SAQ 2, you must have appreciated that the internal temperature of the Sun can be estimated without making any detailed calculations. Further, it is obvious from Eq. (8.17) that if two stars have the same mass, the denser one will be hotter. For a sun-like star, the effective surface temperature is $T_e \approx 5780$ K. And from the solution of SAQ 2, we find that the mean solar temperature, $< T > \approx 4 \times 10^6$ K. This means that the internal temperature must be much higher. You may ask: **What causes such high internal temperature in stars?** To answer this, we must investigate the sources of energy generation in stars. The stars can have possibly three kinds of energy sources: **gravitational**, **chemical** and **nuclear**. You will study about them now.

8.4 SOURCES OF STELLAR ENERGY

Let us first consider the gravitational energy as the source of energy for stars. From Eq. (8.14), note that when the whole matter of the star is scattered at infinity, its gravitational potential energy is zero. When the stellar matter is assembled to make a star of radius R, its potential energy becomes $-\frac{3}{5}\frac{GM^2}{R}$. This means that during the gravitational contraction of a star, that is, during the formation of a star, an energy equal to $\frac{3}{5}\frac{GM^2}{R}$ is released.

It can be shown (TQ 2) that the Sun's energy would last only for about 10^7 yrs if gravitational energy was its only source. But, the results obtained on the basis of radioactive dating of different types of meteorites, deep terrestrial oceanic sediments and lunar rocks suggest an age of $\approx 5 \times 10^9$ years for the Sun. Thus, *the gravitational potential energy cannot be the source for solar luminosity*. We must look for some other source of energy for stars like the Sun.

We are now left with two other possible processes, namely, chemical and nuclear.

The possibility of a chemical process as the source of energy in stellar interior is also ruled out on the basis of results obtained in the following example.

Example 1

Assume that the Sun consists of hydrogen and oxygen and the proportion of these elements is such that the entire solar material could be burned and transformed to water vapour. Show that the total energy available from this process would last only for ~ 10^4 yr given that 10 eV is liberated in the formation of each water molecule.

Solution

The molecular weight of water (H_2O) is 18 u = $18 \times 1.6 \times 10^{-27}$ kg = 28.8×10^{-27} kg. The mass of the Sun is 2×10^{30} kg. Therefore, the total number of water molecules present in the Sun is given by:

$$\frac{2\times10^{30}\,\text{kg}}{28.8\times10^{-27}\,\text{kg}} \approx 6\times10^{55}.$$

Thus, total energy liberated due to the formation of water vapours in the Sun is:

$$6 \times 10^{55} \times 10\ \text{eV} \times 1.6 \times 10^{-19}\ \text{J (eV)}^{-1} \approx 9.6 \times 10^{37}\ \text{J}.$$

Since the Sun radiates energy at the rate ~ 4×10^{26} Js^{-1}, the duration over which the Sun would radiate all its energy generated due to formation of water molecules is:

$$\approx \frac{9.6\times10^{37}\,\text{J}}{4\times10^{26}\,\text{Js}^{-1}} \approx 2.4\times10^{11}\ \text{s} \sim 10^{4}\ \text{yr}.$$

In view of the fact that the Sun has an estimated age of ~ 5×10^9 yr, the result of the above example clearly shows that chemical process cannot be responsible for generation of energy in stars.

Let us now look at the possibility of nuclear processes for generating energy in a sun-like star. You may recall from your school physics that nuclear reactions are of two types: **fission reactions** and **fusion reactions**. In nuclear fission, large unstable nuclei like ^{238}U break into smaller nuclei and energy is released. Since the abundance of such nuclei is negligible in stars, such a process is also ruled out as a source of stellar energy. We are, therefore, left with only fusion process to be considered as possible energy source.

In nuclear fusion process, two lighter nuclei combine and form a new nucleus and energy is released. The amount of energy released depends on the binding energy per nucleon of the elements involved in the reaction. You may recall that the binding energy of a nucleus is the energy required to separate its constituent nucleons by a large distance. Refer to Fig. 8.2 which depicts binding energy curve. Note that as the mass number increases from zero, the value of binding energy per nucleon increases. This implies that if two or more, lighter nuclei (such as hydrogen) are fused together to create a relatively heavier nuclei (such as ^{3}He or ^{4}He), we will have surplus of energy. This is precisely what happens in fusion reactions: enormous amount of energy would be released due to fusion of hydrogen nuclei and consequent production of helium. Further, Fig. 8.2 also shows that the value of binding energy per nucleon saturates at around mass number 50 and shows a very slow decrease beyond. The nature of the binding energy curve, therefore, indicates that small-mass as well as large-mass nuclei are less tightly bound than medium mass (such as Fe) nuclei.

Before proceeding further, you may like to convince yourself whether or not the energy generated due to nuclear fusion can account for the observed luminosity of the Sun. To do so, solve the following SAQ.

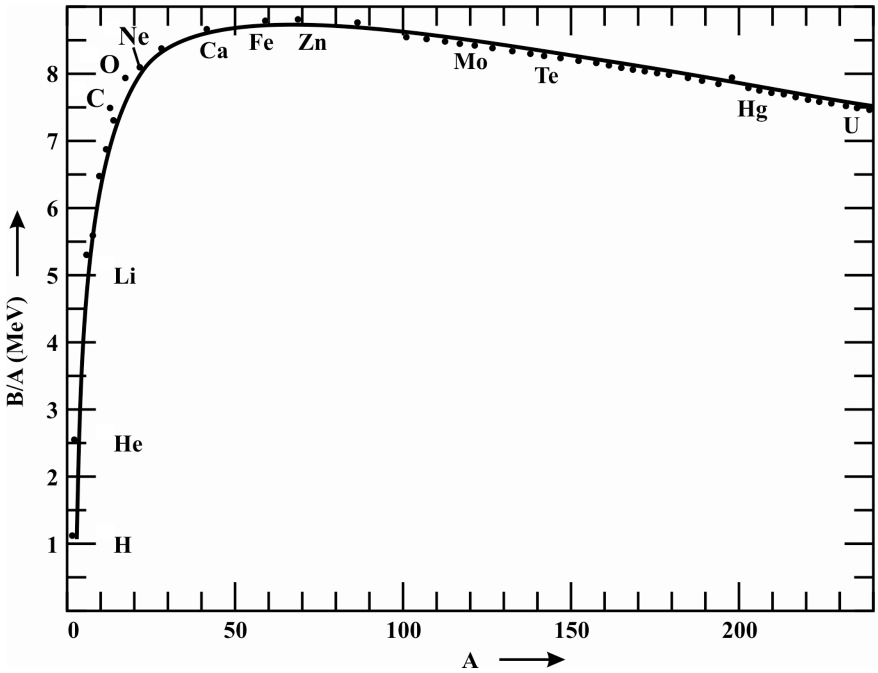

Fig.8.2: Variation of the binding energy per nucleon with mass number

Spend 5 min.

