UNIT 7 FUNCTIONS OF A COMPLEX VARIABLE — ANALYTICITY

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7.1 INTRODUCTION

From the basic calculus course in earlier classes, you are familiar with the concepts of limit, continuity and differentiability of the functions of a real variable. We now turn to a study of these aspects for functions of a complex variable. As pointed out in the Block Introduction, complex analysis is a powerful tool as it facilitates solutions of many problems in different areas of physics, engineering and other allied sciences. In Block 1 of PHE-02 course on Oscillations and Waves, you learnt how conveniently we could obtain the resultant of n superposed harmonic oscillations using complex numbers. Similarly, calculation of the intensity distribution of Fraunhofer diffraction from N identical slits (Block 3 of PHE-09 course on Optics) became possible in a compact form using the complex numbers. While analysing a.c. circuits, this theory simplifies derivations considerably. For such elementary problems, a basic knowledge of complex numbers is sufficient. However, for slightly involved but interesting practical problems in fluid dynamics, the theory of heat, electrostatics and quantum mechanics, a detailed knowledge of the theory of complex analytic functions is necessary.

We begin this unit by introducing complex numbers. You will then learn the properties of functions of a complex variable; their limit and continuity. You will note that differentiability of a function of complex variable needs careful attention as the conditions for the existence of the derivative of a function of a complex variable are different from those of a real variable. In this connection you will learn to derive the Cauchy-Riemann conditions and state the conditions under which a function of a complex variable is analytic.

Some elementary functions of a complex variable and their properties are discussed in Sec. 7.4. And singularities of a complex function form the subject matter of discussion of Sec. 7.7.

Objectives

After completing this unit, you should be able to:

- calculate the derivative of a function of a complex variable;
- identify complex analytic functions and domains of their analyticity;
- work with elementary complex functions; and
- locate the singularities of a complex function.

7.2 COMPLEX VARIABLES

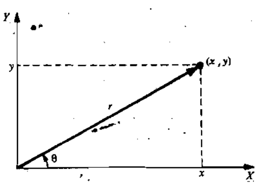
You know that a complex number is an ordered pair of two real numbers a and b written as $a+i\,b$, where $i\left(=\sqrt{-1}\right)$. Similarly, a complex variable is an ordered pair of two real variables

$$z = x + i\,y$$

The ordering is important as $a + i\,b$ is not equal to $b + i\,a$ and $x + i\,y \neq y + i\,x$, in general. Sometimes we use j instead of i. It is called the imaginary unit. You will recall that x is real part of z and y is imaginary part of z:

$$x = \text{Re}(z)$$
$$y = \text{Im}(z)$$

It is useful to represent complex numbers and variables graphically. For this we generally plot x, the real part of z as the abscissa, and y, the imaginary part of z as the ordinate and have the complex xy-plane, as shown in Fig. 7.1(a). We can plot a complex number as a point in the plane if we assign specific values to x and y. The magnitude of the complex number is the distance from the origin to the point representing the number.



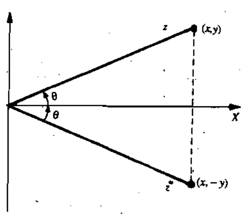


Fig.7.1: (a) A complex number in rectangular and polar coordinate systems; (b) complex conjugate points

In polar coordinates you can write

$$x = r\cos\theta$$

and

$$y = r\sin\theta$$

Then we get a useful polar representation of a complex number:

$$z = x + iy = r\,(\cos\theta + i\sin\theta)$$
$$= r\,e^{i\theta}$$

where

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is called the **Euler identity**. In this representation, r is called the modulus of z and the angle θ is the argument or phase of z:

$$r = |z| = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}(y/x)$$

Fig. 7.1(a) depicts a complex number in rectangular and polar coordinate systems.

You should note here that the choice of coordinate system is dictated by mathematical convenience and/or geometry. For instance, addition and subtraction of complex numbers are easier in the Cartesian representation, whereas multiplication, division, powers and roots are easier to handle in polar form.

The complex conjugate z^* of a complex number z is obtained by changing i to $-i$ everywhere in its expression. Thus $z^* = x - iy$. Notice that

$$zz^* = (x + i\,y)(x - i\,y)$$
$$= x^2 + y^2 = r^2$$

or

$$r = \sqrt{zz^*}$$

Hence $\sqrt{zz^*} = |z|$ is the magnitude of z.

You should note that a complex variable and its conjugate are mirror images of each other reflected in the x-axis (Fig. 7.1b).

So far we have confined ourselves to complex variables. But complex analysis concerns itself with functions of complex variables or complex function. In particular we will focus on complex analytic function. You will learn about these in the following section.

7.3 COMPLEX ANALYTIC FUNCTIONS

A function of a complex variable, $f(z)$ is a prescription for assigning a **unique** complex number to each z. The set of values for which the function is defined is called the domain of z.

If you express $f(z)$ in terms of x and y by putting $x + i\,y$ for z, the complex function can be expressed in terms of real and imaginary parts; each being a function of x and y:

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y) \tag{7.1}$$

where $u(x, y)$ and $v(x, y)$ are respectively the real and imaginary parts of $f(z)$.

We illustrate this through an example.

Example 1: Complex analytic function

Express $f(z) = z^2$ in the form $u(x, y) + i\,v(x, y)$.

Solution

$$f(z) = z^2 = (x + i\,y)^2 = x^2 + (i\,y)^2 + 2x(i\,y)$$

$$= x^2 - y^2 + 2i\,xy$$

Hence $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

7.3.1 Limit, Continuity and Derivative of a Complex Function

Neighbourhood: The set of all points z such that $|z-z_0| < \delta$, where δ is any given positive number, is called a **δ-neighbourhood** of the point z_0. When the point z_0 is omitted from the set, i.e., $0 < |z-z_0| < \delta$, the set of points is called a **deleted δ-neighbourhood**. Geometrically, a δ-neighbourhood is represented by all points lying inside a circle of radius δ drawn around z_0 in the complex plane.

