UNIT 9 FRAUNHOFER DIFFRACTION

Structure

9.1 INTRODUCTION

In the previous unit you studied Fresnel diffraction and learnt that the diffraction pattem depends on the distance between aperture and screen as well as the source. As the observation screen is moved away from the aperture, the diffraction pattern passes from the forms predicted in turn by geometrical optics, Fresnel diffraction and Fraunhofer diffraction. When a plane wavefront is incident at the diffracting aperture, the transition from Fresnel to Fraunhofer pattern is determined by the ratio of the size of the diffracting obstacle to its distance from the source and/or the observation screen. You will now learn about Fraunhofer diffraction in detail.

In Sec. 9.2 we have described the experimental arrangement and salient features of the observed Fraunhofer diffraction pattern from a single slit illuminated by a point source. This is followed by a simple theoretical analysis of observed results. When we deal with plane wavefronts, you will find that theoretical analysis is fairly simple. In Sec. 9.3 we have described Fraunhofer diffraction by a circular aperture because of its importance for optical devices. You will learn that the diffraction pattern consists of a central bright disc (called Airy disc) surrounded by concentric dark and bright rings. As a corollary, you will see that a random array of small and nearly circular obstacles gives overlapping diffraction patterns called halos. You may have observed brilliant halos while driving a car whose fogged window is illuminated by a motorcycle at the back. We shall discuss the physical basis for diffraction halos at the end of this unit

Objectives

After going through this unit you will be able to

- describe experimental arrangement for observing Fraunhofer diffraction pattern from a narrow vertical slit and a circular aperture
- explain observed irradiance on the basis of simple theoretical analysis
- solve numerical problems, and .
- explain formation of diffraction halos.

9.2 DIFFRACTION FROM A SINGLE SLIT: POINT SOURCE

From the previous unit, you may recall that to observe Fraunhofer diffraction pattern, we require a point source, which is far away (almost at infinity) from the diffracting aperture (a single slit in the present discussion). The wavefronts of light approaching the diffracting aperture can be assumed to be essentially plane. The observation screen should also be at infinite distance from the aperture. You may now like to ask: Is it practical to put the source of tight and the observation screen at infinite distance from the diffracting aperture? This definitely is not practical because (i) the intensity of diffracted light reaching the observation screen would be reduced infinitesimally (inverse square law) and (ii) we will require infinitely big laboratory rooms. Do these limitations suggest that we cannot observe Fraunhofer diffractibn? These difficulties arc readily overcome by using converging lenses in an actual experiment.

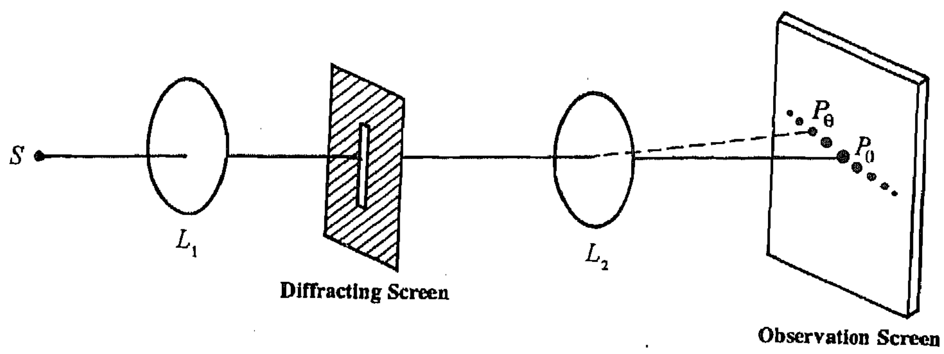


Fig.9.1: Producing *Fraunhofer* diffraction *pattern*

The experimental arrangement for producing Fraunhofer diffraction pattern is shown in Fig. 9.1. A monochromatic point source (S) of light is placed in the focal plane of a converging lens L_1, so that a plane wave is incident on a long narrow slit. Another convergent lens L_2 is placed on the other side of the slit. The observation screen is placed at the second focal point of this lens. Then light focussed at any point on the observation screen is due to parallel diffracted wavelets emanating from different portions of the slit. You must note that the observation screen and diffraction screen are kept parallel. Moreover, both the screens are perpendicular to the common axis of L_1 and L_2. The slit is so adjusted that the common axis of these lenses is perpendicular to the length of the slit and passes through the middle of the slit, both in height and width.

In a physics laboratory this arrangement is easily achieved by using an ordinary spectrometer. We hope that you got an opportunity to work with a spectrometer in your second level laboratory course. To observe the diffraction from a point source, the slit of the collimator should be replaced by a fine pinhole, which should be carefully positioned at the focal point of the collimator lens. The observation screen can be placed at the back focal plane of the telescope. Alternatively, wc may observe the back focal plane of lens L_2 with an eyepiece. The diffracting screen with slit aperture is placed between the two lenses suitably on the turn table.

9.2.1 Observed Pattern

Let us pause for a minute and think what would the diffraction pattern of the vertical slit look like? Or what would be the distribution of intensity in this pattern? You may think that the diffraction pattem would be a single vertical line or a series of vertical lines on the observation screen. This line of thought is wildly off-target. The actual diffraction pattern is astonishingly different; it consists of a horizontal streak of light composed of bright elongated spots connected by faint streaks. In other words, after passing through the vertical slit, light spreads along a horizontal line. This means that diffraction pattern is along a line perpendicular to the length of the diffracting slit. You may interpret this horizontal diffraction as a spread out image of the point source. The extent of horizontal spreading is controlled by the width of the slit; as the width increases, the spreading decreases. And in the extreme case of a very wide slit, the (horizontal) diffraction streak reduces to a bright point. Physically, very wide slit means that the slit has effectively been removed.

The salient features of the observed Fraunhofer diffraction pattern of a single vertical slit from a point source are shown in Fig. 9.2. These are summarised below:

i) The diffraction pattern consists of a horizontal streak of light along a line perpendicular to the length of the slit.

ii) The horizontal pattern is a series of bright spots. The spot at the central point P_0, which lies at the intersection of the axis of L_1 and L_2 with the observation screen, is the brightest. On either side of this spot, we observe a few more bright spots symmetrically situated with respect to P_0.

