UNIT 11 DIFFRACTION AND RESOLUTION

Structure

11.1 INTRODUCTION

In the preceding two units of this block you have learnt that due to diffraction, the image of an object is fringed evcn if an aberration-free converging lens is used. That is, image of a point object is spread over a small area on the observation screen. Does this mean that no optical device can form a perfect image? The answer to this question is: The image of a point source is not geometric point. And diffraction does place a limit on the ability of optical devices to transmit perfect information (quality image) about any object. Such optical systems are said to be diffraction limited.

Broadly speaking, diffraction limited systems can be classified into two categories: (i) Human eye, microscope and telescope which enable us to see two objects (near or distant) distinct and (ii) Grating and prism which form a spectrum and enable us to see two distinct wavelengths (colours). In principle, in both types of instruments two close fringed (diffraction) patterns are formed on the screen. The question that should logically come to your mind is: How to characterise the ability of an optical instrument to distinguish two close but distinct diffraction images of two objects or two wavelengths? This ability is measured in terms of resolving power. You may now like to know: What criterion enables us to compute resolving power? The most widely used criterion is due to Rayleigh. According to this, two diffraction images are said to be just resolved when the first minimum of diffraction pattern of one object falls at the same position where the central maximum of the diffraction pattern of the other lies. When the patterns come closer than this, the objects are not resolvable. When the patterns do not overlap, the images are distinct and hence objects are resolved. It is also important to know whether the same criterion applies to both types of optical devices? How can we improve resolution and see deeper in space even during the day? We have addressed all these aspects in this unit.

Objectives

After studying this unit, you should be able to

- explain how diffraction limits image forming ability of optical devices
- use Rayleigh criterion to compute expressions for resolving power of a telescope, a microscope and a diffraction grating
- solve numerical problems based on resolution, and
- describe how Michelson stellar interferometer helps in improving resolution.

11.2 DIFFRACTION AND IMAGE FORMATION BY AN OPTICAL INSTRUMENT

Image of a single point source

You will recall that in geometrical optics, it is assumed that a well corrected lens, which does not suffer from aberrations, produces truly a point image of a point source. According to this, the image of a distant star (which may be considered as a point source) will be a sharp point-like image in the focal plane of the objective of the telescope. In reality, geometrical optics is only an approximation. Since as you know that even with a well-corrected objective lens of a telescope, the image of a star is not a point, but a bright circular disc (called Airy disc) surrounded by a number of alternate bright & dark rings. Neither the central disc nor the other fainter rings are sharply limited, but gradually shade off at the edges. However, between the central disc and the first faint ring (and similarly between the first and second faint rings etc.) there is a circle of zero intensity. This has been discussed in Sec. 9.3 and the objective of the telescope acting as a circular aperture produces the Fraunhofer's diffraction pattern of the star.

Disregarding the outer rings which are quite faint, the central bright disc represents the image of the star. We have seen that the angular radius of the disc (the angle subtended by the line joining the centre of the disc and a point on the circle of zero intensity at the centre of the objective) is $1.22 \frac{\lambda}{D}$ where λ is the effective wavelength of the light of the star and D the diameter of the circular aperature (i.e. the objective of the telescope). Therefore the angular diameter of the central disc is $2 \times (1.22 \frac{\lambda}{D})$. Obviously for small aperature sizes, the size of the Airy disc inscreases. But the larger the aperature size, the smaller is the Airy disc and therefore, the more true is the image of a point source. Therefore no matter how free from aberrations an astronomical telescope objective be, what is observed at best is not a point image of a star, but a small bright disc of finite size. Similar arguments are true for a microscope. We therefore conclude that diffraction constrains an optical divice in the formation of sharp point-like image of a point source due to finite sizes of its compoents.

Image of two point objects

The actual manifestation of this restriction due to diffraction is experienced in imaging when we observe two point sources. Since the objective of every optical instrument acts as a circular aperture and the two point sources are mutually incoherent, each point source is imaged as an Airy pattern. Fig. 11.1 shows *Airy* patterns of two stars formed by a telescope. In this figure the sizes of the Airy discs are such that hey are quite far apart from each other and therefore arc seed distinctly as two patterns. In other words, we are able to see these two stars easily as two when they are so far apart and subtend an angle equal to a as shown in the figure.

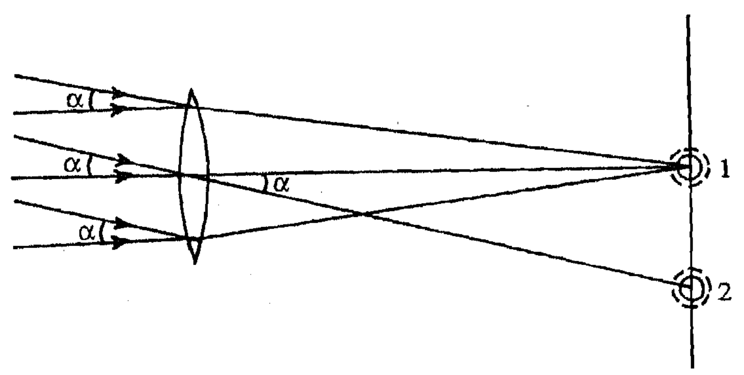


Fig. 11.1: Formation of Airy patterns *in* imaging of *two* stars by a telescope

11.3 RESOLVING POWER OF OPTICAL INSTRUMENTS

Rayleigh Criterion

The qucstion now arises; How close can the two point objects be brought, so that when their Airy discs come closcr together in the image plane we can still infer from their resultant intcnsity pattern that there are two Airy discs ? In other words when we bring the sources & corrcspondingly their Airy discs closer and closer, a stage will be reached, whcn the overlap is so much that we will not be able to distinguish between the two discs and it will appcar as a single disc. What is that critical stage whcn we can just infer that there as two discs ? At this critical stage we say that the two sources are just resolved. We would likc to know, what is the criterion for deciding the critical stages ?

Lct us pcrform a simple experiment to clarify this point. Draw on a paper two parallel lines onc millimeter apart and view thcm from a distance. Move backward or forward till thcsc bccomc blurrcd and just merge into one another. Experience tells us that lmm is barcly rcsolvcd from a distance of 3m. The angle subtcndcd by lmm at 3m is $\frac{1}{3000}$ radian or about 1 minute of arc. We say that the eye just resolves two bright points with an angular separation of about 1 minutc of arc. It is said that the resolution limit of the eye is about 1 minute of arc. Sometimes this is also referred to as the resolving power.

This qucstion was considered by Lord Rayleigh who arrived at a criterion for deciding the rcsolution limit and this conventional specification is therefore called Rayleigh Criterion.

