# UNIT 2 GAUSS'S LAW

## Structure



## 2 . INTRODUCTION

Michael Faraday carried out an interesting experiment. A large metallic box was mounted on insulated supports. Faraday went into the box and got the box charged with a powerful electrostatic generator. He was completely safe. Can you believe? No. But it is true. Let us find out the reason for this strange phenomenon. Well, after going through this unit, you will be able to find out the reason for it yourself because the explanation of this phenomenon is facilitated by the Gauss's law, which we discuss in this unit.

Gauss's law is a consequence of Coulomb's law, so though it contains no additional information, its mathematical form enables us to solve many problems of electric field calculation far more conveniently than through the use of Coulomb's law. In the preceding unit, you learnt that electric field at any point is given by the force experienced by a unit positive charge placed at that point. In this unit, we will develop the concept of flux of an electric field and then arrive at the Gauss's law. We will also see how this law allows us to calculate the electric field far more easily than we could using Coulomb's law.

In mechanics, in addition to the concept of force, we introduce the concepts of work and energy, Similarly, in electrostatic phenomena, we would discuss notions of work and energy. For this purpose, we need to discuss the concept of electric potential which provides a link between the concepts of electric field, work and energy. Therefore, the next unit deals with electric potential.

### Objectives

After studying this unit, you should be able to:

- appraise yourself that the number of lines of force crossing a closed surface is proportional to the net charge enclosed by that surface,
- relate the electric flux through any surface to: (i) the field strength, (ii) surface area, and (iii) orientation of surface relative to the field,

- write the relation between the electric flux and the charge enclosed within the surface,
- compute the electric flux through any closed surface placed in the electric field,
- use the Gauss's law to compute electric fields in case of spherical, linear and planar symmetry,
- discover how the Gauss's law enables the calculation of the electric field due to an infinitely long charged wire in a much easier way than the method of the preceding unit,
- explain that if a conductor carries, a net charge, this charge must be distributed over its surface and there is no electric field inside a closed metallic box, and
- show that Gauss's law can be written in terms of divergence of an electric field.

## 2.2 WHAT GAUSS'S LAW IS ALE ABOUT

Gauss's law expresses the relation between an electric charge and the electric field that it sets up. In the preceding unit, you learnt that the electric field of any charge distribution extends throughout space. You also saw how the electric field is visually represented by means of electric lines of force. It might strike you that there exists some relation between the charge and the number of lines of force. Do you remember that in Unit 1, we said that "number of lines of force" is a vague term because we can draw as many lines of force as we want? Here we shall define a quantity called flux of the electric field which is a very precise mathematical notion and can be physically thought of in terms of lines of forces. So, while the concept of lines of force is useful for picturising the electric field, the notion of electric flux is very useful for reformulating the laws of electric field.

### 2.2.1 Counting Lines of Force

Before proceeding, let us do some analytical exercises. Fig. 2.1 shows some charge distributions.

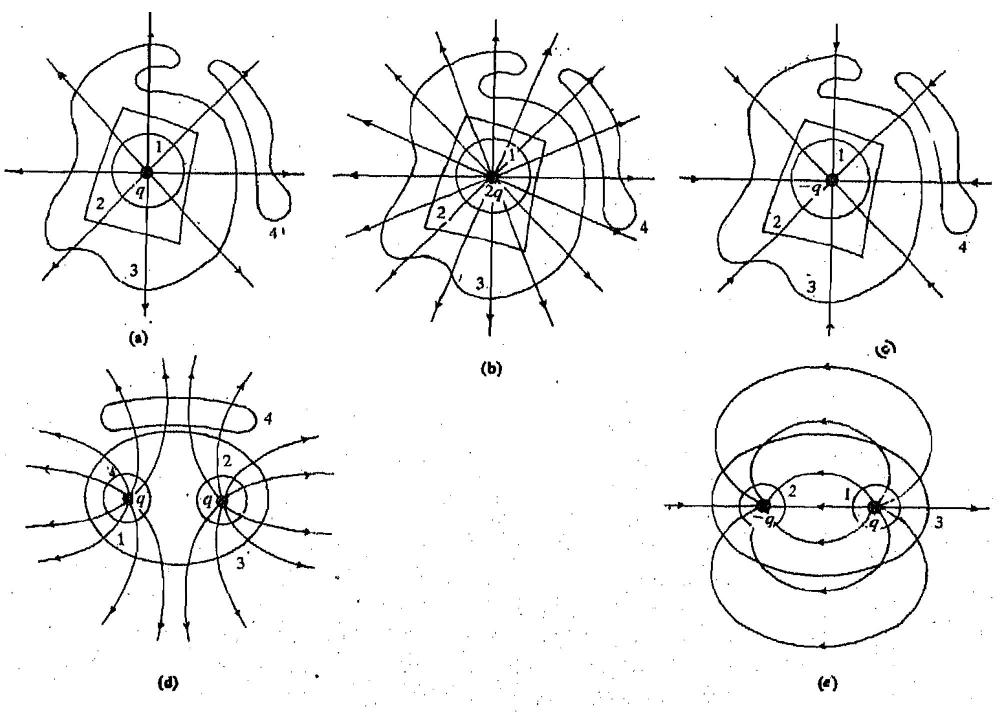



**Fig. 2.1: In all cases, the number of lines of force emerging from a closed surface is proportional to the net charge enclosed.**

**Those surfaces are closed for which there can be a clear distinction between points that are inside the surface, on the surface and outside the surface. In other words, a closed surface has an enclosed volume. For. such a surface, you can tell its inside from its outside; there is no ambiguity.**

For each distribution, a number of closed surfaces are indicated by coloured lines. Count the lines of force and say how many lines of force cross each surface. Adopt the convention that a line of force crossing the surface from inside to outside is counted as positive and the one crossing from outside to inside as negative. Further, we draw by convention 8 lines emerging from charge q, 16 from $2q$ and so on.

Consider Fig. **2.1a.** For surfaces 1 and 2, the answer is eight. Surface 3 is a little ambiguous but the number of lines of force crossing this surface is also eight. This is because one line of force (shown bold) which crosses the surface three times actually does so twice while going out and once while going in for a net gain of one crossing. In fact, any closed surface you might draw that encloses the charge $q$ would have **eight** lines of force crossing it. For surface 4, two lines of force cross it while going in and two while going out, making zero net crossings. So, no lines of force cross surface 4. Although, surface 4 lies in the field of the charge $q$ but q is not inside the surface.

Figure **2.1b** is identical except that now the surfaces 1, 2 and 3 enclose the charge $2q$ and hence sixteen lines of force cross these surfaces. Surface 4, which does not enclose the charge, still has zero net crossings. Figure **2.1c** is similar-to Fig. **2.1a** except for the sign of the charge. So, –8 lines of force cross the surfaces enclosing the charge $-q$.

In Fig. **2.1d,** surfaces 1 and 2 each enclose the charge q and hence eight lines of force cross each of these surfaces. But how many lines of force cross surface 3. Your answer should be sixteen. See what is the total charge enclosed by surface 3. The charge is $q+q = 2q$ which is expected. The surface 4 has zero net lines of force crossings. The answer is obvious because it encloses no charge.

Let us see Fig. **2.1e,** which shows two charges **equal** in magnitude but opposite in sign. Eight lines of force cross the surface 1. It is obvious because it encloses charge $q$. Similarly, surface 2 encloses $-q$ and has $-8$ lines of force crossing it. Surface 3 is interesting. It encloses both charges q and $-q$. The net enclosed charge is zero. Count the lines of force. As many go out as come in, so the total number of **lines** of force crossing the surface is zero.

This simple exercise of counting lines of force crossing the surfaces for various distributions tells you a simple statement about the electric field: **The number of lines-of force crossing a closed surface is proportional to the net charge enclosed by that surface.**

**SAQ 1**

What would happen if we were to bring an enormous charge Q close to the surface 3 in Fig. **2.1e?** How many lines of force will cross the surfaces 1, 2 and **3?**

To describe rigorously this very interesting observation about lines of force crossing a surface, we will develop a new concept, that of the electric flux.

### 2.2.2 Electric Flux

In the last subsection, we considered closed surface. Let us now consider an open surface such as a flat sheet of area $S$ placed in a uniform electric field E which is represented, say, by means of four lines of force as shown in Fig. **2.2a.**

Now if we have another uniform electric field $2E$, then such electric field will be represented by eight lines of force according to our earlier convention. Now bring the same flat sheet into this field (shown in Fig. **2.2b)** and count the lines of force crossing this area. Answer will be eight. In **Fig. 2.2c,** the field is the same as in Fig. **2.2b,** but the **area** of the sheet is halved and so the number of lines of force crossing the sheet is **halved.** If the flat sheet of Fig. **2.2b** is turned to position shown in Fig. **2.2d,** then it remains no longer perpendicular to the lines of force and hence the number of lines of force crossing the sheet is reduced.

We see that the number of lines of force crossing any surface depends on three things: Field strength E, surface area $S$ and orientation of surface relative to field.

