$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\right]_0^\pi - \frac{1}{n^2\pi}\left[\cos nx\right]_0^\pi$$

$$= -\frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\cos nx}{n}\right]_0^\pi + \frac{1}{\pi}\left[\frac{\sin nx}{n^2}\right]_0^\pi$$

$$= \frac{1}{n}$$

Therefore $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}$

This series converges to the periodic extension of $f(x)$ onto the entire x-axis (Fig. 7.23). At the points of discontinuity ($x = 0, \pm 2\pi, \pm 4\pi \ldots$) the series converges to the value

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi}{2}$$

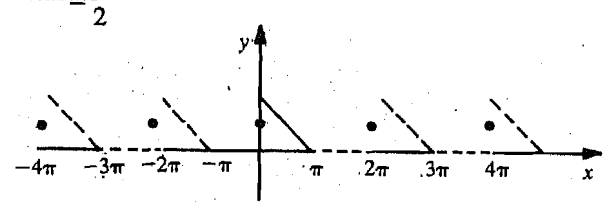

Fig. 7.23

These are shown by the solid dots in the figure. At $n = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots.$ the series will converge to the value

$$\frac{f(\pi^-) + f(-\pi^+)}{2} = 0$$

which is the value of the function at these points.

UNIT 8 APPLICATIONS OF FOURIER SERIES TO PDEs

Structure

8.1 INTRODUCTION

In Unit 7 you have studied the technique of expanding an arbitrary function in terms of Fourier series. In this unit we will use this technique' to solve some important BVPs in physics. Specifically, we will illustrate the application of Fourier series to solve BVPs involving the diffusion equation, wave equation and Laplace equation. For example, we will study heat conduction along a cylindrical rod as well as diffusion of particles using the diffusion equation. Such BVPs arise in engineering and industrial applications, viz. modelling heat flow in the fuel rods in a nuclear reactor, evaporation of water, drying of granular products, etc.

Using the wave equation we shall solve the 'plucked string' problem which models the motion of a string in a variety of musical instruments like the sitar, guitar, violin etc. We shall also study torsional vibrations which arise in several mechanical systems having a rotating shaft such as the axle in a car, propeller in a ship, drill pipe in an oil well, etc.

We will solve Laplace's equation for steady state heat flow in a rectangular plate (which can be used to model heat flow across refrigerator doors). Finally we will solve Laplace's equation for determining the potential at a point due to a circular disc. This problem will demonstrate the fact that Fourier series can be used to solve problems involving non-Cartesian geometry. We hope that after studying this unit you will be able to appreciate the fact that Fourier series can be used for solving a wide variety of real-world problems.

Objectives

After studying this unit you should be able to apply the Fourier series to

- solve the diffusion equation, wave equation and Laplace's equation for a given BVP
- solve similar BVPs involving other PDEs

8.2 DIFFUSION EQUATION

You have studied the one-dimensional diffusion equation (for heat flow) in Unit 5. You have solved it under specific initial and boundary conditions for a given physical problem in Unit 6. You have also solved the two-dimensional heat flow equation in Unit 6.

In its most general form, the diffusion equation is expressed as

$$\nabla^2 u + G(x, y, z, t) = \frac{1}{k}\frac{\partial u}{\partial t} \tag{8.1a}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian and G is an arbitrary function of x, y, z and t. As you know the function $u(x, y, z, t)$ could represent temperature in a body so that Eq. (8.1a) models heat flow in that body. For example, the temperature T of a current-carrying metallic wire can be modelled, by adding another term in Eq. (5.10c) which accounts for the heat generated due to current conduction. If I is the current in the wire and R its resistance, an additional amount of heat $(\alpha I^2 R\Delta x)$ will be accumulated in the portion of the wire between x and $x + \Delta x$. Thus, you can add the term $I^2 R\Delta x$ in Eq. (5.10c) and repeat the remaining steps to obtain the following heat flow equation for a current-carrying wire:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t} - \frac{I^2 R}{KA}, \tag{8.1b}$$

For the two-dimensional diffusion equation, we have to represent the arbitrary function $u(x, y)$ by Fourier series in two variables. It is of the form :

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y$$

This is beyond the scope of this course.

where k is the thermal diffusivity and K, the thermal conductivity of the wire. When Eq. (8.1a) is used to model the diffusion of dissolved substances in a solution, $u(x, y, z, t)$ represents the concentration of the liquid. For example, the PDE

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u \tag{8.1c}$$

can be used to model the loss (diffusion) of moisture from a porous object through its surface. Here γ is constant and u represents the moisture concentration. You can see that Eq. (8.1) is a nonhomogeneous PDE. Now, in Unit 6, you have learnt to solve only homogeneous PDEs. Further in Unit 7, you have learnt to use the Fourier series of a single variable only (read the margin remark). Therefore, in this section we shall restrict the application of Fourier series to one-dimensional homogeneous diffusion equation. Let us consider two specific applications of this equation: in heat conduction and in diffusion of particles.

8.2.1 Heat Conduction

In Unit 7, we introduced the idea of using Fourier series in the solution of a one-dimensional diffusion equation. You had also completed the solution for a specified boundary-value problem. Let us consider another example of heat flow, where the Fourier series can be applied. This is a slightly different application.

Example 1

Consider the flow of heat in a uniform bar of length L, insulated along its length. As you know the temperature of the bar is modelled by the diffusion equation

$$\frac{\partial T(x, t)}{\partial t} = k\frac{\partial^2 T(x, t)}{\partial x^2}, \quad (0 < x < L,\ t > 0) \tag{8.2a}$$

One end of the block is immersed in a block of ice, maintained at 0°C, while the other end is insulated (Fig. 8.1a). This gives rise to the boundary conditions

$$T(0, t) = 0 \quad \text{and} \quad \frac{\partial T(L, t)}{\partial x} = 0, \ t \geq 0 \tag{8.2b}$$

If the initial temperature distribution is given by

$$T(x, 0) = \frac{x}{2}(2L - x) \qquad (0 < x < L) \tag{8.2c}$$

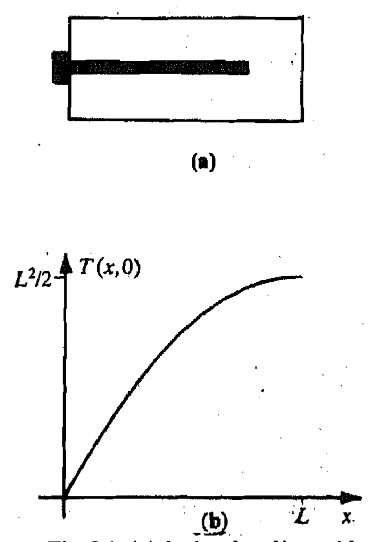

Fig. 8.1: (a) An insulated bar with its left end immersed in ice (b) the initial temperature distribution of the bar

(see Fig. 8.1b), then solve the heat equation (8.2a). (Note that the initial condition is physically consistent with the boundary conditions at $x = 0$ and $x = L$).