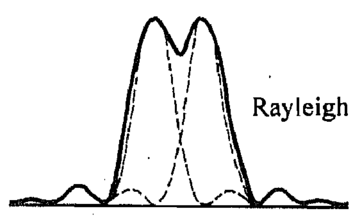
Wc have sccn (Scc. 9.3) that thc intcnsity distribution through the Airy disc is similar to thc one obtained from a point source with a single slit. However the angle θ between the ccntral maximum and the first minimum on either side is given by

$$\text{Sin } \theta \equiv \theta = 1.22 \frac{\lambda}{D}$$

With two closc Airy discs, Raylcigh laid down the criterion that the two patterns are just rcsolvcd when thc ccntral maximum of one diffraction pattern coincides with the first minimum of the othcr diffraction pattern. This criterion apcars arbitrary but has the virtue of bcing particularly simple. In Raylcigh's own words :

> "Thc rulc is convcnicnt on account of its simplicity and is sufficiently accurate in view of the uncertainty as to what exactly is mcant by rcsolution".

Bcforc applying this critcrion to specific instruments e.g. tclescope wc will consider the intcnsity of the rcsultant pattern of two stars of equal intensity, whcn thc two discs are so close that they satisfy the Rayleigh's criterion as in Fig. 11.2. We can show (see example 3) that whcn the maximum of onc intensity curve falls at the minimum of the other curve, the two curves cross each other at 0.4 of the maximum intcnsity. Thercfore at this point thc rcsultant intcnsity will be equal to sum of the two intcnsities and equal to 0.8. This means that the resultant intensity will show a dip of 20%. This dip is easily visible to evcn unaided human eye. But one can argue that even a smaller dip in intensity could be dctcctcd by a scnsitive instrument and therefore fixing the dip at 20% by Raylcigh's criterion is arbitrary. But this critcrion is accepted because of its simplicity, which is based on the positions of maximum and minimum.



Barely Resolved

Fig. 11.2: Rayleigh criterion for imaging two stars of small angular separation

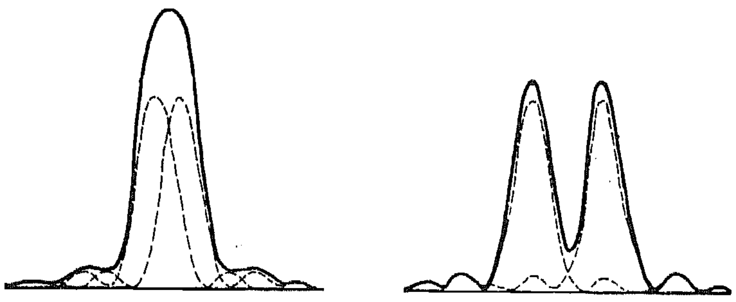
11.3.1 Astronomical Telescope

We again consider Fig. 11.1 in which a telescope points towards two close luminous stars which subtcnd an angle α on the objective, We assume that these two stars are equally bright. The plane waves from thcsc stars fall on the objective and give rise to Airy diffraction patlems. Since the stars are effectively at an infinite distance from us, the diffraction patterns (images) arc formed in the back focal plane of the telescope objective, where it is examined with the aid of the eye piece. The angle subtcnded by thc mid points of central discs is equal to the angle subtcndcd by the stars at the objcctivc. And for thcsc stars to be just resolved, Rayleigh criterion demands that maximum (centre) of the Airy disc due to one star should fall on the minimum (periphery) of the disc due to the other star, as in the right hand part of the photograph shown in Fig. 9.8 and the intcnsity distribution curve shown in Fig 11.2. We know that for the two stars to be just resolved, the angle subtended by the two stars at the objective should be equal to the angular half-width of the Airy disc. Using Eq. (9.13) we can say that the **minimum resolvable angular separation or angular limit of resolution** for two close stars which can be resolved by a telescope is

$$\theta_{min} = \frac{1.22\,\lambda}{D} \qquad (11.1)$$

This angle θ_{min} is also called the resolving limit.

Two stars subtending an angle a at the objective will be resolved for a > θ_{min} and unresolved for a < θ_{min}. The intensity plot for more than resolved, and unresolved stars are shown in Fig. 11.3.



Not Resolved

Fully Resolved

Fig.11.3: Plot of Inlensities of two resolved, and unresolved stars

The ccntre-to-centre linear separation of two just rcsolvcd stars is givcn by

$$s = f\,\theta_{min} = \frac{1.22\,\lambda f}{D} \qquad (11.2)$$

where f is the focal length.

The **resolving power,** often abbreviatcd as R.P. for an optical devicc is gencrally dcfincd as the reciprocal of resolving limit, i.e., as θ_{min}^{-1}. This mcans that resolution ability of diffraction-limited systems dcpends on the sizc of thc aperturc and thc wavclcngth. For a given wavelength, the resolving powcr of a tclcscopc can bc incrcascd by using larger diamcter objectives. To give you some appreciation of numcrical valucs, wc now give a solved example. You should go through it carcfully.

Example 1

An astronomical observatory has a 40 inch telescope. Calculalc the angular half-width of Airy disc for this telescope. Take λ = **6000 Å.**

Before the S.I. units wcrc adopted, thc objective sizes wcrc expressed in inches.

Solution

From Eq. (11.1) we recall that

$$\theta_{min} = 1.22\,\lambda/D$$

On substituting the given data, you will find lhat

$$\theta_{min} = \frac{1.22 \times (6 \times 10^{-5}\ \text{cm})}{40 \times 2.54\ \text{cm}}$$

$$= 7.2 \times 10^{-7}\ \text{rad}$$

$$= 0.15\ \text{sec. of arc}$$

This tclcscope will resolve two close stars subtcnding an anglc of 0.15" at its objcctive.

SAQ 1

Spend 5 min

An astronaut orbiting at an height of 400 km claims thal hc could scc thc individual houscs of his city as they passed bcneath him. Do you bclicvc him. If not, why?

You now know that a 40 inch telescope has a minimum angle of resolution equal to 7.2×10^{-7} rad. We have seen that the minimum angle of resolution of the eye is about 1 minute of arc. We take the minimum angle of resolution of the eye about 3.35×10^{-4} rad. An important question that should come to our mind is: What should be the magnifying power of the eyepiece of the telescope to take full advantage of the large diameter of the objective? The telescope must magnify about $\frac{3.35 \times 10^{-4}\ \text{rad}}{7.2 \times 10^{-7}\ \text{rad}} = 465$ times. Note that any further magnification will only make the image bigger but it would not be accompanied by increase in details which are not available in the primary image. (The resolution is determined by diffraction at the objective, i.e. the magnitude of θ_{min}.) To get some idea about these details, you should carefully go through the following example.

Example 2

Compare the performance of two telescopes with objectives of apertures 100 cm and 200 cm. Take their focal lengths to be equal.