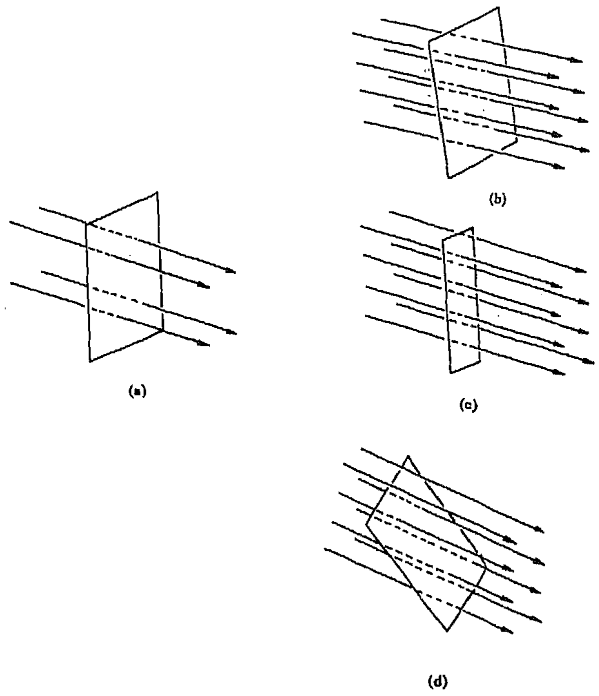



Fig. 2.2: (a) The number of lines of force crossing a flat surface depends on, (b) the field strength, (c) the surface area, and (d) the orientation of the surface relative to the field direction.

The surfaces shown in Fig. 2.2 are open surfaces because they do not define an enclosed volume.

To specify the orientation of the surface, we draw a perpendicular to the surface. If $\theta$ is the angle between the electric field and the perpendicular as shown in Fig. 2.3, then the number of lines of force passing through the surface range from **maximum** to minimum depending on $\theta$. That is,

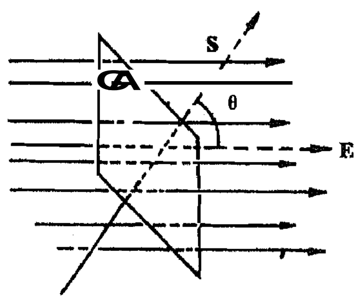



Fig. 2.3: The number of lines of force crossing the area is proportional to cos $\theta$, where $\theta$ is the angle between the field and the perpendicular to the surface.

when $\theta = 0°$ ; lines of force crossing the surface is maximum

when $\theta = 90°$ ; lines of force crossing the surface is zero.

Clearly the number of lines of force crossing a surface is proportional to the projection of the field on to the perpendicular to the surface, **i.e.,** cosine of $\theta$. [Note $\cos 0° = 1$ and $\cos 90° = 0$]. Putting together the three quantities on which the number of lines of force depends gives

number of lines of force crossing a surface $\propto ES\cos\theta$.

By including $\cos\theta$ in the dot product you can write

number of lines of force crossing a surface $\propto \mathbf{E}\cdot\mathbf{S}$ **...(2.1)**

where **E** is the electric field vector and S is a vector whose magnitude is equal to the area of the surface and whose direction is that of the perpendicular to the surface.

The quantity on the left side of Eq. (2.1) is a vague term because we can draw as many lines of force as we like. But the quantity on the right side of the equation has a definite value and is called **electric flux** denoted by $\Phi$. Hence,

$$\Phi = \mathbf{E}\cdot\mathbf{S} \qquad ...(2.2)$$

**Eq.** (2.2) shows that **flux** being the scalar product of two vectors, is itself a scalar quantity. Since E is measured in $NC^{-1}$, SI unit of flux is $Nm^2C^{-1}$.

Dot (scalar) product of two vector quantities A and B is given by $\mathbf{A}.\mathbf{B} = AB\cos\theta$ where $\theta$ is the angle between the vectors A and B.

To find the total **flux** through any closed surface placed in a **non-uniform** electric field as shown in Fig. 2.4, we divide the surface into many small patches so that the electric field is nearly uniform over each patch. Then from Eq. **(2.2),** the **flux** $d\Phi$ through each patch will be

$$d\Phi = \mathbf{E}\cdot d\mathbf{S} \qquad ...(2.3)$$

where **E** is the electric field at the patch and $d\mathbf{S}$ is a vector whose magnitude is the area $dS$ and the direction is that of the **outward drawn normal** to surface.

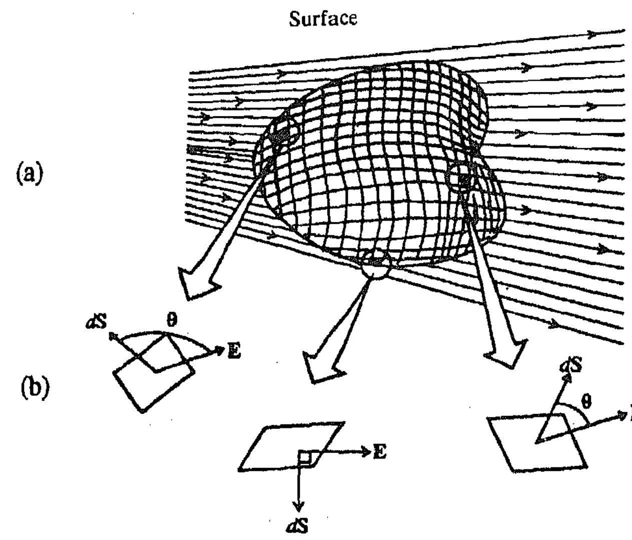



Fig. 2.4: (a) A closed surface of arbitrary shape immersed in an electric field. Its surface is divided into small patches of area $dS$. (b) The electric field vectors E and the area vectors $d\mathbf{S}$ for three representative patches marked x, y and z.

Total flux through the surface will be obtained by adding the fluxes through all the patches. Suppose each patch becomes infinitesimally small and the number of such patches is arbitrarily large, then the sum becomes a surface integral. Hence, the total flux is given by:

$$\Phi = \iint \mathbf{E} \cdot d\mathbf{S}$$

If the surface is closed, one often indicates this in the integral as

$$\Phi = \oiint \mathbf{E} \cdot d\mathbf{S} = \oint \mathbf{E} \cdot d\mathbf{S} \qquad ..(2.4)$$

**In order to simplify the writing** of integrals, surface integrals are written only with one integral sign. It will be understood that **when the integral is taken over an area $dS$, the surface integral** or **double** integral is **implied.**

The circle on the integral sign indicates that the integration is to be taken over the entire (closed) surface. Before explaining **how** the above expression leads to the Gauss's law, **you** would like to know the meaning of the surface integral.

A **surface integral** of **any** vector function $\mathbf{F}$, over a surface $S$ means **just** this: Divide $S$ into **small patches** (surface elements). Each **patch** is a **vector quantity,** represented by a **vector pointing** towards outkard **normal** and having **a** magnitude equal to the patch area. At every patch, take the scalar product of the patch area vector and the vector function F. Sum all **these** products. The limit of this sum, as the patches shrink, is the surface integral.

Let us see how Eq. (2.4) is used to find out the total flux through any closed surface.

**Example 1**

Fig. 2.5 shows a closed surface S in the form of a cylinder of radius R **immersed** in a uniform electric field $\mathbf{F}$, the cylinder axis being parallel to the field. What is the flux $\Phi$ of the electric field through this closed surface?

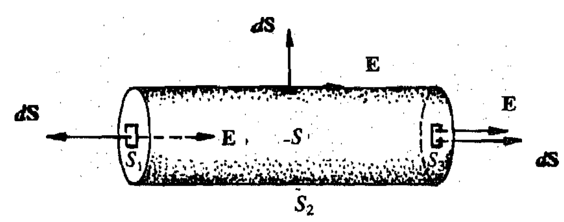



Fig. 2.5: A cylindrical surface, closed by **end** caps, is immersed in a uniform electric field. The cylinder **axis** is parallel to the field direction.

**Solution**

We can write the total electric flux through the surface S as the sum of three terms, an integral over the surfaces: $S_1$, i.e., the left cylinder cap, $S_2$, i.e., the cylindrical surface, and $S_3$ the right cap. Thus, from Eq. (2.4) we have

$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{S}$$

$$= \int_{S_1} \mathbf{E} \cdot d\mathbf{S} + \int_{S_2} \mathbf{E} \cdot d\mathbf{S} + \int_{S_3} \mathbf{E} \cdot d\mathbf{S}$$

For the left cap, angle $\theta$ for all points is $180°$, E is constant, and all the vectors $d\mathbf{S}$ are parallel. Thus,

$$\int_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int E (\cos 180°)\, dS = -E \int dS = -\pi E R^2$$

because $\pi R^2$ is the cap area. Similarly, for the right cap,

$$\int_{S_3} \mathbf{E} \cdot d\mathbf{S} = +\pi E R^2$$

the angle $\theta$ for all points being zero there. Finally, for the surface $S_2$,

$$\int_{S_2} \mathbf{E} \cdot d\mathbf{S} = 0,$$

the angle $\theta$ being $90°$ for all points on the cylindrical surface. Total flux through the cylindrical surface $S$ becomes

$$-\pi E R^2 + 0 + \pi E R^2 = 0.$$

Therefore, the net outward flux of the electric field through this closed surface is zero.