Solution

Using the method of separation of variables we write $T(x, t)$ as a product of two terms: $T(x, t) = X(x).Y(t)$. Taking $-\lambda^2$ as the separation constant, we get

$$\frac{X''}{X} = \frac{Y'}{kY} = -\lambda^2 \qquad \text{(i)}$$

or

$$X'' + \lambda^2 X = 0 \qquad \text{(ii)}$$

and $$Y' + k\lambda^2 Y = 0 \qquad \text{(iii)}$$

The solutions of (ii) and (iii) for $X(x)$ and $Y(t)$ are well known

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \qquad \text{(iv)}$$

and $$Y(t) = C_3\, e^{-k\lambda^2 t} \qquad \text{(v)}$$

From the boundary conditions for $T(x, t)$ we have

$$T(0, t) = X(0)\, Y(t) = 0$$

and $$\frac{\partial T}{\partial x}(L, t) = \left[\frac{dX(L)}{dx}\right] Y(t) = 0$$

Since $Y(t) \neq 0$, X must satisfy the conditions

$$X(0) = X'(L) = 0$$

Application of the first of these condition gives us $C_1 = 0$. Thus

$$X(x) = C_2 \sin \lambda x$$

The second boundary condition gives us

$$X'(L) = C_2 \cos \lambda' L = 0$$

For a non-trivial solution, for which $C_2 \neq 0$, we have

$$\cos \lambda L = 0$$

or $$\lambda L = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots.$$

We call these values of λ as λ_n. The solutions can thus be written as

$$X_n(x) = C_{2n} \sin\left[\frac{(2n+1)\pi}{2L} x\right], \qquad n = 0, 1, 2, 3, \ldots.$$

From (v) we have

$$Y_n(t) = C_{3n} \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right]$$

Thus

$$T_n(x, t) = X_n(x)\, Y_n(t) = b_n \exp\left[-\left(\frac{(2n+1)\pi}{2L}\right)^2 k\,t\right] \sin\left[\frac{(2n+1)\pi x}{2L}\right], \qquad n = 0, 1, 2, 3, \ldots.$$

where we have put $b_n = C_{2n}\, C_{3n}$. From the principle of superposition, the most general solution is

$$T(x, t) = \sum_{n=0}^{\infty} b_n \exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right) \sin\left[\frac{(2n+1)\pi x}{2L}\right] \qquad (8.3)$$

Applying the initial condition (8.2c) we have

$$T(x, 0) = \sum_{n=0}^{\infty} b_n \sin\left[\frac{(2n+1)\pi x}{2L}\right] = f(x), \quad 0 < x < L \qquad (8.4a)$$

Notice that $f(x)$ in Eq. (8.4a) cannot be expanded in the Fourier series we have introduced in Unit 7 because the argument of the sine function is different. However, we can use the same technique to expand a given function in terms of any sinusoidal series provided its terms satisfy the orthogonality condition.

where $f(x)=\frac{x}{2}(2L-x)$.

We can determine b_n in Eq. (8.4a) using the half-range expansion of $f(x)$. Since $f(x)$ is defined on $0 < x < L$, and $T(x, 0)$ is the sum of a sine series, we can take $g(x)$ to be the odd extension of $f(x)$. Multiplying the LHS of Eq. (8.4a) by $\sin\left[\frac{(2m+1)\pi x}{2L}\right]$ and integrating from $-L$ to L, we have

$$\sum_{n=0}^{\infty} b_n \int_{-L}^{L} \sin\left[\frac{(2m+1)\pi x}{2L}\right]\sin\left[\frac{(2n+1)\pi x}{2L}\right]dx = \int_{-L}^{L} g(x)\sin\left[\frac{(2m+1)\pi x}{2L}\right]dx$$

Applying the technique of Sec. 7.2 we get

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left[\frac{(2n+1)\pi x}{2L}\right]dx$$

$$= \frac{2}{L}\int_0^L \frac{x}{2}(2L-x)\sin\left[\frac{(2n+1)\pi x}{2L}\right]dx$$

You can integrate by parts twice and show that

$$b_n = \frac{16L^2}{(2n+1)^3\pi^3}$$

Hence the solution is given by

$$T(x,t) = \frac{16L^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right)\sin\left[\frac{(2n+1)\pi x}{2L}\right] \quad (8.4b)$$

You may now like to work out an SAQ yourself by applying Fourier series to the homogeneous heat equation.

Spend 5 minutes

SAQ 1

In Example 1, let the initial temperature of the bar be a constant T_0 °C. Then solve the diffusion equation (8.2a),

with $T(0,t) = \frac{\partial T}{\partial x}(L,t) = 0$, $(t \geq 0)$

and $T(x,0) = T_0$, $(0 < x < L)$

Determine the expression for $T(x, t)$ and discuss its behaviour at large values of times.

Note that in this case the initial condition does not match the boundary conditions at $x = 0$ and $x = L$. In reality, when the ends of the bar are put into the ice, it would melt to match the temperature of the ends of the bar, which cool rapidly. Only later would we have $T(0, t) = 0$.

Let us consider another application of Fourier Series for solving the diffusion equation.

8.2.2 Diffusion of **Particles**

Many of our day-to-day experiences involve diffusion of particles. For example, when sugar is added to a cup of tea, it dissolves and then diffuses throughout the tea. Water evaporates from ponds and increases the humidity of the passing air stream. Diffusion of particles plays an important role in many industrial applications. Some examples are: the removal of pollutants from plant discharge streams, the stripping of gases from waste water, acid concentration, salt production and sugar solution concentration through continuous evaporation, drying of industrial products, such as concrete slabs., wood, etc. Here we will apply the diffusion equation; to a typical example of drying of a porous material.

A porous rod containing moisture with one of its ends (for which $x = 0$) sealed is left to be dried. The other end of the rod is in contact with a dry medium, and it loses moisture through its surface to dry air. The concentration of moisture, $u(x, t)$, satisfies the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u, \qquad 0 < x < L, \quad t > 0 \qquad (8.5a)$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad u(L, t) = 0, \qquad t > 0 \qquad (8.5b)$$

$$u(x, 0) = u_0, \qquad 0 < x < L \qquad (8.5c)$$

Let us find $u(x, t)$ and determine the concentration at $x = 0$, i.e., $u(0, t)$ explicitly.

Using the method of separation of variables, we seek a solution of the form $u(x, t) = X(x)\,T(t)$. The PDE becomes

$$X''(x)\,T(t) = \frac{1}{k} X(x)\,T'(t) + \gamma^2 X(x)\,T(t)$$

Dividing by $X(x)T(t)$ we get

$$\frac{X''(x)}{X(x)} = \frac{T'(t) + k\gamma^2 T(t)}{k\,T(t)} = -\lambda^2$$

Thus, we get two ODEs

$$X''(x) + \lambda^2 X(x) = 0, \qquad 0 < x < L \qquad \text{(i)}$$

$$T'(t) + k(\gamma^2 + \lambda^2)\,T = 0, \qquad t > 0 \qquad \text{(ii)}$$

The solution of (i) is

$$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

Applying the boundary conditions, we have

$$X'(0) = 0; \quad X(L) = 0$$

$$X'(0) = \lambda C_2 = 0, \text{ i.e., } C_2 = 0$$

and $\quad X(L) = C_1 \cos \lambda L = 0$

Since $C_1 \neq 0$, this gives

$$\cos \lambda L = 0$$

or.

$$\lambda_n L = \frac{(2n+1)\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

or $\quad \lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$

Thus $X_n(x) = C_{1n} \cos \lambda_n x$, where $\lambda_n = \frac{(2n+1)\pi}{2L}, \quad n = 0, 1, 2, \ldots$

The solution of (ii) is

$$T_n(t) = C_{3n} \exp\left[-k(\gamma^2 + \lambda_n^2)t\right] = C_{3n}\, e^{-\gamma^2 kt}\, e^{-\lambda_n^2 kt}$$