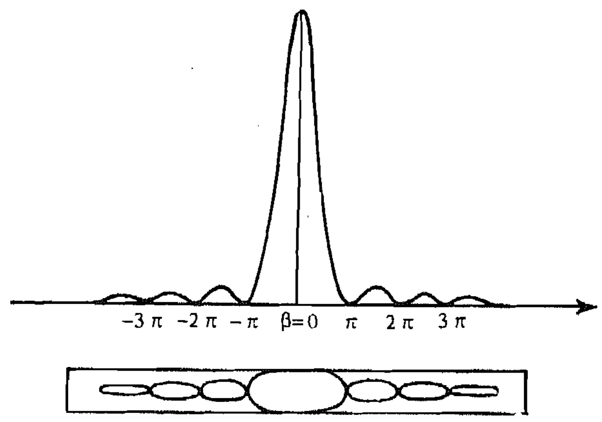


Fig.9.2: Observed Fraunhofer diffraction pattern of a diffracting slit

iii) The intensity of the central spot is maximum and its peak is located at P_0. The peak intensities of other spots, on either side of the central spot, decrease rapidly as we move away from P_0. The central maximum is called principal maxima and other maxima as secondary maxima.

The width of a spot is specified by the distance between two consecutive minima.

iv) The width of the central spot is twice the width of other spots.

v) A careful examination of the diffraction pattern shows that the central peak is symmetrical. But on either side of the central maximum, secondary maxima are asymmetrical. In fact, the positions of the maxima arc slightly shifted towards the observation point P_0.

Let us now learn the theoretical basis of these result?.

9.2.2 Calculation of Intensity Distribution

We take the plane of the paper as horizontal. The plane of the paper is defined by the diffraction streak and the axis of the lens L_2.

The first step in the calculation of intensity distribution is to realise that the observed diffraction pattern is focussed on the observation screen placed at the back focal planc of lens L_2. We know that only parallel rays are brought.to focus in the back focal planc of the lens. Therefore, diffracted light must be emerging as a series of parallel light. The beam of rays parallel to the axis of the lens are focussed at the focal point. However, the beam inclined to the axis of the lens is brought to focus on the back focal planc but away from the focal point. We can as well describe this observation in terms of the wavefront; the two being perpendicular to each other. Since diffraction pattern lies on a horizontal line (which is at right angles to the common axis of L_1 and L_2), the diffracted wavefmnts will be vertical planes perpendicular to the plane of the paper. That is, after passing through the vertical slit, the incident plane waves are replaced by a system of vertical plane waves which proceed in different directions. Therefore, for our theoretical analysis it is sufficient to assume that when a plane wavefront falls on the diffracting slit, each point of the aperture such as A A, A, A, ... B (Fig. 9.3) becomes a source of secondary wavelets, which propagate in the direction of the point P_θ under consideration. These are diffracted plane waves. (You should realise that diffracted waves have no existence in the domain of geometrical optics. The diffracted waves arise due to interaction between light and matter. In the present case, the interaction is between light and the jaws of the slit.)

Refer to Fig. 9.3a which shows the geometry for the irradiance at point P_θ (on the distant screen) which makes an angle θ with the axis. In order to sum up the contributions of different wavelets at P_θ, we must know their amplitudes and phases. The amplitudes of the disturbances from A. A,, A_2, ... will be very nearly equal. Do you know why? This is because the distance of point P_θ from the diffracting screen is very large compared to the width (b) of the aperture.

Now let us consider the phases of the disturbances reaching the point P_θ. You will agree that the points A, A_1, A_2, A, ... B within thc aperture form a series of coherent sources since they havc originated from the same point source. Also points A, A_1, A_n, ..., B are in the samc phase since thcy lie on the same plane wavefront. The phase difference between different diffracted rays reaching the point. P, arises due to the difference in path lengths travcllcd by thcm to reach P_θ. To know the phase diffcrence, we draw a plane normal to thc parallel diffractcd rays. Thc trace of this plane in the planc of the paper is AD (Fig. 9.3a). Though the disturbances are in phase at points A, A_1, A_n ..., B when they start, they reach thc trace AD in diffcrent phases because of thc unequal path lengths travelled by

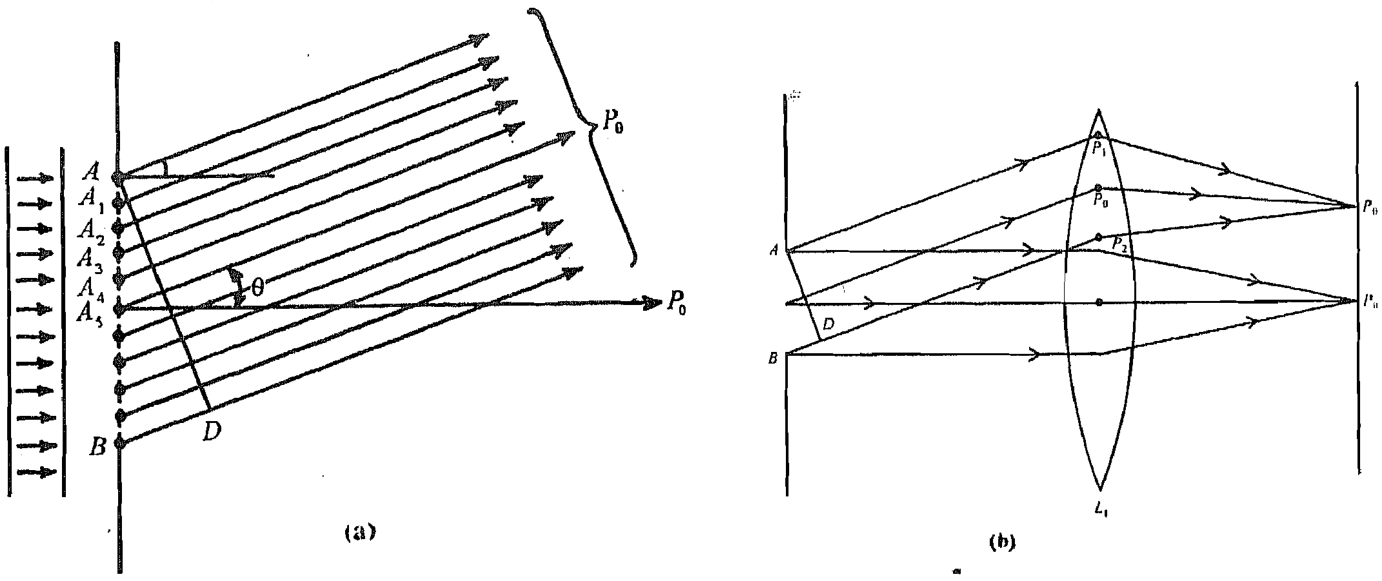


Fig.9.3: (a) Cross-sectional vlcw of the geometry for single slit diffraction
(b) Trace of optical paths between slit and screen

thcm. Thc optical paths of diffracted wavcs from the plane AD to thc focal point P_θ are cqual. This is bccausc in a welt corrcctcd convcrging lcns the optical paths of all rays bctwccn any plane intersecting thc parallel bcam of light perpendicularly and thc point whcrc rays convcrgc aftcr traversing thc lens, arc cqual. That is, optical paths Ap, P_θ and $Dp_2\,P_\theta$ arc cqual, as may be seen from Fig. 9.3(b). Thcrcforc, thc wavclcts arrive at P_θ with thc same relative phase diffcrcncc as the oncs existing at thc trace AD.