## 2.3 GAUSS'S LAW

'In the last sectioq, we found two simple results: (i) the number of lines of force crossing through any closed surface is proportional to the net charge enclosed by that surface, and (ii) the concept of flux which quantifies the physical notion of lines of force crossing a surface. The net result is: The electric flux through any closed surface is proportional to the net charge enclosed by that surface. Mathematically,

$$\Phi \propto q_{enclosed} \qquad \text{...(2.5)}$$

or

$$\Phi = \int \mathbf{E} \cdot d\mathbf{S} \propto q_{enclosed} \qquad \text{...(2.6)}$$

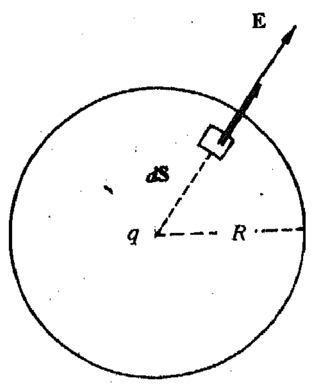



Fig. 2.6: The electric field of a point charge q has the same magnitude over a spherical surface centered on the charge and is everywhere perpendicular to the surface.

To evaluate the proportionality constant in Eq. (2.5) or (2.6), consider a positive point charge q placed in free space and a spherical surface of radius R centered on q (Fig. 2.6). The flux through any surface is given by Eq. (2.4), i.e.,

$$\Phi = \int \mathbf{E} \cdot d\mathbf{S} = \int E\,dS \cos\theta, \qquad \text{...(2.7)}$$

where $\theta$ is the angle between the direction of the electric field and the outward drawn normal to the surface. You have seen in Unit 1 that the magnitude of the electric field at a distance $R$ due to a point charge q is given by

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2}$$

where $\epsilon_0$ is permittivity of free space.
The field points radially outward so that the electric field is everywhere parallel to the outward drawn normal to the surface.

Then 8 = 0, so that $\cos\theta = 1$. Putting the values of E and cos8 in Eq. **(2.7), the** flux through the spherical surface of radius $R$ becomes

$$\Phi = \int_{\text{sphere}} \frac{1}{4\pi\epsilon_0} \frac{q}{R^2}\, dS = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \int dS,$$

**The** expression. for the magnitude of electric field has been taken outside the integral sign because it **has** the same value (or in other words it is constant) everywhere on the spherical surface. The remaining integral is just the sum of the areas of all the infinitesimal elements, $dS$, on the surface of the sphere – in other words the remaining integral is the surface area of the sphere, **i.e.,** $4\pi R^2$. Then the **flux** becomes

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} 4\pi R^2 = \frac{q}{\epsilon_0} \qquad \text{...(2.8)}$$

Comparison of Eq. (2.8) and Eq. (2.5) shows **that** the proportionality constant is $1/\epsilon_0$. The value of $\epsilon_0$ is $8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2}$. So Eq. (2.6) becomes

$$\boxed{\Phi = \int \mathbf{E} \cdot d\mathbf{S} = \frac{q_{\text{enclosed}}}{\epsilon_0}} \qquad \text{...(2.9)}$$

This is known as **Gauss's** law. It tells us Chat the! electric **flux** through the sphere is propoftional to the charge and independent of the radius of the surface. In order to prove Eq. (2.9) for any arbitrary closed surface, we will first define what is meant by solid angle.

A solid **angle** is the space included inside a conical surface, as shown in Fig. **2.7a.** Its value is expressed in steradians (abbreviated sr). Its value is obtained by drawing, with arbitrary radius R and centre at the vertex $O$, a spherical surface and applying the relation

$$\Omega = \frac{S}{R^2}$$

where $S$ is the area of the spherical cap intercepted by the solid angle. Since the surface area of a sphere is $4\pi R^2$, we **conclude** that the complete solid angle around a point' is $4\pi$ steradians.

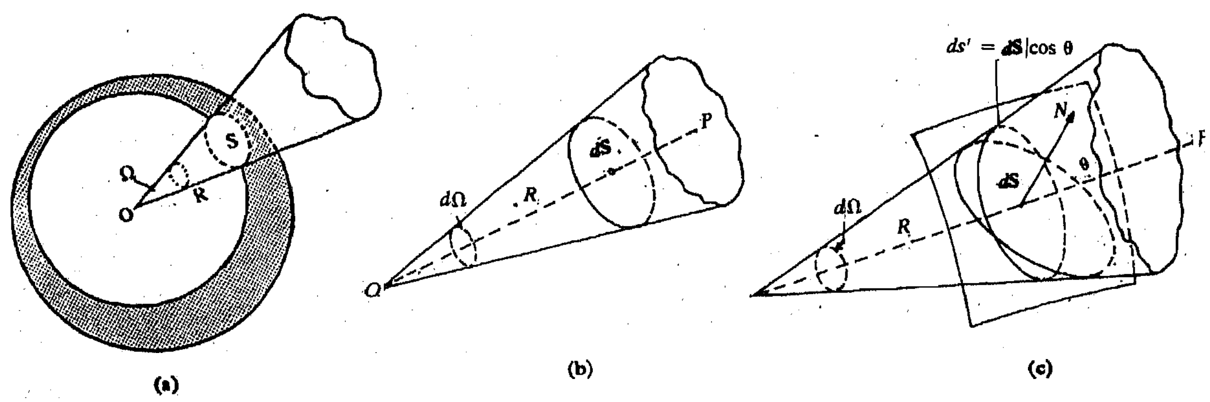



**Fig. 2.7: Solid angle**

When the solid angle is **small** (Fig. **2.7b),** the surface area S becomes $dS$ and is not necessarily a spherical cap, but may be a small plane surface perpendicular to OP so that

$$d\Omega = \frac{dS}{R^2}$$

In some instances, the surface $dS$ is not perpendicular to $OP$, but its normal N makes an angle 8 with OP as shown in Fig. **2.7c.** Then it is necessary to project $dS$ on a plane perpendicular to OP, which gives us the area $dS' = dS\cos\theta$. Thus

$$d\Omega = \frac{dS\cos\theta}{R^2}$$

.The definition of solid angle puts no limitation on the shape of the cone. What matters is the area of the projected surface area, not the shape of the projected area.

Now let us consider a charge q inside an arbitrary closed surface S as shown in Fig. 2.8. The electric field E at every point, of the surface is directed radially outward from the charge. Let us consider any sufficiently small area $dS$ on the surface for which E can be considered to have same magnitude and direction. If $\theta$ be the angle between E and the outward normal to the surface $d\mathbf{S}$, then according to Eq. (2.3), the flux $d\Phi$ through the area $d\mathbf{S}$ is given by

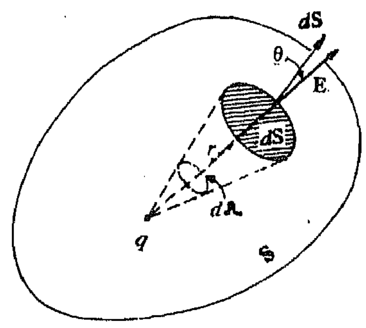



**Fig. 2.8: The electric** flux through a closed surface surrounding a charge is independent of the shape of the surface.

$$d\Phi = \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0\, r^2}\cos\theta\, dS$$

Here $dS\cos\theta/r^2$ is the solid angle $d\Omega$ subtended by the surface element $dS$ as viewed from the charge q. Hence,

$$d\Phi = \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0}\, d\Omega$$

To obtain the total flux through the surface S, integration is done over the entire ,closed surface as follows:

$$\oint d\Phi = \oint \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0}\oint d\Omega$$

**The total** solid angle around any point is $4\pi$. **Therefore,**

$$\Phi = \oint \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0}\cdot 4\pi = \frac{q}{\epsilon_0}$$

This result is the same as the previous result for a spherical surface concentric with the charge. Thus, the relation expressed by Eq. (2.9) is valid for any closed surface irrespective of the position of the charge within the surface,

SAQ 2

**Can** we use Gauss's law for the **surface** shown in Fig. **2.2?** Give reasons.

**Note** that q in Eq. (2.9) is the net charge, taking its algebraic sign into account. If a surface encloses equal and opposite charges, the flux is zero. Charge outside the surface makes no contribution to the value of $q$. However, E on left side of Eq. (2.9) is the electric field resulting from all charges, both those inside and outside the surface.

**SAQ 3**

Fig. 2.9 shows three objects each carrying an electric charge and a coin carrying no charge. The cross-sections of two surfaces $S_1$ and $S_2$ are indicated. What is the flux of the electric field through each of these surfaces? Assume $q_1 = +3.1$ **nC**, $q_2 = -5.9$ **nC** and $q_3 = -3.1$ nC.