Therefore, the general solution of Eq. (8.5a) is

$$u(x, t) = e^{-\gamma^2 kt} \sum_{n=0} a_n \cos \lambda_n x\, e^{-\lambda_n^2 kt}$$

where we have put $a_n = C_{1n} C_{3n}$. To determine these unknown constants, we note from the initial condition that

$$\sum_{n=0} a_n \cos \lambda_n x = u_0, \qquad 0 < x < L$$

Again we can use the even extension of u_0 (Sec. 7.5.1) to obtain its Fourier series expansion, i.e., the coefficients a_n.

$$a_n = \frac{2}{L} \int_0^L u_0 \cos \lambda_n x \, dx$$

$$= \frac{2u_0}{L}\left(\frac{\sin \lambda_n L}{\lambda_n}\right) = \frac{4u_0 \sin \lambda_n L}{(2n+1)\pi}, \qquad n = 0, 1, 2, \ldots .$$

Thus,

$$u(x, t) = \frac{4u_0}{\pi} e^{-\gamma^2 kt} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left[(2n+1)\frac{\pi}{2}\right] \cos\left[\frac{(2n+1)\pi x}{2L}\right] e^{-(2n+1)^2\pi^2 kt/4L^2}$$

We can find the value of $u(0, t)$ by putting $x = 0$ in this solution

$$u(0, t) = \frac{4u_0}{\pi} e^{-\gamma^2 kt} \sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)\frac{\pi}{2}\right]}{(2n+1)} e^{-(2n+1)^2\pi^2 kt/4L^2}$$

$$= \frac{4u_0}{\pi} e^{-\gamma^2 kt}\left[e^{-\tau} - \frac{e^{-9\tau}}{3} + \frac{e^{-25\tau}}{5} - \ldots + \ldots\right]$$

where $\tau = \pi^2 kt/4L^2$.

You should now work out an SAQ to know whether you have grasped the application of Fourier series to the one-dimensional diffusion equation.

Spend 10 minutes

SAQ 2

Solve the **heat/diffusion** problem stated below in terms of dimensionless variables

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1, \qquad t > 0$$

$$\frac{\partial u}{\partial x}(0, t) = 0, \qquad \frac{\partial u}{\partial x}(1, t) = 0, \qquad t > 0$$

$$u(x, 0) = 1 + 2x, \qquad 0 < x < 1.$$

Let us now consider some applications of Fourier series to the wave equation.

8.3 THE WAVE EQUATION

For simplicity we shall **restrict** our discussion to the solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L, \qquad t > 0 \tag{8.6}$$

with given initial and boundary conditions. Let us solve this equation by applying Fourier series to two specific categories of physical problems related to (i) vibrating strings, and (ii) torsional vibrations.

8.3.1 Vibrating Strings

When a sitarist plucks the sitar string, several other tones called overtones or harmonics, are generated along with the fundamental frequency (Recall Eq. (6.39) of Unit 6). The richness of musical sound is related to the number of harmonics that can be detected by the human ear. The larger the amplitude of each harmonic, the more likely it is to be detected. The amplitude of each harmonic depends, in turn, on where exactly the string is plucked. So if we know the point at which a sitar string is plucked, we can get a fair idea of the richness of the sound produced. To mathematically model this physical situation, we have to determine a unique solution of Eq. (8.6) for the "plucked string" problem which we consider in the following example.

Example 2 : **The 'plucked string' problem**

A string is plucked at its mid-point and then released from rest from this position (Fig. 8.2). The resulting vibrations are modelled by Eq. (8.6) along with the following boundary and initial conditions.

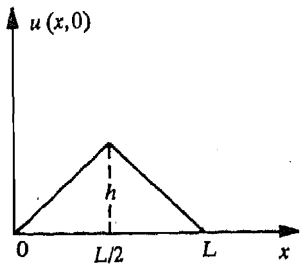

Fig. 8.2

$$u(0,t)=0, \quad u(L,t)=0$$

$$u(x,0)=\begin{cases} \dfrac{2hx}{L}, & 0<x<\dfrac{L}{2} \\[2ex] 2h\left(1-\dfrac{x}{L}\right), & \dfrac{L}{2}\le x<L \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=0$$

where h is a positive constant which is small compared to L.

These conditions correspond to an initial triangular deflection and zero initial velocity.

In Unit 6 you have already obtained the general solution of the wave equation for given boundary conditions. The general solution given by Eq. (6.40) is

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi vt}{L}+b_n\sin\frac{n\pi vt}{L}\right)\sin\frac{n\pi x}{L} \qquad \text{(i)}$$

Let us now apply the initial conditions to (i) :

$$u(x,0)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}=\begin{cases} \dfrac{2hx}{L}, & 0<x<\dfrac{L}{2} \\[2ex] 2h\left(1-\dfrac{x}{L}\right), & \dfrac{L}{2}\le x<L \end{cases} \qquad \text{(ii)}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=\left[\sum_{n=1}^{\infty}\left(-a_n\frac{n\pi v}{L}\sin\frac{n\pi vt}{L}+b_n\frac{n\pi v}{L}\cos\frac{n\pi vt}{L}\right)\sin\frac{n\pi x}{L}\right]_{t=0}$$

$$=\sum_{n=1}^{\infty}b_n\frac{n\pi v}{L}\sin\frac{n\pi x}{L}=0 \qquad \text{(iii)}$$

Eq. (iii) will be satisfied only if $b_n = 0$ for all n, as you have obtained in Eq. (6.43). So now you have to determine a_n, i.e., you have to expand $u(x, 0)$ in a Fourier sine series. In effect, you have to obtain the odd periodic extension of $u(x,0)$ and hence its half-range expansion in a Fourier sine series. Recall that you have worked out a problem for an even periodic extension of the same function in the terminal question 1 of Unit 7.

So you may like to solve this part of the problem yourself.

Spend 10 minutes

SAQ 3

Show that the solution of the "plucked String" problem specified by Eq. (8.6) is

$$u(x, t) = \frac{8h}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi vt}{L} + \ldots\right]$$

Another way to start any string vibrating is to strike it (a piano string, for example). In this case, the initial conditions would be $u(x, t) = 0$ at $t = 0$ and the velocity $\frac{\partial u}{\partial t}$ will be given as a function of x (i.e., the velocity of each point of the string is given at $t = 0$). Now that you have practised determining Fourier series you should feel confident enough to be able to solve any such **BVP** related to vibrating strings.

Another interesting application of the wave equation is in torsional vibrations. Such vibrations can result from unbalanced torques on shafts in a wide variety of machinery in cars, aircraft, turbines, railway engines, etc. You may know that a shaft is a bar that is usually cylindrical and solid. It is used to support rotating pieces in machines or to transmit power or motion by rotation. Some common examples of shafts are axles connecting the wheels of a car, spindles on a spinning wheel, propeller shafts used for ship propulsion and shafts in belt and pulley arrangements. So let us now consider a typical problem involving torsional vibrations of a shaft.