Solution

We know that for a telescope, the minimum angle of resolution

$$\theta_{min} = \frac{1.22\,\lambda}{D}$$

For the first telescope $\theta_{min} = \frac{1.22\,\lambda}{100\ \text{cm}}$, where A is in cm. Therefore. rhe radius of central diffraction disc $r = f\theta_{min} = f\,\frac{1.22\,\lambda}{100\ \text{cm}}$ and the area of Airy disc

$$A_1 = \pi r^2 = \pi \left(f \frac{1.22\,\lambda}{100\ \text{cm}} \right)^2$$

The area of the telescope objective which collects light is $\pi \left(\frac{100\ \text{cm}}{2} \right)^2$. This light is largely concentrated in the central maximum and gradually decreases as shown in Fig. 9.8. If we assume that light is uniformly disuibuted over the disc, its brightness i.e. light per unit area

$$I_1 \propto \pi \left(\frac{100\ \text{cm}}{2} \right)^2 \div \pi \left(f \frac{1.22\,\lambda}{100\ \text{cm}} \right)^2$$

$$= (50)^2 \times \frac{(100)^2}{f^2\,(1.22)^2\,\lambda^2}\ \text{cm}^4$$

$$= \frac{100^2 \times 100^2}{4f^2\,(1.22)^2\,\lambda^2}\ \text{cm}^4$$

For the second telescope, $\theta_{min} = \frac{1.22\,\lambda}{200\ \text{cm}}$. That is, h e minimum angle of resolution for the second telescope is half of that for the first telescope. In other words, the R.P. of 200 cm telescope is twice as large. To compare their relative performances, let us compare the brightness. As before, the area of central diffraction disc

$$A_2 = \pi \left(f \frac{1.22\,\lambda}{200\ \text{cm}} \right)^2$$

and brighmcss

$$I_2 = \frac{(200)^2\,(200)^2}{4f^2\,(1.22)^2\,\lambda^2}\ \text{cm}^4$$

$$= 16\, I_1$$

In words, the area of the central diffraction disc of the first telescope is four times bigger. And the light collected by the second telescope in four times greater. Hence the brightness of image by the second telescope is 16 times. In other words by making the aperture 2 times, the intensity of image becomes $2^4 = 16$ times. So we may conclude that

(i) The ability of a telescope to resolve two close stars depends on the diameter of its objective.

(ii) The intensity of the image formed by doubling the size of aperture is sixteen times since the objective collects four times more light and concentrates it over an area of the Airy disc which is only one fourth. This means that a distant star, which is too faint to be observed by a smaller objective (of the smaller telescope), becomes visible by a larger telescope. That is, a bigger telescope can see farther in the sky. Therefore, the deeper we want to penetrate the space, the greater should be the aperture of the objective of telescope.

You may now like to pause and ponder for a while. Then you should answer SAQ 2.

SAQ 2

Spend 5 min

We can see the stars at night but as sun rises they gradually fade away and are not visible during the day. What measure would you suggest to enable researchers to make astronomical observations in the day time itself?

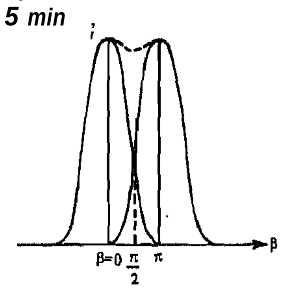


Fig. 11.4: Resolution of two single slit patterns: Rayleigh's criterion

Example 3

Calculate the dip in the resultant intensity of two $\left(\frac{\sin\beta}{\beta}\right)^2$ curves according to Rayleigh criterion, i.e. when the maximum of one curve falls on the minimum of the other curve.

Solution

We assume that the two curves have equal intensity. These curves are symmetrical and will cross at $\beta = \pi/2$, as shown in Fig. 11.4.

At the point of intersection, both curves have equal intensity:

$$I = \left(\frac{\sin\frac{\pi}{2}}{\pi/2}\right)^2 = \frac{4}{\pi^2} = 0.4053$$

At this point the resultant intensity will be equal to the sum of the two intensities and therefore equal to 0.8106. This means that according to Rayleigh criterion, the resultant intensity will show a dip of about 20%. And this dip is easily visible to even unaided human eye. If these two curves are brought closer, the dip will gradually decrease and it becomes difficult to resolve the images. Moreover, if these intensities were unequal, the dip will not be 20%.

In the above example we have taken the intensity of both the curves to be equal. This essentially means that in Rayleigh criterion we take both the stars to be equally luminous. Another important point to note is that the curves are of finite angular (or lateral) width. In case of grating (or prism), two spectrum lines, though assumed to be of equal intensity, are very sharp. Now the question arises: Can we use the Rayleigh criterion even for a grating? From your second level physics laboratory you may recall answer to this question; we do use the same criterion. Is the dip 20% or so even in this case? To discover answer to this question, you should answer the following SAQ.

SAQ 3

Spend 5 min

What is the dip in the resultant intensity of two $\left(\frac{\sin N\gamma}{\sin\gamma}\right)^2$ curves according to Rayleigh criterion?

Another more realistic criterion for resolving power has been proposed by Sparrow. We know that at the Rayleigh limit there is a central dip or saddle paint between adjacent peaks. As the distance between two point sources is decreased beyond Rayleigh limit, the central dip will grow shallower and may ultimately disappear such that the resultant curve has a broad flat top. The angular separation corresponding to thal configuration is said to be Sparrow limit. However, we will not discuss it any further.

Another useful image forming device is a microscope. Let us now learn to calculate its resolving power.

11.3.2 Microscope

We know that an astronomical telescope is used to view far off objects whose exact distances are usually unknown. However, we were chiefly interested in their smallest permissible angular separation at the objective. In case of an optical microscope, the self-luminous objects being examined are very close to the objective and **subtend** a large angle. For this reason, by resolving power of a microscope we mean the smallest distance, rather than the minimum angular separation, between two point objects (O and O') when their fringed images (I and I') are just resolved. Each image consists of a central Airy disc (surrounded by a system of rings which are very faint and not considered). According to Rayleigh criterion, the central maximum of I should be at the same position where the minimum of I' lies. The angular separation between the two discs

