UNIT 13 APPLICATIONS OF LAPLACE TRANSFORMS

Structure

13.1 INTRODUCTION

In the previous unit you have studied the definition of the Laplace transform and its properties. You have also learnt to calculate the Laplace transforms and inverse Laplace transforms of several functions. In this unit, you will study how Laplace transforms are used to solve differential equations. The process is very simple. Using Laplace transforms, the differential equation is converted to an algebraic equation which can be solved easily. The inverse Laplace transform of this solution gives the solution of the differential equation.

This method is widely used in physics and engineering, for example, in the study of the motion of a particle under a dissipative force, with given initial conditions, or in the study of radioactive disintegration. This method is particularly useful in physical problems where the functions that occur possess discontinuities, for example, if the driving force acts for a short time, or is periodic but not a sine or cosine function. Such forces act in systems experiencing forced oscillations, e.g., in electrical circuits subjected to short pulses. For the sake of simplicity, in this unit we shall restrict ourselves mostly to cases where the physical system is modelled by ordinary differential equations.

In order to study this unit effectively you require a sound knowledge of integral calculus and a practice of evaluating definite integrals. You will also need to reduce an algebraic fraction into partial fractions. If you have thoroughly studied Unit 12 and worked out the various SAQs and TQs, then you already have enough practice and you will find this to be an easy and interesting unit. In fact, in this unit, there are hardly any new concepts to be dealt with; we would only apply the previous knowledge to new situations.

Objectives

After studying this unit, you should be able to:

- determine the Laplace transforms of derivatives of a function;
- obtain the Laplace transform of the integral of a function;
- calculate the inverse Laplace transform of the derivative and integral of a Laplace transform of some function; and
- use Laplace transforms to solve an ordinary differential equation with given initial and boundary conditions.

13.2 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The general method of using Laplace transforms to solve differential equations for physical systems consists of the following steps:

1. Express the given physical phenomenon in terms of a differential equation.

2. Convert the differential equation into an algebraic equation by the method of Laplace transformation.

3. Solve the algebraic equation.

4. Obtain the inverse Laplace transform of the algebraic solution, which will give the desired solution.

5. Interpret the result.

We will explain all these steps with the help of examples.

Let us apply this method to a linear, second order, ordinary differential equation with constant coefficients, of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c\,y=f(x) \tag{13.1}$$

Now in order to convert this equation to an algebraic equation, we require the Laplace transforms of the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and integrals. So you must learn this aspect first.

13.2.1 Laplace Transforms of Derivatives and Integrals of a Function

You might say that we could obtain $L[f'(t)]$ by differentiating with respect to the variable t, the right hand side in Eq. (12,1). But we cannot, because t is the variable of integration in a definite integral.

To obtain the Laplace transform of the derivative of a function $f(t)$ directly, we begin from the basic definition. Thus, if $f(t)$ is continuous for all $t \geq 0$, the condition for the existence of the Laplace transform is satisfied for some M and a, and $f'(t)$ exists for $t \geq 0$, then $L[f'(t)]$ exists and is given by

$$L[f'(t)]=\int_0^{\infty} e^{-st} f'(t)\,dt \tag{13.2}$$

where the prime now denotes differentiation with respect to t. [Hereafter, a prime will denote differentiation with respect to whatever independent variable there may be.] We now integrate the right hand side by parts (taking e^{-st} as the first function and $f'(t)$ as the second function) and obtain

$$L[f']=e^{-st} f(t)\Big|_0^{\infty}-(-s)\int_0^{\infty} e^{-st} f(t)\,dt \tag{13.3}$$

The first term on the right vanishes at the upper limit when $s > 0$, and yields $f(0)$ at the lower limit. The integral in the second term is just the Laplace transform of $f(t)$. We therefore get

$$\boxed{L[f'(t)]=sL[f(t)]-f(0),\;\; s>0} \tag{13.4}$$

Let us now obtain the Laplace transform of the second order derivative. We have, by definition,

$$L[f''(t)]=\int_0^{\infty} e^{-st} f''(t)\,dt. \tag{13.5}$$

We may again integrate Eq. (13.5) by parts twice and obtain the result in terms of $L[f(t)]$. But we don't need to follow this process. If we denote the first derivative $f'(t)$ by some other symbol such as $g(t)$, that is $g(t) = f'(t)$, then $g'(t)$ would be equal to $f''(t)$. Using this in Eq. (13.5), we have an integral similar to that in Eq. (13.2) with f replaced by g.. Comparing it with Eq. (13.4), we can write

$$L[g'(t)] = sL[g(t)] - g(0). \qquad (13.6)$$

Now substituting for g and g' we get

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

$$= s[sL(f) - f(0)] - f'(0)$$

or

$$\boxed{L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0), \qquad s > 0} \qquad (13.7)$$

where we have again used Eq. (13.4) in going from the first step to the second. You can easily continue the process to higher derivatives. Here you should understand that $f(0)$ denotes the values of $f(t)$ at $t = 0$, and $f'(0)$ is the value of the derivative df/dt at $t = 0$. Let us consider a couple of examples to illustrate these ideas.