SAQ 3

Assume that originally the Sun comprised only of hydrogen and that the inner 10 percent of the Sun's mass could be converted into helium. For how long would the Sun be able to radiate at the rate given by $L = 4\times10^{26}$ Js^{-1}. The mass of the Sun is $M_\Theta = 2\times10^{30}$ kg.

On solving SAQ 3 you find that the Sun would indeed be able to radiate energy at the present rate for another 5 billion years by fusing hydrogen into helium.

Now the next logical question is: **How is the fusion of hydrogen nuclei into helium nuclei made possible?** In other words, you may like to know what the pre-conditions for nuclear fusion to take place are and how these conditions are obtained in the stellar interiors. Nuclear fusion involves fusion of two positively charged nuclei against Coulomb repulsion.

To know the amount of energy released due to fusion of four hydrogen nuclei and formation of a helium nucleus, note that the total mass of four hydrogen atoms is 4.031280 u while that of a helium atom is 4.002603 u. The mass defect is 0.02867u and in accordance with the Einstein equation, $E = mc^2$, the energy released due to fusion of four hydrogen nuclei into a helium nucleus is equal to (0.02867 u) $\times c^2$ where c is the velocity of light.

The possibility of such a process can be understood on the basis of the potential energy curve that an atomic nucleus would experience when it approaches another atomic nucleus (Fig. 8.3). Note that the potential energy curve consists of two distinct regions. Region I corresponds to the situation when separation between the two nuclei is such that their potential energy is due to Coulomb repulsion. Region II represents the situation when two nuclei are very close to each other and the curve is in the form of a potential well. The potential well illustrates the strong nuclear forces that bind the nuclei. From Fig. 8.3, it is obvious that the two nuclei can fuse only when the approaching nuclei are able to overcome the repulsive Coulomb barrier. This means that the approaching nuclei have sufficient energy to overcome the barrier.

The question is: **At what temperature, the approaching nuclei will have sufficient energy?** Using classical dynamics, we find that the temperature required to provide sufficient energy to two nuclei so that they overcome Coulomb's barrier is much higher than the core temperature T_c (~ 1.5×10^7 K) of the Sun. Thus, classical physics fails to explain the possibility of fusion reactions in stars. You must convince yourself about the inadequacy of classical physics to explain nuclear fusion by solving the following SAQ.

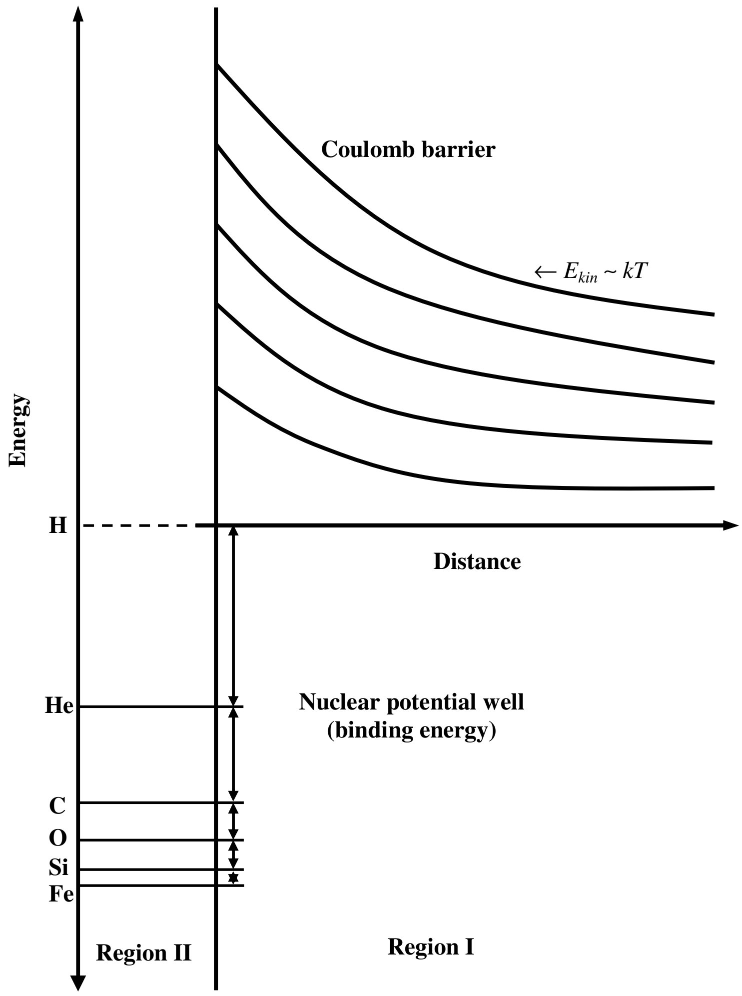

Fig.8.3: Schematic representation of the potential energy barrier experienced by an atomic nucleus approaching another atomic nucleus

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SAQ 4

Spend 5 min.

Suppose that two nuclei have charges Z_1e and Z_2e and in order to interact, they must be separated by a distance ~ 10^{-13}m. Calculate their mutual potential energy. If their relative kinetic energy is 3 k_BT, calculate the temperature required by two hydrogen nuclei to overcome this potential barrier.

The temperature in the core of the Sun is ~ 1.5×10^7K. At this temperature, only a few proton-proton fusion can take place. There is, however, a finite probability that particles with insufficient energy can tunnel thorough the potential barrier and react. Now, let us look at some thermonuclear fusion reactions through which lighter nuclei fuse and release energy.

In stars like the Sun, one of the prominent thermonuclear reactions is the so-called **proton-proton chain** or **p – p chain**. This reaction proceeds sequentially through three steps as given below:

$$\mathrm{H}^1 + \mathrm{H}^1 \rightarrow \mathrm{H}^2 + \overline{e} + \nu, \ \ (1.19\,\mathrm{MeV})$$

$$\mathrm{H}^1 + \mathrm{H}^2 \rightarrow \mathrm{He}^3 + \gamma, \ \ (5.49\,\mathrm{MeV})$$

$$He^3 + He^3 \rightarrow He^4 + H^1 + H^1, \ (12.85\,MeV)$$

Note that in these reactions, four hydrogen nuclei combine to form a helium nucleus. The amount of energy liberated at each step is given within brackets. The p – p chain produces most of the energy in Sun and other such stars.