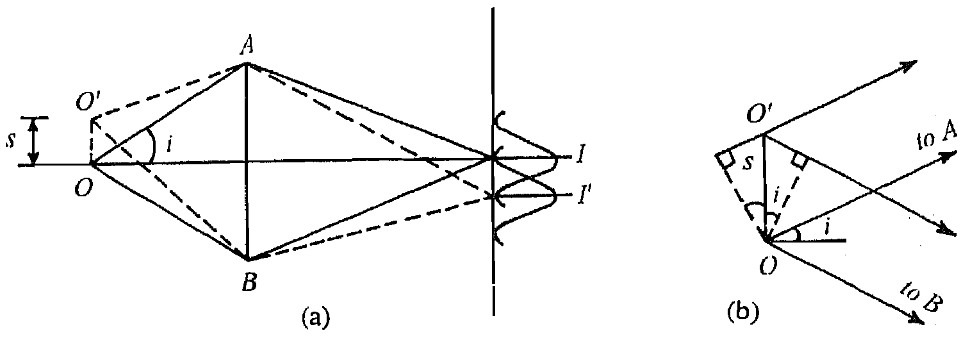


Fig. 11.5: The optical microscope (a) Airy pattern images of two objects O and O' separated through a distance s (b) Ray diagram for computation of path difference $O'B - O'A$

or the limit of resolution $\theta_{min} = \dfrac{1.22\,\lambda}{D}$. When two images are just resolved, the wave from O' diffracted to I has zero intensity (first dark ring) and extreme rays $O'BI$ and $O'AI$ differ in path by 12211 , i.e. $O'BI - O'AI = 1.22\lambda$ since $BI = AI$. Therefore, the path difference $O'B - O'A = 1.22\lambda$ (Fig.11.5(a)). We show an enlarged part in the Fig. 11.5(b) from which we see that $O'B$ is longer than OB by s sin i and $O'A$ is shorter by the same amount. Here the pint O subtends an angle 2i at the objective of the microscope. Thus the path difference of the extreme rays from O' to the objective is $2s$ sin i. Upon equating this to 1.22λ, we find that the minimum separation between two points in an object that can be resolved by a microscope is given by

$$2\, s \sin i = 1.22\,\lambda$$

or

$$s = \frac{1.2211}{2 \sin i} = \frac{0.61\,\lambda}{\sin i}$$

In high power microscopes, the space between the object and objective is filled with oil of refractive index μ. For an oil immersed objective, the above expression becomes

$$s = \frac{0.61\,\lambda}{\mu \sin i} \qquad (11.3)$$

You may now like to answer an SAQ.

Spend 5 min

SAQ 4

In the above discussion we assumed that the two point objects were self-luminous. Suppose two objects are illuminated by the same source. Will Eq. (11.3) still hold?

Abbe investigated this problem of image formation in detail and found that the resolving power depends on the mode of illumination of the object. In the above treatment both O and O' were treated as self-luminous objects and thus the light given out by these had no constant phase relationship. For all practical modes of illumination, the resolving power may be taken simply as

$$R.P = \frac{0.61\,\lambda}{\mu \sin i}$$

The term $\mu \sin i$ is termed as numerical aperture (N.A) of the microscope objective. The maximum value of i is 90°. This gives the microscopic limit on R.P approximately as $\frac{\lambda}{2\mu}$. This shows that smaller the N. A greater will be the distance between two resolvable points. In practice, good objectives have N. $A \approx 1$ so that the smallest distance that can be resolved by a microscope is of the order of the wavelength of light used. Obviously, with light of shorter wavelength, say ultraviolet rather than visible light, microscopy allows for perception of finer details. (We may have to take the photographs and then examine the images.)

In your school physics curriculum you have learnt that electrons exhibit diffraction effects. The deBroglie wavelength of an electron is given by

$$\lambda\,(\text{Å}) = \frac{12.3}{\sqrt{V}} \qquad (11.4)$$

For electrons accelerated to 100 kV, the wavelength is

$$\lambda\,(\text{Å}) = \frac{12.3}{\sqrt{10^5}} = 0.039 \times 10^{-10}\ \text{m} \qquad (11.5)$$

This wavelength is 10^{-5} times that for visible light. The resolving power of an electron microscope will therefore be very high. This makes it possible to examine objects that would otherwise be completely obscured by diffraction effects in the visible spectrum. In this connection we may mention tremendous utility of electron microscope in the study of minute objects like viruses, microbes and finer details of crystal structures. It is better than even ultraviolet microscope for high resolution applications.

The de Bmglie wavelength of an electron is given by

$$\lambda = \frac{h}{m_e v}$$

where h is Planck's constant, m_e is electronic mass and v is electron speed. When an electron beam is accelerated through a potential difference V, we can write

$$v = \sqrt{\frac{2eV}{m_e}}$$

On combing these relations we find that

$$\lambda = \frac{h}{\sqrt{2 m_e e}} \cdot \frac{1}{\sqrt{V}}$$

Substituting the values

$h = 6.6 \times 10^{-34}$ Js,
$m_e = 9.11 \times 10^{-31}$ kg and
$e = 1.6 \times 10^{-19}$ C,

you will find that

$$\lambda\,(\text{Å}) = \frac{12.3}{\sqrt{V}}$$

11.3.3 Diffraction Grating,

You are familiar with a sodium lamp. It emits out two close wavelengths of light, the so-called D, and D_2 lines, with wavelengths λ_1 = 5890 Å and λ_2 = 5896 Å. With a given grating, each wavelength will form a spectral line. We wish to know whether these two spectrum lines will be resolved i.e.. be seen as two distinct spectrum lines. The ability of a grating (or any spectroscope) to resolve two close wavelenths λ and $\lambda + \Delta\lambda$ is called its resolving power and is defined as

$$R.P. = \frac{\lambda}{(\Delta\lambda)_{min}} \qquad (11.6)$$

where $(\Delta\lambda)_{min}$ is the least resolvable wavelength difference or limit of resolution and λ is the mean wavelength. It is sometimes also called chromatic resolving power.

We know that the grating forms a principal maximum corresponding to wavelength λ at the diffraction angle θ. Similarly, the principal maxima corresponding to $\lambda + \Delta\lambda$ will be at $\theta + \Delta\theta$. At first thought you may argue that the two colours will be separated and always appear to be resolved since the two angles are different. This could be so if the principal maxima, i.e. the spectrum lines in the experimental arrangement, were truly sharp like an ideal geometric line. But we know that the principal maximum has a finite (angular) width. Therefore, the question is: How close can these be brought so that they are seen distinct? Obviously, sharper the lines, the closer these can be brought and still be seen as two.

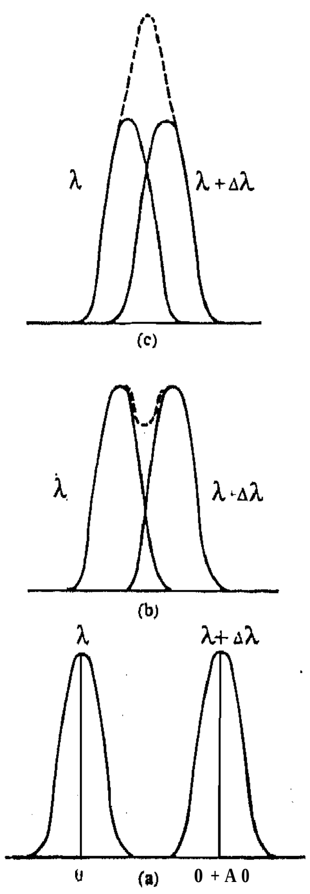


Fig. 11.6: Resolution of two spectral lines

This question was also carefully examined by Rayleigh. In Fig. 11.6(a) we show plots of two widely separated principal maxima. In Fig. 11.6(b) we have brought these closer so that the principal maximum of $\lambda + \Delta\lambda$ is situated at the position where the minimum of λ falls. The dotted line defines resultant intensity, which shows a dip. You will recall that according to Rayleigh criterion, this is the closest we can bring these curves and still regard them as separate. If we bring them still closer, as in Fig. 11.6(c), the resultant intensity (shown by the dotted line) signifies a single enhanced principal maxima.