Example 1

Obtain the Laplace transforms of the functions:

a) $f(t) = t^2$
b) $f(t) = \sin^2 t$

Solution

a) Since $f(t) = t^2$, $f(0) = 0$
$f'(t) = 2t$, $f'(0) = 0$
$f''(t) = 2.$

Hence, using Eq. (13.7), we get

$$L[f''(t)] = L(2) = s^2 L[t^2]$$

or

$$L(t^2) = \frac{L(2)}{s^2} = \frac{2}{s}\frac{1}{s^2} = \frac{2}{s^3}$$

We have taken this example deliberately to show you that in general there are several ways of obtaining the transforms of given functions and you have to use the easiest one.

b) Since

$$f(t) = \sin^2 t \qquad , \quad f(0) = 0,$$

$$f'(t) = 2\sin t \cos t = \sin 2t, \quad f'(0) = 0.$$

Eq. (13.4) yields

$$L(\sin 2t) = sL(\sin^2 t)$$

or

$$L\left(\sin^2 t\right) = \frac{1}{s} . L(\sin 2t) = \frac{1}{s} . \frac{2}{s^2 + 4} \qquad \text{(from Table 12.1)}$$

Thus

$$L\left(\sin^2 t\right) = \frac{2}{s\left(s^2 + 4\right)}$$

Before studying further, you should work out an SAQ to obtain some practice of Eqs. (13.4) and (13.7).

SAQ 1

Spend 10 min

Obtain $L[f(t)]$ for

(a) $f(t) = t \cos \omega t$,
(b) $f(t) = t \sinh at$.

Let us now obtain the Laplace transform of the integral of a function. We shall not prove this result here and you should also not worry about its proof. But the result is interesting and important for calculating Laplace transforms and inverse Laplace transforms. What is important for you is to use this result.

If you are particularly interested in knowing the proof of this formula, you will find it in any book mentioned for extended reading at the end of this unit. But you must practice applying this formula to obtain Laplace transforms of various functions.

If the Laplace transform of $f(t)$ is $F(s)$, and it exists, then

$$L\left[\int_0^t f(x)\,dx\right] = \frac{F(s)}{s}, \quad \text{and} \qquad (13.8a)$$

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(x)\,dx. \qquad (13.8b)$$

Let us apply this result to an example.

Example 2

Determine the inverse Laplace transform of

$$F(s) = \frac{1}{s(s-a)}$$

where a is a non-zero constant.

Solution

You will notice that the given function is obtained by dividing $1/(s-a)$ by s. Using Eq. (13.8b) we get

$$L^{-1}\left[\frac{1}{s(s-a)}\right] = \int_0^t f(x)\,dx \qquad (13.9)$$

Thus $F(s) = \frac{1}{s-a}$. We know that the inverse Laplace transform of $\frac{1}{s-a}$ is $f(x) = e^{ax}$. On substituting this in Eq. (13.9) and solving the integral we get

$$L^{-1}\left[\frac{1}{s(s-a)}\right] = \int_0^t e^{ax}dx = \left.\frac{e^{ax}}{a}\right|_0^t = \frac{e^{at}-1}{a} \tag{13.10}$$

We could also have obtained the inverse Laplace transform of the function in Example 2 by the method of partial fractions described in Unit 12. But using Eq. (13.8a and b) we could avoid all that algebra. This example shows how useful and powerful the above technique is.

You should apply this result to another problem to get some practice with it.

SAQ 2

Spend 5 min

Obtain $f(t)$ given $L[f(t)] = \dfrac{1}{s^2(s^2+a^2)}$.

We can now use these results for solving ordinary differential equations. Let us now learn the method.

13.2.2 The Method

Let us first consider an initial value problem involving a linear, second order, homogeneous ordinary differential equation having constant coefficients:

$$ay'' + by' + cy = 0, \quad y(0) = c_1, y'(0) = c_2 \tag{13.11}$$

where prime denotes the derivative with respect to x.

Let $Y(s) = L(y)$ be the Laplace transform of the solution $y(x)$.

Step 1: Then the first step is to apply Laplace transformation to Eq. (13.11) and use Eqs. (13.4) and (13.7) alongwith the linearity property and initial conditions. Thus we can transform Eq. (13.11) and write

$$aL[y''] + bL[y'] + cL(y) = L(0) = 0.$$

since a, b and c are constants.

or

$$as^2\,Y(s) - as\,y(0) - a\,y'(0) + bs\,Y(s) - b\,y(0) + c\,Y(s) = 0$$

or

$$Y(s)\left[as^2 + bs + c\right] - c_1(as+b) - ac_2 = 0$$

or

$$Y(s)\left[as^2 + bs + c\right] = c_1(as+b) + ac_2 \tag{13.12a}$$

The equation (13.12a) for the transform $Y(s)$ of the unknown function $y(x)$ is called the **subsidiary equation** of the given differential equation.

Step 2: We now solve Eq. (13.12a) for $Y(s)$, which is very simple:

$$Y(s) = \frac{c_1(as+b) + ac_2}{as^2 + bs + c} \tag{13.12b}$$

Step 3: Obtaining the inverse transform of $Y(s)$ yields $y(x)$:

$$y(x) = L^{-1}[Y(s)] \tag{13.12c}$$

In Sec. 12.4 you have learnt how to calculate the inverse transforms of various functions. You can now apply this method to a concrete problem.

SAQ 3

Spend 10 *min*

Use the Laplace transforms to solve the initial value problem

$$y'' + 4y' + 3y = 0, \quad y(0) = 3, \; y'(0) = 1$$

We are now ready to apply the method of Laplace transforms to solve differential equations in physics. This is the major application of the technique of Laplace transforms. We may even say that the entire Unit 12 and the sections of this unit so far were developed and the properties of Laplace transforms studied with a view to apply them eventually to the solution of differential equations in physics.