Two sources arc snid to bc cohcrcnt if thcy emit in-phase wavcs of thc same frequency.

Lct us consider thc apcrture AB to bc divided into n cqual parts so that $AA_1 = A_1A_2 = A_2A_3 = b/n = \Delta$. It mcans that wc have a serics of point sources from A to B. Actually, thc aperture contains a continuous distribution of points from A to B, and thcrcforc in thc limiting casc, $n \to \infty$ and $\Delta \to 0$ such that $n\Delta \to b$. Consider thc two rays starting from two neighbouring points A and A_1. Thc path diffcrcncc bctwccn them is AA_1 sine, where θ is the angle bctwccn thc diffracted rays and thc normal to the slit. Hcncc the corresponding phase diffcrcncc is givcn by

$$\phi = \frac{2\pi}{\lambda}(AA_1 \sin \theta) = \frac{2\pi}{\lambda}\left(\frac{b}{n}\sin\theta\right) = \frac{2\pi}{\lambda}\Delta \sin\theta \qquad (9.1)$$

Lct thc ficld at P, duc to thc disturbance originating from A bc $a_0 \cos \omega t$. Then, thc ficld duc to thc disturbance from A_1 is $a_0 \cos(\omega t - \phi)$. Hcrc wc havc assumed that thc amplitudes of disturbances from diffcrent points arc cqual. The ficld duc to disturbances from succcssivc points A_2, A_3, ..., B arc $a_0 \cos(\omega t - 2\phi)$, $a_0 \cos(\omega t - 3\phi)$, ..., $a_0 \cos(\omega t - n\phi)$, rcspectivcly. Thc magnitude of resultant ficld at P_θ is cqual to thc sum of thcsc disturbances. Hcncc wc can write

$$E = a_0 \cos \omega t + a_0 \cos(\omega t - \phi) + a_0 \cos(\omega t - 2\phi) + \ldots + a_0 \cos(\omega t - n\phi)$$

In Unit 2 of PHE-02 course on Oscillations and Waves, wc summed up this scrics (Eq. (2.38)). Wc will just quote thc result here:

$$E = a_0 \left[\frac{\sin \dfrac{n\phi}{2}}{\sin\left(\dfrac{\phi}{2}\right)}\right] \cos\left(\omega t - \frac{n\phi}{2}\right)$$

$$= E_\theta \cos\left(\omega t - \frac{n\phi}{2}\right) \tag{9.2}$$

where E_θ is the amplitude of the resultant field at P_θ :

$$E_\theta = a_0 \frac{\sin\left(\frac{n\phi}{2}\right)}{\sin(\phi/2)} \tag{9.3}$$

In the limit $n \to \infty$ and $\Delta \to 0$, $n\Delta \to b$. Then from Eq. **(9.1)** we have

$$\frac{n\phi}{2} = \frac{n}{2}\frac{2\pi}{\lambda}\Delta \sin\theta = \frac{\pi}{\lambda}(n\Delta)\sin\theta = \frac{\pi}{\lambda} b \sin\theta$$

so that $\phi = \frac{2\pi}{\lambda}\frac{b\sin\theta}{n}$ will be very small for $n \to \infty$. We may therefore write

$$\sin\left(\frac{\phi}{2}\right) \cong \frac{\phi}{2} = \frac{\pi b \sin\theta}{n\lambda}$$

Substitute this result in Eq. **(9.3)**. On simplification you will find that

$$E_\theta = a_0 \frac{\sin\left(\frac{n\phi}{2}\right)}{\sin(\phi/2)} \cong a_0 \frac{\sin(n\phi/2)}{(\phi/2)} = na_0 \frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\left(\frac{\pi b \sin\theta}{\lambda}\right)}$$

$$= na_0\left(\frac{\sin\beta}{\beta}\right) = A\left(\frac{\sin\beta}{\beta}\right) \tag{9.4}$$

where we have written

$$A = n\, a_0$$

and

$$\beta = \pi \frac{b\sin\theta}{\lambda} \tag{9.5}$$

You will note that for a given wavelength, β signifies half of the phase difference between disturbances originating from the extreme points A and B. The expression for resultant field at P_θ takes the form

$$E = A\frac{\sin\beta}{\beta}\cos(\omega t - \beta) = E_\theta \cos(\omega t - \beta) \tag{9.6}$$

The corresponding intensity distribution at P_θ is given by

$$I_\theta = A^2\left(\frac{\sin\beta}{\beta}\right)^2 \tag{9.7}$$

Let us pause for a while and ponder as to what we have achieved. This result suggests that the intensity is maximum at $\theta = 0$. This readily follows by noting that when we substitute $\theta = 0$ we have both β and $\sin\beta$ equal to zero but

$$\lim_{\beta \to 0} \frac{\sin\beta}{\beta} = 1$$

Therefore

$$I_{\theta=0} = A^2$$

This result is expected on geometrical considerations. In the limit of a distant screen, the central point becomes equidistant from each point on the slit. All diffracted waves arrive in phase at P_0 and interfere constructively. A^2 is then the value of the maximum intensity at the centre of the pattern. This maximum is also termed **principal maximum.**

For brevity wc write $I_{\theta=0} = A^2 = I_0$. Then intensity at any point at an angle θ with the horizontal axis is given by

$$I_\theta = I_0\left(\frac{\sin\beta}{\beta}\right)^2$$

Positions of maxima and minima

A plot of Eq. (9.7) for intensity distribution is shown in Fig.9.4. You will note that the intensity is maximum for θ = 0: $I_{\theta=0} = I_0 = A^2$. The intensity gradually falls on either side of the principal maximum and becomes zero when β = + π or β = − π since sin (± π) is zero. This is the first minimum. So we can say that the angular half width of principal maximum is from β = 0 to β = π. The second minimum on either side occurs at β = ± 2π. Thus we get the minima when

$$\beta = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$$

$$= m\pi \qquad m = \pm 1, \pm 2, \pm 3, \ldots \qquad (9.8)$$

Note that the value m = 0 is excluded because it corresponds to the principal maximum (for β = 0). Substituting the value of β from Eq. (9.8) in Eq. (9.5) we find that the condition for minima can also be expressed as

$$b\sin\theta = \pm\lambda, \pm 2\lambda, \pm 3\lambda\ldots$$

$$= m\lambda, \qquad m = \pm 1, \pm 2, \pm 3,\ldots \qquad (9.9)$$

That is, θ depends on the wavelength of light and the slit width. For a given slit width, the spread in diffraction pattern depends directly on the wavelength. Accordingly you should expect that red light will be diffracted through a larger angle than the blue or violet light.