8.3.2 Torsional Vibrations

Consider a uniform, undamped torsionally vibrating shaft of finite length, subject to given initial conditions of angular displacement and angular velocity (Fig. 8.3). This means that we have to find solutions of the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\theta}{\partial t^2} \qquad (8.7)$$

where θ is the angle of twist of the shaft and $v^2 = E_s/\rho$. Here E_s is the modulus of elasticity in shear, and ρ, the mass per unit volume of the shaft. Once again we use the method of separation of variables to express $\theta(x, t)$ as

$$\theta(x, t) = X(x)\,T(t)$$

Just as in the case of a vibrating string (Eq. 6.15 of Unit 6) we reduce Eq. (8.7) to a set of two ODEs:

$$T'' = \frac{\lambda^2}{v^2}T, \qquad X'' = -\lambda^2 X$$

where $(-\lambda^2)$ is the separation constant. The solutions of these ODEs are

$$T = A\cos\lambda vt + B\sin\lambda vt$$

and $$X = C\cos\lambda x + D\sin\lambda x$$

Thus the solution is

$$\theta(x, t) = X(x)\,T(t) = (C\cos\lambda x + D\sin\lambda x)(A\cos hut + B\sin Avt) \qquad (i)$$

You can see that the solution is periodic, repeating itself for every increase in time t by $\frac{2\pi}{\lambda v}$. In other words, $\theta(x, t)$ represents a torsional motion of period $\frac{2\pi}{\lambda v}$ or frequency $\lambda v/2\pi$.

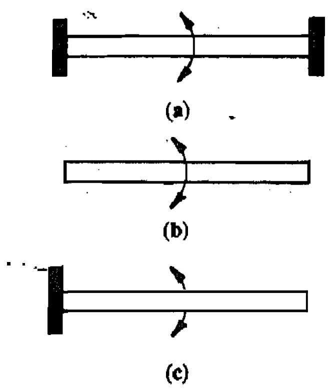

Fig. 8.3: A torsionally vibrating (a) both ends fixed (b) free (c) one end fixed and one end free

It remains now to find the values of λ, A, B, C and D. The values of λ are determined by the given boundary conditions which define how the shaft is constrained at its ends. There are three cases which occur most often in physical systems:

1) Both ends of the shaft are fixed so that no twisting can take place (Fig. 8.3a)
2) Both ends of the shaft are free to twist (Fig. 8.3b)
3) One end of the shaft is fixed, while the other is free to twist (Fig. 8.3c).

Case 1

The boundary conditions in this case are

$$\theta(0, t) = \theta(L, t) = 0, \qquad t > 0$$

Applying these conditions you get the general solution which is the familiar result obtained for the vibrating string (Eq. 6.40 of Unit 6):

$$\theta(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi vt}{L} + b_n \sin \frac{n\pi vt}{L} \right) \sin \frac{n\pi x}{L}$$

This solution has to satisfy the given initial conditions on angular displacement and angular velocity. If we set $t = 0$ in the equation for $\theta(x, t)$ and its derivative we get

$$\theta(x, 0) \equiv f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

and
$$\left. \frac{\partial \theta}{\partial t} \right|_{t=0} \equiv g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi v}{L} b_n \right) \sin \frac{n\pi x}{L}$$

where $f(x)$ and $g(x)$ are some functions of x, representing the initial angular displacement. and angular velocity of the shaft. We can then use the half-range sine expansions of $f(x)$ and $g(x)$. This gives

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

and

$$b_n = \frac{L}{n\pi v} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{n\pi v} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx$$

Thus, a uniform shaft with both ends restrained against turning vibrates torsionally at any one of the infinite number of natural frequencies

$$f_n = \frac{nv}{2L} \text{ cycles per unit time}, \qquad n = 1, 2, 3, \ldots .$$

Case 2

When both ends of the shaft are free. no torque acts through the end section (i.e., at $x = 0$ and at $x = L$) since there is no shaft material beyond these points. Thus, the torque transmitted through these ends is zero, i.e.,

$$\tau = E_s I \left. \frac{\partial \theta}{\partial x} \right|_{\text{end points}} = 0$$

where I is the moment of inertia of the rod, and E_s is the shear modulus of elasticity, Both E_s and I are non-zero. Thus for a free-end, the boundary conditions are

$$\frac{\partial \theta}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

The subsequent solution proceeds on familiar lines. In fact, you may like to complete the solution for a specific problem. Try the following SAQ.

The torque transmitted through nny cross-section of a twisted shaft is proportional to the twist per unit. length, i.e., the slope of the (θ, x) curve, at that cross-section,

Thus $\tau \propto \frac{\partial \theta}{\partial x}$

For solid shafts the proportionality constant is equal to $E_s I$ where E_s is the shear modulus of elasticity and I the moment of inertia of the shaft.

SAQ 4 *Spend 10 minutes*

A uniform shaft free at each end is twisted so that it rotates through an angle proportional to $(2x - L)/2$. If the shaft is released from rest in this position, what will its subsequent angular displacement as a function of x and t be?

Have you noted that the natural frequencies of the vibrating shafts in Cases (1) and (2) are the same? However, the amplitudes of vibration are not the same. For the shaft fixed at both ends, the amplitudes along the shaft are proportional to $\sin \frac{n\pi x}{L}$ whereas for the shaft free at the ends, the amplitudes are proportional to $\cos \frac{n\pi x}{L}$

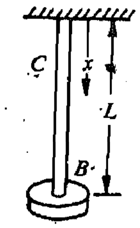

Fig. 8.4

Case 3

A typical example of a shaft fixed at one end and free at the other is the drill pipe used in oil wells. A drill collar (C) containing the cutting bit (B) is attached to the lower end of the pipe (Fig. 8.4). The boundary conditions for such a shaft are

$$\theta(0, t) = 0 \text{ and } \left.\frac{\partial \theta}{\partial x}\right|_{L, t} = 0, \qquad t > 0$$

When we impose these conditions on Eq. (i) we get

$$C = 0, \cos \lambda L = 0, \text{ which gives}$$

$$\lambda_n L = (2n-1)\frac{\pi}{2}, \qquad n = 1, 2 \ldots$$

and $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2 \ldots$$

The general solution is, therefore,

$$\theta(x, t) = \sum_{n=1}^{\infty} (\sin \lambda_n x)(A_n \cos \lambda_n vt + B_n \sin \lambda_n vt)$$

Now suppose we have the initial conditions that

$$\theta(x, 0) = f(x) \quad \text{and} \quad \left.\frac{\partial \theta}{\partial t}\right|_{x, 0} = g(x)$$

These initial conditions yield

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

and

$$g(x) = \sum_{n=1}^{\infty}\left[\frac{(2n-1)\pi v}{2L} B_n\right] \sin\left[\frac{(2n-1)\pi x}{2L}\right]$$

Recall that in Example 1, we have written $\lambda_n L = (2n+1)\frac{\pi}{2}$ but n takes the values of 0, 1, 2 . . . , In this case $\lambda_n L = (2n-1)\frac{\pi}{2}$, where n = 1, 2 . . . So the expression for λ_n is the same in both the cases.

Recall that we have obtained the coefficients A_n and B_n in the heat conduction problem. You can easily verify that the half-range sine expansions of f (x) and $g(x)$ yield

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left[\frac{(2n-1)\pi x}{2L}\right] dx$$

and

$$B_n = \frac{4}{(2n-1)\pi v}\int_0^L g(x) \sin\left[\frac{(2n-1)\pi x}{2L}\right] dx$$

These coefficients can be obtained for any function f (x) and $g(x)$ integrable on the interval $0 < x < L$.

Finally, we will take up the application of Fourier series to Laplace's equation.

8.4 LAPLACE'S EQUATION

You know the three dimensional Laplace's equation: $\nabla^2 u = 0$. The function u may repsresent many physical qualities: it may be the gravitational potential in a region containing no matter or the electrostatic potential in a charge-free region. The steady-state temperature (i.e., temperature not changing with time) in a region containing no source of heat also satisfies Laplace's equation. In Unit 5, we have derived this equation for the velocity potential for an incompressible and irrotational fluid. Of all these diverse areas where Laplace's equation applies, we have selected two to illustrate the applications of Fourier series. These are the steady-state heat flow and the potential problem. You can extend the procedure explained here to other specific problems.