13.3 APPLICATIONS IN PHYSICS

We shall illustrate the applications with the help of several examples.

Example 3

Damped harmonic oscillator

Consider an oscillator of mass m, and the spring constant k which is subject to damping. Let b be the damping constant. If $y(t)$ denotes the instantaneous displacement of the oscillator at time t from its equilibrium position, then from Unit 3 of PHE-02 you know that the motion of the oscillator is governed by the differential equation

$$my''(t) + by'(t) + ky(t) = 0, \tag{13.13a}$$

when it is not acted upon by any external driving force. Here primes denote differentiation with respect to time t. Let us solve this equation with the particular values $m = 5$, $b = 1$; $k = 50$, and the initial conditions

$$y(0) = 1, \; y'(0) = -3. \tag{13.13b}$$

Eq. (13.13a) models a mass m resting on a horizontal surface, attached to a spring (Fig. 13.1a), and then allowed to vibrate. The **initial conditions** of Eq. (13.13b) tell us that we start measuring time when the displacement of the particle is +1 and at this position, its velocity is –3. If we take the positive y-axis to the right of the origin as per the normal convention, then this means that at time $t = 0$, the particle is to the right of the origin (at +1) and is moving towards the origin (velocity, –3) (Fig. 13.1b). To solve this equation means to obtain the position y of the particle at any later time t, that is, to find the functional dependence of y on t.

You must understand the significance of each term in the differential equation, Eq. (13.13a), and also the initial conditions in Eq. (13.13b). In this problem, each term in Eq. (13.13a) represents force. (Note that, with y denoting displacement, y' is the velocity and y'' is the acceleration.) Eq. (13.13a) can also be written as

$$my'' = -ky - by'.$$

The left hand side is the force according to Newton's law, that is the product of mass and acceleration. The first term on the right hand side is the restoring force which is given by the product of the spring constant and the displacement. The negative sign of this term indicates that the restoring force is always directed towards the centre. The second term is the damping force which is proportional to the velocity. The negative sign here indicates that the damping force is opposite to the direction of velocity. The left hand side of Eq. (13.13a) represents the total external force acting on the oscillator, and the right hand side tells that it is zero.

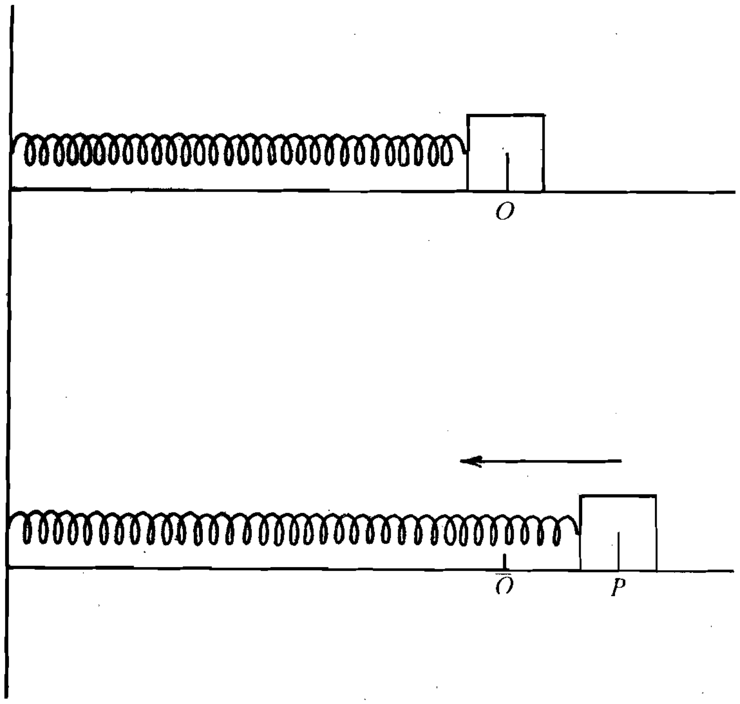

Fig.13.1: (a) A spring mass system at its equilibrium position; (b) the oscillating system at the instant $t = 0$.

Solution

Step 1: To solve Eq. (13.13a) subject to the initial conditions of Eq. (13.13b), we take the Laplace transform of the entire Eq. (13.13a):

$$L[my''(t)] + L[by'(t)] + L[ky(t)] = L(0) = 0 \qquad (13.14)$$

What we really do in going from Eq. (13.13a) to Eq. (13.14) is that we multiply every term of Eq. (13.13a) by e^{-st} and integrate over t from 0 to ∞. Since m, b and k are constants independent of t, they can be taken out of the operation L by the linearity property of the Laplace transforms. Let us denote the Laplace transform of $y(t)$ by $Y(x)$, that is,

$$L[y(t)] = Y(x) \qquad (13.15)$$

We now use Eqs. (13.4) and (13.7), respectively, to write Eq. (13.14) with particular given values of the parameters, as

$$5\left[x^2 Y(x) - xy(0) - y'(0)\right] + \left[xY(x) - y(0)\right] + 50Y(x) = 0. \qquad (13.16)$$

Here you must remember that $y(0)$ and $y'(0)$ are known while $Y(x)$ is the unknown for which we must solve the above equation.