You may now like to know: What will happen when white light illuminates a single slit? We expect that each wavelength will be diffracted independently. This gives rise to a white central spot surrounded by coloured fringes. The outer part of this pattern would tend to be reddish. You can easily observe this diffraction pattern by looking through the tines of a dinner fork at a candle in a dimly illuminated room. On twisting the fork about its handle, you will observe the diffraction pattern as soon as the cross-sectional area becomes small enough.

The expression $I_\theta = I_0\left(\frac{\sin\beta}{\beta}\right)^2$ gives the diffraction intnsity in different directions. In order to determine the directions (and positions) of secondary maxima, we differentiate this equation with respect to β and equate the result to zero. This gives

$$\frac{dI_\theta}{d\beta} = 2I_0\left(\frac{\sin\beta}{\beta}\right)\left[\frac{\beta\cos\beta - \sin\beta}{\beta^2}\right]$$

$$= 2I_0\sin\beta\left[\frac{\cos\beta}{\beta^2} - \frac{\sin\beta}{\beta^3}\right] = 0$$

so that $\qquad \sin\beta\,(\beta - \tan\beta) = 0$

From this we get the conditions sin β = 0 and β − tan β = 0.

The condition sin β = 0 implies that β = ± mπ, where m is any integer. This is a trivial condition as it signifies minima and is of no interest.

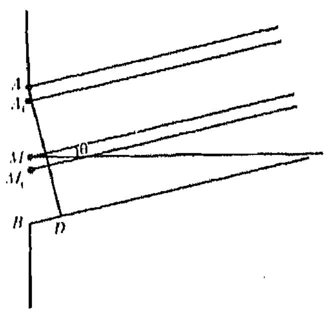
The condition β = tan β therefore gives the positions of secondary maxima. This is a transcendental equation. The roots of this equation can be found by a graphical method. All you have to do is to recall that an angle equals its tangent at intersections of the straight line

$$y = \beta$$

and the curve

$$y = \tan\beta \qquad (9.10)$$

A very clear idea of the single slit pattern can be obtained from the following simple qualitative argument. The path difference between waves diffracted by extreme points in the slit is BD = b sin θ (see fig. below). Suppose that light is focussed at some point P_0. If BD is an integral multiple of λ, we will show that the resultant intensity at P_0 will be zero. For m = 1, the angle θ satisfies the equation b sinθ = λ. We divide the slit into two equal halves AM and MB as shown in the figure below. Consider the waves starting from the two point sources A and M. The path difference between them is AM sinθ = (b/2) sinθ = λ/2. The corresponding phase difference will be π. Therefore the two waves on superposition lead to zero intensity at P_0. Similarly, for a point A_1, just below A, there will be a corresponding point M_1 just below M such that the path difference between disturbances generated by them is again λ/2. On superposition this pair also leads to zero intensity at P_0. We can thus pair off all the points in the upper half (AM) with corresponding points in the lower half (MB) and the disturbances due to upper half of the slit will be cancelled by disturbances due to the lower half. So the resultant intensity at P_0 will be zero. This explains why we get a minimum intensity at P_0 when the path difference between the rays from extremes is equal to λ.



Let us now consider the case m = 2 so that the path difference b sinθ between the extreme rays is equal to 2λ. We can now imagine that the slit is divided into four equal parts. We can, by similar pairing, show that the first and second quarters have a path difference of λ/2 and cancel each other. Third and fourth quarters cancel each other by the same argument so that the resultant intensity in the focal plane at P_0 is again zero. For m = 3 the path difference between the two extreme rays is b sinθ = 3λ. In this case, the slit should be divided into six equal parts to show similar pairing and cancellation and then leading to zero intensity. By this simple qualitative argument, we have shown that when the path difference between the extreme parallel diffracted rays in a particular direction is an integral multiple of λ, the resultant diffracted intensity in that direction is zero.

Plots of these curves arc also shown in Fig. 9.4. The points of intersection excluding $\beta = 0$ (which corresponds to principal maximum) occur at $\beta = 1.43\pi, 2.46\pi, 3.47\pi$ etc. and give the position of the first, second, third maxima on either side of the central maximum. You should note that these maxima do not fall midway between two minima. For instance, the first maximum occurs at 1.43π rather than 1.50π. Similarly the second maximum occurs at 2.46π rather 2.50π and so on. This means that the intensity curves are asymmetrical. The plot clearly shows that the positions of maxima are slightly shifted towards the centre of the pattern. You may recall that this asymmetry is observed experimentally as well.

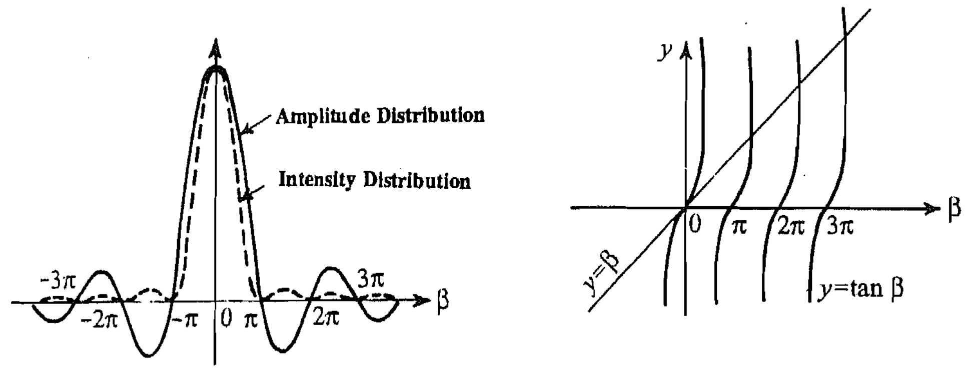


Fig. 9.4: Amplitude and intensity contours for Fraunhofer diffraction of a single slit showing positions of maxima and minima

Let us now calculate the intensities at these positions of maxima. The intensity of first secondary maximum is given by

$$\left(\frac{\sin 1.43\,\pi}{1.43\,\pi}\right)^2 = 0.0496$$

This means that the intensity of the first secondary peak (nearest to the central peak) is about 4.96% of the central peak. Similarly, you can calculate and convince yourself that the intensities of the second and third secondary maxima are about 1.68% and 0.83% of the central maximum. We call these maxima as secondary maxima.

