If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ and $\lim_{(x,y)\to(x_0,y_0)} g(x,y)$ exist, then

i) $$\lim_{(x,y)\to(x_0,y_0)} (af+bg)(x,y) = a \lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm b \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad (A.2)$$

where a and b are constants.

ii) $$\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad (A.3)$$

iii) $$\lim_{(x,y)\to(x_0,y_0)} (f/g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \Big/ \lim_{(x,y)\to(x_0,y_0)} g(x,y) \qquad (A.4)$$

In Example 1, we have determined the limits of some functions using these formulas.

UNIT 6 PARTIAL DIFFERENTIAL EQUATIONS IN PHYSICS

Structure

6.1 Introduction

Objectives

6.2 The Method of Separation of Variables

6.3 Solving Initial and Boundary Value Problems in Physics

6.4 Summary

6.5 Terminal Questions

6.6 Solutions And Answers

6.1 INTRODUCTION

In Unit 5 you have learnt the basic concepts of order, degree, linearity, and type of partial differential equations (PDEs). Such equations arise for systems whose behaviour is governed by more than one independent variable. We come across PDEs in such diverse fields as meteorology, structural engineering, fluid mechanics, elasticity, heat flow, pollutant and neutron diffusion, wave propagation, aerodynamics, electromagnetics and nuclear physics. Most applied problems in physics are formulated in terms of second-order PDEs. From PHE-02 course on Oscillations and waves, you are familiar with the wave equation which governs wave propagation—a phenomenon responsible for hearing, seeing, music and our communication with the world at large. In your course on electric and magnetic phenomena, you would have come across Laplace's and Poisson's equations. These equations can also be used to determine gravitational potential, steady-state temperature etc.

A particularly useful method employed frequently to solve several second-order partial differential equations is the method of **separation of variables.** Depending on the number of independent variables, this method facilitates to reduce a linear PDE to two or more ordinary differential equations, which you already know to solve. This method is illustrated in Sec. 6.2. Boundary value problems in physics invariably exhibit rectangular, spherical or cylindrical symmetry in one or more dimensions. In Sec. 6.3 we illustrate the above said method to obtain a unique solution, subject to the given initial and boundary conditions. Since the same PDE may apply to many problems, the method discussed here can be used to solve many more problems than are illustrated here.

The term separation of variables was used in Unit 1 of this course in a completely different context.

Objectives

After studying this unit you should be able to

- solve a given PDE using the method of separation of variables
- obtain a unique solution to a given physical problem.

6.2 THE METHOD OF SEPARATION OF VARIABLES

Linear second order PDEs form the backbone of theoretical physics, Apart from Laplace's equation and Poisson's equation, the most important of these are the Helmholtz equation, Telegraph equation, wave equation, Klein-Gordon equation, Schrödinger equation and Dirac's equation.

Nonlinear PDEs are encountered in the study of shock wave phenomenon, atmospheric physics and turbulence. Higher order PDEs occur in the study of viscous fluids and elasticity..

The first question that should logically come to your mind is: How to solve a PDE? As a first strategy, we would like to reduce the given PDE to simpler differential equations containing fewer variables. (The process may be continued until a set of ordinary differential. equations is obtained), Next, we put the ODEs so obtained in easily solvable form using methods discussed in Block 1 of this course. The simplest and most widely used method for reducing common and physically important PDEs is the **method** of **separation of variables.** Let us now learn how it works..

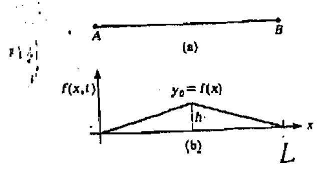

Fig. 6.1 : Vibrations of a string fixed at both ends

You will learn other analytical methods for solving PDEs, such as Green's function technique or numerical methods in later courses.

To illustrate the method of separation of variables, we consider a finite string *AB* of length L fixed at both ends, as shown in Fig. 6.1(a). Suppose that the string is plucked (initial displacement $h(x)$) and then released from rest, as shown in Fig. 6.1(b). If we choose x-axis along the length of the string, you may recall from Unit 5 of PHE-02 course Oscillations and Waves that the motion of the string is described by the 1- D wave equation :

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2} \tag{6.1}$$

You would note that there is no term containing mixed partials like $\frac{\partial^2 f}{\partial t \partial x}$ or $\frac{\partial f}{\partial t}$ $\frac{\partial f}{\partial x}$ in Eq. (6.1). This is because this equation is obtained under the assumption that the string is displaced only slightly from its equilibrium position;

We now assume that the solution of Eq. (6.1) can be written in the form of a product as

$$f(x, t) = X(x)\ T(t) \tag{6.2}$$

Physically, it means that the dependence of the unknown function on one variable is in no way affected by its dependence on other variable. Does this imply that there is no connection at all between X and *T*? No, it only means that the function X does not depend on t and the function T does not depend upon x. For instance, the function

$$f(x, t) = x \sin \omega t \tag{6.3a}$$

is completely separable in x and t. On the other hand, the function

$$f(x, t) = x + t \tag{6.3b}$$

is inseparable in that the function cannot be written as a product of two functions.