Step 2: Substituting the initial values in Eq. (13.16) from Eq. (13.13b) and collecting terms, we get the equation

$$\left(5x^2 + x + 50\right) Y(x) - 5x(1) - 5(-3) - 1 = 0$$

or

$$\left(5x^2 + x + 50\right) Y(x) = 5x - 14, \qquad (13.17)$$

giving

$$Y(x) = \frac{5x-14}{5x^2+x+50} = \frac{x-2.8}{x^2+0.2x+10} \tag{13.18}$$

Step 3: We must now obtain the inverse Laplace transform of Eq. (13.18). From the experience of Unit 12, we would rewrite the denominator of the function $Y(x)$ as

$$x^2+0.2x+10=(x+0.1)^2+9.99$$

Thus Eq. (13.18) becomes

$$Y(x) = \frac{(x+0.1)-2.9}{(x+0.1)^2+9.99} \tag{13.19}$$

If we replace $x + 0.1$ by x, we notice that the function on the right hand side of Eq. (13.19) becomes

$$F(x) = \frac{x-2.9}{x^2+9.99} = \frac{x}{x^2+(3.161)^2} - \frac{0.918\times 3.161}{x^2+(3.161)^2}. \tag{13.20}$$

The inverse Laplace transform of $F(x)$ can be easily found by referring to the standard Laplace transforms given in Table 12.1 of Unit 12. You will find that

$$L^{-1}[F(x)] = \cos(3.161t) - 0.918\sin(3.161t), \tag{13.21}$$

Since $Y(x)$ can again be obtained from $F(x)$ by replacing x by $x + 0.1$, here we can apply the first shifting theorem discussed in Section 12.2 of Unit 12. Thus the inverse Laplace transform of $Y(x)$ is straightaway found to be

$$L^{-1}[Y(x)] = e^{-0.1t}[\cos(3.161t) - 0.918\sin(3.161t)]. \tag{13.22}$$

But, from Eq. (13.15), the inverse Laplace transform of $Y(x)$ is just the unknown displacement $y(t)$ which we are trying to determine, that is

$$L^{-1}[Y(x)] = y(t).$$

or

$$y(t) = e^{-0.1t}[\cos(3.161t) - 0.918\sin(3.161t)]. \tag{13.23}$$

This is finally the solution of the differential equation (13.13a), with given values of m, s, k, subject to the initial values given in Eq. (13.13b).

To complete the solution, we must also interpret Eq. (13.23). You can see that the solution $y(t)$ is a product of two factors, one of which is $e^{-0.1t}$ and the other is $\cos(\omega t) - 0.918\sin(\omega t)$, where we have used $\omega = 3.161$. Note that ω is the angular frequency of the oscillatory motion. The second factor is a combination of the trigonometric functions cos (ωt) and sin (ωt), while the first factor is an exponentially decreasing function of time.

Let us consider only the second factor first. We can write it as

$$\cos(\omega t) - 0.918\sin(\omega t) = a\sin(\omega t + c),$$

where a and c are constants that can be determined by simple algebra to be

$$a = \sqrt{1.842} = 1.357, \quad \tan c = -1/0.918 = -1.089 \quad \text{or} \quad c = 2.313\,\text{rad}.$$

Thus we have

$$f(t) = \cos(\omega t) - 0.918\sin(\omega t) = 1.357\sin(\omega t + 2.313).$$

The function $f(t)$ is a sinusoidal curve with a period $2\pi/\omega = 1.988$ and amplitude 1.357, with a starting phase of 2.313 rad at $t = 0$ (Fig. 13.2).

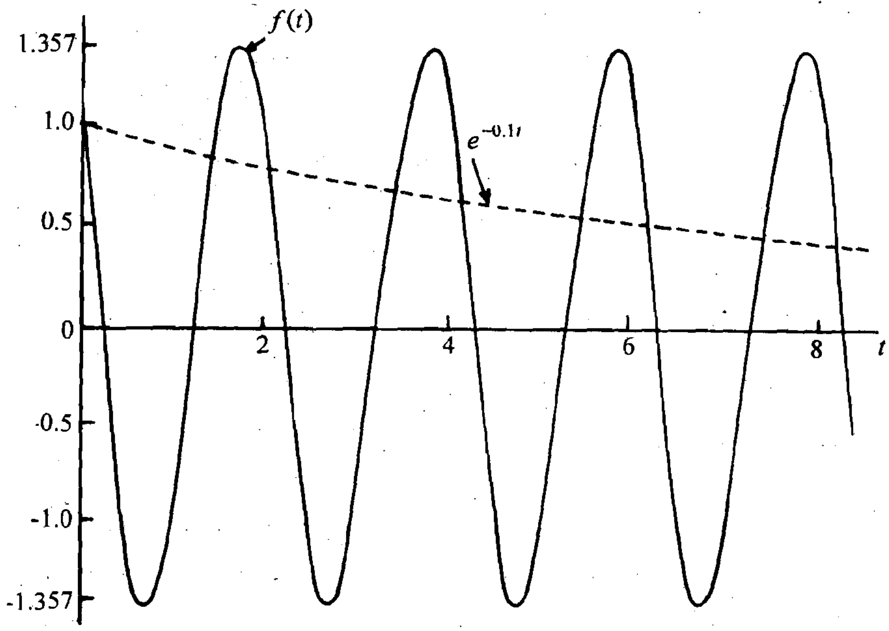

Fig.13.2: The solid line shows the function $f(t) = a\sin(\omega t + c)$ with the given values of a and c, and the dotted line shows the function $e^{-0.1t}$

However, the complete solution $y(t)$ is a product of this function and $e^{-0.1t}$. Thus the motion of the mass attached to the spring would be like that shown in Fig. 13.3, a sinusoidal function of decreasing amplitude and of period 1.988. The decrease in the amplitude is a result of damping which may be due to friction. This completes the solution of Example 3, with specific values of parameters as given in the problem. You could have also worked out the solution with the general parameters m, s, k, without using any particular values, and arrived at a general solution.