$y = \beta$ is a straight line passing through the origin. $y = \tan\beta$ is represented by a family of curves having for asymptotes $\beta = \frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

The intensities of the secondary maxima can be calculated to a fairly close approximation by finding the values of β at halfway positions i.e. at $\beta = \frac{n + 2\pi}{2}, \frac{2x + 3\pi}{2}, \frac{3\pi + 4\pi}{2}, \ldots$ etc. The intensities at these positions are $\frac{4}{9\pi^2}, \frac{4}{25\pi^2}, \frac{4}{49\pi^2}, \ldots$ or $\frac{1}{22.1}, \frac{1}{61.7}, \frac{1}{121}, \ldots$ of the central maximum which are very close to the above calculated values. We thus see that most of the light is concentrated in the central maximum.

Another important characteristic of the principal maximum is that its width is double of the width of secondary maximum. We have left its mathematical proof as an exercise for you. Before you proceed, you should solve SAQ 1.

Spend 5 min

SAQ 1

Show that the principal maximum is twice as wide as the secondary maxima.

To give you a feel for numerical values and fix the ideas developed in this section, we now give a few solved examples. You should go through these carefully.

Example 1

In the experimental set up used to observe Fraunhofer diffraction of a vertical slit (width 0.3mm), the focal length of lens L_2 is 30 cm. Calculate (a) the diffraction angles and positions of the first, second and third minima, and (b) the positions of the first, second and third maxima on either side of the central spot. The slit is illuminated with yellow sodium light which is a doublet. You may take $\lambda = 6000$ Å.

Solution

You have seen that the conditions for minima are given by b sin 0 = $m\lambda$; $m = \pm 1, \pm 2, \pm 3, ...$ For small values of θ, we may write sin $\theta \cong 0$. Then

$$\theta = m\frac{\lambda}{b}$$

and the distance $P_0 P_\theta$ is f θ, where f is the focal length. Therefore,the diffraction angles $\theta_1, \theta_2, \theta_3$ for the first, second and third minima are $\frac{\lambda}{b}$, $2\frac{\lambda}{b}$, and $3\frac{\lambda}{b}$, respectively..

On substituting the numerical values of λ and b we find that

$$\theta_1 = \frac{6000 \times 10^{-8}\,\text{cm}}{0.3 \times 10^{-1}\,\text{cm}} = 2 \times 10^{-3}\ \text{rad}$$

$$\theta_2 = 28, = 4 \times 10^{-3}\ \text{rad}$$

$$\theta_3 = 3\theta_1 = 6 \times 10^{-3}\ \text{rad}$$

The distances d_1, d_2, d_3 of these minima from the central spot are

$$d_1 = f\,8, = (30\ \text{cm}) \times 2 \times 10^{-3} = 6 \times 10^{-2}\ \text{cm} = 0.06\ \text{cm}$$

$$d_2 = 2f\,\theta_1 = 2 \times 0.06\ \text{cm} = 0.12\ \text{cm}$$

$$d_3 = 3f\,\theta_1 = 3 \times 0.06\ \text{cm} = 0.18\ \text{cm}$$

You will note that these minima are separated by a distance of 0.06 cm on the focal plane of the lens. We know that the first three secondary maxima occur at $\beta = 1.43\pi, 2.46\pi$ and 3.47π, respectively. The corresponding diffraction angles in radians for these three maxima are

$$(\theta_1)_{max} = 1.43\frac{\lambda}{b},\ (\theta_2)_{max} = 2.46\frac{\lambda}{b}\ \text{and}\ (\theta_3)_{max} = 3.47\frac{\lambda}{b}$$

$$(\theta_1)_{max} = (1.43)(2 \times 10^{-3}),\ (\theta_2)_{max} = 2.46\,(2 \times 10^{-3}),$$

and

$$(\theta_3)_{max} = (3.47)\,(2 \times 10^{-3})$$

and the corresponding distances from the central point (Po) are

$$d_1 = f(\theta_1)_{max} = (30\ \text{cm}) \times 1.43 \times 2 \times 10^{-3} = 0.086\ \text{cm}$$

$$d_2 = f(\theta_2)_{max} = (30\ \text{cm}) \times 2.46 \times 2 \times 10^{-3} = 0.16\ \text{cm}$$

$$d_3 = f(\theta_3)_{max} = (30\ \text{cm}) \times 3.47 \times 2 \times 10^{-3} = 0.21\ \text{cm}$$

Example 2

In the above experiment, we change slit widths to 0.2mm, 0.1mm, and 0.06mm. Calculate the positions of the first and second minima.

Solution

For slit width b = 0.2 mm, we have

$$d_1 = f\theta_1 = (30\ \text{cm}) \times \frac{6000 \times 10^{-8}\,\text{cm}}{0.2 \times 10^{-1}\,\text{cm}} = 0.09\ \text{cm}$$

Similarly

$$d_2 = f\theta_2 = 2 \times 0.09\text{cm} = 0.18\ \text{cm}$$

These minima are separated by 0.09 cm. Recall that the corresponding value for a slit of width 0.03 cm was 0.06 cm. This means that for a given wavelength, the spread of secondary maximum increases as slit width decreases. This conclusion is brought out in the following calculations as well.

For a slit of width b = 0.1 mm, we have

$$d_1 = (30\text{ cm}) \times \frac{6000 \times 10^{-8}\text{ cm}}{0.1 \times 10^{-1}\text{ cm}}$$

$$= 0.18\text{ cm}$$

and $$d_2 = 2 \times 0.18\text{ cm} = 0.36\text{ cm}$$

For slit width b = 0.06 mm, we have

$$d_1 = (30\text{ cm}) \times \frac{6000 \times 10^{-8}\text{ cm}}{0.06 \times 10^{-1}\text{ cm}}$$

$$= 0.3\text{ cm}$$

and $$d_2 = 2 \times 0.3\text{ cm} = 0.6\text{ cm}$$

We thus find that for slits of widths 0.3mm, 0.2mm, 0.1mm, and 0.06mm, the first minimum on either side of the principal maximum occurs at distances of 0.06 cm, 0.09 cm, 0.18 cm, 0.3 cm. In these four cases, the corresponding principal maximum extends over 0.12 cm, 0.18 cm, 0.36 cm and 0.6 cm.