Limit: A function $f(z)$ is said to have the limit ℓ as z approaches z_0, if for a positive real number ε, (no matter how small, but not zero), there is a positive real number δ depending on ε and z_0 such that $|f(z) - \ell| < \varepsilon$ for all z for which $0 < |z-z_0| < \delta$. This means that $f(z)$ can be made to approach ℓ as close as desired by bringing z close to z_0 (Fig. 7.2). We write this condition as

$$\lim_{z \to z_0} f(z) = \ell.$$

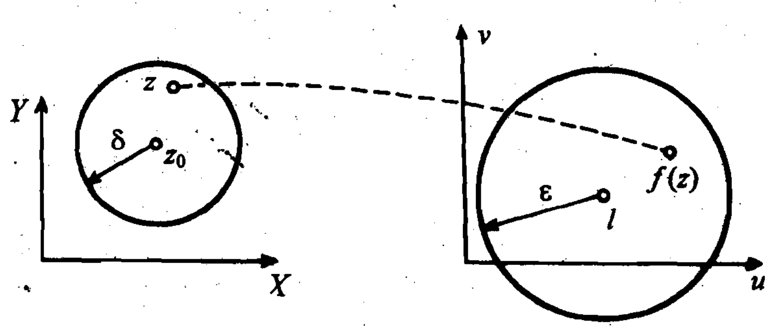


Fig.7.2:

Continuity: In the definition of limit, you must have noted that z may approach z_0 from any direction in the complex plane. Moreover, nothing is said about the value of $f(z)$ at z_0. In particular, $f(z_0)$ need not be equal to ℓ. But if $f(z_0) = \ell$, the function $f(z)$ is said to be **continuous** at z_0:

$$\lim_{z \to z_0} f(z) = f(z_0) = \ell$$

This definition implies that $f(z)$ may be said to be continuous in some domain if it is continuous at each point of that domain.

Having established functions of a complex variable and discussed their limit as well as continuity, we now proceed to calculate the derivative of a complex function.

Derivative of a complex function

A function $f(z)$ is said to be **differentiable** at the point $z = z_0$ if the limit

$$\lim_{\Delta z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. Before we proceed further, let us pause for a moment and recall what is implied by the definition of a limit. It implies that $f(z)$ is defined in a neighbourhood of z_0. Moreover, z may approach z_0 from any direction. Hence, differentiability at z_0 means that $f(z_0)$ has a unique value independent of the path followed to approach this point.

The derivative of a function of a complex variable $f(z)$ is defined in the same way as that for a function of a real variable:

$$\frac{df(z)}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{7.2}$$

If Δx and Δy are increments in the variables x and y, respectively, we can write $\Delta z = \Delta x + i\Delta y$. Then $\Delta z \to 0$ means that the real part Δx and the imaginary part Δy each must tend to zero separately. There are an infinite number of ways of taking this limit. Refer to Fig. 7.3 and consider a point z at which you are calculating the derivative of the function $f(z)$. You take another adjacent point $z + \Delta z$. Depending upon the arg Δz, the vector Δz will orient in the complex plane. So when the limit $\Delta z \to 0$ is considered, you may think that you

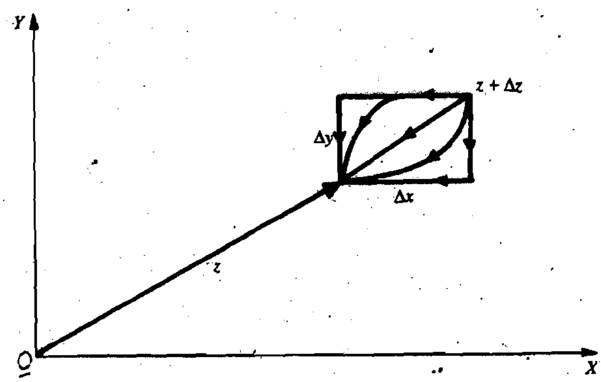


Fig.7.3: The point z is approached in many different ways as $|\Delta z| \to 0$

can approach z from all possible directions. In other words, $|\Delta z| \to 0$, but the arg Δz can have arbitrary values. Then according to Eq. (7.2), the limit may assume an infinite number of values depending on the value of arg Δz chosen (which is any value between 0 to 2π). This is very unsatisfactory. It is therefore important to know as to under what conditions the limit given by Eq. (7.2) does not depend on the path chosen to approach z and is therefore unique.

To enable you to appreciate the problem better, we consider a specific example. Let $f(z) = x + 3iy$. Then

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{x+\Delta x+3i(y+\Delta y) - (x+3iy)}{\Delta x + i\,\Delta y}$$

$$= \frac{\Delta x + 3i\,\Delta y}{\Delta x + i\,\Delta y}$$

If we first put $\Delta y = 0$, and let $\Delta x \to 0$, the limit is 1. But if we let $\Delta x = 0$, and $\Delta y \to 0$, the limit is +3. That is, we obtain a different value for each path and the limit is not unique. Then we say that the derivative of $f(z)$ does not exist.

This example clearly illustrates that differentiability of a complex function is a rather stronger requirement than in real analysis.

SAQ 1

Spend 5 min.

Show that the function $f(z) = |z|^2$ is differentiable only at $z = 0$.

7.3.2 Cauchy-Riemann Conditions

Consider a function $f(z)$ whose first derivative $f'(z)$ exists. If we write from Eq. (7.1) $f(z) = u(x, y) + i\,v(x, y)$ and note that

$$z + \Delta z = x + i\,y + (\Delta x + i\,\Delta y) = (x + \Delta x) + i(y + \Delta y)$$

then we find that

$$f(z+\Delta z) = u(x+\Delta x,\ y+\Delta y) + i\,v(x+\Delta x,\ y+\Delta y)$$

$$\therefore \quad \frac{df(z)}{dz} = f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\,\Delta y} \qquad (7.3)$$

In evaluating this double limit, you can take $\Delta x \to 0$ and $\Delta y \to 0$ in any order. But the derivative $f'(z)$ should be independent of the path you take in making z approach zero.