We apply Rayleigh criterion, for the resolution of two spectral lines by a diffraction grating for the common diffraction angle θ, From Eqn. (10.19), the condition for the principal maxima of $(\lambda + \Delta\lambda)$ is :

$$d \sin \theta = n (\lambda + \Delta \lambda)$$

and the condition for the fust minimum adjacent to the principal maximum for wavelength λ is :

$$d \sin \theta = n \lambda + \frac{\lambda}{N}$$

On simplifying these we get

$$\frac{\lambda}{\Delta \lambda} = nN \tag{11.7}$$

We note that in a given order n, the R. P is proportional to the total number of slits. Does this mean that R. P increases indefinitely with N? It is not so. Think why? Does it have some connection with the width of the grating? You will also note that the resolving power is independent of grating constant. It means that resolving powers of two gratings having equal number of lines but different grating constants will be equal.

To enable you to grasp these concepts and appreciate the numerical values, we now give some more solved examples.

Example 4

For D_1 and D_2 sodium lines, λ_{D} = 5890 Å and λ_{D_2} = 5896 Å. Calculate the **minimum** number of lines in a grating which will resolve the doublet in the first order.

Solution

Let us take the average wavelength as 5893 Å. From Eq. (11.6) the resolving power is

$$\frac{\lambda}{\Delta \lambda} = \frac{5893 \times 10^{-8}\,\text{cm}}{6 \times 10^{-8}\,\text{cm}} = 982.2$$

Therefore, we must have a grating with more than 983 lines to resolve sodium doublet in fust order. A grating of 1000 lines will serve the purpose.

Example 5

Suppose that to observe sodium doublet we use a grating having $d = 10^{-3}$cm and a lens of focal length 2m. Calculate the linear separation of the two lines in the 1st and 2nd order.

Solution

We know that

$$d \sin \theta = n\lambda$$

Let us first consider the 1st order. For the D_1 line

$$\sin \theta_1 = \frac{5890 \times 10^{-8}\,\text{cm}}{10^{-3}\,\text{cm}} = 5890 \times 10^{-5}$$

or

$$\theta_1 \cong 5890 \times 10^{-5} \text{ rad}$$

Similarly for the D_2 line

$$\sin\theta_2 = \frac{5896 \times 10^{-8}\,\text{cm}}{10^{-3}\,\text{cm}} = 5896 \times 10^{-5}$$

or

$$\theta_2 \cong 5896 \times 10^{-5} \text{ rad}$$

$$\therefore \quad \Delta\theta = \theta_2 - \theta_1 = 6 \times 10^{-5} \text{ rad}$$

With a lens of focal length 200 cm, we find that linear separation between D_1 and D_2 lines is

$$l = f \Delta \theta$$

$$= (200 \text{ cm}) \times (6 \times 10^{-5} \text{ rad})$$

$$= 12 \times 10^{-3} \text{cm} = 0.12 \text{ rnrn}$$

This shows that 6 Å are separated by 0.12 mm in 1st order. Alternatively we may say that linear separation is nearly 50 Å per millimeter in the first order. You can readily check that in the second order this linear separation will be 25 Å per millimeter.

11.4 IMPROVING RESOLUTION

You now know that with the help of a telescope we can view a faint star, resolve two close stars and measure the angle subtended by the double star at the objective of the telescope. However, it is worth noting that from the Fraunhofer diffraction pattem of a star, we cannot measure its angular diameter. To overcome this limitation, Fizeau suggested a slight modification that we should use a two slit adjustable aperture (with provision for lateral adjustment), in front of the objective of the telescope. As a result, the plane wavefront falling on the double slit is diffracted and collected by the objective. The Fraunhofer diffraction pattern of the double slit is formed in the back focal plane of the objective. The measurements to determine angular diameter are made from the observations on these interference fringes.

Rcfcr to Fig. 11.7. Two slit apertures S_1 and S_2 are at a distance d apart. The telescope is first pointed towards the double stars, which act as two point sources O and $O_,$. The two point sources are separated by an angle 8 in a direction at right angles to the lengths of the slits. Such objects emit white light and because of intensity considerations, the observations have to be made with white light fringes. It is therefore customary to assume an effective value of the wavelength emitted by the source. This depends on the distribution of intensity of light and the colour response of the eye. The interference patterns, due to O and O_1 have the same fringe spacing since this spacing depends upon separation between slit apertures and the focal length of the objective. Moreover, these fringe patterns are shifted with respect to each other by an angle 8. Therefore, as shown in the figure, the central maximum of the pattem due to O is at P and that due to O_1 is at P'. If O and O_1 are two incoherent sources, the combined pattern is formed by summing the **intensities** of these two patterns at each point. Assuming that both O and O_1 have equal brightness, we can plot two $\cos^2\gamma$ curves on the same scale and shift them suitably and then get the resultant curve.

The intensity of the double slit pattem is given by

$$I = 4R^2 \frac{\sin^2 \beta}{\beta^2} \cos^2 \gamma$$

where $\beta = \frac{\pi a \sin\theta}{X}$ **and** $\gamma = \frac{\pi d \sin 0}{\lambda}$

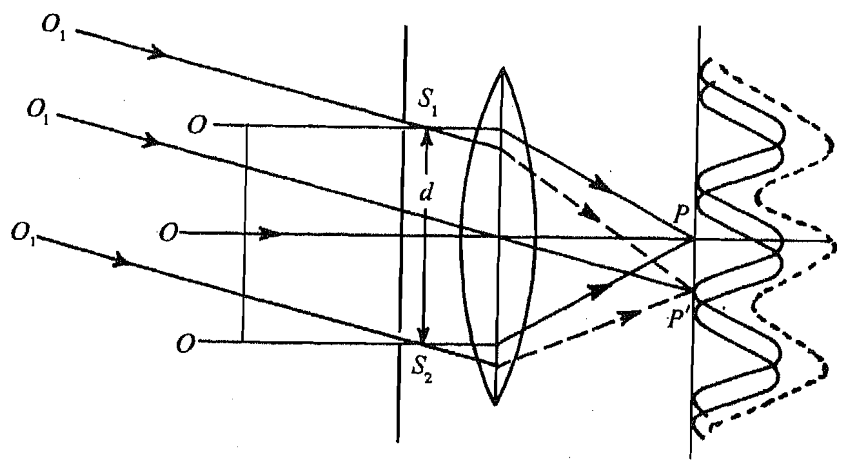
in which a is the slit width and d is h e slit separation. The positions of h e maxima arc given by

$$d \sin 0 = n\lambda$$

where $n = 0, 1, 2, 3, \ldots$ When θ is small, the successive maxima occur at

$$\theta = 0, \frac{\lambda}{d}, \frac{2\lambda}{d}, \frac{3\lambda}{d}, \ldots$$

so that the angular separation between successive maxima is given by $\theta_1 = \frac{X}{d}$. Further, if a is small, the interference pattern will be essentially a $\cos^2\gamma$ curve near the centre.