This shows that as the slit becomes narrower, the spread of central maximum increases. Conversely, the wider the slit, the narrower is the central diffraction maximum.

We now consider an inlcresting case where width of the slit is varied in comparison to the wavelength of light.

Example 3

Consider a slit of width b = 10λ , 5λ and λ . Calculate the spread of the central maximum.

Solution

From Eq. (9.9), we note that for a slit of width b = 10λ , the first minimum is located at

$$10\lambda \sin\theta = \lambda$$

or

$$\sin\theta = 0.10$$

and

$$\theta = 5.7°$$

For a slit of width 5λ , we have

$$5\lambda \sin\theta = \lambda$$

or

$$\theta = 11.5°$$

That is, as the aperture of the slit changes from 10λ to 5λ , the diffraction pattern spreads out about twice as far. For b = λ,

$$\sin\theta = 1$$

or

$$\theta = 90°$$

The first minimum falls at 90°. That is, the central maximum spreads out and the diffraction pattern shows no ripple. These features are shown in Fig. 9.5.

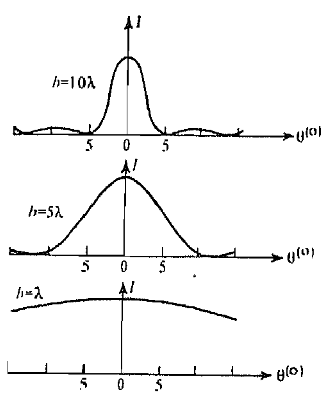


Fig. 9.5: Single-slit diffraction irradiances as the slit width varies

You may now like to answer an SAQ.

Spend 5 *min.*

SAQ 2

We illuminate the slit of Example 1 with violet light of wavelength 4358 Å from a mercury lamp. Show that the diffraction pattern shrinks correspondingly.

Diffraction Pattern of a Rectangular Aperture

So far we have described Fraunhofer diffraction pattern of a slit aperture. Let us now consider as to what will happen if both dimensions of the slit are made comparable. We now have a rectangular aperture of width a and height b as shown in Fig. 9.6(a). We expect that the emergent wave will spread along the length as well as the width of the slit. Can you depict the diffraction pattern? It is shown in Fig.9.6(b). Mathematically, the intensity is given by $I = \frac{I_0 \sin^2 \beta \sin^2 \alpha}{\alpha^2 \beta^2}$ where $\beta = \frac{b \pi \sin \theta}{\lambda}$ and $\alpha = \frac{\pi a \sin \theta}{\lambda}$.

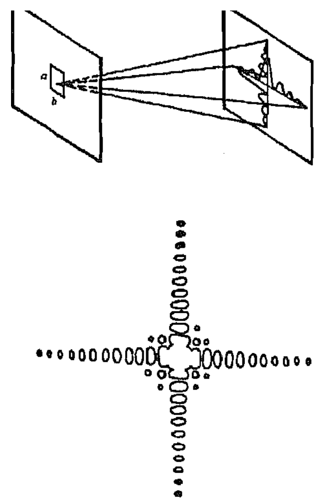


Fig.9.6: Diffraction from a rectangular aperture. Both dimensions of the rectangular aperture are small and a two-dimensional diffraction pattern is discernible on the screen. The lower part shows diffraction pattern of a single square aperture

Slit Source

The experimental arrangement shown in Fig. 9.1 is modified as shown in Fig. 9.7 so that instead of the point source of light we have a slit source of light (Fig. 9.7(a)).

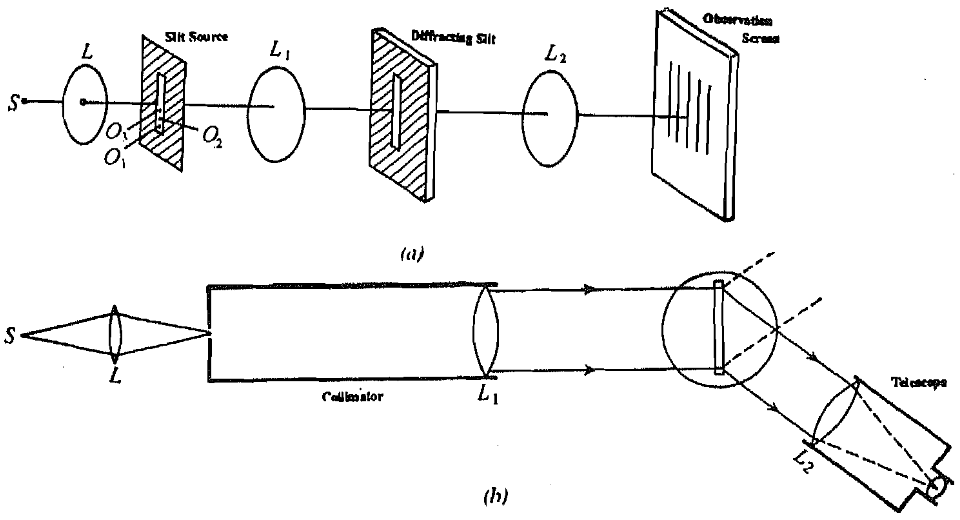


Fig.9.7: (a) Experimental arrangement for diffraction from a vertical narrow single slit illuminated by a slit source (b) Experimental arrangement in a physics laboratory.

As a matter of fact, the experimental arrangement, which is commonly employed in most experiments, uses a spectrometer (Fig. 9.7(b)). The slit of the collimator arm is illuminated so that each point of the slit source acts as an independent source. You know that a point source gives a horizontal streak of light as the diffraction pattern of a vertical slit. Now when we use a slit as a source, we can imagine a series of point sources O_1, O_2, O_3, ... etc. one above the other to form the slit source (Fig. 9.7(a)). Each point source will give its own diffraction pattern since each point is to be regarded as an independent point source. With the same diffracting slit and the same lenses L, and L_2, the central diffraction maximum due to all point sources will lie above one another and give a central bright vertical fringe. Similarly from secondary maxima and minima points, we will obtain a series of vertical fringes, which will be situated at equal intervals on either side of the central fringe. The resulting pattern arises by superposition of a series of horizontal diffraction streaks stacked on each other in a vertical direction. The intensity along any horizontal line will be the same as in Fig.9.2. This is because each point of the slit source acts as an independent and effectively as a non-coherent source.