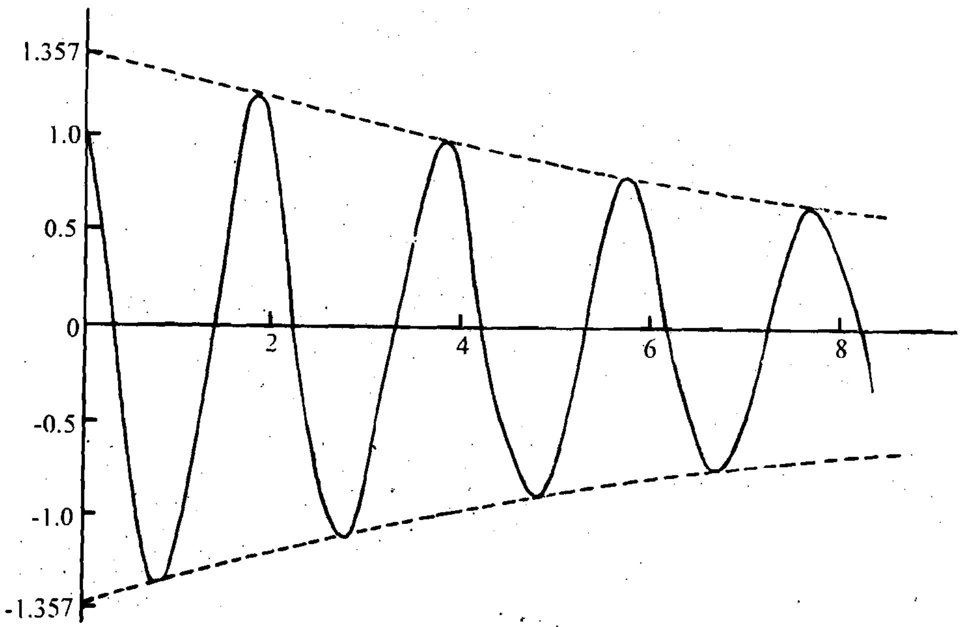

Fig.13.3: The displacement of a damped harmonic oscillator

Differential equations like Eq. (13.13a) (along with initial conditions) arise in several branches of physics – in solving problems in electrical networks, in radioactivity where one element decays into another and it decays into a third one and so on, in problems involving mechanical systems, etc. Another typical problem is that of forced vibrations. To bring home the power of this method we now consider a damped vibrating system subjected to a force that acts for a short time.

Example 4: A damped vibrating system subjected to a single square pulse

Solve the initial value problem

$$y'' + 5y' + 4y = f(t), \qquad y(0) = 0, y'(0) = 0 \tag{13.24}$$

with

$$f(t) = \begin{cases} 1 \ , & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

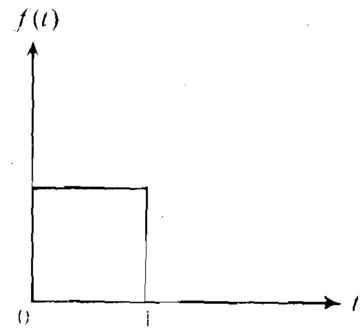

Fig.13.4: The forcing function $f(t)$

The forcing function $f(t)$ is shown in Fig. 13.4. This equation can model the response of an *LCR* circuit to an applied square pulse.

Solution

Step 1: We take the Laplace transform of Eq. (13.24) and set up the subsidiary equation:

Let $Y(s) = L[y(t)]$.

Then we have

$$s^2 Y + 5sY + 4Y = \frac{1}{s}\left(1 - e^{-s}\right)$$

where we have used the result for SAQ 6 of Unit 12 with appropriate changes.

Step 2: Solving for Y, we obtain

$$Y(s) = \frac{\left(1 - e^{-s}\right)}{s\left(s^2 + 5s + 4\right)} = \left(1 - e^{-s}\right)F(s)$$

where

$$F(s) = \frac{1}{s(s+1)\ (s+4)}$$

$$= \frac{1}{4s} - \frac{1}{3(s+1)} + \frac{1}{12(s+4)}$$

Step 3: Hence by Table 12.1 we get

$$f(t) = L^{-1}(F) = \frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-2t}$$

Therefore, using second shifting theorem, we have

$$L^{-1}\left\{e^{-s}F(s)\right\} = f(t-1)\ H(t-1) = \begin{cases} 0 & 0 \le t < 1 \\ \frac{1}{4} - \frac{1}{3}e^{-(t-1)} + \frac{1}{12}e^{-2(t-1)} & t > 1 \end{cases}$$

This yields the solution

$$y(t) = L^{-1}(Y) = f(t) - f(t-1)H(t-1) = \begin{cases} \frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-2t} & 0 \le t < 1 \\ \frac{1}{3}e^{-t}(e-1) - \frac{1}{12}e^{-2t}\left(e^2 - 1\right) & t > 1 \end{cases} \quad (13.25)$$

Example 5: Coupled pendulums

The motion of coupled pendulums shown in Fig. 13.5 is governed by the equations.

$$m\ddot{x}_1 = -m\omega^2 x_1 + k(x_2 - x_1) \quad (13.26a)$$

$$m\ddot{x}_2 = -m\omega^2 x_2 + k(x_1 - x_2) \quad (13.26b)$$

where $\omega^2 = g/l$. The initial conditions are

$$x_1 = x_2 = 0, \dot{x}_1 = v, \dot{x}_2 = 0 \text{ at } t = 0$$

i.e., the pendulum 1 is set into motion with speed v at $t = 0$. Solve the equations for x_1 and x_2.

