UNIT 11 APPLICATIONS OF FOURIER TRANSFORMS

Structure

11.1 INTRODUCTION

In Unit 10, we have defined the Fourier transform and evaluated Fourier transforms of simple aperiodic functions. You have also studied the Fourier integral theorem and learnt how to obtain the Fourier sine and cosine transforms of functions.

In physics, we often come across boundary value problems which can be modelled by ordinary or partial differential equations with some **initial** or **boundary conditions** on the unknown function. You have learnt several methods of solving such problems in the course PHE-05 entitled Mathematical Methods in Physics-II. For example, in PHE-05, you have learnt how to solve the wave equation, the heat conduction equation and the Laplace equation for steady state heat flow or electric potential of a charged body.

In this unit we will consider the applications of Fourier transforms in physics. We will first obtain the Fourier transforms of some functions of special interest in physics. Fourier transforms are especially useful in solving partial differential equations in physics. Therefore, as a first step you will learn how to determine the Fourier transforms of derivatives. Then we will use these results to solve a variety of partial differential equations, e.g., the wave equation, the diffusion equation and Laplace's equation. In the next unit you will study another important integral transform, namely, the **Laplace transform**.

Objectives

After studying this unit, you should be able to:

- determine the Fourier transforms of derivatives of a function;
- solve a given partial differential equation using Fourier transforms; and
- apply the method of Fourier transforms to solve problems in physics.

11.2 FOURIER TRANSFORMS IN PHYSICS

In Unit 10, we have obtained Fourier transforms of some functions and considered only a few examples from physics. We would now like to widen the scope and include some more examples from physics. This will give you some idea of the wide range of applications of Fourier transforms in physics.

Example 1: Integral Representation of Dirac Delta Function

The Dirac delta function is an important mathematical tool used extensively in physics. The one-dimensional Dirac delta function is conventionally defined as

$$\delta(x) = 0 \qquad x \neq 0 \tag{11.1a}$$

$$\int_{-a}^{b} \delta(x)\, dx = 1, \qquad a, b > 0 \tag{11.1b}$$

Thus this is a function sharply peaked in the neighbourhood of the point $x = 0$ and is zero elsewhere. Many physical quantities are conveniently represented by the Dirac delta function, e.g., point charge located at a point $x = 0$.

Eqs. (11.1a and b) also imply that

$$\int f(y)\,\delta(x-y)\,dy = f(x) \tag{11.2}$$

Now recall Eqs. (10.20a and b) from Unit 10 defining the pair of Fourier transforms:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{ixy}\, dy \tag{11.3a}$$

$$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy}\, dx \tag{11.3b}$$

Notice that we have used the variable y instead of k here.

Substituting Eq. (11.3b) in Eq. (11.3a) and changing the variable of integration to x', we can write

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy\, e^{ixy} \int_{-\infty}^{\infty} f(x') e^{-ix'y}\, dx'$$

or

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-x')y}\, dy \tag{11.4}$$

Eq. (11.4) holds for any function $f(x)$. Notice that the left hand side of Eq. (11.4) refers to the value of f at a particular value of x whereas the right hand side contains an integral over **all** values of the argument of f.

Now comparing Eq. (11.4) with Eq. (11.2), we get

$$\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(x-x')y}\, dy \tag{11.5}$$

This is the **integral representation of the Dirac delta function**. This representation of the Dirac delta function brings out its special property: Notice that the left hand side of Eq. (11.4) is independent of the value of $f(x')$ for all $x' \neq x$ and f is an arbitrary function. Then the effect of Dirac delta function is to make the integral over x' receive zero contribution from all x' except in the immediate neighbourhood of $x' = x$. At $x' = x$, the contribution is so large that we get a finite value of the integral.

We can thus write Eq. (11.4) as:

$$f(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, \delta(x-x') \tag{11.6}$$

We can visualize the Dirac delta function as a very sharp narrow pulse (in space, time, density etc.). The Dirac delta function can be used to present a discrete quantum as if it were a continuum distribution. This is very useful in effects which take place in an interval which is much shorter than the response interval of the system on which they act. For example, the impulse at time t_0 can be represented as

$$j(t) = J\,\delta(t-t_0) \tag{11.7}$$

Or a point charge q at position x_0 can be written as the following charge distribution

$$\rho(x) = q\,\delta(x - x_0) \tag{11.8}$$

In terminal question 3(b) of Unit 10, you have obtained the Fourier transform of a function which we will now interpret physically.

Example 2: Fourier Transform of a Finite Wave Train

The function

$$f(t) = \begin{cases} \sin \omega_0 t & |t| < \dfrac{N\pi}{\omega_0} \\ 0 & |t| > \dfrac{N\pi}{\omega_0} \end{cases} \tag{11.9}$$

is an infinite wave train $\sin \omega_0 t$ which has been clipped to a finite wave train having N cycles by some device such as the Kerr cell (Fig. 11.1). Thus it represents a finite wave train. Obtain its Fourier transform.