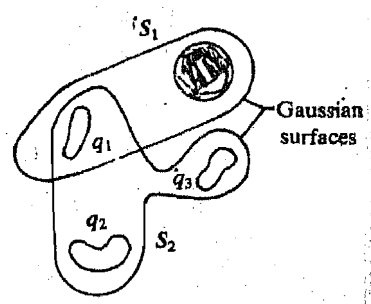



**Fig.** 2.9: Three objects each **carrying** an electric charge; and a coin which carries no charge.

Instead of assuming point charges to be **contained** within the closed surface, let us assume that the charge is continuously distributed throughout the volume enclosed by the surface. Let the chargedensity be $\rho$. If $dV$ be a small volume element then the charge contained within this volume **element** will be $\rho\, dV$. Therefore, total charge enclosed within the entire volume is $\iiint \rho\, dV$. Then Gauss's law is expressed as:

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\iiint \rho\, dV$$

For simplicity, also written as

$$\oint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho\, dV \qquad \ldots (2.10)$$

Instead of **writing** multiple integral signs, for simplicity, integration is represented by **one integral sign.** When the integral is taken over a volume $dV$, **it** will **be implied as volume Integral or triple integral.**

where $\int_V$ indicates volume integral through the region enclosed within the closed surface. Eq. (2.10) is an integral form of the statement of Gauss's law.

Now you would like to know the meaning of volume **integral.** Let the region enclosed within the closed surface be ***V***. It is a region in three dimensional space. Let $\rho$ be a quantity, say, charge which is defined for unit volume. Divide the region $V$ into $n$ elementary volumes $dV_1, dV_2, \ldots, dV_n$. Then the product of the quantities $\rho$ and $dV_1$ will give the charge enclosed within elementary volume $dV_1$. Sum all these products, **i.e.,**

$$\sum_{r=1}^{n} \rho \, dV_r$$

Then the limit of this sum, as $n$ tends to infinity and the dimensions of each sub-division tend to zero, is called the volume integral of $\rho$ over the region $V$. It represents the total charge enclosed within the entire region $V$ and is denoted by $\iiint \rho \, dV$.

The solution of Eq. (2.10) is often difficult to perform mathematically, although the physical meaning of the law in this form is more comprehensive. A differential form of Gauss's law is useful for the solutibn of many problems. We shall arrive at this form of Gauss's law at a later stage in Section 2.5. Now let us see how Gauss's law is useful in determining field distribution about objects having symmetrical **geometry**. We shall discuss a few such cases in the next section.

## 2.4 GAUSS'S LAW – SOME APPLICATIONS

Gauss's law applies to any hypothetical closed surface (called a **Gaussian surface)** and enclosing any charge distribution. However, evaluation of surface integral becomes simple only when the charge distribution has sufficient symmetry. In such situation, Gauss's law allows us to calculate the electric field far more easily than we could using Coulomb's law. Since Gauss's law is valid for an arbitrary closed **surface,** one uses this freedom to choose a surface having the same symmetry as that of charge distribution to evaluate the surface integral. We will illustrate the use of Gauss's law for three important symmetries.

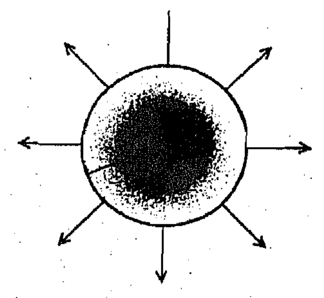


**Fig. 2.10: For a spherically symmetric, charge distribution. electric field vectors at a given radius, all have the same magnitude and point in the radial direction.**

### 2.4.1 Spherical Symmetry

A charge distribution is spherically symmetric if the charge density (that is, the charge per unit volume) at any point depends only on the distance of the point from a central point (also called centre of symmetry) and not on the direction. Fig. 2.10 represents a spherically symmetric distribution of charge such that the charge density is high at the centre and zero beyond r. Spherical symmetry of charge distribution implies that the magnitude of electric field also depends on the distance r from the centre of **symmetry.** In such situation, the only possible direction of the field consistent with the symmetry is the radial direction–outward for a positive charge (Fig. 2.10) and inward for a negative charge. The examples of spherically symmetric charge distributions are: (i) a point charge, (ii) a uniformly charged sphere, and **(iii)** a **uniformly** charged thin spherical shell.

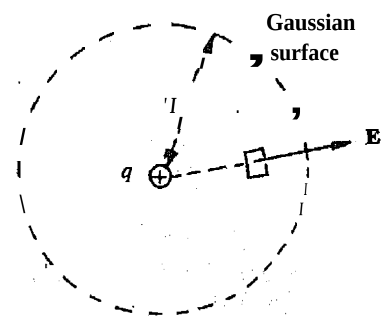



**Fig. 2.11: A spherical Gaussian surface centered on a point charge.**

(i) E of a point charge

Fig. 2.11 shows a positive point charge q. Using Gauss's law, let us find out electric field at a distance of $r$ from the charge q. Draw a **concentric** spherical Gaussian surface of radius $r$. We **know** from symmetry that E points radially outward. If we divide the Gaussian **surface** into differential areas $dS$ then, both $E$ and $d\mathbf{S}$ will be at right angles to the **surface,** the angle $\theta$ between them being zero. Thus, the quantity $\mathbf{E} \cdot d\mathbf{S}$ becomes **simply** E $dS$ and Gauss's law (see Eq. 2.9) becomes

$$\epsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon_0 \oint E \, dS = q.$$

Because E has the same magnitude for all points on the Gaussian surface, we can factor it out of the integral leaving

$$\epsilon_0 E \oint dS = q. \qquad \ldots(2.11)$$

However, the integral in Eq. (2.11) is just the area of the spherical surface, **i.e.,** $4\pi r^2$, so that the equation becomes

$$\epsilon_0 E(4\pi r^2) = q$$

or

$$E = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \qquad \ldots(2.12)$$

which is Coulomb's law in the form in which we have written in Unit 1. This shows that Gauss's law and Coulomb's, law are not two independent physical laws but the same law expressed in different ways.

(ii) E of a spherical charge distribution

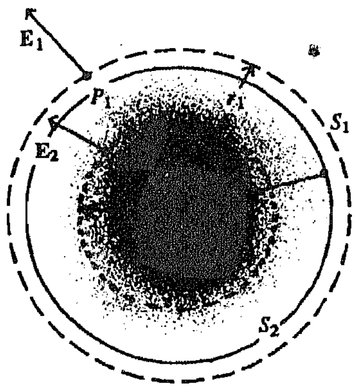



**Fig. 2.12: The electric field of a spherical charge distribution. The surface $S_1$ encloses the entire charge Q; while surface $S_2$ encloses only some of the charge.**

Suppose a total charge Q is spread uniformly throughout a sphere of radius $R$ as shown in Fig. 2.12. Let us find the electric field at some point such as $P_1$ outside the distribution and at point $P_2$ inside it. If you use Coulomb's law to find out the field, you have to carry out an integration which would sum the electric field vectors at $P_1$ arising from each elementary volume in the charge distribution. Let's try a different method using Gauss's law.

(a) Field **for points** outside the charge distribution: Let us draw an imaginary Gaussian surface $S_1$ of radius $r_1$ through the point $P_1$, where we wish to find electric field. Let the magnitude of the field be denoted by $E_1$. Because of the spherical symmetry, the electric field is same at all points on this Gaussian surface. Also at any point on the Gaussian surface, the field is radially directed, **i.e.,** perpendicular to the surface so that $\cos\theta = 1$. (Here it is assumed that the sphere has net positive charge; if there is net negative charge, the field will point radially inward and $\cos\theta = -1$). Then the flux through this Gaussian sphere $S_1$ becomes

$$\Phi = \oint \mathbf{E}_1 \cdot d\mathbf{S}_1 = \oint E_1 \cos\theta \, dS_1$$

$$= E_1 \oint dS_1 \qquad (\cos\theta = 1)$$

$$= 4\pi r_1^2 E_1 \qquad \ldots(2.13)$$

because $\oint dS_1$ is just the surface area of the sphere $S_1$, **i.e.,** $4\pi r_1^2$.

According to Gauss's law, the flux through the sphere $S_1$ is given by $\frac{q}{\epsilon_0}$, where q is the net charge **enclosed** by the sphere $S_1$. Equating the **flux in** Eq. **(2.13)** to $\frac{Q}{\epsilon_0}$ gives

$$4\pi r_1^2 E_1 = \frac{Q}{\epsilon_0} \quad \text{(because charge enclosed within the sphere } S_1 \text{ is Q)}$$

so that

$$E_1 = \frac{1}{4\pi \epsilon_0} \frac{Q}{r_1^2} \qquad \ldots(2.14)$$

This shows that the **field** at all points on surface $S_1$ is the same as if **all** the **charges** within the surface $S_1$ were concentrated at the centre.