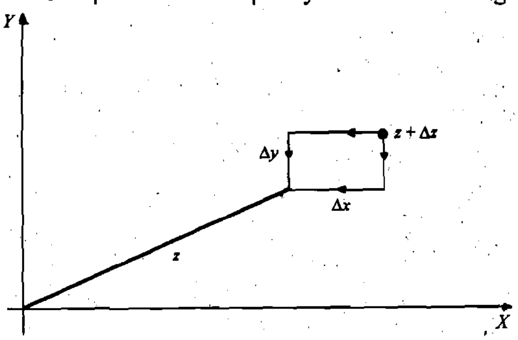


Fig.7.4: Two alternative approaches to the point z

Refer to Fig. 7.4. It shows two different paths. Let us now find the conditions necessary for the existence of $f'(z)$.

Path 1: Let $\Delta y = 0$ so that $\Delta z \to 0$ parallel to the x-axis. With $\Delta z = \Delta x$, we can write Eq. (7.3) as

$$f'(z) = \lim_{\Delta z\to 0} \frac{\{u(x+\Delta x, y) + i\,v(x+\Delta x, y)\} - \{u(x,y) + i\,v(x,y)\}}{\Delta x}$$

Now let $\Delta x \to 0$. The expression for $f'(z)$ takes the form

$$f'(z) = \lim_{\Delta x \to 0} \left\{ \left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right) + i \left(\frac{v(x+\Delta x, y) - v(x, y}{\Delta x} \right) \right\}$$

$$= \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right) + i \lim_{\Delta x \to 0} \left(\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right)$$

Since $f(z)$ exists, we can safely assume that the real limit is non-zero. Hence, we obtain

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \tag{7.4}$$

where $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are the partial derivatives of u and v respectively with respect to x.

Path 2: Put $\Delta x = 0$ so that $\Delta z \to 0$ parallel to the y-axis. Then Δz becomes $i\Delta y$ and let $\Delta y \to 0$ to obtain

$$f'(z) = \lim_{\Delta y \to 0} \left\{ \frac{u(x, y+\Delta y) + i\, v(x, y+\Delta y) - u(x,y) - i\, v(x,y)}{i\, \Delta y} \right\}$$

$$= \lim_{\Delta y \to 0} \left(\frac{u(x, y+\Delta y) - u(x,y)}{i\, \Delta y} \right) + \lim_{\Delta y \to 0} \left(\frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right)$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{7.5}$$

You should note that the existence of the derivative $f'(z)$ implies the existence of four partia derivatives. Further, uniqueness of the derivative requires that the expressions for $f'(z)$ be equal:

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

By equating the real and the imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{7.6a}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{7.6b}$$

These basic relations are known as **Cauchy - Riemann conditions**. These were derived by Cauchy and used extensively by Riemann. We may now summarise the important result of this sub-section.

> If $f(z) = u(x,y) + i\, v(x,y)$ is defined and is differentiable in the neighbourhood of a point z_0, then $f'(z)$ exists at a point z_0, if Cauchy - Riemann conditions are satisfied and the first partial derivatives of u and v are continuous at z_0.

For a polynomial function of degree n,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ a_n z^n$$

it follows that

$$f'(z) = a_1 + 2a_2 z + \ldots\ n\ a_n z^{n-1}.$$

That is, $f'(z)$ exists everywhere for a polynomial function of z.

Conversely, if the Cauchy-Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative $f'(z)$ exists.

7.3.3 Analytic Functions

A function $f(z)$ of a complex variable z is said to be **analytic** at the point z_0, if it is differentiable at that point as well as at all points in its neighbourhood. The function $f(z) = z^*$ is not an analytic function of z as its derivative exists no-where in the complex plane. To enable you to appreciate this statement, we note that by definition

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

If this limit exists independent of the manner in which Δz approaches zero and is unique, we say that the function is analytic. Otherwise, the function will be non-analytic. Here

$$\frac{dz^*}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^* - z^*}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - i\,y + \Delta x - i\,\Delta y - (x - iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(x + i\,y + \Delta x + i\,\Delta y)^* - (x + iy)^*}{\Delta x + i\,\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + iy}$$

From this you will note that if we take $\Delta y = 0$, then $\frac{dz^*}{dz} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$. On the other hand, if we put $\Delta x = 0$, $\frac{dz^*}{dz} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{\Delta y} = -1$. Since the required limit is not unique, i.e. it depends on the path in which $\Delta z \to 0$, the derivative does not exist and $f(z) = z^*$ is non-analytic.

Next we consider the function $f(z) = |z|^2$. While solving SAQ 1, you have shown that it is non-analytic except at the origin. Now we use Cauchy-Riemann equations to illustrate this point. From Sec. 7.2 you will recall that

$$|z|^2 = zz^* = x^2 + y^2$$

so that

$$u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0$$

$$\therefore \frac{\partial u}{\partial y} = 2x, \ \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = 2y, \ \frac{\partial v}{\partial x} = 0$$

From these results, you will note that the Cauchy-Riemann equations will hold only if x and y are zero. This is true only at the origin and not in the neighbourhood, howsoever small, of the origin. That is, $f(z) = |z|^2$ is differentiable only at $z = 0$ and not at any point in the neighbourhood of $z_0 = 0$. So $|z|^2$ is not an analytic function at $z_0 = 0$.

Next, let us consider the function $f(z) = z^2$. We can write it as $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$. Therefore, $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

and

$$\frac{\partial u}{\partial y} = -2y, \quad -\frac{\partial v}{\partial y} = -2y$$

Since the Cauchy-Riemann equations are satisfied at all points in the xy-plane, the function z^2 is analytic everywhere.

Analytic functions hold the centre stage in complex analysis. You will learn more about these functions as you progress along this course.

A function $f(z)$ that is analytic at every point of the z-plane is called an **entire function**. You can easily verify that every polynomial in z is an entire function. You will come across other examples of entire functions in Sec. 7.4.

7.3.4 Harmonic Functions

You have studied the Laplace's equation in electrostatics in PHE-07 as well as in the PHE-07 course on differential equations. Starting from Cauchy - Riemann conditions, we now show that the function $u(x,y)$ and $v(x,y)$ separately satisfy the Laplace's equation in x and y.