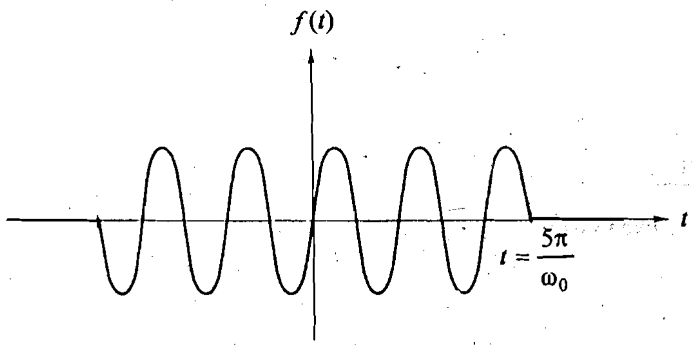

Fig.11.1: Finite wave train

Solution

In terminal question 3(b) of Unit 10 you have obtained the Fourier sine transform of this function as

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{N\pi/\omega_0} \sin \omega_0 t \sin \omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin[(\omega_0 - \omega)(N\pi/\omega_0)]}{2(\omega_0 - \omega)} - \frac{\sin[(\omega_0 + \omega)(N\pi/\omega_0)]}{2(\omega_0 + \omega)} \right] \tag{11.10}$$

How does $g(\omega)$ depend on frequency? For large ω_0 and $\omega_0 \approx \omega$, only the first term in Eq. (11.10) will be important. It is plotted in Fig. 11.2. Do you recognise this pattern? It is the amplitude curve for the single slit diffraction pattern (Recall Block 3 of the course PHE-09 entitled Optics).

The function $g(\omega)$ has zeroes at

$$\frac{\omega_0 - \omega}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \pm \frac{1}{N}, \pm \frac{2}{N} \text{ and so on.}$$

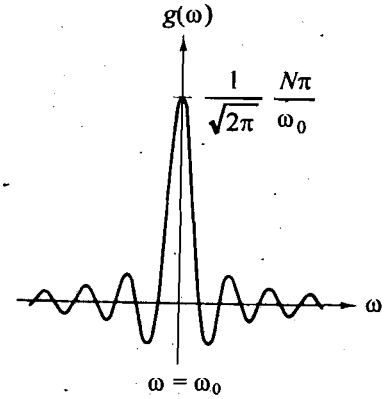

Fig.11.2: Fourier transform of finite wave train

The Fourier transform of the finite wave train may also be interpreted as a Dirac delta distribution. Since the contributions outside the central maximum are small, we may take

$$\Delta\omega = \frac{\omega_0}{N}$$

as a good measure of the spread in the frequency of the pulse. If N is large, i.e., if we have a long pulse, the frequency spread will be small. If, however, N is small, i.e., the wave train is clipped short, the frequency distribution will be wider.

You may now like to work out an exercise yourself.

SAQ 1

Spend 10 min

In a resonant cavity, an electromagnetic oscillation of frequency ω_0 dies out as

$$f(t) = \begin{cases} A_0 e^{-\omega_0 t/2Q} e^{+i\omega_0 t}, & t > 0 \\ 0 & t < 0 \end{cases}$$

The parameter Q is a measure of the ratio of the stored energy to energy loss per cycle. Determine the frequency distribution of the oscillation $g^*(\omega)\, g(\omega)$, where $g(\omega)$ is the Fourier transform of $f(t)$. Interpret your result physically.

An important function which appears in many areas of physical science is the **Gaussian** or **normal distribution**. You have encountered this function in the physics electives PHE-04 (Units 5 and 7) and PHE-06. Its Fourier transform is of importance both in itself and because, when interpreted statistically, it yields a form of uncertainty principle independent of quantum mechanics. We now take up this function.

Example 3: Gaussian distribution function

We take the Gaussian distribution in the normalized form:

$$f(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\tau} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad -\infty < t < \infty \tag{11.11}$$

This distribution has zero mean and a root-mean-square deviation $\Delta t = \tau$. Its Fourier transform is

$$g(\omega) = \frac{1}{2\pi\tau} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\tau^2}\right) \exp(-i\omega t)\, dt$$

We now add and subtract the quantity $(\tau^2 i\omega)^2$ to express the exponent in the integrand as a complete square. Thus

$$g(\omega)=\frac{1}{2\pi\tau}\int_{-\infty}^{\infty} dt\,\exp\left\{-\frac{1}{2\tau^2}\left[t^2+2\tau^2 i\omega t+(\tau^2 i\omega)^2-(\tau^2 i\omega)^2\right]\right\}$$

or

$$g(\omega)=\frac{\exp(-\tau^2\omega^2/2)}{2\pi\tau}\int_{-\infty}^{\infty}\exp\left[-\frac{(t+i\omega\tau^2)}{2\tau^2}\right]dt$$

Now changing the variable of integration to $u = t + i\omega\tau^2$, we obtain

$$g(\omega)=\frac{\exp(-\tau^2\omega^2/2)}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}\,\tau}\int_{-\infty}^{\infty}\exp\left[-\frac{u^2}{2\tau^2}\right]du$$