Further, a relatively smaller amount of energy in the Sun is also generated by the thermonuclear reaction known as **carbon-nitrogen-oxygen (CNO)** cycle as given below:

$$\begin{aligned} H^1 + C^{12} &\rightarrow N^{13} + \gamma, \ (1.45\,MeV) \\ N^{13} &\rightarrow C^{13} + \bar{e} + \nu, \ (2.22\,MeV) \\ H^1 + C^{13} &\rightarrow N^{14} + \gamma, \ (7.54\,MeV) \\ H^1 + N^{14} &\rightarrow O^{15} + \gamma, \ (7.35\,MeV) \\ O^{15} &\rightarrow N^{15} + \bar{e} + \nu, \ (2.71\,MeV) \\ H^1 + N^{15} &\rightarrow C^{12} + He^4, \ (4.96\,MeV). \end{aligned}$$

You may note that in the CNO cycle, the carbon destroyed in the first step gets regenerated in the last step. Further, similar to the p – p chain, CNO cycle also produces a helium nucleus from four hydrogen nuclei. *A* CNO *cycle, however, cannot begin unless carbon is present.*

The thermonuclear energy generated in the stars depends on the abundance of fusionable matter and the interior temperature of the stars. This provides us a basis to link the luminosity of stars with its mass. Let ε denote the rate of energy generated per unit mass in thermonuclear reactions. Then, the luminosity dL caused by an element of mass dM can be written as:

$$dL = \varepsilon \, dM \qquad (8.18)$$

Since $dM = 4\pi r^2 \rho(r)\, dr$, Eq. (8.18) can be expressed as:

$$\frac{dL}{dr} = \varepsilon 4\pi r^2 \rho(r) \qquad (8.19)$$

Eq. (8.19) is one of the **basic equations of stellar structure**.

The energy generated at the core of a star must flow towards its surface because the temperature of the core is very high compared to the star's surface. Further, since the star's surface continuously radiates energy, it will cool off unless the radiated energy is replaced. Energy transport in the star has important consequences for its structure because the transport process determines the temperature (and pressure) of different layers of the star's interior. Would you not like to know what the different mechanisms of energy transport in a star are? This is the subject matter of the next section.

8.5 MODES OF ENERGY TRANSPORT

There are three basic energy transport processes: **conduction**, **radiation** and **convection**. Fig. 8.4 shows a schematic diagram for these three processes.

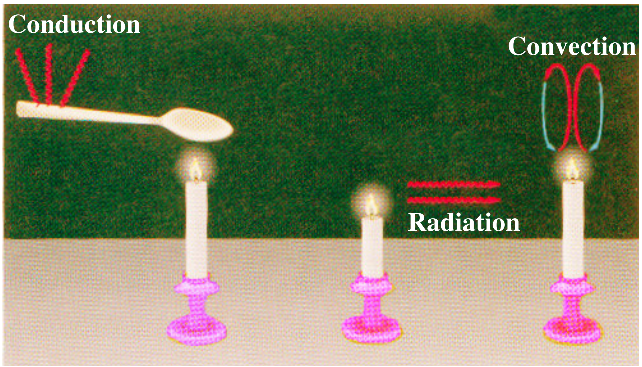

Fig.8.4: Three modes of energy transport: conduction, radiation and convection

Conduction is the most familiar form of heat flow and this process works through vibrating atoms. The energy of the vibrating atoms is transferred to the nearby cooler atoms by collisions and energy transport through conduction works better in solids and not in gases. You will learn later in this Unit that conduction is responsible for energy transfer in stars like white dwarfs whose interior is in a crystallised form having density ~ 10^6g cm^{-3}. For ordinary gaseous stars, conduction process is not important.

Radiation is the next most familiar mode of energy transport and is responsible for energy transport in some layers of the interiors of almost all the stars. The process of energy generation in the central region of a star produces very high energy photons which are γ-rays. As these photons travel outwards, they collide with matter. At each collision, γ-ray photon loses energy and when it reaches the surface, its frequency lies in the visible range. The progress of photons is extremely slow as they travel outward and this, in fact, regulates the solar luminosity at the level of ~ 10^{26} Js^{-1}. The absorption of the energy of γ-rays by the stellar gas, is characterized by the **absorption coefficient**, also known as the **opacity**, k_λ of the gas. The subscript λ indicates that the absorption depends on the wavelength. The important sources of opacity at high temperatures inside a star are:

i) *Electron scattering:* The scattering of photons by free electrons.
ii) *Photoionisation*: The energy of photon is used for successive ionisation of atoms/ions.

We can obtain an expression for the opacity on the basis of qualitative arguments.

Consider a slab of the stellar gas of thickness dx. Let F_λ denote the flux of radiation that strikes at one end. If the mass density of the gas is ρ, the amount of radiation absorbed by the slab is proportional to i) the density of matter in the slab ii) the incident flux and iii) the thickness of the slab. Thus, the amount of flux absorbed is:

$$dF_\lambda \propto \rho F_\lambda dx$$

$$= -k_\lambda \rho F_\lambda dx, \qquad (8.20)$$

where the minus sign indicates absorption. Photons generated inside the Sun do not reach the solar surface directly; they are scattered by the electrons and nuclei. This scattering is isotropic and thus their forward and backward scattering is equally likely. The travel of photons inside a star is, therefore, like that of a drunken person. It is also called the random walk. It takes thousands of years for these photons to reach the star's surface. To have an idea about the time taken by a photon to reach star's surface from the core, go through the following example carefully.

Example 2

Suppose that the energy transport due to radiation process is analogous to the random walk. Compute the time taken by a photon, generated in the core of the Sun, to reach the solar surface. Given that for the Sun, the mean free path is $\ell \sim 0.5$ cm for photon at an average density and temperature of 1.4 g cm^{-3} and 4.5×10^6 K, respectively.

Solution

You may recall from Thermodynamics and Statistical Mechanics (PHE-06) course that, according to the theory of random walk, we can write the mean square distance, $< D^2 >$ moved in N steps as:

$$< D^2 > = N\ell^2$$

This gives $N = \dfrac{< D^2 >}{\ell^2}$, the number of steps required to travel a distance $< D^2 >$ in steps of size ℓ in one dimension. For three dimensional space, $N \sim 3\dfrac{< D^2 >}{\ell^2}$. Now, for the photon at the core which has to reach the surface, we have $< D^2 >= R^2$. Hence, the time taken for the photon to reach the solar surface is given by

$$t = \frac{3R^2}{c.\ell}$$

since, in each step, the time taken is ℓ/c where c is the velocity of light. Substituting the values of R, c and ℓ, we get

$$t \cong 30{,}000 \text{ yr.}$$

Now, you may like to know: **How does the radiative transport of energy give rise to temperature gradient in stars?** To find out, let us consider a thin spherical shell around a spherical surface of radius r as shown in Fig. 8.5. Let the temperature of the sphere be T. Since the spherical region within radius r acts as a source of black body radiation, the radiative flux incident on the inner side of the shell can be expressed by (Stefan Law):