Pig. 11.7: Principle of double slit to observe a double star

We can show graphically that if this shift is a small fraction of the angular separation θ, the resultant intensity distribution resembles a $\cos^2\gamma$ curve. The resultant intensity does not fall to zero at the minimum. By successive increase in angle 8, a stage can come when the maximum of one pattern, say due to O, coincides with the minimum of O_1. Then we have $\frac{\theta_1}{2} = \frac{\lambda}{2d}$ **.** And the paths from the two sources differ by $\frac{h}{2}$ **.** We can show graphically that the resultant curve shows a uniform intensity and the fringes have disappeared. If we displace the two curves further, the fringes will reappear becoming sharp when the fringes are displaced by a whole fringe width **i.e.,** $8 = \theta_1$. They will disappear again when $8 = \frac{3\theta_1}{2}$ or $\frac{5\theta_1}{2}$ **.** Therefore, with two point sources **subtending an** angle θ at the double slit, the condition for the disappearance of fringes is

$$\theta = \frac{\lambda}{2d}, \frac{3\lambda}{2d}, \frac{5\lambda}{2d}, ...$$

To measure angular separation of a **double** star, the double slit is mounted in front of the objective of the telescope which points towards the double star. (We should remember that the line joining the stars should be perpendicular to the length of the slits.) We expect interference pattern due to the double slit. If on adjusting the separation between the slits, the interference fringes can be made to disappear, we can infer that the star is a double star. The first disappearance should take place when the angular separation is $\frac{\lambda}{2d}$ **.** Let us compare this with the expression for the resolving limit of a telescope (0 = 1.22 $\frac{\lambda}{a}$ **,** where a is the diameter of the objective). If the double slits are a apart and the **first** disappearance occurs for d = a, the angle 8 between the double stars is 0 = $\frac{It}{2d} = \frac{It}{2a}$.
This angle is effectively half of the resolving limit of the telescope. It explains the genesis of the **statement** : The R.P **of a** telescope may be **doubled** by placing a **double** slit in front **of** it. You must however note that with a double slit, we can only infer the presence of a double star (from the disappearance of the fringes); we neither get the images of the stars nor resolve them. Indeed, even before the disappearance of the fringes, a blurring of **fringes starts.** This angle is only a small fraction of θ_1. You may have **realised** that this **method** enables us to measure the angular diameter of the disc of the star and Michelson **successfully** used it in 1920.

Angular Diameter of a Star

For measuring **the** angular diameter of the disc of a star we should first know the condition for the disappearance of fringes for a double slit placed in Front of a telescope. In contrast to two point sources, the disc of a star consists of a series of points extending from one end O, to another end O,. In Fig. 11.7, we see that when O, and **the** central point O satisfy **the** condition for disappearance of fringes, the point just next to O_1 will have a similar point next to O and so on. Thus all the points between O, and O will have corresponding points lying between O and O, satisfying the condition for **disappearance** of fringes and **h e** disc of stars will show no fringes. Since the angle between O_1 and O for the **first** disappearance of fringes is $\frac{\lambda}{2d}$, the angle between O, and O_2 (which is for the total disc) equals $\frac{h}{d}$ **.** Thus the angular diameter θ of the disc of the star, **computed** from the first disappearance of fringes, is given by $8 = \frac{A}{d}$ **.** The fringes from extreme points O_1 and O, **are** displaced by a whole fringe. For successive disappearances 8 is given by 8 = $\frac{2\lambda}{d}, \frac{3\lambda}{d}, ...$ We know that if **the** source is a circular disc, the correct condition for **the first** disappearance is $\theta = 1.22 \frac{\lambda}{d}$ **.** This method was successfully used to measure angular **diameters** of planetary satellites. But attempts to apply it for single stars failed because of their small angular diameters. Even with the largest slit separation possible with the available telescopes, the fringes remained distinct; no disappearance was achieved. To overcome this difficulty, Michelson devised stellar interferometer in 1890. We will discuss it now.

11.4.1 Michelson Stellar Interferometer

The principle of Michelson Stellar Interferometer is **illustrated** in the Fig.11.8. The slit **apertures** S_1 and S_2 in front of the telescope are fixed. Light from a star reaches them **after** reflection from a **symmetrical** system of mirrors M,, M_2, M_3 and M_4 **mounted** on a rigid

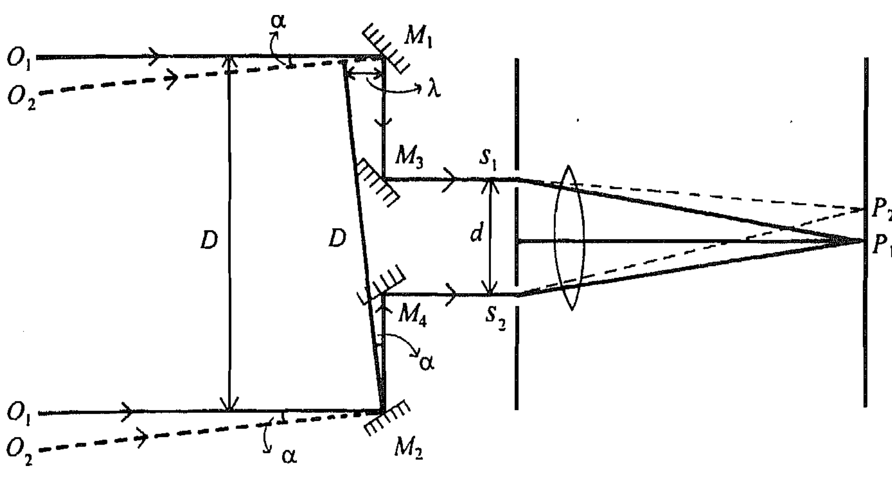


Fig. 11.8: Schematics of Michelson Stellar Interferometer

girder in front of the telescope. The inner mirrors M_3 and M_4 are fixed but the outer mirrors M_1 and M_2 can be separated out symmetrically in a direction perpendicular to the lengths of the slit apertures. Therefore light from one edge of the star (shown as solid line) reaches the point P in the focal plane via the paths $O_1 M_1 M_3 S_1 P_1$ and $O_1 M_2 M_4 S_2 P_1$.