Since the Gaussian function is normalized, we have

$$\frac{1}{\sqrt{2\pi}\,\tau}\int_{-\infty}^{\infty}\exp\left[-\frac{u^2}{2\tau^2}\right]du=1$$

Thus

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\tau^2\omega^2}{2}\right) \quad (11.12a)$$

Notice that $g(\omega)$ is itself a Gaussian distribution with zero mean and a root-mean-square deviation $\Delta\omega=\frac{1}{\tau}$. Recall that $\Delta t=\tau$ so that we get

$$\Delta\omega.\Delta t=1$$

That is, the spreads in ω and t are related inversely and the relation is independent of the value of τ (Fig. 11.3). Notice from Fig. 11.3 that as τ decreases, the peak of $f(t)$ becomes sharper while that of $g(\omega)$ becomes broader, and vice versa.

In physical terms of time and frequency, the narrower an impulse (say an electrical pulse) is in time, the greater the spread of frequency components it must contain. Thus, if $f(t)$ is sharply peaked, $g(\omega)$ will be flattened and vice-versa.

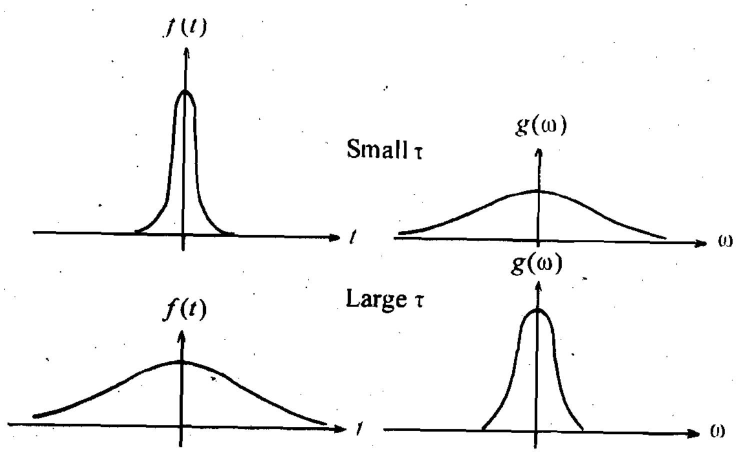

Fig.11.3: The Fourier transform of Gaussian distribution

This result is valid for any pair of Fourier related variables, e.g., spatial position and wave number,

$$\Delta k . \Delta x = 1$$

Now you may be wondering: What does all this mean? Suppose for a pair of Fourier transforms $f(x)\left(\equiv e^{-bx^2}\right)$ and $C(k)$ similar to $f(t)$ and $g(\omega)$, we take travelling waves of the form e^{ikx} with all possible values of k from $-\infty$ to ∞. The values of k are such that the amplitude of each wave is determined by the value of the Fourier transform $C(k)$. Now suppose we superpose all these waves (that is, add them algebraically or integrate over k since k is a continuous variable here). Then the result would just be the function $f(x)$. In fact, we could see this by substituting the Fourier transform $C(k)$ back into the expression for $f(x)$. For $f(x) = \exp(-bx^2)$,

$$C(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-bx^2\right) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2b}} \exp\left(-\frac{k^2}{4b}\right) \quad (11.12b)$$

Therefore, from Eq. (11.3a)

$$f(x) \equiv \exp\left(-bx^2\right) = \frac{1}{\sqrt{4\pi b}} \int_{-\infty}^{\infty} \exp\left(-\frac{k^2}{4b}\right) e^{ikx} dk \quad (11.12c)$$

The factor $\left(1/\sqrt{4\pi b}\right)\exp\left(-k^2/4b\right)$ in the right hand side of this equation is the amplitude of the wave e^{ikx}. The integral in Eq. (11.12c) is known as the **integral representation** of the function $\exp\left(-bx^2\right)$.

This result can be related to Heisenberg's uncertainty principle in quantum mechanics (refer to Unit 5 of PHE-11). In quantum mechanics, a particle such as an electron is represented by a wave packet. You have studied in PHE-11 that the position and the momentum of the particle are both uncertain and the product of these uncertainties has a certain minimum attainable value. Let us consider a wave packet whose position uncertainty is Δx.

This wave packet can now be thought of as being made up of an infinite number of waves with all possible wave numbers k, such that each wave has a known amplitude. This mixing will result in an uncertainty in its wave number which will be inversely proportional to the uncertainty in position. The product of the uncertainties in the position and the wave number of the particle would thus be a constant of the order of unity, that is $\Delta x\ \Delta k \approx 1$ as we have said above. The momentum of the particle is proportional to its wave number ($p = \hbar k$ where $\hbar$ is the Planck's constant) and hence $\Delta p = \hbar \Delta k$. This immediately shows that $\Delta x\ \Delta p \approx \hbar$, which is nothing but Heisenberg's uncertainty principle.

Again if we use the relation

$$E = \hbar\omega$$

and take $f(t)$ to represent the wavefunction, we can write the result $\Delta\omega\ \Delta t = 1$ as

$$\Delta E\ \Delta t \approx \hbar$$

Let us consider another application of Fourier transforms in quantum mechanics.