(b) Field for points inside the charge distribution: The field inside the charge distribution depends on how charge is distributed, This is **because any Gaussian** sphere with $r < R$, such as surface $S_2$ of Fig. 2.12, does not enclose the entire charge Q. The charge enclosed depends on the charge distribution. Suppose, a Gaussian sphere $S_2$ of radius $r_2$ is drawn passing through the **point** $P_2$, where we wish to find the electric field. Let the field be denoted by $E_2$. Inside the sphere $S_2$ Eq. (2.9) **for** the **flux** still holds, but now the oharge enclosed is some fraction of

Q. The volume of the charged sphere is $\frac{4\pi}{3} R^3$ and it contains a total charge Q. Since charge is spread uniformly throughout the sphere, the volume charge density $\rho$ is constant and is given by:

$$\rho = \frac{Q}{\frac{4\pi}{3} R^3}.$$

Therefore, the charge enclosed by the sphere $S_2$ will be just the volume of that sphere multiplied by the volume charge density, that is,

$$q_{enclosed} = \frac{4\pi}{3} r_2^3 \times \frac{Q}{\frac{4\pi}{3} R^3} = Q \frac{r_2^3}{R^3}$$

From Eq. (2.9). we have

$$4\pi r_2^2 E_2 = Q \frac{r_2^3}{R^3}$$

so that

$$E_2 = \frac{1}{4\pi \epsilon_0} \frac{Q r_2}{R^3} \qquad \text{...(2.15)}$$

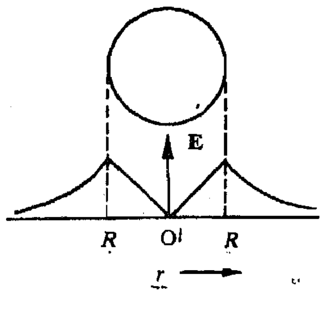



**Fig. 2.13: Field strength versus radial distance for a uniformly charged sphere of radius $R$. For $r > R$, the field has the inverse-square dependence of a point charge field.**

The electric field inside the charge distribution increases linearly with distance from the centre ($E \propto r$) whereas outside the charge distribution, the electric field falls off as $\frac{1}{r^2}$ as clear from Eq. *(2.14)*. Fig. *2.13* shows the combined results for the fields both inside and outside the sphere.

**(iii). Field** of a thin spherical shell

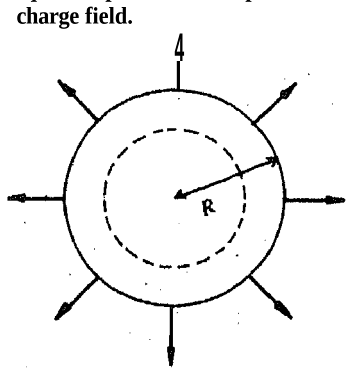



**Fig. 2.14: A charged spherical shell. Any Gaussian sphere inside the shell encloses zero net charge and the field inside is zero.**

Consider a thin spherical shell of radius $R$ carrying a total charge Q distributed uniformly over its surface as shown in Fig. *2.14.* Since this distribution is spherically symmetric, we already know that the electric field outside the shell is the point charge field of Eq. *2.14.* To find the electric field inside the shell, a Gaussian sphere is drawn inside the shell. The charge enclosed within this Gaussian surface is zero. Equating the flux from Eq. *(2.13)* to this zero enclosed charge gives:

$$4\pi r^2 E = 0 \ .$$

Since $r \neq 0$, the field is zero everywhere inside the shell. Remember that the zero field inside the shell did not follow only from the fact that the charge enclosed within it is zero; but it was possible because of the spherical symmetry which leads to Eq. (2.13). Now consider a spherical surface $S$ ground a dipole as shown in Fig. *2.15.* Here the ***net*** charge enclosed by the surface is zero but field within the surface is not zero. This is because here the charge distribution—a dipole—is not spherically symmetric, so that Eq. *(2.13)* cannot be obtained although the flux (**i.e.,** net number of lines of force) crossing the surface is zero.

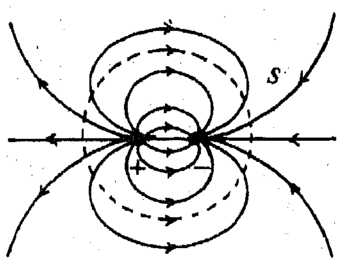



**Fig. 2.15: The net charge enclosed by the sphere is zero, but the field within the sphere is not zero. Here the charge distribution—a dipole—is not spherically symmetric, so that Eq. (2.13) is not a valid expression for the flux.**

The above examples have shown that, to calculate the electric field using Gauss's law, the following steps are needed:

*1)* Study the symmetry to see if you can construct a Gaussian surface on which the field magnitude and its direction relative to the surface are constant. If this is not possible, then Gauss's law, although true, will not provide a simple **calculation** of the field. Gaussian surface can be of any size or shape as long as it is closed.

2) Evaluate the flux, The choice of the Gaussian surface makes the term $E\cos\theta$ constant so that this term can come outside the flux integral leaving an integral equal to the **surface** area.

3) Evaluate the enclosed charge. If the **Gaussian** surface is within the charge distribution, then the enclosed charge will not be the same as total charge.

*4)* Equate the flux to $q_{enclosed}/\epsilon_0$ and solve E. The direction of E can be determined from the symmetry,

### 2.4.2 Line Symmetry

A charge distribution has the cylindrical symmetry when it is infinitely long and **has** a charge density that depends only on the perpendicular distance from a line called symmetry axis (Fig. 2.16). By symmetry, the electric field will point radially outward from the axis and its magnitude will depend only on perpendicular distance from the axis. (Here, we assume positive charge, for negative charge the field points inwards.) Let us find an expression for E at a distance $r$ from the line charge (say a wire).

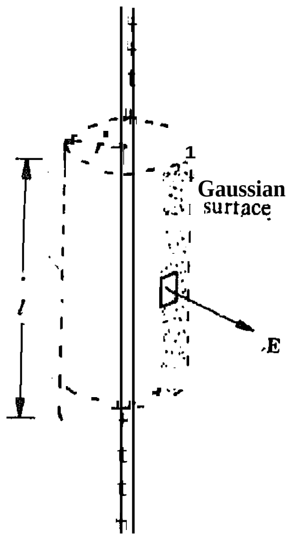



**Fig. 2.16: A cylinder of length $l$ and radius r is the Gaussian surface. It encloses a portion of an infinitely long line charge.**

Draw a Gaussian surface which is a circular cylinder of radius r and length $l$ closed at each end by plane caps normal to the axis as shown in Fig. 2.16. To calculate the flux, note that electric field also has the cylindrical symmetry, which implies its magnitude at a point depends only on the perpendicular distance of the point from the symmetry axis, and its direction has to be radially outwards. (You should try out other possibilities for field direction and convince yourself that there is only one possibility compatible with the cylindrical symmetry.) The flux through the cylindrical surface is

$$\Phi = \int \mathbf{E}\cdot d\mathbf{S} = \int E dS = E \int dS = 2\pi r l E \qquad \text{... (2.16)}$$

where $2\pi rl$ is the area of the curved surface. The flux through the end of the cylinder is zero because the field lines are parallel to the plane caps of the Gaussian surface. Mathematically, the vector **E** and $d\mathbf{S}$ are perpendicular, so that $\cos\theta = 0$ in the dot product E. $d\mathbf{S}$. Therefore, the only flux is through the curved part of the cylinder given by Eq. (2.16). Gauss's law tells **us** that the **flux** is proportional to the **charge** enclosed within the cylindrical Gaussian surface, **i.e.,**

$$2\pi r l E = \frac{q_{enclosed}}{\epsilon_0}.$$

so that
$$E = \frac{q_{enclosed}}{2\pi\, \epsilon_0 r l} \qquad \text{... (2.17)}$$

If the line charge density is A, then the charge enclosed by the Gaussian cylinder of length $l$ is $\lambda l$. Using this expression for $q_{enclosed}$ in Eq. (2.17) gives:

$$\boxed{E = \frac{\lambda l}{2\pi\, \epsilon_0 r l} = \frac{\lambda}{2\pi\, \epsilon_0 r}} \qquad \text{...(2.18)}$$

This is the same-result we found in Example 4 of Unit 1 through a tedious Coulomb's law method. Notice how much simpler is the Gauss's law.

In order to find the electric field inside the wire, we consider two cases:

i) Suppose the charge is distributed uniformly within the wire and charge density is $\rho$. Let the radius of the wire is R. To find $E$ at an inner point $P$, a distance r apart from the axis of the wire, draw a Gaussian cylinder of radius $r$ and length I passing through P as shown in Fig. 2.17. As explained earlier, the flux is due to the curved surface only. Hence, from Gauss's law

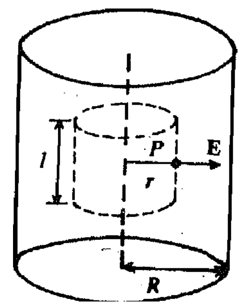



**Fig. 2.17: Enlarged view of the wire is shown to calculate the electric field at any point within it.**

$$\int \mathbf{E}\cdot d\mathbf{S} = E 2\pi r l = \frac{q'}{\epsilon_0}$$

The charge $q'$ inside this Gaussian surface $= \pi r^2 \rho l$.

$$E\, 2\pi r l = \frac{\pi r^2 \rho l}{\epsilon_0}$$

or
$$E = \frac{r\rho}{2\epsilon_0} \qquad \text{...(2.19)}$$

Thus, the electric field at a point inside an infinite uniformly charged wire is radially directed and varies as the distance from its axis.

ii) When the charge is on its surface only, the electric field at any point inside it is zero because the net charge in the Gaussian surface through this point is zero.

Eqs. **(2.18)** and **(2.19)** show that the electric field due to a charged wire or cylinder does not depend upon its radius. Hence, we can say that it is same as though the charge on the wire or cylinder were concentrated in a line along its axis. (Remember it while solving Terminal Question 4.)