This will form interference fringes with the angular separation equal to $\frac{\lambda}{d}$, the central fringe being at P_1. The other edge of the star sends light along the path shown by dotted lines O_2M_1 an O_2M_2, and produces a similar system displaced slightly so that its central fringe is at P_2. You now know that from a continuous series of points when two extreme fringe systems are displaced by a whole fringe width, the resultant intensity pattern will show uniform intensity and the fringes will disappear. We will show that the angular diameter of the star $\alpha = 1.22 \frac{\lambda}{D}$, where D is the separation of outer mirrors M_1 and M_2.

You can easily convince yourself by noting that the optical paths $M_1 M_3 S_1$ and $M_2 M_4 S_2$ have been maintained equal so that the optical path difference for light from the two edges of the star is the same at S_1 and S_2 as at M_1 and M_2. If the path difference at M_1 and M_2 is one whole wavelength, the path difference at S_1 and S_2 is also one whole λ and fringe shift is equal to one fringe width. This leads to disappearance of fringes.

We now proceed to show that when $\alpha = \frac{\lambda}{D}$, the two dotted rays O_2M_1 and O_2M_2 have a path difference of one λ when they reach the mirrors M_1 and M_2. The path difference of one wavelength arises because as shown in Fig. 11.8, the telescope is so directed that the two rays O_1M_1 and O_1M_2 travel equal paths to arrive at mirrors M_1 and M_2 and hence arrive in phase at mirrors M_1 and M_2. However the two dotted rays O_2M_1 and O_2M_2 are inclined to the axis of the objective of the lens at an angle α, so that this edge of the star sending rays O_2M_1 and O_2M_2 is one wavelength farther from M_1 than from M_2. This explains the path difference of one λ However one may argue that the dotted rays will not be reflected by ordinary laws of reflection. But because angle α is so small there will be enough diffraction at mirrors M, and M_2 so that we still can select a pair of rays which reach the centres of S_1 and S_2 by paths which are identical to those traversed by solid lines. For small angle α we have $\alpha = \frac{\lambda}{D}$.

In this arrangement the smallest angular diameter that can be measured is determined by the separation of the outer mirrors M_1 and M_2 rather than the diameter of the objective of the telescope. Therefore, the stellar interferometer magnifies the effective resolving power of the telescope in the ratio $\frac{D}{d}$. We may emphasize that for a circular star disc, the fringes will disappear when $\alpha = 1.22 \frac{\lambda}{D}$. This implies that the outer mirrors have to be moved out a little farther. You should note that the fringe separation is controlled by the small distance d, rather than the bigger distance D, and hence is easily seen.

The interferometer was mounted on the large reflecting telescope (diameter 100 inch) of the Mount Wilson observatory, which was used because of its mechanical strength. The first star whose angular diameter was measured by this method was Betelegeuse (α–Orions) whose fringes disappeared when the separation between M_1 and M_2 was equal to 121 inches. Assuming $\lambda = 5700$ Å, we find that

$$\alpha = \frac{1.22\,\lambda}{D} = \frac{1.22 \times 5700 \times 10^{-8}\ \text{cm}}{121 \times 2.54\ \text{cm}}$$

$$= 22.7 \times 10^{-8}\ \text{rad}$$

$$= 0.047\ \text{seconds of arc}$$

The distance of Betelegeuse was measured by parallax method. Its linear diameter was then found to be 4.1×10^8 km, which is about 300 times the diameter of the sun. The maximum separation of the outer mirrors was 6.1m so that the smallest measurable angular diameter with $\lambda = 5500$ Å was about 0.02 seconds of arc. This is insufficient for most of the stars. The smallest star for which measurements were made was Arcturus. Its actual diameter is 27 times that of the sun.

At the surface of the earth, the sun disc has an angular diameter of about $32' \cong 0.018$ rad. If we imagine the sun to be at a distance of the nearest star, its disc would subtend an angle of 0.007 seconds of arc. This will require a mirror separation of 20m for disappearance of fringes. It is difficult to achieve this in practice since we require a rigid mechanical connection between mirrors and eye piece.

Let us now summarise what you have learnt in this unit

11.5 SUMMARY

- Diffraction constrains an optical device in the formation of a sharp point-like image of a point source.
- Rayleigh's criterion for resolution of two imnges demands that the first minimum of diffraction pattern of one object and the central maximum of the diffraction pattern of the other should fall at the same position.
- The minimum resolvable angular separation or angular limit of resolution of two close objects by a telescope is given by

  $$\theta_{min} = \frac{1.22\lambda}{D}$$

  where λ is the wavelength and D is diameter of the telescopic objective.
- The resolving power of a telescope is inverse of angular limit of resolution. The deeper we want to penetrate the space, the greater should be the aperture of the objective of telescope.
- The resolving power of a microscope is defined as the smallest distance between two point objects when their fringed images are just resolved:

  $$R.P = \frac{0.61\,\lambda}{\mu \sin i} = \frac{0.61\,\lambda}{N.A.}$$

  where i is the angle of incidence. μ sin i is known as numerical aperture and is approximately equal to one for good objective.
- The resolving power of a diffraction grating is defined as

  $$R.P = \frac{\lambda}{(\Delta\lambda)_{min}} = nN$$

  where A λ is the least resolvable wavelength difference, n is order of spectrum and N is the total number of slits.

11.6 TERMINAL QUESTIONS

1. A diffraction-limited laser beam (λ = 6300 Å) of diameter 5mm is directed at the earth from a space laboratory orbiting at an altitude of 500km. How large an area would the central beam illuminate?

2. Refraction takes place in a prism. The face of the prism limits the refracted beam to a rectangular section. By using Rayleighs criterion, it can be shown that the resolving power of a prism

$$\frac{\lambda}{d\lambda} = t\frac{d\mu}{d\lambda}$$

where t is the length of the base of the prism, μ is the refractive index of the material of prism for wavelength λ. A prism is made of dense flint glass for which refractive indices for λ = 6560 Å and 4860 Å are 1.743 and 1.773 respectively. Calculate the length of the base of the prism.

11.7 SOLUTIONS AND ANSWERS

SAQs

1. The minimum angle of resolution of eye

$$\theta = \frac{1.22\,\lambda}{D} = \frac{1.22 \times (5.5 \times 10^{-5}\,\text{cm})}{0.2\,\text{cm}} = 3.36 \times 10^{-4}\,\text{rad}$$

The lateral width for resolution

$$l = r\,\theta = (4 \times 10^5\text{m}) \times (3.36 \times 10^{-6}\text{ rad}) = 1.34\text{m}$$

Since it is much less than the width of individual houses, it is not wise to believe the astronaut.