Example 4: The Quantum Mechanical Wave Function in Momentum Space

In Blocks 2 and 3 of the course PHE-11, you have studied the Schrödinger equation and learnt to solve it for simple systems. You have obtained the wave functions for a particle in a box, the simple harmonic oscillator, the hydrogen atom etc. All these wave functions $\psi(x)$

or $\psi(r)$ which depended on position were solutions in what is known as the **configuration space**. For a particular quantum mechanical system it is also possible to find the wave function in an equivalent space called the **momentum space** in which the wave function is the function of momentum. The wave functions in the two spaces are related by the equations

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) e^{ipx/\hbar} dp \tag{11.13a}$$

and

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx \tag{11.13b}$$

Clearly, but for the presence of $\hbar$, $\psi(x)$ and $\phi(p)$ are Fourier transforms of one another. The representation of a wave function in the momentum space is found to be useful in problems such as Compton scattering from atomic electrons, the wavelength distribution of scattered radiation, which depends on the momentum distribution of the target electrons.

We hope that these examples have given you a glimpse of the range of applications of Fourier transforms in physics. We now turn our attention to the solution of differential equations. But before that you would like to attempt an exercise.

SAQ 2

Spend 10 min

Show that if the wave function $\psi(x, t)$ in configuration space is normalized to unity, then the wavefunction in momentum space is also normalized to unity.

Let us now study the application of Fourier transforms in solving differential equations.

11.3 SOLVING DIFFERENTIAL EQUATIONS THROUGH FOURIER TRANSFORMS

In order to use Fourier transforms to solve a given differential equation we need to know Fourier transforms of derivatives. So to begin with we will study this aspect.

11.3.1 Fourier Transforms of Derivatives

The Fourier transform of a function $f(x)$ is

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{11.14a}$$

and of the derivative $df(x)/dx$, it is

$$g_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df(x)}{dx} e^{-ikx} dx \tag{11.14b}$$

Integrating Eq. (11.14b) by parts, we get

$$g_1(k) = \frac{e^{-ikx}}{\sqrt{2\pi}} f(x) \Bigg|_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Now $f(x)$ must vanish as $x \to \pm\infty$, in order that the Fourier transform exists. Thus we obtain

$$g_1(k) = ik\, g(k)$$

Fourier transform of the derivative of a function

$$FT\{f'(x)\} = ik\, g(k), \quad \text{where } g(k) = FT\{f(x)\} \qquad (11.15)$$

Thus the Fourier transform of the derivative of a function is (ik) times the transform of the original function. We can readily generalise Eq. (11.15) to obtain the relation for the n^{th} derivative.

Notice that we are using the notation $FT\{f\}$ for Fourier transform of the function *f*.

Fourier transform of the n^{th} derivative of a function

$$FT\left\{\frac{d^n f(x)}{dx^n}\right\} = (ik)^n g(k) \qquad (11.16)$$

provided all integrated parts vanish as $x \to \pm\infty$.

This is the power of Fourier transform, the reason why it is so useful in solving differential equations: **The operation of differentiation is replaced by multiplication.**

You will appreciate the power of this method when you apply the results obtained so far to solve some partial differential equation in physics.

11.3.2 Applications to Partial Differential Equations in Physics

To illustrate the procedure, we consider a familiar example from elementary physics.

Example 5: The Wave Equation

Suppose an infinitely long string is vibrating freely. You know that the amplitude of vibrations satisfies the wave equation (see Unit 6, PHE-05)

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad (11.17a)$$

Solve the wave equation assuming the initial condition

$$y(x,0) = f(x) \qquad (11.17b)$$

Solution

Applying the Fourier transform to Eq. (11.17a), i.e., multiplying by $e^{-i\alpha x}$ and integrating over x, we obtain

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2} e^{-i\alpha x} dx = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial t^2} e^{-i\alpha x} dx$$

Using Eq. (11.16) for $n = 2$, we have

$$(i\alpha)^2 Y(\alpha,t) = \frac{1}{v^2}\frac{\partial^2 Y(\alpha,t)}{\partial t^2} \qquad (11.18)$$

where we have used the fact that x and t are independent of each other and the definition

$$Y(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y(x,t)\, e^{-i\alpha x} dx$$

Since in Eq. (11.18) no derivatives with respect to α appear, it is an ordinary differential equation. In fact it is the linear oscillator equation with the solution

$$Y(\alpha, t) = C\, e^{\pm iv\alpha t} \qquad (11.19)$$

Thus, an **application of Fourier transform allows us to transform a partial differential equation to an ordinary differential equation.** Applying the initial condition at $t = 0$, we get

$$Y(\alpha,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

$$= F(\alpha) \qquad (11.20)$$

where $F(\alpha)$ is the Fourier transform of $f(x)$. Thus

$$Y(\alpha, t) = F(\alpha)\; e^{\pm iv\alpha t} \qquad (11.21)$$

The inverse Fourier transform of $Y(\alpha,t)$ is

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(\alpha, t)\, e^{i\alpha x}\, dx$$

and by Eq. (11.21) we have

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha(x \pm vt)}$$

Since $f(x)$ is the inverse Fourier transform of $F(\alpha)$, we obtain the solution of Eq. (11.17a) as

$$y(x,t) = f(x \pm vt) \qquad (11.22)$$

corresponding to the waves advancing in the $-x$ and $+x$ directions.