You will observe that clear fringes are obtained only when the width of the source slit is small. Suppose that the width of the source slit is gradually increased. This will lead to an increase in the width of its image on the observation screen. A stage will come when the width of its image, becomes comparable with the distances between successive vertical fringes. This will gradually make the vertical fringes less clear and indistinct. For a similar reason, we obtain clear fringes only when the source slit is parallel to the diffraction slit.

9.3 DIFFRACTION BY A CIRCULAR APERTURE

Fraunhofer diffraction by a circular aperture is of particular interest because a lens in an optical device like microscope, telescope, eye etc. can be regarded as a circular aperture. For this case, the experimental arrangement is shown in Fig. 9.8(a). A plane wave is incident normally on the aperture and a lens whose diameter is much larger than that of

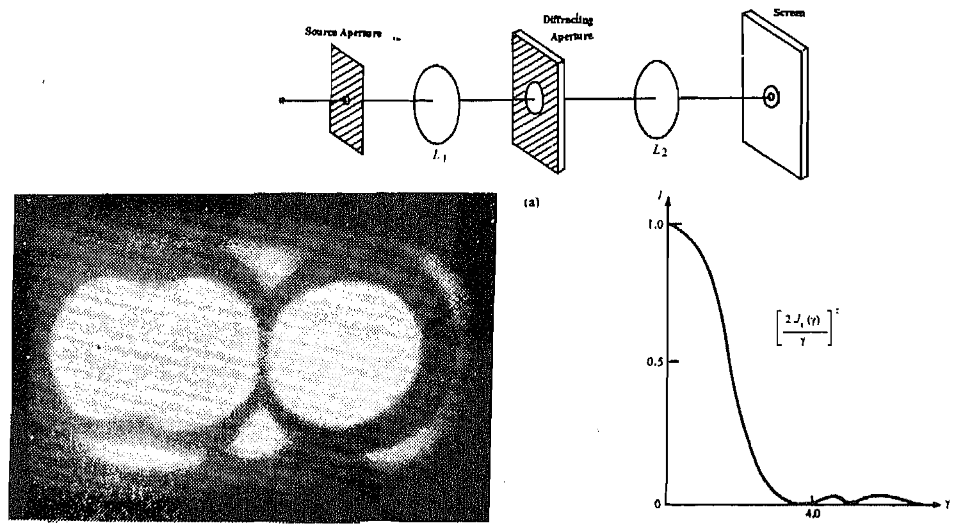


Fig.9.8: (a) Experimental arrangement for observing the Fraunhofer diffraction pattern by a circular aperture. (b) The Airy pattern of three stars; the circle of light at the left of the figure corresponds to zeroth order (c) The corresponding intensity distribution.

the aperture is placed close to it. The Fraunhofer diffraction pattern is observed on the back focal plane of the lens. Because of the rotational symmetry of the system, we expect that the diffraction pattern will consist of concentric dark and bright rings known as Airy pattern. Fig. 9.8 (b) shows the Airy pattern of three stars. The detailed derivation of the diffraction pattern for a circular aperture involves complicated mathematics. So we just quote the final result for the intensity distribution:

$$I = I_0 \left[\frac{2J_1(\gamma)}{\gamma}\right]^2 \qquad (9.11)$$

where

$$\gamma = \frac{\pi D}{\lambda}\sin\theta \qquad (9.12)$$

Here D is the diameter of the aperture, λ is the wavelength of light and θ is the angle of diffraction, I_0 is the intensity at $\theta = 0$ (which represents the central maximum) and $J_1(\gamma)$ is the Bessel function of the first order. (We know that you are not very familiar with Bessel functions.) We may just mention that the variation of $J_1(\gamma)$ is somewhat like a damped sine curve. Moreover, the intensity is maximum at the centre of the pattern since

$$\lim_{\gamma \to 0} \frac{2J_1(\gamma)}{\gamma} \to 1$$

similar to the relation

$$\lim_{\beta \to 0} \frac{\sin\beta}{\beta} \to 1$$

Other zeros of $J_1(\gamma)$ occur at $\gamma = 3.832, 5.136, 7.016, \ldots$ which correspond to the successive dark circles in the Airy pattern. Thus the first dark ring appears when

$$\sin\theta = \frac{3.832\,\lambda}{\pi D} \cong \frac{1.22\,\lambda}{D} \qquad (9.13)$$

Let us compare this result with the analogous equation for the narrow slit. We find that the angular half-width of the central disc, i.e. the angle between the central maximum and the first minimum of the circular aperture, differs from that for the slit pattern through the weird number 1.22. The intensity distribution of Eq. (9.11) is plotted in Fig. 9.8(c). The pattern is similar to that for a slit, except that the pattern for circular aperture now has rotational symmetry about the optical axis. The central maximum is consequently a circular disc of light, which may be regarded as the diffracted "image" of the point source by the circular aperture. It is called the Airy disc. It is surrounded by a series of alternate dark and bright Fringes of decreasing intensity. However, the pattern is not sharply defined.

If you consider any section through the circular aperture intensity distribution is very much the same as obtained from a point source with a single slit. Indeed, the circular aperture pattern will be obtained if you rotate the single slit pattern about an axis in the direction of the light and passing through the central point of the principal maximum.

We now give an example to enable you to have a feel for the numerical values.

Example 4

Plane waves from a helium-neon laser with wavelength 6300 Å are incident on a circular aperture of diameter 0.5 mm. What is the angular location of the first minimum in the diffraction pauern? Also calculate the diameter of Airy disc on a screen 10m behind the aperture.

Solution

We know from Eq. (9.13) that

$$D \sin\theta = 1.22\lambda$$

On substituting the given values, we get

$$(0.5 \times 10^{-3}\text{m}) \sin\theta = 1.22 \times 630 \times 10^{-9}\text{m}$$

or

$$\sin\theta = \frac{1.22 \times 630\ 10^{-9}\text{m}}{0.5 \times 10^{-3}\ \text{m}}$$

$$= 1.54 \times 10^{-3}$$

In the small angle approximation, sin 8 ≅ 8 so that

$$\theta = 1.54 \times 10^{-3}\ \text{rad} = 0.087°$$

On a screen placed 10m away, the linear location of the first minimum is

$$x = D \tan\theta \cong D \sin\theta \cong D\theta$$

Hence

$$x = (10\text{m}) \times (1.54 \times 10^{-3}\ \text{rad})$$

$$= 15.4 \times 10^{-3}\text{m} = 1.54\ \text{cm}$$

This value of x signifies the radius of the Airy disc so that the diameter is about 3 cm.