8.4.1 Steady-state Heat Flow

In Unit 6 you have solved Laplace's equation for determining the steady-state temperature of a Circular cylinder (Example 2) and a semi-circular plate (Example 3). However, in both these examples we need not use Fourier series to determine the particular solution. So let us consider a specific BVP for Laplace's equation which involves the use of Fourier series. A modified version of this problem can be used to model the flow of heat across a refrigerator door.

Example 3 : Steady-state temperature of a rectangular metal plate

A thin rectangular metal plate is sandwiched between sheets of insulation (Fig. 8.5a). Since the plate is very thin and insulated at two of its surfaces, one may assume that the temperature does not vary in the z-direction. In the steady state, the temperature of the plate obeys the two-dimensional Laplace equation:

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < B \qquad (8.8)$$

where L is the length and B the width of the plate.

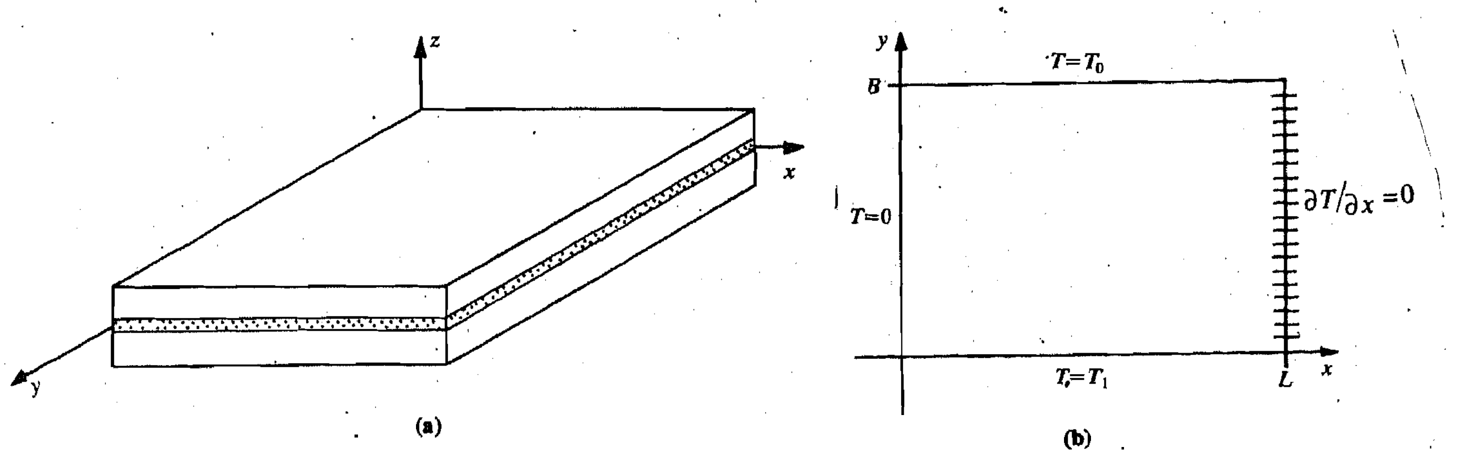

Fig. 8.5: (a) A thin plate between sheets of insulation (b) the boundary conditions for $T(x, y)$

Suppose that the temperature of the plate is held at T_0 at its top edge, T_1 at its bottom edge and 0°C on the left edge. The plate is insulated on the right edge, so that no heat flows in that direction, and the partial derivative of T in the x-direction is zero (See Fig. 8.5b). Can you write down these boundary conditions mathematically? These are

i) $T(0, y) = 0, \quad \frac{\partial T(L,y)}{\partial x} = 0, \quad 0 < y < B$

ii) $T(x, 0) = T_1 \quad T(x, B) = T_0, \quad 0 < x < L$

We wish to determine $T(x, y)$ by solving Laplace's equation subject to these boundary conditions.

Solution

For a non-trivial solution we write

$$T(x, y) = X(x)\, Y(y)$$

and use the method of separation of variables to get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

Since $X(x)$ vanishes at the boundaries the ratio $\frac{X'(x)}{X(x)}$ cannot be positive. Hence, we get the two ODEs :

$$X'' + \lambda^2 X = 0 \qquad 0 < x < L$$

and $$Y'' - \lambda^2 Y = 0 \qquad 0 < y < B$$

The solutions are

$$X(x) = A \cos \lambda x + B \sin \lambda x$$

and $$Y(y) = C \cosh \lambda y + D \sinh \lambda y$$

Applying the boundary conditions (i) and (ii) we get

$$A = 0, \quad \cos \lambda L = 0$$

or $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2 \ldots$$

which gives $X_n(x) = B_n \sin \lambda_n x$

and $$Y_n(y) = C'_n \cosh \lambda_n y + D'_n \sinh \lambda_n y$$

The general solution for $T(x, y)$ is

$$T(x, y) = \sum_{n=1}^{\infty} (C_n \cosh \lambda_n y + D_n \sinh \lambda_n y) \sin \lambda_n x$$

with $$\lambda_n = \frac{(2n-1)\pi}{2L}, \qquad n = 1, 2, \ldots$$

and $$C_n = C'_n B_n, \quad D_n = D'_n B_n$$

The coefficients C_n and D_n are determined by applying the boundary condition **(ii).** At y = 0

$$T(x, 0) = C_n \sin \lambda_n x = T_1, \qquad 0 < x < L$$

from which you can determine C_n to be

$$C_n = \frac{2}{L} \int_0^L T_1 \sin \lambda_n x \, dx = \frac{4T_1}{(2n-1)\pi}$$

At $y = B$,

$$T(x, B) = \sum_{n=1}^{\infty} (C_n \cosh \lambda_n B + D_n \sinh \lambda_n B) \sin \lambda_n x = T_0, \qquad 0 < x < L$$

Now we have to choose D_n so that the quantity within brackets is the Fourier sine coefficient of the function representing the given boundary value (T_0 in this case). Let us put

$$C_n \cosh \lambda_n B + D_n \sinh \lambda_n B = G_n$$

Then, the coefficients G_n are given by the relation

$$G_n = \frac{2}{L}\int_0^L T_0 \sin\lambda_n x\, dx = \frac{4T_0}{(2n-1)\pi}$$

This gives the coefficients D_n in terms of the known coefficients C_n and G_n :

$$D_n = \frac{G_n - C_n\cosh\lambda_n B}{\sinh\lambda_n B} = \frac{4}{(2n-1)\pi}\;\frac{T_0 - T_1\cosh\lambda_n B}{\sinh\lambda_n B}$$

Thus, the unique solution of this problem is

$$T(x, y) = 4\sum_{n=1}^{\infty}\left(\frac{T_1\cosh\lambda_n y}{2n-1} + \frac{T_0 - T_1\cosh\lambda_n B}{(2n-1)\sinh\lambda_n B}\sinh\lambda_n y\right)\sin\lambda_n x$$

The solution for the case $B = 2L$, $T_1 = 10°C$, $T_0 = 20°C$ is shown in Fig. 8.6. The curves shown are the isotherms $T(x, y) = T_c$ for various values of T.