You have studied the heat conduction equation in Unit 6 of PHE-05. You know that this equation can also be used to model the diffusion of any substance in a medium. Let us take up the example of diffusion of salt in water.

Example 6: The One-dimensional diffusion equation

Let us take up the problem of diffusion of a certain quantity of salt in water. Suppose at time $t = 0$, M g of salt is introduced at a point x_0 in water flowing in a long and narrow pipe (Fig. 11.4). Obtain the concentration of salt at a later time t.

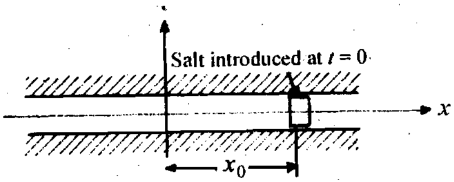

Fig.11.4: Diffusion of salt in water

Solution

Do you realise that this problem is similar to the diffusion of heat in an insulated rod which you have studied in Units 6 and 8 of PHE-05? We can idealize this problem by assuming an infinitely long pipe. Further, we can treat it as a one-dimensional problem since the pipe is very narrow. The concentration of salt $\rho(x, t)$ then satisfies the one-dimensional diffusion equation

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial x^2} \tag{11.23a}$$

with the initial condition

$$\rho(x,0) = \frac{M}{A}\delta(x-x_0) \tag{11.23b}$$

The initial condition expresses the fact that at $t = 0$, $\rho(x,0)$ is zero everywhere except at $x = x_0$ where M amount of salt has been introduced over a cross-section A of the pipe. The boundary condition is

$$\lim_{x\to\pm\infty} \rho(x,t) = 0 \qquad \text{(for all } t\text{)}$$

The boundary condition reflects the fact that the total amount of salt is conserved.

Now let $R(k,t)$ be the Fourier transform of $\rho(x,t)$. Then

$$R(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \rho(x,t)\,e^{-ikx}\,dx \tag{11.24}$$

Now

$$\frac{dR(k,t)}{dt} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial\rho(x,t)}{\partial t}e^{-ikx}\,dx \tag{11.25}$$

Using Eq. (11.23a) in Eq. (11.25) and Eq. (11.16)

$$FT\left(\frac{\partial^2\rho}{\partial x^2}\right) = (ik)^2 R(k,t)$$

we get

$$\frac{dR(k,t)}{dt} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} D\frac{\partial^2\rho(x,t)}{\partial x^2}e^{-ikx}\,dx$$

$$= -Dk^2R(k,t)$$

This is an ordinary differential equation and its solution is

$$R(k,t) = R(k,0)\,e^{-Dk^2t} \tag{11.26}$$

where $R(k,0)$ is the constant of integration.

Applying the initial condition and using the relation $R(k,0) \equiv FT[\rho(x,0)]$ alongwith Eq. (11.23b), we get

$$R(k,0) \equiv FT\left\{\frac{M}{A}\delta(x-x_0)\right\} = \frac{M}{A}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\delta(x-x_0)dx$$

$$= \frac{M}{A}\frac{1}{\sqrt{2\pi}}e^{-ikx_0},$$

where we have used Eq. (11.2).

Therefore

$$R(k,t) = \frac{M}{\sqrt{2\pi}A}e^{-ikx_0}e^{-Dk^2t}$$

We can now take the inverse Fourier transform of $R(k, t)$ to obtain

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(k,t)\, e^{ikx} dk$$

$$= \frac{M}{A}\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx_0}\, e^{-Dk^2 t}\, e^{ikx} dk$$

$$= \frac{M}{A}\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)}\, e^{-Dk^2 t} dk$$

We can express the exponent as a square and write

$$-Dt\,k^2 + ik(x-x_0) = -\left[\left(\sqrt{Dt}\,k + \frac{x-x_0}{2i\sqrt{Dt}}\right)^2 - \frac{(x-x_0)^2}{4Dt}\right]$$

Then changing the variable of integration to $k' = \left(\sqrt{Dt}\,k + \frac{x-x_0}{2i\sqrt{Dt}}\right)$, we get

$$\rho(x,t) = \frac{M}{A}\frac{1}{2\pi}\frac{1}{\sqrt{Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right] \int_{-\infty}^{\infty} e^{-k'^2} dk'$$

Using the result $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$, we obtain the expression for the concentration of salt at a later time t as

$$\rho(x,t) = \frac{M}{A}\frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right] \tag{11.27}$$

The solution $\rho(x, t)$ is shown in Fig. 11.5. Notice that for small t the spread in $\rho(x, t)$ is less, and vice versa.

