TABLE TOP EXPERIMENT 1

FOURIER ANALYSIS OF PERIODIC WAVEFORMS

Structure

T1.1 INTRODUCTION

A simple method of recording sound is by converting it into an electrical signal using a microphone-amplifier assembly. The signal waveform can be displayed on the cathode ray oscilloscope. Fig. T1.1 shows signal waveforms corresponding to some characteristic musical sounds. The pure harmonic tone produced by a tuning fork has a simple periodic waveform. The signatures of other sounds are more complex and might be unfamiliar. The physicist's way of understanding music and musical instruments is to analyse the structure of these waveforms. How do we go about this task?

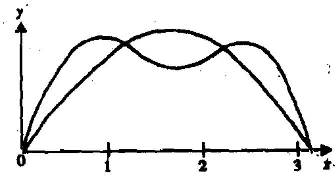

Fig. T1.1 : Some characteristic musical sounds.

You may have already encountered the idea that a complex waveform can be regarded as being made up from several simple vibrations of different frequencies [Refer to Unit 7 of PHE-05 (Mathematical Methods in Physics II]. If you have studied this course then you know that if a function is periodic, it is usually possible to express it in terms of a Fourier series of its harmonic components. Thus the problem of analysing a given waveform becomes one of determining the component frequencies, their relative amplitudes and phase factors.

In this section we will establish a practical numerical procedure for analysing a recorded periodic waveform. We will briefly discuss how to express a periodic function as a Fourier series. You may like to go through Unit 7 of PHE-05 to refresh these concepts. You should be able to determine the frequency, relative amplitude and phase of a finite number of harmonic components of a periodic waveform.

Objectives

After doing this exercise, you should be able to

- answer questions about the convergence of results,
- analyse the relationship between the number of data points specifying the waveform and the number of harmonic components that can be determined reliably.
- determine the frequency, relative amplitude and phase of a finite number of harmonic components of a periodic waveform.

T1.2 FOURIER SERIES OF A PERIODIC WAVEFORM

You will recall that if $f(t)$ is a periodic function of period T, it can be expressed as a Fourier series. We write this in the form

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos \frac{2k\pi t}{T} + b_k \sin \frac{2k\pi t}{T} \right] \qquad \text{(T1.1)}$$

where the coefficients a_k and b_k are the amplitudes of the various harmonic components and are given by

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt \qquad \text{(T1.2a)}$$

$$a_k = \frac{2}{T} \int_0^T f(t) \cos \frac{2k\pi t}{T}\, dt \qquad \text{(T1.2b)}$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin \frac{2k\pi t}{T}\, dt \qquad \text{(T1.2c)}$$

You may have some experience of evaluating these integrals for special periodic functions. For instance for the square wave (Fig. T1.2),

$$f(t) = \begin{cases} V & 0 < t < T/2 \\ -V & T/2 < t < T \end{cases}$$

it is easy to determine the Fourier coefficients analytically and write the corresponding Fourier series as

$$f(t) = \frac{4V}{\pi} \left[\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \ldots\right] \qquad \text{(T1.3)}$$

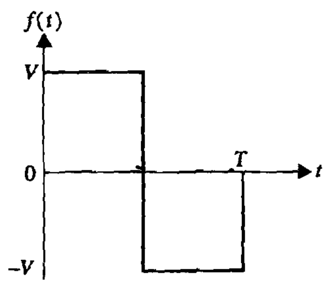

Fig. T1.2: Square Wave

where $\omega = 2\pi/T$ is the angular frequency of the square wave. Of course, the complete series has an infinite number of terms and these need to be added to obtain the function precisely. In actual practice, the amplitudes of higher harmonic components gradually decrease and it is enough to consider the sum of the first few terms. An illustrative exercise would be to plot one cycle of the square wave using Fourier series keeping the first 3 terms, first 5 terms and first 9 terms, respectively.

If the function $f(t)$ can be expressed algebraically, it is usually possible to obtain closed form analytical solutions for the Fourier coefficients. On the other hand, for an experimentally recorded signal, the waveform is specified as a finite set of data points either read off from a graph or recorded directly by a measuring instrument. In such cases we have to determine a finite number of unknown Fourier coefficients using these data points. The integrals (Eqs. T1.2 a, b, c) then have to be evaluated numerically.

T1.2.1 Numerical Evaluation of Fourier Coefficeints

Consider the signal in Fig. T1.3. This has been traced from the CRO. To Fourier analyse this, we undertake the following steps :

I. Reading the data

1. Identify a single cycle of the waveform. Measure the time period T. In this case, T = 10s. Assign the values 0° and 360° to the start and end of the cycle.

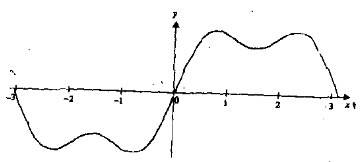

Fig. T1.3

Observation Table T1.1: Nine Point Harmonic Analysis of a Waveform

θ°	f(θ)	cosθ	fcosθ	sinθ	fsinθ	cos2θ	fcos2θ	sin2θ	fsin2θ	cos3θ	fcos3θ	sin3θ	fsin3θ
0													
40													
80													
120													
160													
200													
240													
280													
320													

2. Decide the number of data points N to be read off.

   We choose $N = 9$.