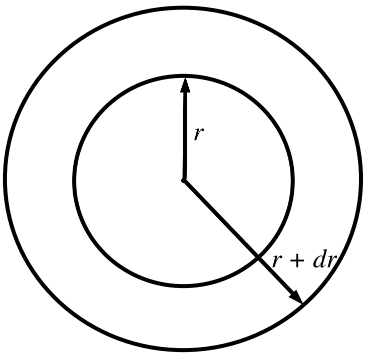

Fig.8.5: Spherical shell of thickness *dr* around a surface of radius *r*

$$F(r) = \sigma T^4, \tag{8.21}$$

where σ is the Stefan's constant. Similarly, the radiative flux emerging outward from the shell surface at $r + dr$ is:

$$F\,(r + dr) = \sigma\,(T + dT\,)^4 \tag{8.22}$$

where $T + dT$ is the temperature at the surface of radius $r + dr$. It is important to mention here that dT is negative because the surface at $r + dr$ is cooler than the inner surface of the shell located at r. Since dT is very small compared to T, we can expand $(T + dT)^4$ using binomial expansion. Doing so, we get:

$$F\,(r + dr) = \sigma\, T^4 + 4\sigma\, T^3\, dT$$

Therefore, the flux absorbed by the shell can be written as:

$$dF = F(r + dr) - F(r) = 4\sigma T^3\, dT \qquad (8.23)$$

Combining Eqs. (8.20) and (8.23), we get:

$$4\sigma T^3 dT = -\, k(r)\, \rho(r)\, F(r)\, dr \qquad (8.24)$$

Since luminosity, $L(r) = 4\pi r^2 F(r)$, we can rewrite Eq. (8.24) as:

$$\frac{dT}{dr} = -\frac{k(r)\rho(r)}{4\sigma T^3} F(r)$$

$$= -\frac{k(r)\rho(r)}{acT^3}\left(\frac{L(r)}{4\pi r^2}\right) \qquad (8.25)$$

since $\sigma = \frac{ac}{4}$, where a is a constant and c is the velocity of light. Eq. (8.25) gives the temperature gradient within a star due to radiative transport of energy. This expression needs to be multiplied by an extra factor of $\frac{3}{4}$ on the right hand side so that it becomes consistent with the one obtained by incorporating the details of such a process. We, therefore, write the temperature gradient as:

$$\boxed{\frac{dT}{dr} = -\frac{3}{4}\,\frac{k(r)\rho(r)}{acT^3}\left(\frac{L(r)}{4\pi r^2}\right)} \qquad (8.26)$$

Eq. (8.26) is yet another basic equation of stellar structure.

Now, let us consider the third mode of energy transport, namely, convection which plays an important role in stars. Convection refers to the process in which heat energy is transported by mass motion, i.e., by transport of the hot/cool matter itself. You are familiar with convection currents or bubbles moving up and down when water in a beaker is gradually heated from below. Such a motion also takes place in certain regions in stars with hot fluid masses rising outward releasing their heat energy and the cooler matter sinking downward to receive more energy. Fig. 8.6 depicts the photospheric granulation which is a strong evidence supporting convective transport of energy at the base of Sun's photosphere. Convection causes mixing of the stellar

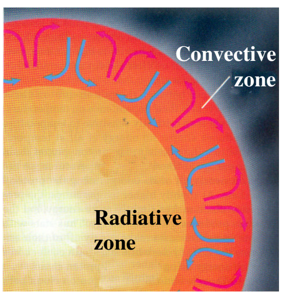

Fig.8.6: Photospheric granulation of the Sun which is caused due to convective transport of energy in which the hot matter comes out to the solar surface from layers below the surface

constituents in certain regions inside a star and produces homogeneity of chemical composition by transferring heavy elements from interior to the surface. You may ask:

Under what condition(s) does convection become the dominant mode of energy transport? *The process is dominant when the temperature gradient becomes too steep*. Steep temperature gradients are generally created in regions with high opacity which restrains the flow of energy through radiative process.

To determine the temperature gradient in a convective region of a star, we consider a situation where hot bubbles of gas rise up and expand *adiabatically*. After rising through a characteristic distance, the bubbles lose extra heat and get mixed up with the surroundings. For such a process, the bubble's adiabatic temperature gradient is given by:

$$\left[\frac{dT}{dr}\right]_{ad} = \left(\frac{\gamma-1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \tag{8.27}$$

Eq. (8.27) can be obtained by using the adiabatic relation $P = K\rho^{\gamma}$ and the equation of state of the gas, $P = \frac{k_B}{\mu m}\rho T$. Also, from the equation of state, we have $P = Nk_BT$, where N is the number of particles per unit volume. In astrophysics, it is usual to take the mass of a particle as μm, where μ is called the *mean molecular weight* and m is the *mass of a proton*. Further, for $\frac{dP}{dr}$ we use Eq. (8.3) in Eq. (8.27). With these substitutions, it is possible to write Eq. (8.27) as:

$$\left[\frac{dT}{dr}\right]_{ad} = -\left(\frac{\gamma-1}{\gamma}\right)\frac{\mu m}{k_B}\frac{GM(r)}{r^2} \tag{8.28}$$

Eq. (8.28) is one of the basic equations of stellar structure. You should note that only if the actual temperature gradient in a star is steeper than the adiabatic gradient given by Eq. (8.27), convective transport of energy can take place. We call the actual temperature gradient in such a case as **superadiabatic**. Therefore, for convection to take place, we must have:

$$\left[\frac{dT}{dr}\right]_{actual} > \left[\frac{dT}{dr}\right]_{ad} \tag{8.29}$$

In fact, it can be shown that convection dominates the radiative transport of energy in a region if $\left[\frac{dT}{dr}\right]_{actual}$ is slightly superadiabatic. In any case, the actual mode of energy transport in a region inside a star depends on the temperature gradient existing there.

Now, you should pause for a moment and think what you have learnt so far. You have learnt to derive certain equations on the basis of the principles of physics connecting various parameters of a star. These equations are known as the basic equations of stellar structure. You may ask: Why did we do all this? How do equations of stellar structure help enhance our understanding of stars? These equations are used to develop theoretical models of stars. If the predictions of these models are in agreement with observations, we can conclude that the assumptions made about the parameters of stellar interior are valid. In case of disagreement between theoretical prediction and observations, the models are 'fine-tuned' by modifying the initial assumptions. You will indeed appreciate that this is the only way to investigate the interior of stars because we simply cannot look into those interiors. In the next section, you will learn to develop a stellar model on the basis of equations of stellar structure.

8.6 SIMPLE STELLAR MODEL

Developing a stellar model essentially involves solving the equations of stellar structure for a star. As such, it is a very complex task because large numbers of equations with several unknowns need to be solved. This does not mean that we cannot get a physical picture of a star. We find that, with some valid approximations, simple stellar models are easier to calculate. Such models help understand the basis of some of the empirical laws, such as, mass luminosity relation. Before discussing any stellar model, let us first list the basic equations of stellar structure.