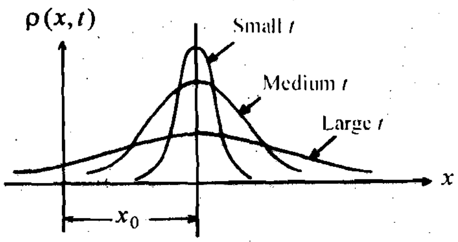

Fig.11.5: The concentration of salt in water at any time t

You may like to stop here and practice this technique yourself. Try the following SAQ.

SAQ 3

Spend 15 min

The one-dimensional Fermi age equation for the diffusion of neutrons slowing down in some medium (such as graphite) is

$$\frac{\partial^2 q(x,\tau)}{\partial y^2} = \frac{\partial q(x,\tau)}{\partial \tau}$$

Here q is the number of neutrons that slow down (falling below some given energy per second per unit volume) and τ, called the Fermi age is a measure of the energy loss. If the source of neutrons is planar, emitting N neutrons at $x = 0$, per unit area per second, i.e., $q(x,0) = S\,\rho(x)$, obtain the solution $q(x,\tau)$.

Finally, we solve the Laplace equation for steady state temperature distribution.

Example 7: Steady State Temperature Distribution of a Metallic Plate

Consider an infinite metal plate (Fig. 11.6) placed in the xy plane. Its edge along y-axis is maintained at temperature 0°C and the temperature in the edge along x-axis is given by

$$T(x,0) = \begin{cases} 100°\text{C} & 0 < x < 1 \\ 0°\text{C} & x > 1 \end{cases}$$

Determine the steady-state temperature distribution of the plate.

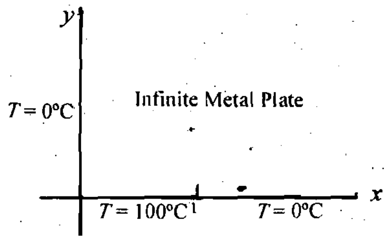

Fig.11.6: Temperature distribution in an infinite metallic plate

Solution

The temperature of the plate satisfies the Laplace equation

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0 \tag{11.28}$$

From Unit 8 of the course PHE-05, we know that the general solution of $T(x, y)$ is

$$T(x,y) = \sum_n b_n e^{-k_n y} \sin k_n x \tag{11.29}$$

where k_n is a constant. Since $T \to 0$ as $y \to \infty$ and $T = 0$ when $x = 0$, we have retained only the e^{-ky} and sine terms. Now we have to allow for all values of k so instead of b_n, we will have a function of k and instead of the summation sign we will have an integral. Note that $k > 0$ since e^{-ky} must tend to zero as $y \to \infty$. Thus we must find a solution of the form

$$T(x,y) = \int_0^\infty B(k) e^{-ky} \sin kx \, dk$$

When $y = 0$, we have

$$T(x,0) = \int_0^\infty B(k) \sin kx \, dk \tag{11.30}$$

This is a Fourier sine transform of $B(k)$. Thus $B(k)$ is the inverse Fourier sine transform of $T(x,0)$.

$$B(k) = \sqrt{\frac{2}{\pi}} g_s(k) = \frac{2}{\pi} \int_0^\infty T(x,0) \sin kx \, dk$$

For $T(x,0)$ given in this problem, we get

$$B(k) = \frac{2}{\pi} \int_0^1 100 \sin kx \, dk$$

$$= -\frac{200}{\pi} \frac{\cos kx}{k} \Big|_0^1$$

$$= \frac{200}{\pi k}(1 - \cos k)$$

Therefore

$$T(x,y)=\frac{200}{\pi}\int_0^\infty \frac{(1-\cos k)}{k} e^{-ky}\sin kx\, dk \quad (11.31)$$

Integrating this expression we obtain the solution as

$$T(x,y)=\frac{200}{\pi}\left[\tan^{-1}\frac{x}{y}-\frac{1}{2}\tan^{-1}\frac{(x+1)}{y}-\frac{1}{2}\tan^{-1}\frac{x-1}{y}\right] \quad (11.32)$$

The method of integration becomes very simple if we use the results of Unit 12 here. $T(x,y)$ is just the Laplace transform of the function

$$f(k)=\frac{(1-\cos k)\sin kx}{k}$$

Using the results

$$L\left(\frac{\sin kx}{k}\right)=\tan^{-1}\frac{x}{y}$$

$$L\left(\frac{\sin kx\cos k}{k}\right)=\frac{1}{2}\left[\tan^{-1}\frac{x+1}{y}+\tan^{-1}\frac{x-1}{y}\right]$$

we get the equation (11.32).

Let us now summarize what you have studied in this unit.