To proceed, we differentiate Eq. (7.6a) with respect to x, to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

provided these second derivatives exist. In fact, $u(x,y)$ and $v(x,y)$ have continuous partial derivatives of all orders since the derivative of an analytic function $f(z) = u + i\,v$ is also analytic.

Similarly, by differentiating Eq. (7.6b) with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Since the mixed second order derivatives of these functions must be equal, we have

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Using this result in the preceding two equations, we obtain the Laplace's equation for u:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (7.7a)$$

By similarly differentiating Eq. (7.6a) with respect to y and Eq. (7.6b) with respect to x, we get Laplace's equation for v:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (7.7b)$$

So we find that the real and imaginary parts of a complex function that is analytic in a domain satisfy Laplace's equation. In fact, a function that has continuous partial derivative of the second order and satisfies the Laplace's equation is said to be a **harmonic function** in the variables x and y. For an analytic function $f(z) = u(x, y) + i\, v(x, y)$, both u and v are harmonic functions; one of these is said to be the **conjugate harmonic** of the other harmonic function.

To illustrate these concepts, we will now like you to go through the following example.

Example 2: Harmonic conjugate function

Show that $u = 2x\,(1-y)$ is harmonic in some domain. Calculate its harmonic conjugate v.

Solution

Since

$$u = 2x\,(1-y)$$

$$\frac{\partial u}{\partial x} = 2\,(1-y) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = 0$$

Similarly

$$\frac{\partial u}{\partial y} = -2x \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 0$$

On combining these results, we find that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So u is a harmonic function.

To find its harmonic conjugate, we use the Cauchy-Riemann condition given by Eq. (7.6a):

$$\frac{\partial u}{\partial x} = 2\,(1-y) = \frac{\partial v}{\partial y}$$

Integrating v with respect to y, we obtain

$$v = 2y - y^2 + g(x) \qquad \text{(i)}$$

Again

$$\frac{\partial u}{\partial y} = -2x = -\frac{\partial v}{\partial x}$$

Integrating v with respect to x, we get

$$v = x^2 + h(y) \qquad \text{(ii)}$$

By equating (i) and (ii), we get

$$2y - y^2 + g(x) = x^2 + h(y)$$

Comparison of terms containing x and y leads us to

$$h(y) = 2y - y^2 \text{ and } g(x) = x^2$$

$$\therefore \quad v(x, y) = x^2 + 2y - y^2$$

On combining this result with the given expression for u, you can construct the complex function:

$$f(z) = u + iv = 2x(1-y) + i(x^2 + 2y - y^2)$$

$$= 2x - 2xy + i(x^2 - y^2) + 2iy$$

$$= 2(x+iy) + i(x^2 + (iy)^2 + 2xiy)$$

$$= 2z + i(x+iy)^2$$

$$= 2z + iz^2$$

You may now solve an SAQ.

SAQ 2

Spend 5 min

Obtain the analytic function whose real part is $u(x, y) = e^x \cos y$.

7.4 SOME ELEMENTARY COMPLEX FUNCTIONS

You are familiar with many important elementary functions involving a real variable e.g. polynomial, exponential, trigonometric, logarithmic etc. and their properties. You may now like to learn about such functions when the independent variable is extended to the complex domain. You will note that some of the elementary functions have interesting properties, which find useful applications in physical sciences and engineering.

7.4.1 The Exponential Function e^z

The exponential function e^z is defined by the same power series in z as e^x is represented by real x:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

$$= 1 + \sum_n \frac{z^n}{n!} \qquad (7.8)$$

From Cauchy-Riemann equations, you can show that e^z is an analytic, in fact an entire function in the z-plane.

The most important property of the exponential function is the functional equation

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

To establish this, we note that

$$e^{z_1} = 1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \ldots$$

and

$$e^{z_2} = 1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \ldots$$

On multiplying these expressions, we have

$$e^{z_1}e^{z_2} = 1 + (z_1 + z_2) + \left(\frac{z_1^2}{2!} + z_1 z_2 + \frac{z_2^2}{2!}\right) + \ldots$$

$$+ \left(\frac{z_1^n}{n!} + \frac{z_1^{n-1} z_2}{(n-1)!} + \ldots + \frac{z_1 z_2^{n-1}}{(n-1)!} + \frac{z_2^n}{n!}\right) + \ldots$$

$$= 1 + (z_1 + z_2) + \frac{(z_1 + z_2)^2}{2!} + \ldots + \frac{(z_1 + z_2)^n}{n!} + \ldots$$

$$= e^{z_1 + z_2}$$

If $z_1 = z$ and $z_2 = -z$, then it readily follows that $e^z . e^{-z} = e^{z-z} = e^0 = 1$ holds even for complex z.

$$\therefore \quad e^{-z} = \frac{1}{e^z}$$

$$\Rightarrow \quad e^z \neq 0$$

Example 3: Exponention function

Show that $\frac{d}{dz} e^z = e^z$.

Solution

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

$$= u(x, y) + i\, v(x, y)$$

where we have put $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$. From Eq. (7.4) you will recall that $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$. Here

$$\frac{\partial u}{\partial x} = e^x \cos y \text{ and } \frac{\partial v}{\partial x} = e^x \sin y$$

$$\therefore \quad f'(z) = \frac{d}{dz} e^z = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x (\cos y + i \sin y)$$

$$= e^x e^{iy} = e^{x+iy} = e^z$$

7.4.2 Trigonometric Functions

The geometrical definition of sin θ (opposite side/hypotenuse) in a right-angled triangle loses its meaning, when angle θ is replaced by a complex number.

We define sin z and cos z in terms of power series:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \qquad (7.9)$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \qquad (7.10)$$

Let us replace z by iz in Eq. (7.8). Then we obtain

$$e^{iz} = 1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} - \frac{z^6}{6!} - \ldots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)$$

$$= \cos z + i \sin z \qquad (7.11a)$$

That is, Euler's formula is valid for complex numbers.