To illustrate the method, we differentiate Eq. (6.2) twice with respect to x. This gives

$$\frac{\partial f}{\partial x} = X'T$$

and
$$\frac{\partial^2 f}{\partial x^2} = X''T \tag{6.4}$$

where prime(s) denote ordinary differentiation with respect to x. This emphasises the fact that the derivative is the total derivative and the function X has only one independent variable. Similarly, if we differentiate Eq. (6.2) with respect to t, we obtain

$$\frac{\partial f}{\partial t} = X\dot{T}$$

and
$$\frac{\partial^2 f}{\partial t^2} = X\ddot{T} \tag{6.5}$$

where dot(s) denote ordinary differentiation with respect to t. We have used primes and dots just to distinguish the independent variables with respect to which differentiation has been carried out.

By inserting results contained in Eqs. (6.4) and (6.5) into Eq. (6.1), you would obtain.

$$X(x)\ \ddot{T}(t) = v^2 X''(x) T(t)$$

Dividing throughout by $v^2 X(x) T(t)$, we get

$$\frac{\ddot{T}(t)}{v^2 T(t)} = \frac{X''(x)}{X(x)} \tag{6.6}$$

The left hand side of this equation involves functions which depend only on t whereas the expression on right-hand side is a function of x only. Thus, if we vary t and keep x fixed, the right-hand side cannot change. This means that $\ddot{T}(t)/v^2 T(t)$ must remain constant for all t. Similarly, if we vary x holding t fixed, the left-hand side must not change. That is, the quantity $X''(x)/X(x)$ must be the same for all x. Mathematically, we express this fact by saying that both sides must be equal to a constant, k say. Is this argument sound? To discover the answer to this question, let us write k to represent either side of Eq. (6.6), i.e.,

$$\frac{\ddot{T}(t)}{v^2 T(t)} = k = \frac{X''(x)}{X(x)} \qquad (6.7)$$

This is really the key to the process of separation of variables.

Then from the right-hand side, we have

$$\frac{\partial}{\partial t}(k) = \frac{\partial}{\partial t}\left[\frac{X''(x)}{X(x)}\right] = 0$$

and from the LHS, we have

$$\frac{\partial}{\partial x}(k) = \frac{\partial}{\partial x}\left[\frac{\ddot{T}(t)}{v^2 T(t)}\right] = 0$$

Since the first order partial derivative of k with respect to t or **x** is zero, k must be **a constant.** It is called the **separation constant.** It means that if $y = a_0 \sin \omega t$ is a solution **of the** ODE

$$\ddot{y} + \omega^2 y = 0$$

we will get **an** identity, for all values of t, on substituting the assumed form of the **solution** in the given equation.

Thus, you can now rewrite the given equation as two ordinary differential equations:

$$X''(x) - k\,X(x) = 0 \qquad (6.8a)$$

and

$$\ddot{T}(t) - k\,v^2\,T(t) = 0 \qquad (6.8b)$$

That is, by assuming a separable solution, we have reduced a partial differential equation in two variables into two equivalent ordinary differential equations.

SAQ 1

Spend 15 minutes

Use the method of separation of variables to reduce the following **PDEs** to a set of **ODEs** :

i) $\dfrac{\partial^2 T}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial T}{\partial r} + \dfrac{\partial^2 T}{\partial z^2} = 0$

ii) $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V(r,\theta)}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial V}{\partial \theta}\right) = 0$

iii) $\dfrac{\partial^2 \psi}{\partial x^2} + \alpha \dfrac{\partial \psi(x,t)}{\partial t} = 0$

The **first** of these equations describes the steady-state temperature distribution in a cylindrical body, like **control/fuel** rods in the reactor core. The second PDE describes the potential in the region on either side of a spherical surface. The third PDE is the one-dimensional **Schrödinger** wave equation.