Basic Equations of Stellar Structure

In the previous sections, we used the following basic physical principles to obtain the equations of stellar structures:

- Hydrostatic equilibrium,
- Equation of state for stellar matter,
- Mechanism of stellar energy generation, and
- Modes of energy transport in stellar interior.

These basic equations are used to compute theoretical stellar models. This is equivalent to "constructing a theoretical star"! Once different models are computed, their location in H-R diagram is found out. It is so because the H-R diagram sets a detailed standard to be met by any theory of stellar structure and evolution. A theoretical model is wrong if the physical characteristics of the computed "star" are such that its location falls in the gap or empty regions of the H-R diagram. The physical parameters of a star at any point in its interior are temperature, $T(r)$, pressure $P(r)$, density $\rho(r)$, and luminosity $L(r)$. The **basic equations of stellar structure** are:

Hydrostatic equilibrium: $$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r) \qquad (8.30)$$

Mass continuity : $$\frac{dM}{dr} = 4\pi r^2 \rho(r) \qquad (8.31)$$

Energy transport: (radiative) $$\frac{dT}{dr} = -\left(\frac{3}{4ac}\right)\frac{k\rho}{T^3}\frac{L(r)}{4\pi r^2} \qquad (8.32)$$

Energy transport: (convective) $$\frac{dT}{dr} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m}{k_B}\frac{GM(r)}{r^2} \qquad (8.33)$$

Energy generation: (Thermal equilibrium) $$\frac{dL}{dr} = 4\pi r^2 \rho(r)\varepsilon(r) \qquad (8.34)$$

Equation of state: $$P(r) = R\rho(r)T(r) \qquad (8.35)$$

In the above equations, $\varepsilon(r)$ is the thermonuclear energy production rate per unit mass. You have learnt about it earlier in relation with luminosity. Further, the opacity k occurring in these equations depends on temperature and density of the gas. In fact, the exact form of the opacity relation depends on the process responsible for it. Computation of stellar opacity is, however, a complex process and is usually approximated by **Kramer's opacity relation** given as

$$k = const.Z(1+X)\frac{\rho}{T^{3.5}} \quad (8.36)$$

where X denotes the amount of hydrogen in a gram of stellar matter (it is also called the **abundance** of hydrogen), Z denotes the abundance of heavier elements. (In astrophysics, elements heavier than helium are called heavier elements.) The opacity relation given by Eq. (8.36) is valid for stars on the main sequence.

To obtain the values of physical parameters by integrating the stellar structure equations, we invoke the following boundary conditions:

$$M(r) = 0 \quad \text{and} \quad L(r) = 0 \quad at\ r = 0 \ \text{(the centre of star),} \quad (8.37)$$

and

$$M(r) = M; \quad L(r) = L \quad \text{and} \quad T(r) = T_{\text{eff}} \quad \text{at} \quad r = R \ \text{(surface of a star)} \quad (8.38)$$

With this background information, we are now in a position to discuss a stellar model.

8.6.1 Polytropic Stellar Model

In such a model, there is no need to know the actual source of energy generation in the star. Further, we assume that any change in the equilibrium structure of a star takes place in such a way that the specific heat remains constant, i.e.,

$$\frac{dQ}{dT} = C = \text{ constant} \quad (8.39)$$

where C denotes the heat capacity when neither pressure (P) nor volume (V) is constant. We call such a change as a **polytropic change**. *An adiabatic or an isothermal change, therefore, represents a polytropic change of zero and infinite heat capacities, respectively*. Instead of the adiabatic relation $dQ = 0$, we now have $dQ = CdT$. In such a situation, the first law of thermodynamics

$$dQ = C_V dT + PdV$$

takes the form

$$CdT = C_V dT + PdV$$

Now, using the equation of state for a perfect gas, $PV = RT$ and the fact that $R = C_P - C_V$, we can write the above expression as:

$$(C - C_V)\frac{dT}{T} = (C_P - C_V)\frac{dV}{V} \quad (8.40)$$

where C_P and C_V are the specific heats at constant pressure and constant volume, respectively. Let us now define an exponent γ' similar to the adiabatic exponent γ:

$$\gamma' = \frac{C_P - C}{C_V - C} \quad (8.41)$$

Thus, we can write from Eq. (8.40) that

$$TV^{\gamma' - 1} = \text{const}, \qquad PV^{\gamma'} = \text{const}. \quad (8.42)$$

SAQ 5

Spend 5 min.

Derive Eq. (8.42).

It is usual to express the physical variables, e.g., density, pressure and temperature in terms of the **polytropic index** *n* defined as:

$$n = \frac{1}{\gamma' - 1} \tag{8.43}$$

Since $PV^{\gamma' - 1} = const.$, we can write:

$$P = K\rho^{1+\frac{1}{n}} \tag{8.44}$$

where K is a constant. Further, the density is expressed in terms of a non-dimensional parameter θ defined in terms of the central density ρ_c as

$$\rho = \rho_c\,\theta^{\,n}. \tag{8.45}$$

We, therefore, get the following expressions for pressure and temperature:

$$P = P_c\,\theta^{\,n+1} \tag{8.46a}$$

$$T = T_c\,\theta \tag{8.46b}$$

where $P_c = K\rho_c^{\frac{n+1}{n}}$ and $T_c = \left(\frac{\mu m}{k_B}\right)K\rho_c^{1/n}$. We shall see below that P_c and T_c are the central pressure and temperature.

SAQ 6

Spend 5 min.

Derive Eq. (8.46a) and (8.46b).

With this formal introduction to polytropic changes, let us consider the following stellar structure equations:

Hydrostatic Equilibrium: $$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r) \tag{8.47}$$

Mass continuity: $$\frac{dM}{dr} = 4\pi r^2 \rho(r) \tag{8.48}$$

Substituting Eq. (8.48) in Eq. (8.47) and rearranging terms we can write:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\,\rho(r) \tag{8.49}$$

Substituting for P and ρ from Eqs. (8.44) and (8.45) in Eq. (8.49), we get:

$$\left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = -\theta^n \tag{8.50}$$

To write Eq. (8.50) in a simpler form, let us introduce a dimensionless variable ξ as follows:

$$r = \alpha\xi; \tag{8.51}$$

where $\alpha = \left[\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right]^{\frac{1}{2}}$. Substituting Eq. (8.51) in Eq. (8.50) and rearranging terms we get:

> **Lane-Emden** equation
>
> $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \tag{8.52}$$

Eq. (8.52) is known as **Lane-Emden** equation. Solution of this equation, for a given n, provides the **density** and **pressure profile** inside a star. The boundary conditions under which this equation must be solved are:

$$\theta = 1 \quad \text{and} \quad \frac{d\theta}{d\xi} = 0 \qquad \text{at} \quad \xi = 0. \tag{8.53}$$

θ is also known as ***Lane-Emden's function***.