From Euler's formula, it follows that $\exp(2\pi i) = 1$. Further, if we let $z = -z$, we can write

$$e^{-iz} = \cos(-z) + i \sin(-z)$$

You will recall that $\cos z$ is an even function of z whereas $\sin z$ is an odd function of z. Therefore, we have $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$. Hence, above equality takes the form

$$e^{-iz} = \cos z - i \sin z \qquad (7.11b)$$

On combining Eqs. (7.11a) and (7.11b), we find that

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

and

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad (7.12)$$

Both $\sin z$ and $\cos z$ are **entire functions** as they are linear combinations of e^{iz} and e^{-iz}, which are entire functions.

You can easily show that trigonometric addition formulae also hold good with complex arguments (SAQ 3);

i) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (7.13a)

ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ (7.13b)

iii) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$ (7.13c)

iv) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$ (7.13d)

Spend 10 min

SAQ 3

Prove Eq. (7.13).

By Putting $z_1 = z_2 = z$ in (iv), you will note that it reduces to

$$\cos^2 z + \sin^2 z = 1 \qquad (7.13e)$$

It is important to note that unlike the inequalities $|\sin x| \leq 1$ and $|\cos x| \leq 1$ (x real), a similar result does not follow for complex argument. To see this, let θ be any positive angle. Then

$$\cos(i\theta) = \frac{1}{2}(e^{-\theta} + e^{\theta}) > e^{\theta}$$

as $e^{-\theta} > 0$. Thus $\cos(i\theta)$ is real and can attain large values if θ is not small.

SAQ 4

Spend 2 min

Show that $\frac{d \sin z}{dz} = \cos z$ and $\frac{d \cos z}{dz} = -\sin z$.

Using the definition of cos z, we can write

$$\cos z = \cos(x+iy) = \cos[i(ix-y)] = \frac{1}{2}(e^{ix-y} + e^{-ix+y})$$

$$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$$

$$= \cos x\frac{(e^{y}+e^{-y})}{2} - i\sin x\frac{(e^{y}-e^{-y})}{2}$$

$$= \cos x \cosh y - i \sin x \sinh y \qquad (7.14a)$$

Similarly you can prove that

$$\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y \qquad (7.14b)$$

The hyperbolic sine and cosine functions are defined by the formulae

$$\sinh y = \frac{e^{y}-e^{-y}}{2}$$

and

$$\cosh y = \frac{e^{y}+e^{-y}}{2}$$

As these functions are connected with the hyperbola, the name hyperbolic has been attributed to this class of functions. This follows from the formula

$$\cosh^2 x - \sinh^2 x = 1$$

These relations are useful in numerical computation of cos z and sin z.

Zeros of sin z and cos z

If $f(z)$ is zero for a value of $z = z_0$, i.e. $f(z_0) = 0$, then z_0 is said to be a zero of the function $f(z)$. For sine function the condition for a zero is $\sin z = 0$.

From Eq. (7.14b), this condition reduces to $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$ to be satisfied simultaneously. For x and y real, $\cosh y > 1$ and the first condition gives $\sin x = 0$ which occurs for $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ For these values of x, cos x does not vanish. So it is also required that $\sinh y = 0$. From the definition of sinh y, you will note that $\sinh y = 0$ is satisfied only for $y = 0$. Therefore

$\sin z = 0$ occurs for $x = 0, \pm\pi, \pm 2\pi, \ldots$ and $y = 0$ or when $z = 0$ or $z = n\pi$; $n = 1, 2, 3, \ldots$

SAQ 5

Spend 2 min

Show that $\cos z = 0$ occurs when $z = \pm(2n-1)\frac{\pi}{2}$; $n = 1, 2, 3, \ldots$

From the zeros of sin z and cos z you will note that the singularities of tan z occur at $z = \pm(2n-1)\frac{\pi}{2}$, where $\cos z = 0$, but sin z is not zero. tan z is analytic at all other points.

7.4.3 Hyperbolic Functions

The hyperbolic sine and cosine functions with complex argument are defined in the same way as they are defined with the real argument:

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

and

$$\tanh z = \frac{\sinh z}{\cosh z}$$

Other hyperbolic functions — coth z, sech z, cosech z — are the reciprocals of tanh z, cosh z and sinh z respectively. As e^{z} and e^{-z} are entire functions, it follows that sinh z and cosh z are also entire functions. tanh z is analytic everywhere except at the zeros of cosh z. Derivatives of hyperbolic functions can be easily calculated using their definitions. Thus

$$\frac{d \sinh z}{dz} = \cosh z$$

$$\frac{d \cosh z}{dz} = \sinh z$$

$$\frac{d \tanh z}{dz} = \operatorname{sech}^2 z$$

and

$$\frac{d \coth z}{dz} = -\operatorname{cosech}^2 z$$

You may work out the derivatives of other hyperbolic functions with little effort.

An important identity is

$$\cosh^2 z - \sinh^2 z = 1$$

Verify it yourself using the definitions of cosh z and sinh z.

7.4.4 The Logarithmic Function

For a given real positive number x, logarithm to the base e, written as $\ln x$, is a number u such that

$$e^{u} = x \qquad (7.15)$$

You should note that only one real number u satisfies this equation for a given x. So $\ln x$ is a single-valued function of x.

By analogy with $\ln x$, logarithm of a complex variable z, $\ln z$ is defined as a function w such that

$$e^{w} = z \qquad (7.16)$$

Writing $w = u + i\,v$ in the above relation, we find that $e^{w} = e^{u+iv} = e^{u}e^{iv} = e^{u}(\cos v + i \sin v)$. So Eq. (7.16) takes the form

$$e^{u}(\cos v + i \sin v) = z \qquad (7.17)$$

Remembering the polar form of $z\ (=r\,e^{i\theta})$, it follows that v is one of the values of arg z and $e^u = |z|$.

$$\therefore\ u = \ln |z|$$

A solution of Eq. (7.16) gives a value of logarithm of z:

$$w = \ln z = \ln |z| + i \arg z$$

You will recall that arg z has an infinite number of values, $\arg z = \theta_p + 2\pi n;\ n = 0, \pm 1, \pm 2, \ldots$ where θ_p is the principal value of arg z defined by $-\pi < \theta_p < \pi$.