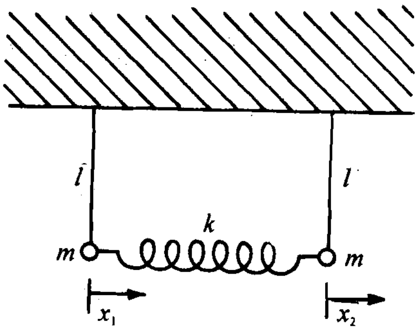

Fig.13.5: Coupled pendulums

Solution

Let

$$L[x_1(t)] = X_1(s) \quad \text{and} \quad L[x_2(t)] = X_2(s)$$

Then the subsidiary equations are

$$m\left(s^2 X_1 - v\right) = -m\omega^2 X_1 + k(X_2 - X_1)$$

$$ms^2 X_2 = -m\omega^2 X_2 + k(X_1 - X_2)$$

Solving for X_1 and X_2, we get

$$X_1 = \frac{v\left(s^2 + \omega^2 + \frac{k}{m}\right)}{\left(s^2 + \omega^2 + \frac{2k}{m}\right)\left(s^2 + \omega^2\right)}$$

$$X_2 = \frac{vk/m}{\left(s^2 + \omega^2 + 2k/m\right)\left(s^2 + \omega^2\right)}$$

or

$$X_1 = \frac{v}{2}\left[\frac{1}{s^2+\omega^2+\frac{2k}{m}} + \frac{1}{s^2+\omega^2}\right]$$

$$X_2 = \frac{v}{2}\left[\frac{1}{s^2+\omega^2+\frac{2k}{m}} - \frac{1}{s^2+\omega^2}\right]$$

Therefore

$$x_1(t) = L^{-1}(X_1) = \frac{v}{2}\left[\frac{\sin\omega^* t}{\omega^*} + \frac{\sin\omega t}{\omega}\right] \tag{13.27a}$$

$$x_2(t) = L^{-1}(X_2) = \frac{v}{2}\left[\frac{\sin\omega^* t}{\omega^*} - \frac{\sin\omega t}{\omega}\right] \tag{13.27b}$$

where $\omega^{*2} = \omega^2 + \frac{2k}{m}$.

Notice that ω^* and ω are the angular frequencies of the two normal modes.

An interesting application of Laplace transforms is to impulsive forces, i.e., forces lasting only a short time. Such forces can be described with the help of the Dirac delta function $\delta(t)$ about which you have studied in Unit 11. The Laplace transform of Dirac delta function is

$$L\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0} \tag{13.28}$$

Thus for

$$t_0 = 0, \quad L\{\delta(t)\} = 1 \tag{13.29}$$

Using this result we can apply the Laplace transformation method to impulsive forces.

Example 6: Impulsive Force

Newton's second law for impulsive force acting on a particle of mass m moving along a straight line is

$$m\frac{d^2x}{dt^2} = F(t)$$

where $F(t) = p\,\delta(t)$, with p constant, is an impulsive force acting for a very short time.

Obtain $x(t)$ given $x(0) = 0$, $x'(0) = 0$.

Solution

Let

$$L[x(t)] = X(s)$$

Applying Laplace transform to the equation of motion we get

$$ms^2X(s) - ms\,x(0) - m\,x'(0) = p, \qquad \text{since } L[\delta(t)] = 1$$

Applying the initial conditions, we have

$$ms^2 X(s) = p$$

or

$$X(s) = \frac{p}{ms^2}$$

The inverse Laplace transform yields

$$x(t) = \frac{p}{m} t$$

so that

$$\frac{dx(t)}{dt} = \frac{p}{m} \text{ is a constant.}$$

Thus the effect of the impulse $p\delta(t)$ is to transfer, instantaneously, p units of linear momentum to the particle.

To conclude this discussion we take up an example from electromagnetic theory, which involves a partial differential equation.

Example 7: Electromagnetic Waves

The electromagnetic wave equation, with $E = E_y$ or $E = E_z$, for a transverse wave propagating along the x-axis is

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0$$

Solve the equation given the initial conditions

$$E(x,0) = 0 \quad \text{and} \quad \left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0} = 0.$$

Solution

Transforming the equation we have

$$\frac{\partial^2}{\partial x^2} L\{E(x,t)\} - \frac{s^2}{v^2} L\{E(x,t)\} + \frac{s}{v^2} E(x,0) + \frac{1}{v^2} \left.\frac{\partial E(x,t)}{\partial t}\right|_{t=0} = 0.$$

Applying the initial conditions we get

$$\frac{\partial^2}{\partial x^2} L\{E(x,t)\} = \frac{s^2}{v^2} L\{E(x,t)\}$$

This is effectively an ordinary differential equation in the variable x. Its solution is

$$L\{E(x,t)\} = c_1\, e^{-sx/v} + c_2 e^{sx/v}$$

The constants c_1 and c_2 are obtained by additional boundary conditions on x. If $E(x,t)$ is finite as $x \to \infty$, then $L\{E(x,t)\}$ is also finite and we then have

$$c_2 = 0.$$

Now let $E(0,t) = f(t)$. Then

$$c_1 = L[f(t)] = F(s)$$

Substituting for c_1 and c_2, we get

$$L\{E(x,t)\} = e^{-sx/v} F(s)$$

Taking the inverse Laplace transform, we obtain

$$E(x,t) = \begin{cases} F\left(t - \dfrac{x}{v}\right) & t \geq \dfrac{x}{v} \\ 0 & t < \dfrac{x}{v} \end{cases}$$

This is the equation of a wave moving in the positive x-direction with velocity v.