Analytical solutions of Eq. (8.52) with the specified boundary conditions are possible only for n = 0, 1 and 5. The analytical expressions for the Lane-Emden functions for these values of n are:

$$\begin{aligned} n = 0;\ \ \theta_0 &= 1 - \frac{\xi^2}{6} \\ n = 1;\ \ \theta_1 &= \frac{\sin(\xi)}{\xi} \\ n = 5;\ \ \theta_5 &= \left(1 + \frac{\xi^2}{3}\right)^{-\frac{1}{2}} \end{aligned} \tag{8.54}$$

Fig.8.7 shows the density profile inside a polytrope for n = 1.5 and 3.

Spend 10 *min.*

SAQ 7

Verify that the Lane-Emden equation (Eq. (8.52)) is satisfied by the solutions given by Eq. (8.54).

The general solution of the Lane-Emden equation is in the form of a series for θ_n as given below:

$$\theta_n = 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 - \ldots \tag{8.55}$$

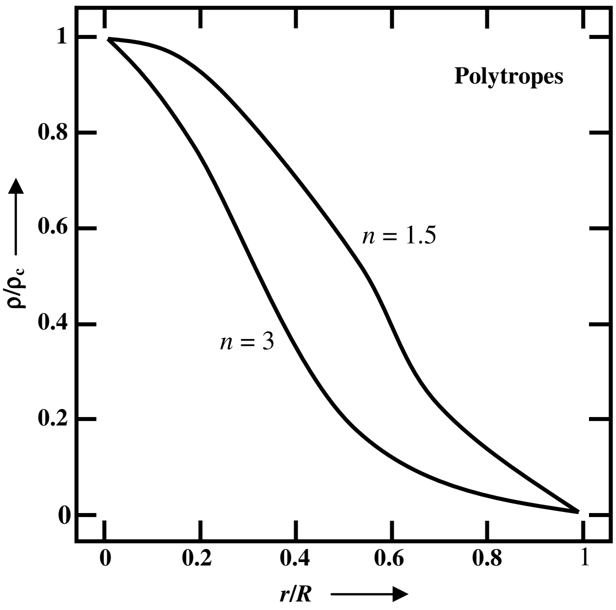

Fig.8.7: Density profile of a polytropic star

Current stellar models calculated for the Sun that fit the observations indicate that a large fraction of hydrogen at the centre of the Sun has been converted to helium (estimated 40 percent hydrogen and 60 percent helium). Energy transport is still radiative in the solar interior upto ~ 0.73 $R_\odot$ and beyond this distance, the temperature gradient reaches a value which sets up convection. Fig. 8.8 illustrates the internal structure of the present Sun.

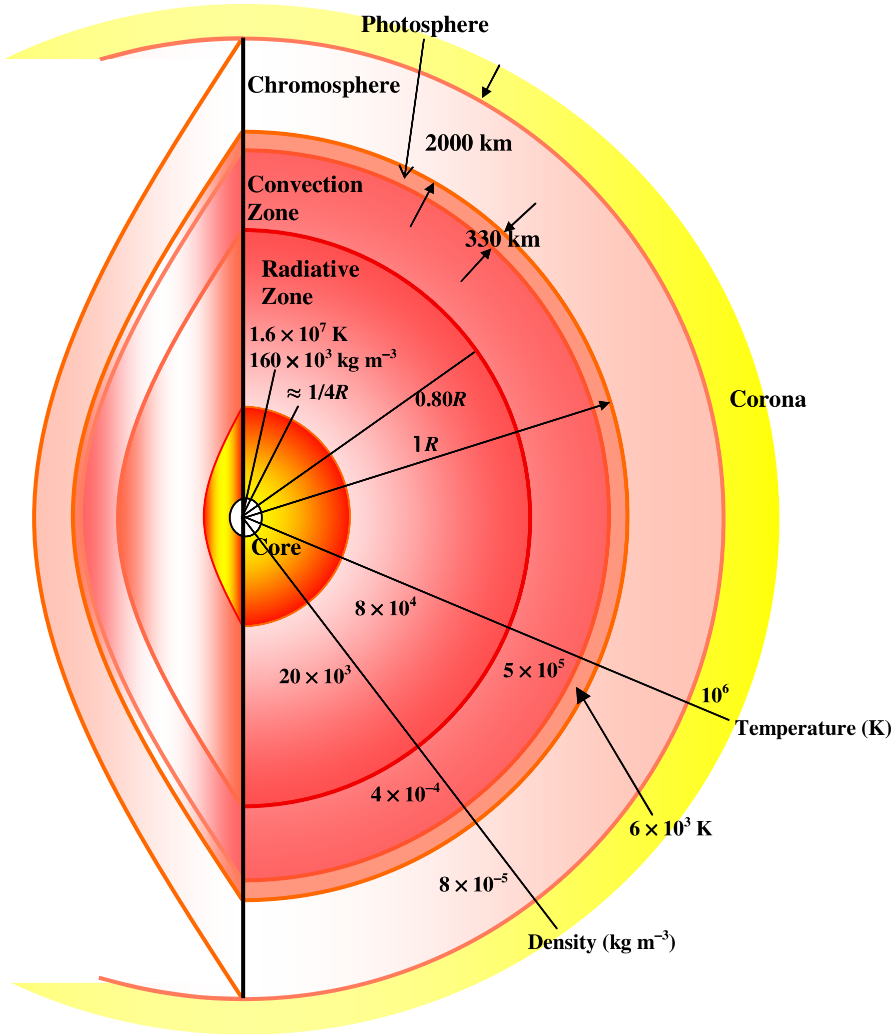

Fig.8.8: The internal structure of the Sun as per a stellar model which shows the main regions of the Sun and values of its important physical parameters

Computation of actual stellar models involves complex mathematics and needs extensive computer resources. It is primarily due to the complexities of the equations of stellar structure.

Now, let us summarise what you have learnt in this Unit.

8.7 SUMMARY

- To understand the internal structure of stars, **theoretical models** of stars are developed and the predictions of these models are compared with observations.

- The **principle of hydrostatic equilibrium** is one of the fundamental principles of physics invoked for investigating stellar structure.

- The **equation of hydrostatic equilibrium** is given as:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r)$$

- The **mass continuity equation** is given as:

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r).$$

- The relation between total thermal energy and the gravitational potential energy of a system of perfect gas particles such as a star is given by the so called **virial theorem**:

$$2U + \Omega = 0.$$

- Of the three possible sources namely **gravitational, chemical** and **nuclear** of energy generation in stars, only nuclear processes can give rise to such high internal temperatures. Energy is generated in stars due to **fusion reactions**, particularly due to fusion of hydrogen nuclei and consequent formation of helium.

- One of the basic equations of stellar structure, as given below, relates the luminosity (an observable parameter) of a star with its density:

$$\frac{dL}{dr} = \varepsilon\, 4\pi r^2 \rho(r)$$

  where ε is the rate of energy generated per unit mass in thermonuclear reactions.