11.4 SUMMARY

- The integral representation of **Dirac delta function** can be obtained with the help of Fourier transforms as

$$\delta(x-x')=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-x')y}\, dy$$

- The Fourier transform of the **Gaussian distribution function** is itself a Gaussian distribution function such that the spreads in the pair of variables of Fourier transformation (ω, t) or (k, x) are related inversely in a classical analogue of the uncertainty principle

$$\Delta\omega\ \Delta t=1$$

$$\Delta k\ \Delta x=1$$

- The wave function of a quantum mechanical system can be represented in equivalent spaces called **momentum space** and **configuration space** using Fourier transforms:

$$\psi(x)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p)e^{ipx/\hbar}\, dp$$

$$\phi(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\, dx$$

- The **Fourier transform of the derivatives** of a function is given by

$$FT\{f'(x)\}=ik\, FT\{f(x)\}$$

$$FT\left\{\frac{d^n f(x)}{dx^n}\right\}=(ik)^n\, FT\{f(x)\}$$

This property gives rise to a powerful way of reducing partial differential equations to ordinary differential equations and solving them. In particular, the wave equation, the one-dimensional diffusion equation and the Laplace equation have been solved for a variety of physical systems.

11.5 TERMINAL QUESTIONS

Spend 30 *min*

1. A metal plate covering the first quadrant of the xy plane has its edge along y-axis insulated. The edge along x-axis is held at an initial temperature

$$T(x,0)=\begin{cases}100\,(2-x) & \text{for}\quad 0<x<2\\ 0 & \text{for}\quad x>2\end{cases}$$

Obtain the steady state temperature distribution as a function of x and y in the form of Eq. (11.31).

2. A rectangular single slit is described by

$$f(x)=\begin{cases}1 & |x|<a\\ 0 & |x|>a\end{cases}$$

Show that the amplitude of the diffraction pattern is given by $F(t)$, the Fourier transform of $f(x)$:

$$F(t)=\sqrt{\frac{2}{\pi}}\,\frac{\sin at}{t}$$

3. Obtain the momentum representation

$$g(\mathbf{p})=\frac{1}{(2\pi\hbar)^{3/2}}\int\psi(\mathbf{r})\,e^{-i\mathbf{p}.\mathbf{r}/\hbar}\,dV$$

for the ground state of the hydrogen atom

$$\psi(\mathbf{r})=\frac{1}{\left(\pi a_0^3\right)^{1/2}}\,e^{-r/a_0}\,.$$

11.6 SOLUTIONS AND ANSWERS

Self-assessment Questions

1. The Fourier transform of $f(t)$ is

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}\,dt$$

Substituting for $f(t)$ in $g(\omega)$ we get

$$g(\omega)=\frac{A_0}{\sqrt{2\pi}}\int_0^{\infty}e^{-\omega_0 t/2Q}\,e^{+i\omega_0 t}\,e^{-i\omega t}\,dt,\qquad t>0$$

$$=\frac{A_0}{\sqrt{2\pi}}\int_0^{\infty}e^{-t\left[\frac{\omega_0}{2Q}+i(\omega-\omega_0)\right]}dt$$

Using the standard result $\int_0^{\infty}t^n e^{-\alpha t}\,dt=\frac{n!}{\alpha^{n+1}}$, we get

$$g(\omega)=\frac{A_0}{\sqrt{2\pi}}\,\frac{1}{\left[\frac{\omega_0}{2Q}+i(\omega-\omega_0)\right]}$$

Now

$$g^*(\omega)\,g(\omega)=\frac{A_0^2}{2\pi}\left[\frac{1}{(\omega-\omega_0)^2+(\omega_0/2Q)^2}\right]$$

Physical interpretation: The larger Q is, that is the smaller the energy loss is, the sharper will the resonance line be in the frequency distribution as $\omega\rightarrow\omega_0$.

2. It is given that the wave function is normalized to unity in the configuration space, i.e.,

$$\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$$

Now

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \phi(p)dp$$

Thus the normalization condition for $\psi(x)$ becomes

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp'\, e^{i[(p-p')x]/\hbar} \phi^*(p')\,\phi(p) = 1$$

where we have used the dummy variable of integration p' in the Fourier transform of $\psi^*(x)$. Using the integral representation Eq. (11.5) of the Dirac delta function, we obtain

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \delta(p-p')\,\phi^*(p')\;\phi(p) = 1$$

where we have put $x' = x/\hbar$, to write

$$\delta(p-p') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i[(p-p')x]} dx$$

We now use the property of the delta function

$$\int_{-\infty}^{\infty} f(x')\,\delta(x-x')\,dx' = f(x)$$

and obtain

$$\int_{-\infty}^{\infty} dp\, \phi^*(p)\,\phi(p) = 1$$

Therefore, the wave function in the momentum space is also normalized to unity.

3. Taking the Fourier transform of the given differential equation, we get:

$$\int_{-\infty}^{\infty} \frac{\partial^2 q(x,\tau)}{\partial x^2} e^{-ikx} dx = \int_{-\infty}^{\infty} \frac{\partial q(x,\tau)}{\partial \tau} e^{-ikx} dx$$

or

$$(-ik)^2 g(k,\tau) = \frac{\partial}{\partial \tau} g(k,\tau)$$

where we have used

$$g(k,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q(x,\tau)\,e^{-ikx} dx$$

or

$$-k^2 g(k,\tau) = \frac{\partial}{\partial \tau} g(k,\tau)$$

The solution is

$$g(k,\tau) = C \exp\left(-k^2\tau\right)$$

where $C = g(k,0)$.