### 2.4.3 Plane Symmetry

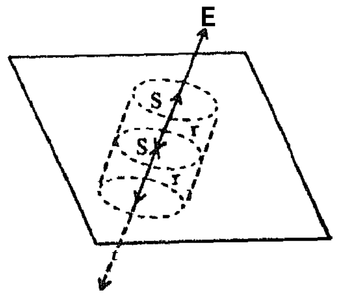



**Fig. 2.18: A charge distribution with plane symmetry showing the electric field. The area of the sheet enclosed by the Gaussian surface is the same as the area $S$ of its ends.**

When the charge density depends only on the perpendicular distance from a plane, the charge distribution is said to have plane symmetry. The electric field is everywhere normal to the plane sheet as shown in Fig. **2.18** pointing outward, if positively charged and inward, if negatively charged. To find the electric field at a distance $r$ in front of plane sheet, it is required to construct a Gaussian surface. A convenient Gaussian surface is a closed cylinder of cross-section area S and length 2r. The sides of Gaussian surface are perpendicular to the symmetry plane and the ends of the surface are parallel to it. Since no lines of force cross the sides, the flux through the sides is zero. But the lines of force cross perpendicular to the ends, so that E and the area element vector $d\mathbf{S}$ on the ends are parallel. The $\cos\theta$ in the product E, $d\mathbf{S}$ is 1 over both ends (–1 if charge is negative). Since the flux through the sides is zero, the total flux through our Gaussian surface then becomes

$$\Phi = \int_{\text{both ends}} E\, dS = 2ES.$$

The factor 2 arises because there are two ends. Then Gauss's law gives

$$2ES = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

If $\sigma$ is the surface charge density, then the charge enclosed is $\sigma S$, then

$$\boxed{E = \frac{\sigma}{2\epsilon_0}} \qquad \text{...(2.20)}$$

### 2.4.4 A Charged Isolated Conductor

Gauss's law permits us to prove an important theorem about isolated conductors.;

> **"If there are any unbalanced, static charges on a conductor, they must reside on the surface of the conductor."**

Consider a solid metallic conductor such as the one shown in Fig. **2.19a** carrying a charge $q$. The dotted line shows a Gaussian surface that lies just below the actual surface of the conductor.

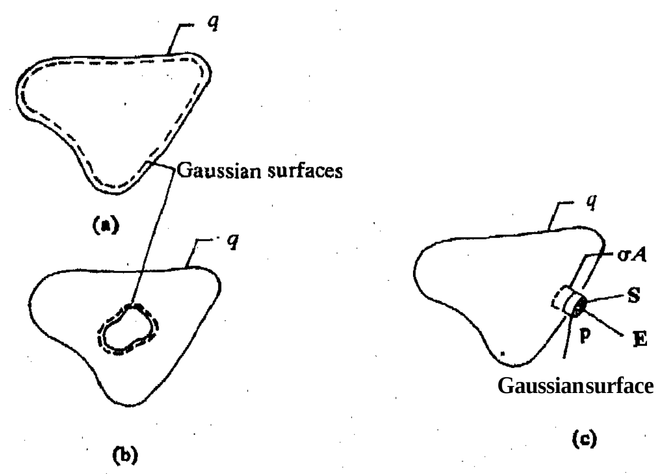



**Fig. 2.19: An insulated solid metallic conductor carrying a charge $q$. (a) A Gaussian surface is drawn within the metal, just below the actual surface. (b) The conductor has an internal cavity. A Gaussian surface lies within the metal close to the cavity wall. (c) A cylindrical Gaussian surface pierces the surface of 'the conductor. It contains a charge $\sigma S$.**

Before proving the theorem, you should realize that, inside the charged metal object, the electric field is zero everywhere. We can see why this must be true without a formal calculation. Suppose, there were an electric field in the interior of the object. Then the charges inside the object that are free to move (electrons in the case of metal) would do so under the influence of the field and internal' currents would be set up. But no currents are observed in a charged conducting object except for a short time after 'the charge is placed on it; since some energy is needed to maintain an electric current, a supply of energy would be needed for currents to continue in such an object. The only conclusion is that the interior'of an isolated conducting object is always free of electric field.

If electric field is zero everywhere inside the conductor, it must be zero for all points on the Gaussian surface because that surface, though close to the surface of the conductor, is definitely inside it. This means that the flux through the Gaussian surface must be zero. Then.according to Gauss's law, the charge inside the Gaussian surface must also be zero. It follows that if a net charge does reside on the body, it can be distributed only over the surface layer of that body.

Since the interior of the solid conductor contains no unbalanced charges, we could scoop out some of the material, leaving a hollow cavity as shown in Fig. **2.19b.** Draw a Gaussian surface surrounding the cavity but inside the conducting body as shown in Fig. **2.19b.** There can be no flux through this new Gaussian surface because $\mathbf{E} = 0$ inside the conductor. Therefore, from **Gauss's** law, that surface can enclose no net charge. We conclude that there is no charge on the cavity walls; it remains on the outer surface of the conductor as in Fig. **2.19a.**

This result has immense practical implications. It tells us that we can shield an object from the influence of electrostatic fields by simply enclosing it within a rnnductor sheath.

**SAQ 4**

Go through the introduction of this unit once again and find out the reason of the strange phenomenon mentioned in it.

Now you would like to know the electric field outside the charged conductor. Let us find out in the next paragraph.

Electric field near a charged conductor

Suppose, magnitude of the electric field at the point **P,** just outside the charged conductor, be E. As shown in Fig. **2.19c,** draw a cylindrical Gaussian surface such that the end caps are parallel to the surface, one lying entirely inside the conductor and the other entirely outside but very close to the surface. The area of its two end caps is $S$ and the point at which electric field is lo be required is within the cap. The cylindrical walls are perpendicular to the surface of the conductor. The direction of the electric field just outside a charged isolated conductor is perpendicular to the surface. This is because if E had a component parallel to the surface, electrons on the surface would be in constant motion. Because they are not so, E must be perpendicular to the surface. The flux through the exterior end cap of the Gaussian surface will be $ES$. The flux through the interior end cap is zero because $E = 0$ for all interior points of the conductor. The flux through the cylindrical walls is also zero because the direction of E is parallel to the surface, so they cannot pierce it. The charge enclosed by the Gaussian surface is $\sigma S$ where $\sigma$ is the surface charge density at the point near which we are to find electric field. Hence, after summing up, the total flux through the entire Gaussian surface is $ES$ and the charge enclosed by that surface is $\sigma S$. Gauss's law then gives

$$\epsilon_0 \Phi = q_{enclosed}$$

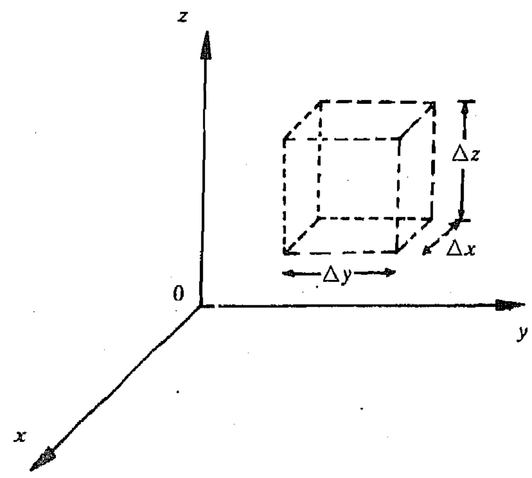
or $\epsilon_0 ES = \sigma S$

$$\therefore E = \frac{\sigma}{\epsilon_0} \quad ...(2.21)$$

At the end of the Section 2.3, we mentioned that Gauss's law can be stated in two forms: integral and differential. In that section, the Gauss's law was written in integral form. Before ending this unit, let us see how differential form is obtained.

## 2.5 DIFFERENTIAL FORM OF GAUSS'S LAW

Consider an infinitesimal element of volume AV with sides ***Ax,*** Ay and $\Delta z$ parallel to the axes of $x$, y and $z$ respectively as shown in Fig. 2.20. Let the electric field be $\mathbf{E}$ at the middle of this element and suppose it has components of magnitude $E_x$, $E_y$ and $E_z$ along the $x$, y and z-axes respectively.




Fig, 2.20 : An infinitesimal element of volume AV having sides $\Delta x$, Ay and $\Delta z$ parallel to thew, y and z-axes.

Consider the two faces of the volume element each with area $\Delta x\, \Delta z$ perpendicular to the y-axis. On the middle of the right-hand face of the volume element which is at a distance $\Delta y/2$ away from the middle of the element, the approximate value of electric field is given by

$$E_y + \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)$$

Since the face area is infinitesimally small, hence the above value **may** be taken as the value all over the face. **The** second term in the above expression is the space rate of change of $E_y$ as we move in the positive y-direction. For your convenience, we tell you how the above expression for electric field is obtained.

It is often necessary to find out what is the increment (or change) in any quantity when the variable on which it depends is changed by a small amount. This can be found at once by **Taylor's theorem:** In'differential calculus, you must have read about the Taylor's theorem.