Since the function ln z has an infinite number of values differing by multiples of $2\pi i$ for a given complex variable z, we say that natural logarithm is an infinitely **many-valued** function.

If we restrict to a partciular choice of arg z, we define a particular ln z function; a branch of the logarithm. The most important branch is the principal value of ln z and is denoted by Ln z. This is obtained by giving arg z its principal value $(-\pi < \arg z \le \pi)$. Thus

$$\text{Ln}\, z = \ln |z| + i\theta,\ -\pi < \theta < \pi$$

Note that Ln z is identical with the ordinary logarithm when z is real and positive.

Branch cut and branch point

Consider now the behaviour of Ln z along the negative real axis. Take two points in the complex z-plane: $z_1 = re^{i(\pi-\varepsilon)}$ and $z_2 = re^{i(-\pi+\varepsilon)}$ which approach the negative real axis from above and below respectively as $\varepsilon \to 0$. Writing

$$f(z) = \text{Ln}\, z = \ln |z| + i\,\theta,\ -\pi < \theta < \pi$$

we find that

$$\lim_{\varepsilon \to 0} f(z_1) = \ln r + i\pi,\ r = |z|$$

and

$$\lim_{\varepsilon \to 0} f(z_2) = \ln r - i\pi$$

Thus Ln z has different limits as the negative real axis is approached from above and from below. Ln z is not continuous along the negative x-axis and so is not differentiable at any point on it. Also Ln z is not defined for $z = 0$. Thus all the points on the negative x-axis including $z = 0$ are singular points of Ln z. (Fig. 7.5).

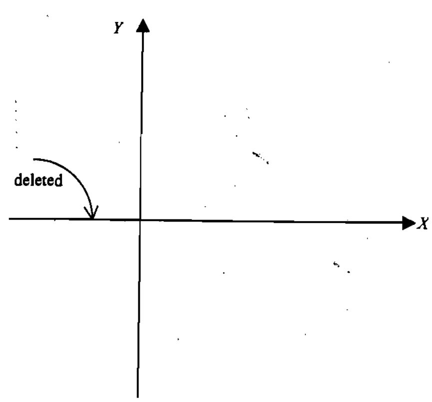


Fig.7.5: Singularities of Ln z on negative x-axis

Let us now introduce a cut extending from $-\infty$ to 0. Then it will not be possible to cross this line. Next we define a bounded domain in the complex plane excluding the cut and given by

$-\pi < \theta < \pi$. Then Ln z will be single-valued and analytic in this domain. Negative x-axis representing the cut is called the "branch cut" for the principal logarithmic function.

Depending on the choice of the value of arg z, we can think of other branches of Ln z. If we define $\ln z = \ln r + i\,\theta$, where $\theta_0 < \theta < \theta_0 + 2\pi$, it corresponds to a different branch of logarithmic function and the straightline $\theta = \theta_0$ from the origin defines the branch cut in this case.

For all the branches of ln z, the point $z = 0$ is a common singular point which is called the branch point of the logarithmic function.

7.5 SINGULARITIES OF A COMPLEX FUNCTION

In Unit 3 of the PHE-07 course on Mathematical Methods in Physics-II, you learnt that a differential equation of the form

$$y'' + p(x)y' + q(x)y = 0$$

is said to have a **singular point** at $x = x_0$ if $p(x)$ or $q(x)$ are not analytic at $x = x_0$. And the singular point $x = x_0$ is said to be **regular** when $p(x)$ and/or $q(x)$ diverge but $\lim\limits_{x\to x_0}(x - x_0)\ p(x)$ and $\lim\limits_{x\to x_0}(x - x_0)^2\ q(x)$ remain finite. If a singular point is not regular, it is called **irregular**. You will now learn these concepts in the context of a complex function.

Suppose that $f(z)$ is a single-valued function of a complex variable. If $f(z)$ is not analytic at a point $z = z_0$, then z_0 is called a singular point or singularity of $f(z)$. To enable you to grasp the idea of singularity, we take an example. Consider a simple function $f(z) = z^2+1$. Its derivative $f'(z)$ at a point z is $2z$ and exists for all finite values of z. So $f(z)$ is an entire function. Now consider the function $g(z) = \dfrac{z^2+1}{z-1}$. At $z = 1$, $g(1) = \dfrac{2}{0} = \infty$ which is undefined and $g'(z)$ does not exist at $z = 1$. That is, the function is not analytic at $z = 1$. This point is a singularity of $g(z)$. But it is an **isolated singularity** because there is no other singularity around $z = 1$.

Definition: The point $z = z_0$ is called an isolated singularity of $f(z)$ if around z_0 we can always draw a circle of suitable radius $\delta > 0$, i.e. $|z - z_0| = \delta$, such that no singular point other than z_0 exists inside the circle. If no such δ exists, then z_0 is called a non-isolated singularity.

For the function $g(z)$ defined above, we note an interesting result. Though $z = 1$ is a singularity of the function, $\lim\limits_{z\to 0}(z-1)\ g(z) = 2 \neq 0$ exists. Therefore, $z_0 = 1$ is called a **simple pole** of $g(z)$.

If z_0 is a singularity of $f(z)$ but $\lim\limits_{z\to z_0}(z - z_0)^n\ f(z) = A \neq 0$ exists (where n is a positive integer), then we say that the singularity of $f(z)$ at z_0 is a **pole of order n.**

Removable singularity: If z_0 is a singularity of $f(z)$ but $\lim\limits_{z\to z_0} f(z)$ exists, then z_0 is called a removable singularity of $f(z)$.

To explain this concept, we consider the function

$$f(z) = \frac{\sin z}{z}$$

You will recall that this function is undefined at $z = 0$ but $\lim\limits_{z\to 0}\dfrac{\sin z}{z} = 1$. So we can say that $z = 0$ is a removable singularity of $\dfrac{\sin z}{z}$.

Next consider the function $f(z) = e^{1/z}$. We can write an exponential function in the form of a power series. For the given function, we have

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots$$

From this we note that $z = 0$ is a singularity. As it is an infinite series in $1/z$, we cannot associate a pole of definite order with the function. So the singularity at $z = 0$ is not a pole; it is an *isolated essential singularity*. You will learn more about it when we discuss Laurent series expansion of a function in the next unit.