You can observe a white light circular diffraction pattern by making a small pinhole in a sheet of aluminum foil. Then look through it at a distant light bulb or a candle standing in a poorly illuminated (dark) room.

Imagine that a random array of small circular apertures is illuminated by plane waves from a white point source. Wc know that each aperture will generate an Airy type diffraction pattern. If the apertures arc small and close together, the diffraction patterns arc large and overlap. The overlapping diffraction patterns produce a readily visible halo, namely, a central white disc surrounded by circular coloured rings. Which colour do you expect to be at the outermost rim? Should it not be red? Similar halos arc also observed when the diffraction is due to a random array of circular obstacles.

Suspended water (n = 1.33) droplets in air (n = 1.00) give rise to diffraction halos: When observed through a light cloud cover around the sun or moon, these diffraction halos arc referred to as coronas. Wc can distinguish between diffraction halos and ice crystal halos. Ice crystal halos are due to refraction and dispersion by the ice crystals; they have red on the inside of the rings.

While driving a car at night, you may have seen brilliant halos through fogged up car windows on which light of a motorcycle following you is incident. These arc diffraction halos. You can easily produce such halos by breathing on the side of a clear glass and then looking through the fogged area at a small source (e.g., match, penlight, or distant bulb).

9.4 SUMMARY

- To observe Fraunhofer diffraction pattern, the distance of the diffracting screen from the source and observation screen should be alomost infinite. Experimentally this condition is achieved by using convergent lenses.
- The Fraunhofer diffraction pattern of a slit formed by a point source consists of a horizontal streak of light. This horizontal diffraction pattern may be regarded as a spread out image of the point source and consists of a series of diffraction spots symmetrically situated with respect to central point.
- The central spot has a maximum intensity and its width is twice compared to other spots which are of equal width. Their intensities decrease rapidly. In fact, most of the light is concentrated in the central maximum.
- The plane wavefront incident on the slit gives rise to a system of vertical plane diffracted wavefronts which originate from each point of the diffracting aperture.
- The intensity at any point P_θ on the screen is computed by taking the phase difference between the successive diffracted waves into account. The intensity at a point P_θ is given by

$$I = I_0 \left(\frac{\sin \beta}{\beta}\right)^2$$

where $\beta = \pi \frac{b \sin \theta}{\lambda}$ and b is width of the slit.

- If the path difference b sine between waves diffracted by extreme ends of the slit is an integral multiple of λ, we obtain zero intensity.
- The Fraunhofer diffraction pattern of a slit (as aperturc) formed by a slit source of light consists of a series of vertical fringes. In this pattern, the central vertical fringe is the brightest and the intensity of other fringes decreases rapidly. The width of central fringe is double of that for other fringes.
- The diffraction pattern of a circular aperture consists of concentric rings with a central bright disc. The first dark ring appears when $\sin \theta = 1.22\, \lambda / D$.

9.5 TERMINAL QUESTIONS

1. A single slit has a width of 0.03 mm. A parallel beam of light of wavelength 5500 Å, is incident normally on it. A lens is mounted behind the slit and focussed on a screen located in its focal plane, 100 cm away. Calculate the distance of the third minimum from the centre of the diffraction pattern of the slit.
2. A helium-neon laser emits a diffraction-limited beam (λ = 6300 Å) of diameter 2 mm. What diameter of light patch would the beam produce on the surface of the moon at a distance of 376 x 10^3 km from the earth? You may neglect scattering in earth's atmosphere.

9.6 SOLUTIONS AND ANSWERS

SAQs

1. We know that angular spread of the central maximum is from

$$\theta = \sin^{-1}\left(\frac{\lambda}{b}\right) \text{ to } \theta = -\sin^{-1}\left(\frac{\lambda}{b}\right).$$

For small θ, we have $\sin \theta \approx \theta$ and we find that principal maximum is spread from $\theta = \frac{\lambda}{b}$ to $\theta = -\frac{\lambda}{b}$.

Similarly, you can show that the first secondary maximum on the positive side extends from $\theta = \frac{h}{b}$ to $\theta = \frac{2h}{b}$ and on the negative side from $\theta = -\frac{h}{b}$ to $\theta = -\frac{2\lambda}{b}$

Thus we see that the central maximum is twice as wide as a secondary maxima.

2. Wc know that

$$b \sin \theta_1 = h$$

$\therefore$ $(0.3 \times 10^{-'}cm) \sin\theta_1 = 4358 \times 10^{-8} cm$

or

$$\sin\theta_1 = 1.45 \times 10^{-3}$$

In the small anglc approximation we can take

$$\theta_1 = 1.45 \times 10^{-3} \text{ rad}$$

and

$$\theta_2 = 2.90 \times 10^{-3} \text{ rad}$$

On comparing these values with those given in Example 1 for the first and second minima, you will note that violet light is diffracted about 27% Icss.

TQs

1. From Eq. (9.9) we know that the conditions for minima are given by

$$b \sin\theta = n\lambda \qquad n = \pm 1, \pm 2, ...$$

Hcrc $b = 0.03 mm = 3 \times 10^{-3} cm$, $n = 3$ and $\lambda = 5500$ Å

$$\therefore \quad \sin\theta = \frac{nh}{b} = \frac{3 \times (5500 \times 10^{-8} cm)}{3 \times 10^{-3} cm} = 5.5 \times 10^{-2}$$

In the small anglc approximation, $\sin\theta \cong \theta \cong \tan\theta$

$$\therefore \quad x = 5.5 \times 10^{-2} \times (100 cm)$$

$$= 5.5 cm$$

2. Suppose that the light patch on the Moon is taken to bc Airy disc of diameter x of a diffraction limited beam of initial diameter 2 mm. Then using Eq. (9.13) wc can write

$$\sin\theta = \frac{1.22\lambda}{D} = \frac{1.22 \times (6300 \times 10^{-8} cm)}{(0.2 cm)}$$

$$= 384.3 \times 10^{-6}$$

In the small angle approximation, $\sin e \cong \theta = 384.3 \times 10^{-6}$ rad. Since $x = 2r\theta$, we find on substituting the numerical values that

$$x = 2 \times (376 \times 10^3 km) \times (384.3 \times 10^{-6})$$

$$= 289 km$$