If $f(x)$ is a quantity which depends on the variable $x$ and, if x changes by a small amount $dx$, then the quantity $f(x)$ will change to $f(x+dx)$ and, according to Taylor's theorem its value will be given by

$$f(x+dx) = f(x) + \frac{df(x)}{dx}(dx) + \frac{1}{2}\frac{d^2f(x)}{dx^2}(dx)^2 + \ldots.$$

If $dx$ is small enough, then second and higher powers of $dx$ can be neglected and we have

$$f(x+dx) = f(x) + \frac{df(x)}{dx}(dx)$$

**Thus** the change, $df(x)$, in the quantity $f(x)$ is given by :

$$df(x) = \frac{df(x)}{dx}(dx)$$

Thus, the derivate $\frac{df(x)}{dx}$ tells you how rapidly the function $f(x)$ varies when $x$ changes by a tiny amount $dx$. In other words, if $x$ changes by an amount $dx$, then $f(x)$ changes by an amount $df(x)$.

Replacing $f(x)$ by $E_y$ ; $\frac{df(x)}{dx}$ by $\frac{\partial E_y}{\partial y}$ and $dx$ by $\frac{\Delta y}{2}$.

The partial differential coefficient of $f(x, y)$ with respect to $x$ is the ordinary differential coefficient of $f(x, y)$ when $y$ is regarded as a constant. It is written as $\partial f/\partial x$.

We get the electric field at the right-hand face as:

$$E_y + \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)$$

Notice that, instead of ordinary differentiation, we have used **partial differentiation**, $\frac{\partial E_y}{\partial y}$. This is because $E_y$ is a function of $x$, y and $z$; and we have to differentiate $E_y$ with respect to only y while keeping $x$ and $z$ as constant.

It is hoped that you have understood the mathematical portion given above. Let us now move further.

The value of the electric field on the left-hand face of the volume element is

$$E_y + \frac{\partial E_y}{\partial y}\left(-\frac{\Delta y}{2}\right)$$

The flux through the right-haxid face of the element in the y-direction is given by

$$\left[E_y + \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\Delta x\,\Delta z.$$

And the flu through the left-hand face of the element in the y-direction is given by

$$\left[E_y - \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\Delta x\,\Delta z.$$

Hence, the net outward flux through these two faces of the element in the y-direction is given by :

$$\left[E_y + \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\Delta x\,\Delta z - \left[E_y - \frac{\partial E_y}{\partial y}\left(\frac{\Delta y}{2}\right)\right]\Delta x\,\Delta z.$$

$$= \frac{\partial E_y}{\partial y}\Delta y\,\Delta x\,\Delta z. \qquad ...(2.22)$$

Likewise, for the other two faces of the element perpendicular to x-direction, the net outward flux is:

$$\frac{\partial E_x}{\partial x}\Delta y\,\Delta x\,\Delta z. \qquad ...(2.23)$$

and for the top and bottom faces, the net outward flux is

$$\frac{\partial E_z}{\partial z}\Delta y\,\Delta x\,\Delta z. \qquad ...(2.24)$$

Therefore, total net outward electric flux through the volume element $\Delta V$ will be the sum of Eqs. (2.22), (2.23) and (2.24), i.e.,

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V \text{ (because } \Delta V = \Delta x\,\Delta y\,\Delta z) \qquad ...(2.25)$$

As already stated, a derivative of the type $\frac{df}{dx}$ is supposed to tell us how fast the quantity $f$ varies if the variable $x$ on which the quantity depends is changed by a small amount. Suppose $f$ is quantity depending on $x$, y and z, then according to the theory of partial derivatives

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz \qquad ...(2.26)$$

This rule tells us how f varies as we go a small distance $(dx, dy, dz)$ away from the point $(x, y, z)$. If $dx$, $dy$ and $dz$ are the components of a vector dl along $x$, $y$ and z-axes, then Eq. 2.26 is reminiscent of a dot product as follows:

$$df = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) \cdot (dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}})$$

$$= (\nabla f) \cdot (d\mathbf{l})$$

where $\nabla$ (called 'del') $= \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$ ..(2.27)

In the definition of $\nabla$ id Eq. (2.27), the unit vectors are written to the left instead of to the right just because one may not confuse with $\frac{\partial f}{\partial x}$ which would be zero, since $\hat{\mathbf{i}}$ is constant.

$\nabla$ is a vector operator which acts on a scalar or vector function via the dot product or cross product. Suppose we want to operate $\nabla$ on a vector function E via the dot product, then from the definition of $\nabla$, as given in Eq. (2.27), we have

$$\nabla \cdot \mathbf{E} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot (E_x\,\hat{\mathbf{i}} + E_y\,\hat{\mathbf{j}} + E_z\,\hat{\mathbf{k}})$$

$$= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \qquad ...(2.28)$$

$\nabla \cdot \mathbf{E}$ has been given the name divergence. It is also written as **div** E. The divergence of a vector function (here' it is electric field) is a scalar quantity and it measures how much the vector E spreads out (diverges) from the point in question. If the vector function E has a large (positive) divergence at any point P as shown in Fig. **2.21a,** then it is spreading out. (If the arrows pointed in, it would be a large negative divergence.) On the other hand, the electric field vector in Fig. **2.21b** has zero divergence at P, which means it is not spreading out at all.

Now using Eq. **(2.28),** we can write Eq. (2.25) as follows:

Total electric flux through volume element $\Delta V$ = (div **E**) $\Delta V$ ...(2.29)

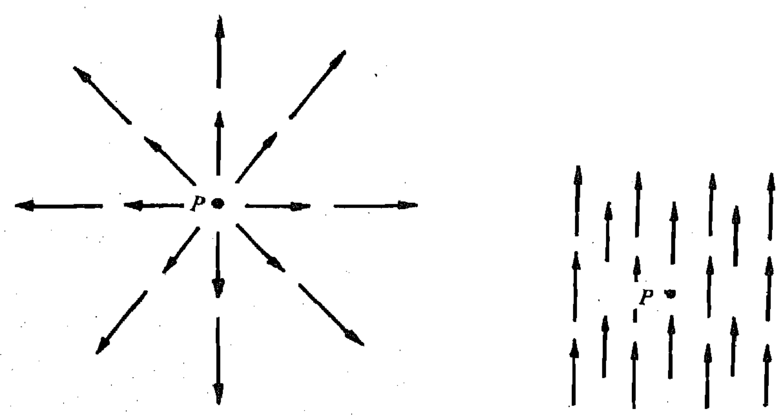



**Fig. 2.21 : (a) Positive divergence at P (*b*) Zero divergence at P.**

Thus, according to Eq. **(2.25),** the flux through **a** small volume is given by (div E) $\Delta V$. Using the original definition of the flux given in Eq. **(2.4),** we may write

$$\oint \mathbf{E} \cdot d\mathbf{S} = \Delta V \,\text{div}\, \mathbf{E} \qquad ...(2.30)$$

If $\rho$ is the 'volume charge density, then the total charge enclosed within the volume element $\Delta V$ will be given by:

$$q = \rho \Delta V \qquad ...(2.31)$$

Now, see Section 2.3 and write down Eq. (2.9)

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q_{enclosed}}{\epsilon_0}$$

Substitute the value of $\oint$ E . $d\mathbf{S}$ and $q_{enclosed}$ from Eqs. (2.30) and (2.31) respectively, we get:

$$\Delta V \text{ div } \mathbf{E} = \frac{\rho \Delta V}{\epsilon_0}$$

or

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0} \quad ...(2.32)$$

Eq. (2.32) is the alternative statement of Gauss's law. This defines the Gauss's **Law** in **differential form.** The divergence of the electric field represents the **net** amount of **flux** coming out of a unit volume element. If **the divergence** of the electric field is positive (or negative) at any point, then the electric flux is emanating (or terminating) from (or on) the closed surface enclosing the charges at that point. Also div E is zero in any region in space having no net charge, Thus, the existence of finite positive value of the divergence at a point shows that there must be a positive charge at the concerned point.

Gauss's Theorem

Eq. (2.32) can be integrated throughout any arbitrary volume V enclosed by a surface S. Thus

$$\int_V \text{div } \mathbf{E} \, dV = \frac{1}{\epsilon_0} \int_V \rho dV \quad ...(2.33)$$

Using Eq. **(2.10),** we get

$$\int_V \text{div } \mathbf{E} \, dV = \int_S \mathbf{E} \cdot d\mathbf{S} \quad ...(2.34)$$

Eqs. **(2.34)** defines the **divergence theorem** (or **Gauss's theorem** as distinguished from **Guass's** law). This theorem is helpful in expressing surface integral of a vector as a volume integral and vice **versa.** Differential form of **Guass's** law is important since it is a starting point for more advanced treatments.

Let us now sum up what we have learnt in this unit.

## 2.6 SUMMARY

- The number of lines of force crossing a closed surface is proportional to the net charge enclosed by that surface.

  The concept of electric flux quantifies the notion "number of lines of force crossing a surface." Flux $\Phi$ is defined as the surface integral of the electric field E over a surface as follows:

  $$\Phi = \int \mathbf{E} \cdot d\mathbf{S}$$

  where $d\mathbf{S}$ is an infinitesimal vector whose direction at any point is towards outward drawn normal to the surface at that point and its magnitude being the area of the surface.