- Energy generated at the core of a star is transported to its surface through one or more than one of the three basic **energy transport processes** namely **conduction, radiation** and **convection**.

- The **temperature gradient** in a star is produced due to radiative transport of energy and is given as:

$$\frac{dT}{dr} = -\frac{3}{4}\frac{k(r)\rho(r)}{acT^3}\left(\frac{L(r)}{4\pi r^2}\right)$$

- **Equations of stellar structure are used to develop theoretical models of stars**. If the predictions of a model are in agreement with observations, we conclude that the assumptions made about the parameters of stellar interior are valid.

- Developing a stellar model basically involves solving equations of stellar structure. This is quite a complex process because a large number of equations with several unknowns are to be solved.

- In polytropic stellar model – a relatively simple stellar model – we do not need to know the source of energy generation. In this model, it is assumed that any change in the equilibrium structure of a star do not alter its heat capacity:

$$\frac{dQ}{dT} = C = \text{constant}$$

- The **Lane-Emden equation** is given as:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

Solution of this equation provides the density and pressure profile inside a star.

8.8 TERMINAL QUESTIONS *Spend 25 min.*

1. Show that when four protons combine to form helium, the energy released is ~ 26.7 MeV.

2. At present, the Sun radiates energy at the rate of ~ 4×10^{26}W. Assuming that the gravitational contraction is the only source of the Sun's radiant energy, how long, since its creation, it would have radiated energy at the present rate? Take $M_\Theta = 2\times10^{30}$ kg and $R_\Theta = 7\times10^8$ m.

3. Show that Eq. (8.55) is consistent with Eq. (8.54).

8.9 SOLUTIONS AND ANSWERS

Self Assessment Questions (SAQs)

1. The left hand side of Eq. (8.8) is:

$$3\int_0^R P4\pi r^2 dr = 3\int_0^R PdV = 2\frac{3}{2}\int_0^R PdV = 2U$$

since the internal or thermal energy can be expressed as $\frac{3}{2}\int PdV$.

2.a) From Eq. (8.11), we can write the total potential energy as:

$$-\Omega = \int_0^R \frac{GM(r)}{r} 4\pi r^2 \rho(r)\, dr$$

Assuming that $\rho(r) \approx <\rho>$, the mean density of stellar matter, we can write:

$$-\Omega = \int_0^R \frac{GM(r)}{r} 4\pi r^2 <\rho> dr = \int_0^R \frac{G}{r}\frac{4\pi}{3} r^3 <\rho> 4\pi r^2 <\rho> dr$$

$$= \int_0^R 3G \frac{4\pi}{3} \frac{4\pi}{3} r^4 < \rho >^2 dr = 3\int_0^R G.r^4 \frac{M^2}{R^6} dr$$

(using Eq. (8.13))

$$= \frac{3GM^2}{R^6} \int_0^R r^4 dr$$

or,

$$\Omega = -\frac{3}{5} \frac{GM^2}{R}$$

This is Eq. (8.14). Further, from the virial theorem (Eq. (8.12)), we can write:

$$U = -\frac{\Omega}{2}$$

$$= \frac{3}{10} \frac{GM^2}{R} \quad \text{(using Eq. (8.14))}$$

Also, from Eq. (8.16), we have:

$$U = \frac{3}{2} \frac{k_B}{\mu m} M < T >$$

Comparing the above expressions for U, we can write:

$$\frac{3}{10} \frac{GM^2}{R} = \frac{3}{2} \frac{k_B}{\mu m} M < T >$$

or,

$$< T > = \frac{1}{5} \frac{GM}{R} \frac{\mu m}{k_B}$$

Further,

$$M = \frac{4}{3} \pi R^3 \rho$$

or,

$$R \propto \left(\frac{M}{\rho} \right)^{\frac{1}{3}}$$

Substituting this value of R, we get

$$< T > \propto M^{\frac{2}{3}} < \rho >^{\frac{1}{3}}$$

which is Eq. (8.17).

b) Eq. (8.17) can be written as:

$$< T > = \frac{1}{5} \frac{GM}{R} \frac{\mu m}{k_B}$$

Substituting the values of G, M, R and k_B, we get:

$$< T > = \frac{1}{5} . \frac{(6.7 \times 10^{-8} \text{ cm}^3 \text{g}^{-1} \text{s}^{-2}) \times (2 \times 10^{33} \text{ g}) \times (1.6 \times 10^{-24} \text{ g})}{(1.4 \times 10^{-16} \text{ erg K}^{-1}) \times (7 \times 10^{10} \text{ cm})}$$

$$\approx 4 \times 10^6 \text{ K.}$$

3. Mass of four hydrogen atoms = 4.031280 u

Mass of a helium atom = 4.002603 u

So, the mass defect = 0.02867 u

Thus, the energy released per gram when four protons combine to form helium is

$$= \frac{(0.02867 \text{ u}) \times (9 \times 10^{20} \text{ cm}^2 \text{s}^{-2})}{4.0328}$$

$$\approx 6 \times 10^{18} \text{ erg.}$$

Thus, we can write:

$$\text{lifetime of the Sun} = \frac{(2 \times 10^{33} \text{ g}) \times (6 \times 10^{18} \text{ erg g}^{-1})}{(4 \times 10^{33} \text{ erg s}^{-1})}$$

$$= 3 \times 10^{17} \text{ s}$$

$$\approx 10^{10} \text{ yr.}$$

4. We can write the potential energy of two nuclei Z_1e and Z_1e separated by a distance r as:

$$\text{P.E.} = \frac{Z_1 Z_2 e^2}{r}$$

$$= \frac{e^2}{r} \qquad \text{if } Z_1 = Z_2 = 1$$

Since the potential barrier will be overcome by the relative kinetic energy of the two nuclei, we can write:

$$3k_B T = \frac{e^2}{r}$$

or,

$$T = \frac{e^2}{3k_B r}$$

$$= \frac{4.8 \times 4.8 \times 10^{-20}}{3 \times 1.4 \times 10^{-16} \times 10^{-11}}$$

$$\approx 5 \times 10^7 \text{ K.}$$

5. From Eq. (8.40), we have:

$$(C - C_V)\frac{dT}{T} = (C_P - C_V)\frac{dV}{V}$$

If we define an exponent γ' as

$$\gamma' = \frac{C_p - C}{C_v - C}$$

we can write:

$$\gamma' - 1 = \frac{C_P - C}{C_V - C} - 1$$

$$= \frac{C_P - C - C_V + C}{C_V - C}$$

$$= \frac{C_P - C_V}{C_V - C}$$

Further, Eq. (8.40) can be written as:

$$(C_V - C)\frac{dT}{T} + (C_P - C_V)\frac{dV}{V} = 0$$

or,

$$\frac{dT}{T} + (\gamma' - 1)\frac{dV}{V} = 0$$

$$TV^{\gamma'-1} = \text{Const.}$$

Further, using the equation of state of a perfect gas:

$$PV = RT$$

we can write:

$$\frac{dP}{P} + \frac{dV}{V} = \frac{dT}{T}.$$

Substituting for $\frac{dT}{T}$, we get:

$$\frac{dP}{P} + \gamma'\frac{dV}{V} = 0$$

or,

$$PV^{\gamma'} = \text{Const.}$$

6. From Eq. (8.44), we have:

$$P = K\rho^{\frac{n+1}{n}}$$

$$= K\left(\rho_c \theta^n\right)^{\frac{n+1}{n}} \qquad \text{(using Eq. (8.45))}$$

$$= K\rho_c^{\frac{n+1}{n}} \theta^{n+1}$$

$$= P_c \theta^{n+1}$$

where $P_c = K\rho_c^{\frac{n+1}{n}}$

Further, we can write pressure as:

$$P = \frac{k_B}{\mu m} \rho T$$

or,

$$T = \frac{\mu m}{k_B} \frac{P}{\rho}$$

$$= \frac{\mu m}{k_B} K \frac{\rho^{\frac{n+1}{n}}}{\rho} \qquad \text{(substituting Eq. (8.44))}$$

$$= \frac{\mu m}{k_B} K\rho^{1/n} = \frac{\mu m}{k_B} K\,(\rho_c \theta^n)^{1/n} \qquad \text{(substituting Eq. (8.45))}$$

$$= \frac{\mu m}{k_B} K\rho_c^{1/n} \theta$$

$$= T_c \theta \quad \text{where} \quad T_c = \frac{\mu m}{k_B} K\rho_c^{1/n}$$

7. Lane-Emden equation (Eq. (8.52)) is:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

For $n = 0$, this equation reduces to:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi}\right) = -1$$

And, for $n = 0$, we have $\theta = 1 - \frac{\xi^2}{6}$

Substituting this value of θ in the left hand side of the above equation, we get:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \left(-\frac{\xi}{3}\right)\right) = \frac{1}{\xi^2} \frac{d}{d\xi} \left(-\frac{\xi^3}{3}\right) = \frac{1}{\xi^2} \left(-\frac{3\xi^2}{3}\right) = -1.$$

Further, for $n = 1$, we have $\theta = \dfrac{\sin\xi}{\xi}$ from Eq. (8.54). So, we get:

$$\frac{d\theta}{d\xi} = \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}$$

Again, substituting this value of $\dfrac{d\theta}{d\xi}$ in the left hand side of Lane-Emden equation, we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi \,.\, \cos\,\xi - \sin\xi)$$

$$= \frac{1}{\xi^2}[-\xi\sin\xi + \cos\xi - \cos\xi] = -\frac{\sin\xi}{\xi} = -\theta$$

as required.

For $\quad n = 5$, we have, $\theta = \left(1 + \dfrac{\xi^2}{3}\right)^{-1/2}$

or,

$$\frac{d\theta}{d\xi} = -\frac{1}{2}\left(1 + \frac{\xi^2}{3}\right)^{-3/2} . \frac{2\xi}{3}$$

Substituting the value of $\dfrac{d\theta}{d\xi}$ in the left hand side of Lane-Emden equation, we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(-\frac{\xi^3}{3}\left(1 + \frac{\xi^2}{3}\right)^{-3/2}\right)$$

$$= \frac{1}{\xi^2}\left[-\frac{3\xi^2}{3}\left(1 + \frac{\xi^2}{3}\right)^{-3/2} + \frac{\xi^3}{3} . \frac{3}{2}\left(1 + \frac{\xi^2}{3}\right)^{-5/2} . \frac{2\xi}{3}\right]$$

$$= -\left(1 + \frac{\xi^2}{3}\right)^{-3/2} + \frac{\xi^2}{3}\left(1 + \frac{\xi^2}{3}\right)^{-5/2} = -\left(1 + \frac{\xi^2}{3}\right)^{-5/2}\left[1 + \frac{\xi^2}{3} - \frac{\xi^2}{3}\right]$$

$$= -\left(1 + \frac{\xi^2}{3}\right)^{-5/2}$$

$= -\theta^5$, as required.

Terminal Questions

1. When four protons combine to form a helium atom, the mass defect is 0.02867 u. Thus, energy released in this process:

$$E = mc^2$$

$$= (0.02867 \times 1.6 \times 10^{-27}\ \text{kg}) \times (9 \times 10^{16}\ \text{m}^2\ \text{s}^{-2})$$

$$= 4.128 \times 10^{-12}\ \text{J}$$

$$= \frac{4.128 \times 10^{-12}}{1.6 \times 10^{-19}}\ \text{eV}$$

$$= 26\ \text{MeV}.$$

2. Assuming the Sun to be a sphere, its gravitational potential energy can be written as:

$$\Omega = \frac{3}{5}\frac{GM^2}{R}$$

So, the time τ for which the Sun will radiate with its luminosity, L can be written as:

$$\tau = \frac{3}{5}\frac{GM^2}{R}.\frac{1}{L}$$

$$= \frac{3}{5}\frac{GM^2}{R}.\frac{1}{L}.\frac{1}{3 \times 10^7}\ \text{yr}.$$

$$= \frac{3}{5} \times \frac{(6.7 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-1}) \times (4 \times 10^{60}\ \text{kg}^2)}{(7 \times 10^8\ \text{m}) \times (4 \times 10^{26}\ \text{kg}\ \text{m}^2 \text{s}^{-1})}\frac{1}{(3 \times 10^7)}\ \text{yr}.$$

$$= 10^8 \times \frac{1}{5}\ \text{yr}$$

$$\approx 2 \times 10^7\ \text{yr}.$$

3. Eq. (8.55) is:

$$\theta_n = 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 - \ldots$$

For $n = 0$, this equation reduces to:

$$\theta_0 = 1 - \frac{\xi^2}{6} \quad \rightarrow \quad \text{a constant}$$

for $\quad n = 1,$

$$\theta_1 = \frac{1}{\xi}\sin\xi = \frac{1}{\xi}\left(\xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \ldots\right)$$

$$= \frac{1}{\xi}\left(\xi - \frac{\xi^3}{6} + \frac{\xi^5}{120} - ...\right) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120}$$

$$= 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 - ...$$

for $\quad n = 5,$

$$\theta_5 = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} = 1 - \frac{1}{2}.\frac{\xi^2}{3} + \frac{-1}{2}.\frac{-3}{2}.\frac{1}{2}.\frac{\xi^4}{9}$$

$$= 1 - \frac{\xi^2}{6} + \frac{\xi^4}{24} = 1 - \frac{\xi^2}{6} + \frac{5}{120}\xi^4 - ...$$

$$= 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 - ...$$

Thus, we find that Eq. (8.55) is consistent with Eq. (8.54).