Meomorphic function: A function which is analytic at all the points of a finite region of the complex plane, except at a finite number of singular points that are poles of the function, is called a **meomorphic function**.

We now summarise what you have learnt in this unit.

7.6 SUMMARY

- A **complex number** has the form $z = x + i\,y$, where x and y are two real numbers and $i = \sqrt{-1}$.
- **The complex conjugate** of a complex number $z = x + i\,y$ is $z^* = x - i\,y$, and $zz^* \left(= x^2 + y^2\right)$ is always a positive real number.
- **The polar form of complex number** is $z = r\,(\cos\theta + i\sin\theta)$, where r is the modulus and θ is the argument of z. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, a complex number can be represented as $z = r\,e^{i\theta}$.
- A function $f(z)$ is said to have a **limit** ℓ for $z \to z_0$ if for a positive real number ε how-so-ever small, there is a positive real number δ such that $|f(z) - \ell| < \varepsilon$ for all z such that $0 < |z - z_0| < \delta$.
- A function $f(z)$ is said to be **differentiable** if the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

  exists.
- A function $f(z)$ of a complex variable can be expressed as

$$f(z) = u(x, y) + i\,v(x, y)$$

  where $u(x, y)$ is the real part of $f(z)$ and $v(x, y)$ is the imaginary part of $f(z)$.
- **Derivative of a function** exists at a point, if it is continuous at that point and satisfies the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- A function of a complex variable is said to be **analytic** at a point z_0 if it is differentiable at that point as well as in some domain about it.
- The real and imaginary parts $u(x, y)$ and $v(x, y)$ of an analytic function $f(z)$, separately satisfy the two-dimensional Laplace equation and are said to be **harmonic functions**.
- **Logarithmic function** $\ln z$ is an infinitely many valued function.

- If a single-valued function of a complex variable is not analytic a point $z = z_0$, then z_0 is **singularity** of the function.

- If z_0 is a singularity of $f(z)$ but $\lim_{z \to z_0} (z - z_0)^n f(z)$ exists, then we say that the singularity of $f(z)$ is a **pole** of order n.

7.7 TERMINAL QUESTIONS

1. Show that whenever $f'(z_0)$ exists, f is necessarily continuous at z_0.

2. Show that in polar form, the Cauchy-Riemann equations can be written as

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

3. Discuss the analyticity of the function

$$f(z) = e^{y}(\cos x + i \sin x)$$

4. For each of the following functions of a complex variable, locate and name the singularities in the finite z-plane.

(a) $$\frac{z^2 - 2z}{z^2 + 2z + 2}$$

(b) $$\frac{\ln (z + 3i)}{z^2}$$

(c) $$\frac{\sin z^2}{z}$$

7.8 SOLUTIONS AND ANSWERS

Self-assessment Questions

1. $$f(z) = zz^*$$

Derivative at z_0:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)\,(z_0 + \Delta z)^* - z_0 z_0^*}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0\, \Delta z^* + z_0^*\, \Delta z + \Delta z\, \Delta z^*}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z_0 \frac{\Delta z^*}{\Delta z} + z_0^* + \Delta z^* \right) = \lim_{\Delta z \to 0} \left(z_0 \frac{\Delta z^*}{\Delta z} + z_0^* \right)$$

If z_0 is approached along the real axis, i.e. $\Delta y = 0$, $\Delta x \to 0$, we have

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} z_0 \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \right) + z_0^* = z_0 + z_0^* \qquad \text{(i)}$$

If z_0 is approached along the imaginary axis, i.e. $\Delta x = 0$, and $\Delta y \to 0$, we get

$$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} z_0 \left(\frac{-i\Delta y}{i\Delta y} \right) + z_0^* = -z_0 + z_0^* \qquad \text{(ii)}$$

From (i) and (ii) you will note that the two expression agree only at $z_0 = 0$.
So $f'(z_0)$ does not exist except at $z_0 = 0$.

2. Let the analytic function $f(z) = u(x, y) + i\ v(x, y)$. The real part is $u(x,y) = e^x \cos y$. To find the imaginary part v, we use the Cauchy-Riemann condition (7.16a):

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

Integrating v with respect to y, we obtain

$$v = e^x \sin y + g(x) \qquad \text{(i)}$$

From the second Cauchy-Riemann condition, we have

$$\frac{\partial u}{\partial y} = e^x \sin y = -\frac{\partial v}{\partial x}$$

Integrating v with respect to x, we get

$$v = e^x \sin y + h(y) \qquad \text{(ii)}$$

On equating (i) and (ii), we find that $g(x) = h(y) = 0$ so that

$$f(z) = u + i\ v = e^x (\cos y + i \sin y) = e^{x+iy}$$

$\therefore\ f(z) = e^z$ is the required analytic function.

3. (i)
$$2 \sin z_1 \cos z_2 = 2 \left(\frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left(\frac{e^{iz_2} + e^{-iz_2}}{2} \right)$$

$$= \frac{1}{2i} \left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} \right]$$

$$= \sin (z_1 + z_2) + \sin (z_1 - z_2) \qquad \text{(a)}$$

On interchanging z_1 and z_2, we get

$$2 \sin z_2 \cos z_1 = \sin (z_2 + z_1) + \sin (z_2 - z_1)$$

$$= \sin (z_1 + z_2) - \sin (z_1 - z_2) \qquad \text{(b)}$$

Adding (a) and (b) we get the required result.