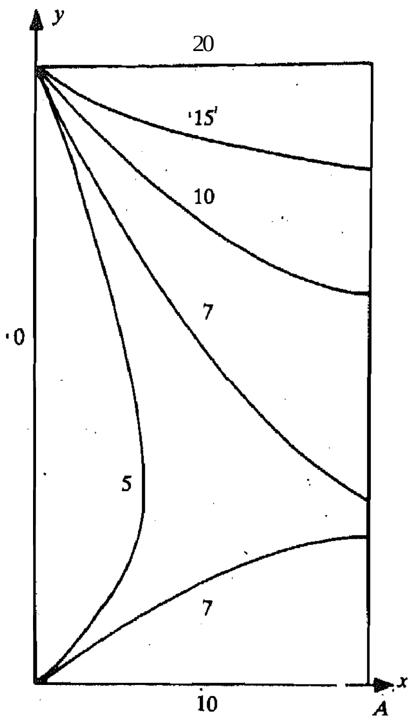

Fig. 8.6: The isotherms $T(x, y) = T_c$ for various values of T when $B = 2L$, $T_1 = 10°C$, $T_0 = 20°C$

You could have solved this problem even if the boundary conditions for $T(x, y)$ on the top and bottom edge had been any piecewise continuous functions, instead of constants, or if the boundary conditions on the left and right edges had been different. Why don't you work out such a problem?

SAQ 5

Spend 15 minutes

Obtain the steady-state temperature for the rectangular plate of Fig. 8.5a given the following boundary conditions :

$$u(0, y) = \frac{U_0 y}{B}, \quad \frac{\partial u}{\partial x}(L, y) = -S, \qquad 0 < y < B$$

$$u(x, 0) = 0, \quad u(x, B) = 0, \qquad 0 < x < L$$

In a certain class of problems involving conductors, all the charge is found on their surfaces. The potential at all points outside the conductor satisfies Laplace's equation. Let us now solve Laplace's equation for the electrostatic potential of a conductor.

So far we have considered problems which require Cartesian coordinate system. In the final section of this unit we are considering an application of Fourier series to a potential problem in a non-Cartesian geometry.

8.4.2 The Potential at a Point due to a Circular Disc

Let us solve Laplace's equation for the potential on a circular metallic disc. For a circular disc it is natural to use plane-polar coordinates (r, θ). The problem is as follows :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2}\right) = 0, \qquad 0 \le r < L, \quad -\pi < \theta \le \pi \qquad (8.9a)$$

$$u(L, \theta) = f(\theta), \qquad -\pi < \theta \le \pi \qquad (8.9b)$$

There are two special features of this problem:

(i) The points $\theta = -\pi$ and $\theta = \pi$ coincide. Therefore, the value of u and its angular derivative should match there:

$$u(r, -\pi) = u(r, \pi), \qquad \frac{\partial u}{\partial\theta}(r, -\pi) = \frac{\partial u}{\partial\theta}(r, \pi), \qquad 0 \le r < L$$

ii) The point $r = 0$ is singular: the coefficient of $\frac{\partial^2 u}{\partial r^2}$ in Eq. (8.9a) is 1, while the coefficients of other terms are $1/r$ and $1/r^2$ We must; therefore, enforce a condition of boundedness :

$u(r, \theta)$ tends to a finite value, i.e., it is bounded, as $r \to 0$.

Keeping these special features in mind, we can solve the potential problem using the method of separation of variables.

Let $u(r, \theta) = R(r)\Theta(\theta)$

Substituting $u(r, \theta)$ in Eq. (8.9a) and taking into account the special continuity conditions, we get

$$\frac{1}{r}[rR'(r)]'\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0, \qquad 0 \le r < L, \quad -\pi < \theta \le \pi \qquad \text{(i)}$$

and $$R(r)\,\Theta(-\pi) = R(r)\,\Theta(\pi), \quad R(r)\,\Theta'(-\pi) = R(r)\,\Theta'(\pi) \quad 0 \le r < L \qquad \text{(ii)}$$

Multiplying Eq. (i) by r^2, dividing it by $R(r)\,\Theta(\theta)$, and eliminating $R(r)$ in (ii) we get

$$\frac{r[rR'(r)]'}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \qquad 0 \le r < L, \qquad -\pi < \theta \le \pi$$

$$\Theta(-\pi) = \Theta(\pi) \text{ and } \Theta'(-\pi) = \Theta'(\pi), \qquad \text{(ii)}$$

Thus

$$\Theta'' + \lambda^2\Theta = 0$$

which gives $\Theta = A\cos\lambda\theta + B\sin\lambda\theta$

The continuity conditions (ii) for Θ give us

$$A\cos\lambda\pi - B\sin\lambda\pi = A\cos\lambda\pi + B\sin\lambda\pi$$

$$A\lambda\sin\lambda\pi + B\lambda\cos\lambda\pi = -A\lambda\sin\lambda\pi + B\lambda\cos\lambda\pi$$

or $2B\sin\lambda\pi = 0$

and $2\lambda A\sin\lambda\pi = 0$

which gives $\lambda_n = n, \quad n = 0, 1, 2, 3, \ldots$

Thus, we have

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \qquad n = 0, 1, 2, 3, \ldots$$

The ODE for $R(r)$ is

$$\frac{r(rR'_n)'}{R_n} = \lambda_n^2 \quad \text{or} \quad r^2R''_n + rR'_n - \lambda_n^2R_n = 0$$

This is an Euler-Cauchy equation with linearly independent solutions (see Example 3 of Unit 6):

$$R_n(r) = r^n \quad \text{and} \quad R_n(r) = r^{-n}$$

The second of these is physically unacceptable as it tends to ∞ in the limit as $r \to 0$. Thus, we have the general solution for $u(r, \theta)$:

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

The boundary condition on $r = L$ yields

$$A_0 + \sum_{n=1}^{\infty} L^n(A_n\cos n\theta + B_n\sin n\theta) = f(\theta) \qquad -\pi < \theta \le \pi$$

This is a Fourier series problem and the coefficients in the series are

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$L^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$L^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

You can solve these three integrals for any given form of f (8) provided they exist.

In this unit, we have considered certain specific applications of Fourier series to some PDEs of special interest in physical problems, viz. the diffusion equation, the wave equation and Laplace's equation. The applications discussed here are, by no means, exhaustive, but only illustrative of this powerful method based on the Fourier series. This method applies to a much wider variety of problems which, of course, cannot all be discussed here for want of time. However, we are sure that you have been able to develop an appreciation of the usefulness of the Fourier method based on Fourier series, from what you have studied in Units 7 and 8.

But we would certainly not like to end this unit on the note that the Fourier method is the ultimate method of solving linear BVPs in PDEs. There are other important methods of solving these problems, such as methods based on Laplace transforms, Fourier transforms and other integral transforms, methods that use the Green's functions and numerical methods. In fact, the development of new methods for solving BVPs is an active area of present-day mathematical research. Most of these methods are usually discussed in advanced level courses on Mathematical Methods in Physics or Differential Equations, at the undergraduate as well as postgraduate level.

We will now summarise what you have studied in this Unit.

8.5 SUMMARY

In this unit you have learnt to apply the Fourier series to solve various boundary-value problems related to :

- the diffusion equation, e.g., the problems of heat conduction and diffusion of liquids in porous solids
- the wave equation, e.g., the problems of vibrating strings and torsional vibrations
- Laplace's equation, e.g., the steady-state heat flow and the potential problem.