We will now summarise what you have studied in this unit.

13.4 SUMMARY

- The **Laplace transforms** of the **first and second order derivatives** of a function $f(t)$ are, respectively, given by:

$$L[f'(t)] = sL[f(t)] - f(0), \quad s > 0$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

- Using Laplace transforms we can **solve differential equations** as per the following procedure:

  Step 1: Convert the differential equation into an algebraic equation by the method of Laplace transformation.

  Step 2: Solve the algebraic equation.

  Step 3: Obtain the inverse Laplace transform which will give the desired solution.

- The above procedure finds many applications in physics, such as damped and forced vibrating systems in mechanics and electricity, impulsive force, electromagnetic theory.

13.5 TERMINAL QUESTIONS

Spend 30 *min.*

1. Solve the initial value problem

$$y'' + 2y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = 8.$$

2. In a radioactive series of n different nuclides, starting with N_1,

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \,,$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \quad \text{and so on.}$$

$$\frac{dN_n}{dt} = \lambda_{n-1} N_{n-1}, \quad \text{stable}$$

Determine N_1), $N_2(t)$ and $N_3(t)$, $n = 3$, with $N_1(0) = N_0, N_2(0) = N_3(0) = 0$.

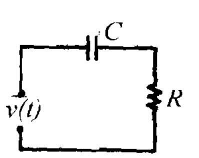

Fig.13.6: An *RC* circuit

3. Obtain the current in the RC circuit in Fig. 13.6 when

$$v(t) = \begin{cases} V_0 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$i(0) = 0.$$

13.6 SOLUTION AND ANSWERS

Self-assessment Questions

1. (a) $L[f(t)] = L(t\cos\omega t)$

$$f(t) = t\cos\omega t$$

$$f'(t) = \cos\omega t - \omega \sin\omega t$$

$$f''(t) = -\omega\sin\omega t - \omega\sin\omega t - \omega^2 t\cos\omega t$$

$$= -2\omega\sin\omega t - \omega^2 f(t)$$

$$f(0) = 0$$

$$f'(0) = 1$$

Using Eq. (13.7) we have

$$L[-2\omega\sin\omega t - \omega^2 f(t)] = s^2 L[f(t)] - 1$$

or

$$(s^2 + \omega^2) L[f(t)] = -2\omega L(\sin\omega t) + 1$$

$$L[f(t)] = \left[-\frac{2\omega^2}{s^2+\omega^2} + 1 \right] \frac{1}{(s^2+\omega^2)}$$

$$\therefore \quad L(t\cos\omega t) = \frac{s^2 - \omega^2}{(s^2+\omega^2)^2}$$

(b) $L[t\sinh at]$

Let

$$f(t) = t\sinh at, \qquad f(0) = 0$$

$$f'(t) = \sinh at + at\cosh at, \quad f'(0) = 0$$

$$f''(t) = a\cosh at + a\cosh at + a^2 t\sinh at$$

$$= 2a\cosh at + a^2 f(t)$$

Using Eq. (13.7), we have

$$L[f''(t) = s^2 L[f(t)]$$

or

$$L\left[2a\cosh at + a^2 f(t)\right] = s^2 L[f(t)]$$

or

$$L[f(t)] = \frac{L(2a\cosh at)}{s^2 - a^2}$$

$$= \frac{2as}{\left(s^2 - a^2\right)^2}$$

2. $$F(s) = \frac{1}{s^2\left(s^2 + a^2\right)^2},$$

Let

$$F_1(s) = \frac{1}{s\left(s^2 + a^2\right)}$$

and

$$F_2(s) = \frac{a}{s^2 + a^2}$$

Then

$$L^{-1}[F_2(s)] = \sin at$$

Since $F_1(s) = \dfrac{F_2(s)}{as}$, therefore

$$L^{-1}[F_1(s)] = \frac{1}{a}\int_0^t \sin ax\; dx$$

$$= \frac{1}{a^2}(1 - \cos at) = g(t)$$

Again

$$F(s) = \frac{F_1(s)}{s}$$

and

$$L^{-1}[F(s)] = \int_0^t g(x)\,dx$$

$$= \frac{1}{a^2}\int_0^t (1 - \cos ax)\,dx$$

$$= \frac{1}{a^2}\left[x - \frac{\sin ax}{a} \Big|_0^t \right]$$

$$= \frac{1}{a^2}\left[t - \frac{\sin at}{a} \right] = \frac{1}{a^3}(at - \sin at)$$