At $\tau = 0$, applying the initial condition we get

$$g(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S\,\delta(x)\, e^{-ikx}\, dx$$

$$= \frac{S}{\sqrt{2\pi}} \quad , \quad \text{since } \int_{-\infty}^{\infty} e^{-ikx}\, \delta(x)\, dx = e^{-ikx}\Big|_{x=0} = 1$$

Thus

$$g(k,\tau) = \frac{S}{\sqrt{2\pi}} \exp\left(-k^2\tau\right)$$

The inverse Fourier transform of $g(k,\tau)$ yields the solution

$$q(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k,\tau)\, e^{ikx}\, dk$$

$$= \frac{S}{2\pi} \int_{-\infty}^{\infty} \exp\left(-k^2\tau + ikx\right) dx$$

Following the procedure of Example 6, we obtain

$$q(x,\tau) = \frac{S}{\sqrt{4\pi\tau}} \exp\left(-x^2/4\tau\right).$$

Terminal Questions

1. We will follow the same procedure as in Example 7. However, in this case we use the cosine solutions as $T \neq 0$ when $x = 0$ but $\frac{\partial T}{\partial x} = 0$ when $x = 0$. Thus the general solution will be of the form

$$T(x,y) = \int_0^{\infty} B(k)\, e^{-ky} \cos kx\, dk$$

When $y = 0$, we have

$$T(x,0) = \int_0^{\infty} B(k) \cos\, kx\, dx$$

This is a Fourier cosine transform of $B(k)$. Thus

$$B(k) = \frac{2}{\pi} \int_0^{\infty} T(x,0) \cos kx.dx$$

For $T(x,0)$ given in this problem, we get

$$B(k) = \frac{2}{\pi} \int_0^{2} (200 - 100x) \cos kx\, dx$$

$$= \frac{400}{\pi} \int_0^{2} \cos\, kx\, dx - \frac{200}{\pi} \int_0^{2} x \cos\, kx\, dx$$

$$= \frac{400}{\pi}\frac{\sin 2k}{k} - \frac{400}{\pi}\frac{\sin 2k}{k} - \frac{200}{\pi}\left(\frac{\cos 2k}{k^2} - \frac{1}{k^2}\right)$$

$$= \frac{200}{\pi k^2}(1 - \cos 2k)$$

Therefore,

$$T(x, y) = \frac{200}{\pi}\int_0^\infty \frac{(1-\cos 2k)}{k^2} e^{-ky} \cos kx \, dk$$

2. In Example 1 of Unit 10 you have obtained the Fourier transform of a similar function for $a = 1$. Following the same procedure we obtain

$$F(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-itx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-itx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{-itx}}{-it}\right|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-ita} - e^{ita}}{-it}$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sin at}{t}$$

or

$$F(t) = \sqrt{\frac{2}{\pi}}\,\frac{\sin at}{t}$$

3. $$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\int \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} dV$$

Substituting $\psi(\mathbf{r})$ in $g(\mathbf{p})$ we get

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\frac{1}{\left(\pi a_0^3\right)^{1/2}}\int e^{-r/a_0} e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} dV$$

Let

$$I = \int e^{-r/a_0} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} dV$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-r/a_0} e^{ipr\cos\theta/\hbar} r^2 dr \sin\theta \, d\theta \, d\phi$$

where we have substituted the volume element in spherical polar coordinates.

Now let us evaluate the integral

$$I_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-ar+ibr\cos\theta} r^2 dr \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \int_0^\infty dr\, r^2 e^{-ar} \int_0^\pi e^{ibr\cos\theta} \sin\theta d\theta$$

Putting $\cos\theta = t$

$$I_1 = 2\pi \int_0^\infty dr\, r^2 e^{-ar} \int_{-1}^1 e^{ibrt} dt$$

$$= \frac{4\pi}{b} \int_0^\infty dr\, r\, e^{-ar} \sin br\, dr$$

$$= \frac{4\pi}{b} \text{Im} \int_0^\infty r\, e^{-ar+ibr} dr$$

where Im (z) denotes the imaginary part of z. Thus

$$I_1 = \frac{4\pi}{b} \text{Im} \int_0^\infty r\, e^{-r(a-ib)} dr$$

$$= \frac{4\pi}{b} \text{Im} \frac{1}{(a-ib)^2} = \frac{8\pi a}{\left(a^2+b^2\right)^2}$$

Therefore with $a = 1/a_0, \quad b = p/\hbar$

$$I = \frac{8\pi / a_0}{\left(\frac{1}{a_0^2} + \frac{p^2}{\hbar^2}\right)^2} = \frac{8\pi a_0^3 \hbar^4}{\left(\hbar^2 + p^2 a_0^2\right)^2}$$

Therefore

$$g(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \cdot \frac{1}{\left(\pi a_0^3\right)^{1/2}} \cdot \frac{8\pi a_0^3 \hbar^4}{\left(\hbar^2 + p^2 a_0^2\right)^2}$$

$$= \frac{2^{3/2}}{\pi} \cdot \frac{a_0^{3/2}\, \hbar^{5/2}}{\left(a_0^2 p^2 + \hbar^2\right)^2}$$