8.6 TERMINAL QUESTIONS

1. A cylindrical elastic bar (e.g., steel bar) of natural length L is initially stretched by an amount cL and is at rest. The initial longitudinal displacement of any section of the bar is proportional to the distance from the fixed end $x = 0$. At the instant $t = 0$, both ends are released and left free. The longitudinal displacement $y(x, t)$ of the bar satisfies the following BVP

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}$$

where $v^2 = E/\rho$, E is the modulus of elasticity and ρ is the density of the material of the bar. Since the ends are free, the force per unit area on the ends of the bar is zero and we get

$$\frac{\partial y}{\partial x}(0, t) = 0, \qquad \frac{\partial y}{\partial x}(L, t) = 0$$

Further $y(x, 0) = cx$, $\quad \frac{\partial y}{\partial t}(x, 0) = 0$

Solve the BVP and obtain $y(x, t)$.

2. The flow of electric current in a pair of telephone wires or power transmission lines

and the emf across the wires can be modelled by equations similar to the diffusion equation provided the loss due to leakage of current is negligible and the inductance of the wires is negligible:

$$\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}, \qquad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Here R is the resistance per unit length and C the capacitance per unit length of the two wires. Solve this equation for a cable of length L for the following boundary and initial conditions

$v(0, t) = 0V, \quad v(L, t) = 0V, \quad t \geq 0$

$v(x, 0) = (6x/L)V.$

8.7 SOLUTIONS AND ANSWERS

SAQs (Self-assessment questions)

1. The general solution of this problem is given by Eq. (8.3). Applying the initial condition we get

$$T(x, 0) = \sum_{n=0}^{\infty} b_n \sin\left[\frac{(2n+1)\pi x}{2L}\right] = T_0, \quad 0 < x < L$$

Using the half-range expansion of $T(x, 0)$ we get

$$b_n = \frac{2}{L}\int_0^L T_0 \sin\left[\frac{(2n+1)\pi x}{2L}\right] dx$$

$$= \frac{2T_0}{L}\left[-\frac{2L}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L}\right]_0^L$$

$$= -\frac{4T_0}{(2n+1)\pi}\left[\cos(2n+1)\frac{\pi}{2} - 1\right]$$

$$= \frac{4T_0}{(2n+1)\pi}, \quad \text{since } \cos(2n+1)\frac{\pi}{2} = 0 \quad \text{for all values of n.}$$

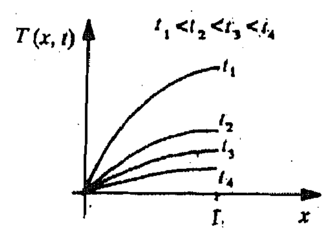

Fig. 8.7

Hence

$$T(x, t) = \frac{4T_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\exp\left(-\left[\frac{(2n+1)\pi}{2L}\right]^2 kt\right)\sin\left[\frac{(2n+1)\pi x}{2L}\right]$$

As t increases, all the exponential terms tend to zero, hence $T(x, t)$ tends to zero. Fig. 8.7 shows the kinds of graphs expected for $T(x, t)$ vs. x for increasingly large values of t.

2) In this problem k = 1, and L = 1. Following Example 1, we apply the given boundary conditions to $X(x)$:

$X'(0) = X'(1) = 0$

This gives us

$C_2 = 0$ and $C_1 \sin\lambda = 0$

Since $C_1 \neq 0$ for a non-trivial solution, we have $\lambda = 0$ or $\lambda_n = n\pi$, n = 0, 1, 2, 3, . .

The solution for $T(t)$ is obtained as in Example 1 and the general solution is

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \exp[-n^2\pi^2 t]\cos n\pi x,$$

At $t = 0$, *we* have

$$u(x, 0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x = 1 + 2x, \qquad 0 < x < 1$$

The coefficients are

$$a_0 = \int_0^1 (1 + 2x)\, dx = \left[x + x^2\right]_0^1 = 2$$

$$a_n = 2 \int_0^1 (1 + 2x) \cos n\pi x\, dx$$

$$= \frac{4}{n^2\pi^2} (\cos n\pi - 1)$$

Thus, the particular solution is

$$u(x, t) = 2 + 4 \sum_{n=1}^{\infty} \left(\frac{\cos n\pi - 1}{n^2\pi^2}\right) \exp(-n^2\pi^2 t) \cos n\pi x$$

$$= 2 - \frac{8}{\pi^2}\left(\cos \pi x\, e^{-\pi^2 t} + \frac{1}{9} \cos 3\pi x\, e^{-9\pi^2 t} + \frac{1}{25} \cos 5\pi x\, e^{-25\pi^2 t} + \ldots\right)$$

3) Half-range expansion of $u(x, 0)$ in a Fourier sine series gives

$$a_n = \frac{2}{L} \int_0^L u(x, 0) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L} \int_0^{L/2} \frac{2hx}{L} \sin \frac{n\pi x}{L}\, dx + \frac{2}{L} \int_{L/2}^{L} 2h\left(1 - \frac{x}{L}\right) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{4h}{L^2}\left(\left[-\frac{xL}{n\pi} \cos \frac{n\pi x}{L}\right]_0^{L/2} + \frac{L^2}{n^2\pi^2}\left[\sin \frac{n\pi x}{L}\right]_0^{L/2}\right)$$

$$- \frac{4h}{L}\frac{L}{n\pi}\left[\cos \frac{n\pi x}{L}\right]_{L/2}^{L} - \frac{4h}{L^2}\left(\left[-\frac{xL}{n\pi} \cos \frac{n\pi x}{L}\right]_{L/2}^{L} + \frac{L^2}{n^2\pi^2}\left[\sin \frac{n\pi x}{L}\right]_{L/2}^{L}\right)$$

$$= -\frac{4h}{L^2}\frac{L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{4h}{L^2}\frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{4h}{n\pi}\left(\cos n\pi - \cos \frac{n\pi}{2}\right)$$

$$+ \frac{4h}{L^2}\left(\frac{L^2}{n\pi} \cos n\pi - \frac{L^2}{2n\pi} \cos \frac{n\pi}{2}\right) - \frac{4h}{n^2\pi^2}\left(\sin n\pi - \sin \frac{n\pi}{2}\right)$$

$$= \frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2}$$

Thus, the solution of the 'plucked string' problem is

$$u(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \cos \frac{n\pi vt}{L} \sin \frac{n\pi x}{L}$$

$$= \frac{8h}{\pi^2}\left[\frac{1}{1^2} \sin \frac{\pi x}{L} \cos \frac{\pi vt}{L} - \frac{1}{3^2} \sin \frac{3\pi x}{L} \cos \frac{3\pi vt}{L} + \ldots\right]$$

4) The solution (i) for the PDE for a vibrating shaft is

$$\theta(x, t) = X(x)\, T(t) = (C \cos \lambda x + D \sin \lambda x)$$

$$(A \cos \lambda vt + B \sin \lambda vt)$$

The boundary conditions for a shaft with both ends free,

$$\frac{\partial \theta}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

imply that for all $t > 0$

$$\frac{\partial X}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and at } x = L$$

This gives

$$D = 0 \quad \text{and} \quad C \sin \lambda L = 0$$

whence

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots .$$

Thus, the general solution for $\theta(x, t)$ is

$$\theta(x, t) = \sum_{n=1}^{\infty} (a_n \cos \lambda_n vt + b_n \sin \lambda_n vt) \cos \lambda_n x$$

where $a_n = A_n C_n$ and $b_n \equiv B_n C_n$.