2. As we increase the aperture of the telescope, the light collected by it from a star gradually increases and gets concentrated in the image (the diffraction disc). Ultimately a stage will come when the image of the star becomes brighter than the background and be visible (This is because the intensity of the image of a star is proportional to fourth power while the background sky light increases as the area of the aperture.) This means that researchers can see stars during the day by using a telescope with larger objective!

3. The maximum is at $Nn\pi$ and minimum at $(Nn + 1)\pi$. The two curves are symmetrical and if they are of equal intensity, they will cross at $N\gamma = Nn\pi + \frac{\pi}{2}$. Therefore, if you evaluate the function $\left(\frac{\sin N\gamma}{\sin\gamma}\right)^2$ at $N\gamma = Nn\pi$ and $N\gamma = Nn\pi + \frac{\pi}{2}$, i.e. $\gamma = n\pi$ and $\gamma = n\pi + \frac{\pi}{2N}$, you will find that

$$\left(\frac{\sin Nn\pi}{\sin n\pi}\right)^2 = N^2$$

and

$$\left[\frac{\sin\left(Nn\pi + \frac{\pi}{2}\right)}{\sin\left(n\pi + \frac{\pi}{2N}\right)}\right]^2 = \frac{1}{\sin^2\left(\frac{\pi}{2N}\right)} = \frac{1}{\left(\frac{\pi}{2N}\right)^2} = \frac{4N^2}{\pi^2}$$

Hence the required ratio is $\frac{4}{\pi^2}$ = 0.4053. This means hat the resultant intensity will show a dip of about 20% as in the case of a telescope.

4. The waves given out by each self-luminous object bear no constant phase relationship so

that the intensities can be added up. The objects viewed with microscopes are illuminated by the same source and there will be some phase relationship between the waves emanating from these. Strictly speaking the intensities will not be additive. But Abbe found that Eq. (11.3) gives the correct order for the limit of resolution.

TQs

1. We know that half angular spread of light beam is given by

$$\theta = \frac{1.22\lambda}{D} = \frac{1.22 \times (6300 \times 10^{-8}\ \text{cm})}{(0.5\text{cm})}$$

$$= 1.54 \times 10^{-4}\ \text{rad}$$

Since the diameter of light patch

$$x = 2r\,\theta$$

the area of the earth illuminated by the beam focussed from the space laboratory at an altitude of 500 km is

$$A = \frac{\pi x^2}{4} = \pi r^2 \theta^2$$

$$= \frac{22}{7} \times (25 \times 10^{10}\ \text{m}^2) \times (1.54 \times 10^{-4})^2$$

$$= 10934\ \text{m}^2 = 0.01\ \text{km}^2$$

2. $$d\mu = 1.773 - 1.743 = 0.03$$

$$d\lambda = 6560 - 4860 = 1700\ \text{Å} = 1700 \times 10^{-8}\ \text{cm}$$

Note that spectral spread is very wide whereas d λ should be a small change. Assuming that μ changes linearly between these two colours, we have

$$\frac{d\mu}{d\lambda} = -\frac{0.03}{1700 \times 10^{-8}\ \text{cm}} = -\frac{3}{17} \times 10^4\ \text{cm}^{-1} \approx -1765\ \text{cm}^{-1}$$

The negative sign signifies inverse value of relationship between λ and μ. The prism is made of dense flint glass and to just resolve D_1 and D_2 lines we require

$$R.P = \frac{5893\ \text{Å}}{6\ \text{Å}} = 982$$

so that

$$\frac{\lambda}{d\lambda} = 982 = 1765\,t$$

and

$$t = \frac{982}{1765\ \text{cm}^{-1}} = 0.556\ \text{cm} \approx 0.6\ \text{cm}$$

Large Telescopes and Image Quality

We have seen that bigger is the aperture of the telescope objective, better is its resolving power. Also bigger apcrture increases the collecting area of the objective, hence enables the telescope to detect very faint objects. Therefore with the aim of seeing remote and faint objects distinctly and clearly, there has been an attempt to build large telescopcs.

A major shift in the 19th century was from lens telescopes to mirror telescopcs which have the added advantage of not suffering from chromatic aberration. Therefore in the early part of this century Mount Wilson Telescope was made with a mirror of 100 inch (2.5 metre). The next big telescopc to be built was Hale Telescope of 5 metre aperture which started functioning in the late forties.

When we make astronomical observations through a ground based telescope, light coming from a distant star is first incident at the top of the atmosphere in the form of a plane wavefront. This light has to travel through the atmosphere to reach the objective of the tclescopc, which brings it to a focus. Therefore when wc look at a star, wc are 'sccing' through the atmosphere. If the atmosphcrc produces no distortion in the plane wavefront, the resolution of the telescopc is $\theta = 1.22 \dfrac{\lambda}{D}$ which is the theoretical limit. This limit is not achieved because of the degradation of the image quality introduced by the atmosphere. Small temperature fluctuations within the atmosphcrc (< 1°) produce fluctuations in the atmospheric refractive index. The light which is initially a plane wave becomes corrugated and distorted after passing through the atmosphcrc. The magnitude of the distortion introduced is of the order of a few wavclcngths of visible light. In the last decades "adaptive optics techniques" have been developed, first to sense the distortion and then make a suitable wavefront correction by controlling the optical path length. If no correction is done, for a telescope larger than 10 cni, the resolution is severely limited by the effect of 'sccing' (which is observnblc as twinkling).

Obviously 'seeing' depends upon observation sites which have therefore to be carefully chosen. Good observation sites would bc high peaks situated in a flat area. Some good examples arc Mauna Kea, Hawai; or La Palma, the Canary Islands.

In addition to the limit imposed by 'seeing' through die atmosphcrc, the image quality of large lclcscopes is affected by the aberrations produced within the tclcscopc. Techniques for improving the image quality have been developed for ensuring precise shape of the primary mirror. This technique is called "active optics".

From the earth we can use only a small part of the electromagnetic radiation. Thus we can use only the visible light and radio waves (and parts of the infrared). A large portion of radiation remains inaccessible because it gets absorbed or scattered while passing through the earths' atmosphcrc. The only way is put up a telescope above the earths' atmosphere like a satellite where it can receive the incoming radiation and the full information can then be beamed to the station on earth. For this reason Humble Space Telescope was launched in -1990. Since it is above the atmosphere it is free from atmospheric distortions in the image. Also, being in a weightless state, it is not subject to distortions of the mirror which arise due to weight for a ground based tclcscopc.

However it is worth mentioning that with the rcccnt advances in active and adaptive optical techniques the ground based telescopes have achieved a high level of accuracy, coming close to that of space telescopc.