When $z_2 = \frac{\pi}{2}$, we see that $\sin \left(\frac{\pi}{2} + z \right) = \cos z$

Similarly it follows that

$$\sin\,(\pi + z) = -\sin z$$

and

$$\cos\,(\pi + z) = -\cos z$$

(ii)

$$2\cos z_1 \cos z_2 = 2\left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right)$$

$$= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} + e^{-i(z_1+z_2)}}{2}$$

$$= \cos\,(z_1 + z_2) + \cos(z_1 - z_2) \qquad \text{(c)}$$

$$2\,\sin z_1 \sin z_2 = 2\left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} - e^{-i(z_1+z_2)}}{2i^2}$$

$$= -\cos\,(z_1 + z_2) + \cos(z_1 - z_2) \qquad \text{(d)}$$

On subtracting (d) from (c), we get

$$\cos\,(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

(iii) We know that

$$\sin\,(z_1 + z_2) = \sin z_1 \cos z_2 - \sin z_1 \sin z_2$$

Using $-z_2$ in place of z_2 we find that

$$\sin\,(z_1 - z_2) = \sin z_1 \cos(-z_2) + \cos z_1 \sin\,(-z_2)$$

Since $\sin(-z_2) = -\sin z_2$ and $\cos(-z_2) = \cos z_2$, we get

$$\sin\,(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$$

(iv) If you replace z_2 by $-z_2$ in the addition formula for $\cos(z_1+z_2)$, you will obtain the required relation.

4. We know that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\therefore \frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{i\left(e^{iz} + e^{-iz}\right)}{2i}$$

$$= \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos z$$

Similarly, we can write

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = i\left(\frac{e^{iz} - ie^{-iz}}{2}\right)$$

$$= -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z$$

7. $\cos z = \cos x \cosh y - i \sin x \sinh y$

We know that $\cos z = 0$ will occur when both the real and the imaginary parts vanish. The real part vanishes when $\cos x \cosh y = 0$. But $\cosh y \neq 0$. Therefore it is required that $\cos x = 0$. This happens when

$$x = \pm\,(2n-1)\,\pi/2; \qquad n = 1, 2, 3, \ldots$$

For the imaginary part to vanish, we note that $\sin x \neq 0$ when $x = \pm\,(2n-1)\,\pi/2$. So we need $\sinh y = 0$. This occurs only when $y = 0$. So the zeros of $\cos z$ will occur for

$$y = 0 \text{ and } x = \pm\,(2n-1)\,\pi/2$$

which means that $z = \pm(2n-1)\,\pi/2;\ \ n = 1,2,3, \ldots$

Terminal Questions

1. Derivative of $f(z)$ at z_0 is defined by

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$f'(z_0)$ exists when the right hand side limit exists. To see whether or not it is so, we write

$$\lim_{\Delta z\to 0}\left[f(z_0+\Delta z)-f(z_0)\right] = \lim_{\Delta z\to 0}\left\{\frac{\left[f(z_0+\Delta z)-f(z_0)\right]}{\Delta z}\Delta z\right\}$$

$$= \lim_{\Delta z\to 0}\frac{\left[f(z_0+\Delta z)-f(z_0)\right]}{\Delta z}\lim_{\Delta z\to 0}\Delta z$$

The first term exists but $\lim_{\Delta z\to 0}\Delta z = 0$.

$$\therefore \quad \lim_{\Delta z\to 0}\left[f(z_0+\Delta z)-f(z_0)\right] = 0$$

or

$$\lim_{\Delta z\to 0} f(z_0+\Delta z) = f(z_0)$$

or

$$\lim_{\Delta z\to 0} f(z) = f(z_0)$$

Therefore, $f(z)$ is necessarily continuous if the derivative $f'(z_0)$ exists.

2. We have $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial\theta}\left(\frac{-x}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial\theta}\sin\theta$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$$

From the Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}.$$

it readily follows that

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad \text{(i)}$$

Similarly, from the Cauchy-Riemann condition

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

we obtain

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \qquad \text{(ii)}$$

On combining (i) and (ii), we obtain the polar form of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

and

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

3. $$f(z) = e^{y}(\cos x + i\sin x) = e^{y}\cos x + i\,e^{y}\sin x$$

If we write $f(z) = u + iv$, we find that

$$u(x, y) = e^{y}\cos x \quad \text{and} \quad v(x, y) = e^{y}\sin x$$

$$\therefore \quad \frac{\partial u}{\partial x} = -e^{y}\sin x$$

$$\frac{\partial u}{\partial y} = e^{y}\cos x$$

$$\frac{\partial v}{\partial x} = e^{y}\cos x$$

and

$$\frac{\partial v}{\partial y} = e^{y}\sin x$$

From the Cauchy Riemann condition (7.13a) it readily follows that

$$-e^{y} \sin x = e^{y} \sin x$$

or

$$e^{y} \sin x = 0 \qquad \text{(i)}$$

Similarly, from Eq. (7.13b) we obtain

$$e^{x} \cos x = -e^{y} \cos x$$

or

$$e^{y} \cos x = 0 \qquad \text{(ii)}$$

You will recognise that (i) and (ii) are **not** simultaneously satisfied at any point in the complex plane. So $f'(z)$ does not exist and $f(z)$ is no-where analytic.

4. a) $$f(z) = \frac{z^2 - 2z}{z^2 + 2z + 2}$$

We note that

$$z^2 + 2z + 2 = 0 \quad \text{for} \quad z = z_1 = -1 + i \quad \text{and} \quad z = z_2 = -1 - i$$

$$\therefore \quad f(z) = \frac{z^2 - 2z}{(z - z_1)(z - z_2)}$$

From this expression it is clear that singularities occur at $z_1 = -1 + i$ and $z_2 = -1 - i$. Moreover, each singularity is a simple pole.

b) $$f(z) = \frac{\ln(z + 2i)}{z^2}$$

At $z = 0$, $\ln(2i) \neq 0$. Therefore $f(z)$ has a pole of order 2 at $z = 0$. $z+2i = 0$ when $z = -2i$. So logarithmic singularity occurs at $z = -2i$, i.e. $z = -2i$ is a branch point.

c) $$f(z) = \frac{\sin z^2}{z} = z\,\frac{\sin z^2}{z^2}$$

At $z = 0, f(z)$ is undefined. But

$$\lim_{z \to 0} f(z) = \lim_{z \to 0}\left(z\,\frac{\sin z^2}{z^2}\right) = \lim_{z \to 0} z \,.\, \lim_{z \to 0}\left(\frac{\sin z^2}{z^2}\right)$$

$\therefore \; f(z) = \dfrac{\sin z^2}{z}$ has a removable singularity at $z = 0$.