At $t = 0$, $\theta(x, 0)$ is proportional to $(2x - L)/2$.

$$\therefore \ \theta(x, 0) = \sum_{n=1}^{\infty} a_n \cos \lambda_n x = k \frac{(2x - L)}{2}, \qquad 0 < x < L$$

Using the half-range expansion technique we get

$$a_n = \frac{2k}{L} \int_0^L \left(x - \frac{L}{2}\right) \cos \frac{n\pi x}{L} dx = \frac{2k}{L} \int_0^L x \cos \frac{n\pi x}{L} dx - k \int_0^L \cos \frac{n\pi x}{L} dx$$

$$= \frac{2k}{L} \left(\left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \right]_0^L + \frac{L^2}{n^2\pi^2} \left[\cos \frac{n\pi x}{L} \right]_0^L \right) - \frac{Lk}{n\pi} \left[\sin \frac{n\pi x}{L} \right]_0^L$$

$$= \frac{2k}{L} \frac{L^2}{n^2\pi^2} (\cos n\pi - 1)$$

$$= \frac{2Lk}{n^2\pi^2} (\cos n\pi - 1)$$

Since the shaft starts vibrating from rest, its initial velocity is zero giving the condition

$$\frac{\partial \theta}{\partial t} (x, 0) = 0$$

$$\text{or} \quad \sum_{n=1}^{\infty} b_n \lambda_n v \cos \lambda_n x = 0$$

This will be satisfied only if $b_n = 0$ for all n. Thus, the solution for the given BVP is

$$\theta(x, t) = \frac{2Lk}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (\cos n\pi - 1) \cos \frac{n\pi vt}{L} \cos \frac{n\pi x}{L},$$

5) We seek non-trivial solutions in the product form $u(x, y) = X(x)\, Y(y)$. Applying the method of separation of variables we get

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \quad 0 < x < L, \quad 0 < y < B$$

with the boundary conditions

$$X(x)\,Y(0) = 0, \qquad X(x)\,Y(B) = 0, \qquad 0 < x < L$$

or $Y(0) = 0$, $Y(B) = 0$

Since Y has to vanish at the boundaries $y = 0$ and $y = B$, the ratio $\frac{Y''}{Y}$ cannot be positive, Thus, we get the ODEs:

$$X'' - \lambda^2 X = 0, \quad Y'' + \lambda^2 Y = 0$$

whence $X(x) = A\cosh\lambda x + B\sinh\lambda x$

$$Y(y) = C\cos\lambda y + D\sin\lambda y$$

The boundary conditions on Y yield the following values of C and λ :

$$C = 0, \quad \lambda_n = \frac{n\pi}{B}, \qquad n = 1, 2, 3, \ldots.$$

Thus $Y_n(y) = D_n \sin\frac{n\pi y}{B}, \quad n = 1, 2, 3, \ldots.$

Thus, the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} (a_n \cosh\lambda_n x + b_n \sinh\lambda_n x)\sin\lambda_n y$$

where $a_n = A_n D_n$, $b_n = B_n D_n$

Applying the remaining boundary conditions we get

At $x = 0$, $\sum_{n=1}^{\infty} a_n \sin\lambda_n y = \frac{U_0 y}{B}, \qquad 0 < y < B$

Using the half-range expansion technique, we get

$$a_n = \frac{2}{B}\int_0^B \frac{U_0 y}{B}\sin\lambda_n y\,dy$$

$$= \frac{2U_0}{B^2}\left(\left[-\frac{y}{\lambda_n}\cos\lambda_n y\right]_0^B + \frac{1}{\lambda_n}\left[\frac{\sin\lambda_n y}{\lambda_n}\right]_0^B\right)$$

$$= \frac{2U_0}{B^2}\left(-\frac{B^2}{n\pi}\cos n\pi + 0\right)$$

$$= -\frac{2U_0\cos n\pi}{n\pi}$$

At $\quad x = L, \quad \frac{\partial u}{\partial x}(L, y) = -S, \qquad 0 < y < B$

Differentiating the series for $u(x,y)$ term by *term* and applying the given boundary conditions we get

$$\frac{\partial u}{\partial x}(L, y) = \sum_{n=1}^{\infty} \lambda_n (a_n \sinh\lambda_n L + b_n \cosh\lambda_n L)\sin\lambda_n y = -S, \qquad 0 < y < B$$

So we must choose b_n such that the coefficient of $\sin\lambda_n y$ will be

$$C_n = \lambda_n (a_n \sinh\lambda_n L + b_n \cosh\lambda_n L)$$

where $$C_n = \frac{2}{B}\int_0^B (-S \sin \lambda_n y)\, dy$$

$$= \frac{2S}{B}\left[\frac{\cos \lambda_n y}{\lambda_n}\right]_0^B = \frac{2S}{B\lambda_n}[\cos n\pi - 1]$$

$$= \frac{2S}{n\pi}(\cos n\pi - 1)$$

Thus

$$b_n = \frac{\frac{C_n}{\lambda_n} - a_n \sinh \lambda_n L}{\cosh \lambda_n L}$$

This completes the solution..

Terminal Questions

1) Using the method of separation of variables we write $y(x, t) = X(x)\, T(t)$ and obtain the ODEs for X and T with the corresponding boundary conditions:

$$X''(x) + \lambda^2 X(x) = 0, \quad X'(0) = 0, \quad X'(L) = 0$$

and $T''(t) + \lambda^2 v^2 T(t) = 0, \quad T(0) = cx, \quad T'(0) = 0$

The solutions are

$$X(x) = A \cos \lambda x + B \sin \lambda x$$

and $T(t) = C \cos \lambda vt + D \sin \lambda vt$

Applying the boundary conditions on X and T we get B = 0, $\sin \lambda L = 0$ which yields the eigen values.

$$\lambda_0 = 0, \quad \lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots \quad \text{and} \quad D = 0.$$

Thus, the general solution is

$$y(x, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi vt}{L}$$

where $a_n = A_n C_n$.

Applying the initial condition we have

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} = cx \qquad 0 < x < L$$

We can expand cx in a Fourier cosine series using the half- range expansion technique. Thus,

$$a_0 = \frac{c}{L}\int_0^L x\, dx = \frac{c}{L}\frac{L^2}{2} = \frac{cL}{2}$$

$$a_n = \frac{2c}{L}\int_0^L x \cos \frac{n\pi x}{L}\, dx$$

$$= \frac{2cL}{\pi^2}\left[\frac{\cos n\pi - 1}{n^2}\right]$$

$$= \frac{2cL}{\pi^2}\left(\frac{(-1)^n - 1}{n^2}\right), \qquad n = 1, 2, \ldots.$$

The solution is

$$y(x,t)=\frac{cL}{2}-\frac{4cL}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)\pi vt}{L}$$

2) Since v satisfies an equation of the fonn of diffusion equation, we can use its solution with given boundary conditions, where k = $1/RC$. The result is

$$v(x,t)=\sum_{n=1}^{\infty} b_n\, e^{-(n\pi/L)^2 t/RC}\sin\frac{n\pi x}{L}$$

Applying the initial condition we have

$$v(x,0)=\frac{6x}{L}=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L},\qquad 0<x<L$$

whence

$$b_n=\frac{2}{L}\int_0^L\frac{6}{L}x\sin\frac{n\pi x}{L}\,dx$$

$$=-\frac{12}{n\pi}(-1)^n$$

$$=\frac{12}{n\pi}(-1)^{n+1}$$

Therefore, the solution is

$$v(x,t)=\frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\exp\left(-\frac{t}{RC}\frac{n^2\pi^2}{L^2}\right)\sin\frac{n\pi x}{L}$$