3. Taking the Laplace transform of the equation we get

$$L(y'') + 4L(y') + 3L(y) = 0$$

Using Eqs. (13.4) and (13.7), we obtain

$$s^2 Y(s) - 3s - 1 + 4sY(s) - 3 + 3Y(s) = 0$$

or

$$Y(s)\left[s^2 + 4s + 3\right] - 3s - 4 = 0$$

or

$$Y(s) = \frac{3s+4}{(s+3)(s+1)} = \frac{5}{2(s+3)} + \frac{1}{2(s+1)}$$

Thus

$$y(t) = \frac{5}{2} L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{5}{2} e^{-3t} + \frac{1}{2} e^{-t}$$

$$= \frac{1}{2}\left(e^{-t} + 5e^{-3t}\right)$$

Terminal Questions

1. Taking the Laplace transform and using Eqs. (13.4) and (13.7) we get

$$s^2 Y(s) - s - 8 + 2s\,Y(s) - 2 - 8\,Y(s) = 0$$

or

$$Y(s) = \frac{s+10}{s^2 + 2s - 8} = \frac{s+10}{(s+4)(s-2)}$$

$$= -\frac{1}{s+4} + \frac{2}{s-2}$$

Therefore

$$y(t) = -L^{-1}\left(\frac{1}{s+4}\right) + 2L^{-1}\left(\frac{1}{s-2}\right)$$

$$= -e^{-4t} + 2e^{2t}$$

2. $$\frac{dN_1}{dt} = -\lambda_1 N_1$$

Taking the Laplace transform and using Eq. (13.4) we get

$$s\,L(N_1) - N_1(0) = -\lambda_1 L(N_1)$$

or

$$L(N_1) = \frac{N_1(0)}{s+\lambda_1}$$

$$= \frac{N_0}{s+\lambda_1}$$

$$N_1 = L^{-1}\left(\frac{N_0}{s+\lambda_1}\right) = N_0 e^{-\lambda_1 t}$$

Thus

$$\frac{dN_2}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2$$

Once again

$$s\,L(N_2) - N_2(0) = \lambda_1 N_0 L\left(e^{-\lambda_1 t}\right) - \lambda_2 L(N_2)$$

or

$$L(N_2)\left[s+\lambda_2\right] = \frac{\lambda_1 N_0}{s+\lambda_1} + N_2(0)$$

or

$$L(N_2) = \frac{\lambda_1 N_0}{(s+\lambda_1)(s+\lambda_2)}$$

$$= \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2}\left[\frac{1}{s+\lambda_1} - \frac{1}{s+\lambda_2}\right]$$

$$N_2 = \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2} L^{-1}\left[\frac{1}{s+\lambda_2} - \frac{1}{s+\lambda_1}\right]$$

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

Now

$$\frac{dN_3}{dt} = \lambda_2 N_2$$

$$= \frac{\lambda_1 \lambda_2 N_0}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

Transforming and using Eq. (13.4):

$$s\,L(N_3) - N_3(0) = C\,L\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right) \quad C = \frac{\lambda_1\lambda_2 N_0}{\lambda_1 - \lambda_2}$$

or

$$L(N_3) = \frac{C}{s}\left[\frac{1}{s+\lambda_2} - \frac{1}{s+\lambda_1}\right] = \frac{C}{s}F(s)$$

Using Eq. (13.8b)

$$N_3 = C\,L^{-1}\left[\frac{F(s)}{s}\right]$$

$$= C\int_0^t \left(e^{-\lambda_2 x} - e^{-\lambda_1 x}\right)dx$$

$$= C\left[\frac{e^{-\lambda_2 x}}{-\lambda_2} + \frac{e^{-\lambda_1 x}}{\lambda_1}\right]_0^t$$

or

$$N_3(t) = C\left[-\frac{e^{-\lambda_2 t}}{\lambda_2} + \frac{1}{\lambda_2} + \frac{e^{-\lambda_1 t}}{\lambda_1} - \frac{1}{\lambda_1}\right]$$

$$= \frac{N_0}{\lambda_1 - \lambda_2}\left[\lambda_1\left(1 - e^{-\lambda_2 t}\right) + \lambda_2\left(e^{-\lambda_1 t} - 1\right)\right]$$

3. The equation of the circuit is

$$R\,i(t) + \frac{1}{C}\int_0^t i(t')\,dt' = v(t)$$

where

$$v(t) = \begin{cases} V_0 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

Taking the Laplace transform, we get

$$R\,I(s) + \frac{I(s)}{sC} = L[v(t)]$$

We can express $v(t)$ in terms of the unit step function as

$$v(t) = V_0[1 - H(t-1)]$$

where

$$H(t-1) = \begin{cases} 1 & t > 1 \\ 0 & t < 1 \end{cases}$$

Therefore

$$L\,v(t) = V_0\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) = \frac{V_0}{s}\left(1 - e^{-s}\right)$$

and

$$I(s)\left[R+\frac{1}{sC}\right]=\frac{V_0}{s}\left(1-e^{-s}\right)$$

or

$$I(s)=F(s)\left(1-e^{-s}\right)=F(s)-F(s)e^{-s}$$

$$F(s)=\frac{V_0/R}{s+1/RC}$$

Now

$$L^{-1}(F)=\frac{V_0}{R}e^{-t/RC}$$

Then applying second shifting theorem we get

$$i(t)=L^{-1}[I(s)]$$

$$=\frac{V_0}{R}\left[e^{-t/RC}-e^{-(t-1)/RC}H(t-1)\right]$$

Thus

$$i(t)=\frac{V_0}{R}e^{-t/RC} \qquad 0<t<1$$

$$=\frac{V_0}{R}\left(1-e^{1/RC}\right)e^{-t/RC} \qquad t>1$$

FURTHER READING

1. Advanced Engineering Mathematics, Kreyszig E., John Wiley & Sons, 6th Edition, 1988.
2. Mathematical Methods in the Physical Sciences, Mary L. Boas, John Wiley & Sons, 1983.