- Gauss's law and Coulomb's law, although expressed in different forms, are the ways of describing the relation between charge and electric field. Remember that Gauss's **law can** be derived from Coulomb's **law,** but not vice versa. So, Gauss's law is not a complete restatement of Coulomb's law.

- Gauss's law is

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0}$$

in which q is net charge inside an imaginary closed surface (called a Gaussian surface) and $\epsilon_0$ is permittivity of free space. Gauss's law expresses an **important** property of the electric field. Unlike Coulomb's law, Gauss's law is not sufficient to determine electric field in all cases.

- The integral form of Gauss's law is:

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int \rho dV$$

where $\rho$ is the volume charge density and $dV$ an infinitesimal volume element.

- The electric field outside a spherically symmetrical **shell with** radius R and total charge q is directed radially and has magnitude

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (r > R)$$

The charge behaves as if it were all concentrated at the centre of the sphere.

- The field inside a uniformly charged spherical shell is exactly zero:

$$E = 0 \quad (r < R)$$

- The electric field due to an infinite line of charge with uniform charge per unit length, $\lambda$, is in a direction perpendicular to the line of charge and has magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

- The electric field due to an infinite sheet of charge is perpendicular to the plane of the sheet and has magnitude

$$E = \frac{\sigma}{2\epsilon_0}$$

where $\sigma$ is the surface charge density.

- In equilibrium, an excess charge on an insulated conductor is entirely on its outer surface

- The electric **field** near the surface of a charged conductor is perpendicular to the surface and has magnitude $E = \frac{\sigma}{\epsilon_0}$.

- Differential form of Gauss's law is div $E = \frac{\rho}{\epsilon_0}$.

- Gauss's theorem states that $\int_V \text{div } E \, dV = \int_S \mathbf{E} \cdot d\mathbf{S}$.

## 2.7 TERMINAL QUESTIONS

1) The electric field in a certain space is given by $E = 200\,\hat{\mathbf{i}}$. How much flux passes through an area $A$ if it is a portion of (a) the $xy$ **plane, (b)** the $xz$ plane, (c) the yz plane?

2) A point charge is placed at the centre of a spherical Gaussian surface. Is flux changed: (a) if the surface is replaced by a cube of the same volume, (b) if the sphere is replaced by a cube of one-tenth the volume, **(c) if** the charge is moved off-centre in the original sphere but still remaining inside, (d) if the charge is moved just outside the original sphere, **(e)** if a second charge is placed near and outside the original sphere, and (f) if a second charge is **placed** inside the Gaussian surface?

3) Suppose that a Gaussian surface encloses no net charge. (a) Does Gauss's law require that E is zero for all points on the surface? (b) if the converse of this statement is true, that is, if E equals zero everywhere on the surface, does Gauss's law require that there be no net charge inside?

4) A thin-walled copper pipe 30.0m long and 2cm in diameter carries a net charge $q = 5.8\mu C$, distributed uniformly. What is the electric field 5.0 mm from the pipe axis? 8cm from the axis? Assume in both cases that the point where you are evaluating the field is not too close to the ends of the pipe ($\epsilon_0 = 8.9 \times 10^{-12}\ C^2N^{-1}\ m^{-2}$).

5) A flat sheet of area $50 cm^2$ carries a uniform surface charge density $\sigma$. An electron 1.5cm from a point near the centre of the sheet experience a force of $1.8 \times 10^{-12}$N directed away from the sheet. Find the total charge on the sheet.

6) Is Gauss's law useful in calculating the field due to three equal charges located at the corners of an equilateral triangle? Explain.

## 2.8 SOLUTIONS AND ANSWERS

**SAQ 1**

The pattern of the lines of force would certainly change. The charge Q will make no contribution to the total number of lines of force crossing any of the surfaces because Q lies outside all three surfaces that we are considering.

**SAQ 2**

No. Gauss's law deals only with closed surfaces. The surfaces shown in Fig. 2.2 are open surfaces, because they do not define an enclosed volume.

**SAQ 3**

For surface $S_1$, the net enclosed charge is $q_1$. The uncharged coin makes no contribution even though the positive and negative charges it contains may be separated by the action of the field in which the coin is immersed. Charges $q_2$ and $q_3$ are outside the surface $S_1$. From Eq. (2.8); we then have

$$\Phi = \frac{q_{enclosed}}{\epsilon_0} = \frac{q_1}{\epsilon_0} = \frac{+3.1 \times 10^{-9}\ C}{8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}} = +350\ N m^2C^{-1}$$

The plus sign indicates that the net charge within the surface is positive ana also that the net flux through the surface is outward.

For surface $S_2$, the net enclosed charge is $q_1 + q_2 + q_3$ so that

$$= \frac{q_{enclosed}}{\epsilon_0} = \frac{q_1 + q_2 + q_3}{\epsilon_0}$$

$$= \frac{+3.1 \times 10^{-9}\ C - 5.9 \times 10^{-9}\ C - 3.1 \times 10^{-9}\ C}{8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}}$$

$$= -670\ Nm^2C^{-1}.$$

The minus sign shows that the net charge within the surface is negative and that the net flux through the surface is inward.

**SAQ 4**

On charging the metal box, the total charge resided on the external surface of the box and hence Faraday was safe:

**Terminal Questions**

1) See Fig. 2.22

a) $\Phi = E.A = 200A\ (\hat{i}\ .\ \hat{k}) = 0$ (direction of $dA$ is towards $\hat{j}$)

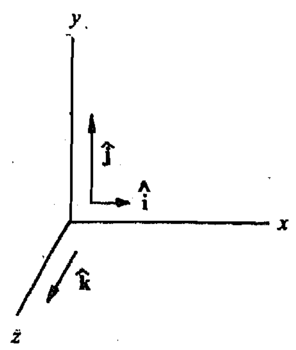



Fig. 2.22

b) $\Phi = 200A\ (\hat{i} . \hat{j}) = 0$

c) $\Phi = 200A\ (\hat{i} . \hat{i}) = 200A$.

2) (a) No (b) No (c) No (d) Yes (e) No (f) Yes

Flux $\Phi = \frac{}{\epsilon_0}$. Flux depends only on charge enclosed. It does not depend on position of charge or charges. If q = 0, $\Phi = 0$. Remember for charge, external to the surface, as much flux enters the'surface as that .leaves the surface as shown in Fig. 2.23.

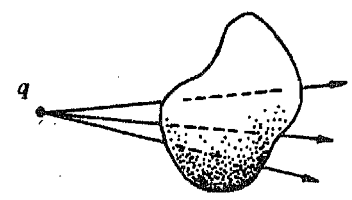



**Fig. 2.23: The net number of lines emerging from a volume due to a charge outside the volume is zero. (The number of lines entering must just equal the number emerging.)**

3) a) When the Gaussian surface contains no net charge, Gauss's law becomes $\mathbf{E} . d\vec{S} = 0$ which does not mean E = 0. Here E and $d\vec{S}$ may be at right angles.

b) When E equals zero everywhere on the surface Gauss's law requires that there be no net charge inside.

4) Here, we do not have a truly infinite line. Rut for points close to the pipe and sufficiently far from the ends, the contribution to the field from distant charges becomes very small, so the field becomes approximately that of an infinite line. A point 5.0mm from the axis lies inside the 2cm diameter pipe. A Gaussian cylinder entirely inside the pipe encloses zero net charge. Therefore, the field is zero everywhere inside the pipe.

For a point outside the pipe, Gaussian cylinder will enclose the entire pipe. Therefore, in Eq. (2.17), length I = 30.0m and the enclosed charge $q$ is $5.8\mu C$. Putting $r = 8.0cm$, we get

$$E = \frac{q_{enclosed}}{2\pi\ \epsilon_0 rl} = \frac{5.8 \times 10^{-6}C}{(2\pi)\ (8.9\times 10^{-12}C^2\,N^{-2}m^{-2})\ (8\times 10^{-2}m)\ (30.0m)}$$

$$= 4.3 \times 10^4\ NC^{-1}.$$

5) The sheet looks effectively infinite for the point which is 1.5cm from it and far from its edges. So the field is given by Eq. (2.20). From the definition of electric field (see Eq. (1.7) of Unit 1), the force $F$ on the electron is just $-eE$. Here $e$ is the electronic charge. Then we can write

$$F = -eE = -e\frac{\sigma}{2\epsilon_0} = \frac{-eq}{2\epsilon_0 A}$$

(because $\sigma$ is the total charge, $q$, on the sheet divided'by the sheet area A).

so that

$$q = \frac{2\epsilon_0 AF}{e} = \frac{2(8.9\times 10^{-12}\ C^2\,N^{-2}\ m^{-2})\ (0.50m)^2\ (1.8\times 10^{-12}N)}{-1.6 \times 10^{-19}C}$$

$$= -50\mu C.$$

The minus sign shows that the sheet carries a negative charge. This is expected because the electron is repelled by the sheet.

6) Gauss's law is not useful for calculating the field due to three equal charges located at the corners of an equilateral triangle because it is difficult to find a surface of appropriate symmetry over which the electric field can be taken constant, and thus to evaluate the integral.