3. Divide the cycle into N equal intervals of width $\theta = 360/N$ (Since $N = 9$ here, $\theta = 360/9 = 40°$

4. Read the values of function as the ordinate at these N points. In this case, read the value for $\theta = 0°$, $\theta = 40°$, $\theta = 80°$, ..., $\theta = 320°$. Note that we have excluded the point at the end of the cycle. Enter your readings in observation Table T1.1.

Computing the Fourier Coefficients

Using the transformations

$$T \rightarrow 2\pi$$

$$2\pi t/T \rightarrow \theta$$

$$(2\pi/T)dt \rightarrow \Delta\theta$$

the Fourier series can be expressed as

$$f(\theta) = \frac{a_0}{2} + (a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \ldots\ldots) + (b_1\sin\theta + b_2\sin 2\theta + b_3\sin 3\theta + \ldots\ldots) \quad \text{(T1.4)}$$

The integrals (T1.2a, b, c) can be evaluated as a sum that approximates the area under the curve so that the coefficients can be estimated as

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{1}{\pi}\sum_{i=1}^{N} f(\theta_i)\,\Delta\theta \quad \text{(T1.5a)}$$

$$a_k = \frac{2}{T}\int_0^T f(t)\cos\frac{2k\pi t}{T}\,dt \approx \frac{1}{\pi}\sum_{i=1}^{N} f(\theta_i)\cos k\theta_i\,\Delta\theta \quad \text{(T1.5b)}$$

and

$$b_k = \frac{2}{T}\int_0^T f(t)\sin\frac{2k\pi t}{T}\,dt \approx \frac{1}{\pi}\sum_{i=1}^{N} f(\theta_i)\sin k\theta_i\,\Delta\theta \quad \text{(T1.5c)}$$

We now outline the method of computing the Fourier components stepwise.

1. Set the number of harmonic components to a finite value. Here, we will retain the first three harmonic components. The problem now is of calculating their amplitudes.

2. Use a calculator to find the values of $\cos\theta_i$, $\sin\theta_i$, $\cos 2\theta_i$, $\sin 2\theta_i$, $\cos 3\theta_i$ and $\sin 3\theta_i$ for θ_i and complete the entries in the corresponding columns of Table T1.1.

3. Calculate the products of these trigonometric functions with $f(\theta_i)$ and complete the table. Find the sum of the product terms in each of the columns.

4. Use Eqs. (T1.5a, b, c) with $\Delta\theta = 2\pi/N$ and determine the harmonic coefficients a_0, a_1, b_1, a_2, b_2, a_3 and b_3

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i) \qquad b_1 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\sin\theta_i$$

$$a_1 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\cos\theta_i \qquad b_2 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\sin 2\theta_i$$

$$a_2 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\cos 2\theta_i \qquad b_3 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\sin 3\theta_i$$

$$a_3 = \frac{2}{N}\sum_{i=1}^{N} f(\theta_i)\cos 3\theta_i \quad \text{(T1.6)}$$

5. Finally, write down the explicit form of the Fourier series. For this signal

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos k\,\theta_k + \sum_{k=1}^{N} b_k \sin k\,\theta_k \qquad \text{(T1.7)}$$

= ....

It is necessary to establish if the functional form actually has a contribution only from the first three harmonic components or this is a false result. After all, the calculation made use of merely 9 data points; these encode very limited information about the signal. We have set this as an excercise for you.

SAQ 1

Repeat the above exercise using $N = 18$ points. You should read at intervals of $\theta = 20°$ Enter your data in observation Table T1.2. Do the values of the coefficients change?

Observation Table T1.2: 18-point Harmonic Analysis of a Waveform

$N = 18$, $\theta = 20°$

θ°	f(θ)	cosθ	fcosθ	sinθ	fsinθ	cos2θ	fcos2θ	sin2θ	fsin2θ	cos3θ	fcos2θ	sin3θ	fsin3θ
0													
20													
40													
60													
80													
100													
120													
140													
160													
180													
200													
220													
240													
260													
280													
300													
320													
340													
SUM													

T1.3 RELIABILITY OF RESULTS

Once you have performed the above exercise, you will immediately appreciate that the correctness of numerical analysis can be established by

1 Checking the convergence of results as the number of data points is increased; and

2. Checking the extent of departure between the recorded signal and its Fourier representation. This is a good way of determining whether higher harmonics are to be included.

We expect the number of harmonic components that can be successfully determined by this procedure to depend upon the number of data points. As a rule of thumb, to correctly estimate the *m*th harmonic component, you must have atleast 4*m* data points, i.e., $N = 4m$.

You may now attempt the following exercise.

SAQ 2

The response curve of a low pass filter to a square input wave is traced as Fig. T1.4. Fourier analyse this wave form and report the values of the amplitudes of the harmonic components present in it.

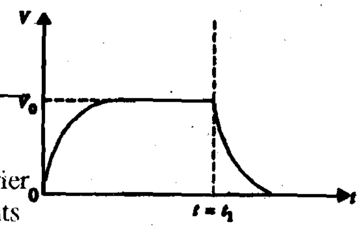

Fig. T1.4