You can now solve these equations without much difficulty using the methods developed in Unit 2, Block 1 of this course. For instance, for a nonzero value of k, the solutions of **Eqs. (6.8a)** and **(6.8b)** are of the form **exp(mx)** and **exp(nt)** respectively. The characteristic equations are

$$m^2 - k = 0 \qquad (6.9a)$$

and

$$n^2 - k\,v^2 = 0 \qquad (6.9b)$$

which have roots

$$m_1 = \sqrt{k} = \mu,\ \ m_2 = -\sqrt{k} = -\mu \qquad (6.10a)$$

and

$$n_1 = v\sqrt{k} = v\mu, \quad n_2 = -v\sqrt{k} = -\mu v \qquad (6.10b)$$

The resulting solutions, therefore, are

$$X(x) = A\exp(\mu x) + B\exp(-\mu x) \qquad (6.11a)$$

and

$$T(t) = C\exp(\mu v t) + D\exp(-\mu v t) \qquad (6.11b)$$

which are sums of growing and decaying exponentials. If you calculate time derivative of $T(t)$, you will obtain velocity, which too will increase or decrease with respect to time. This means that the kinetic energy of an element of the string will increase and decrease with time simultaneously, which is physically unacceptable.

You can now write the general solution as

$$f(x, t) = X(x)\,T(t) = [A\exp(\mu x) + B\exp(-\mu x)]\,[C\exp(\mu v t) + D\exp(-\mu v t)] \qquad (6.12)$$

However, in view of the argument given before Eq. (6.12), this solution does not give the desired wave motion. So k cannot have positive values. Similarly the value $k = 0$ leads to a trivial solution and is not acceptable. However, for $k < 0$, $\sqrt{k}$ will be imaginary. Therefore, we can write

$$\sqrt{k} = i\beta$$

where β is a real number and $i = \sqrt{-1}$. Then, Eq. (6.11a) becomes

$$X(x) = A\exp(i\beta x) + B\exp(-i\beta x) \qquad (6.13a)$$

and Eq. (6.11b) takes the form

$$T(t) = C\exp(i\beta v t) + D\exp(-i\beta v t) \qquad (6.13b)$$

$\exp(i\theta) = \cos\theta + i\sin\theta$

Using the Euler's relation, you can rewrite Eqs. (6.13a) and (6.13b) as

$$X(x) = A_1\sin\beta x + A_2\cos\beta x \qquad (6.14a)$$

and

$$T(t) = G_1\sin\beta v t + G_2\cos\beta v t \qquad (6.14b)$$

where A_1, A_2, G_1 and G_2 are new constants, You can easily verify that $A_1 = i(A - B)$, $A_2 = A + B$, $G_1 = i(C - D)$ and $G_2 = C + D$. The solutions given by Eqs. (6.14a, b) are periodic in space and time. You can now write the general solution of 1 - D wave equation as

$$f(x, t) = X(x)\,T(t) = (A_1\sin\beta x + A_2\cos\beta x)\,(G_1\sin\beta v t + G_2\cos\beta v t) \qquad (6.15)$$

In the above example we have illustrated the method of separation of variables by considering PDEs in two variables (x, t). Can you think of a physical situation where the PDE of interest involves more than two variables? The music produced by a drum used in folk dances involves the vibrations of a circular membrane. The wave motion is two dimensional and the PDE involves three variables (r, θ, t). Similarly, in the heat flow in a rectangular plate, the number of independent variables is three : (x, y, t). This is of particular interest to a reactor physicist since plate type fuel elements may be used in the reactor core. It is, therefore, important for us to extend the method of separation of variables to three (or more) variables. For simplicity, let us first consider a rectangular membrane whose edges are fixed at $x = 0$, $x = a$, $y = 0$ and $y = b$, as shown in Fig. 6.2.

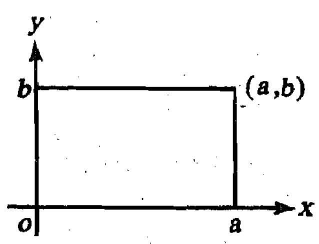

Fig. 6.2 : A rectangular membrane fixed at edges

Rectangular Membrane

The function $f(x, y, t)$ satisfies the wave equation

$$\frac{\partial^2 f(x, y, t)}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y, t) \qquad (6.16)$$

While solving this equation by the method of separation of variables, we expect to reduce it to three second order ODEs, which possess periodic solutions in space and time. You can do so in two ways :

i) By partial separation of Eq. (6.16) in space variables (x, y) taken together and the time variable by writing

$$f(x, y, t) = F(x, y)\, T(t) \tag{6.17a}$$

where $F(x, y)$ is a function of space and T depends only on time.

This will result in an **ODE** in time and a **PDE in** space variables, which may then be further split to arrive at ODEs in x and y. This two-stage process is worthwhile to attempt as it invariably facilitates mathematical steps.

ii) Separate all the three variables by writing

$$F(x, y, t) = X(x)\, Y(y)\, T(t) \tag{6.17b}$$

How do we know that **this** is valid? The answer is simple. We do not say that it is valid; We only wish to discover if it works. But we expect that both substitutions should lead us to the same result. Why? Because a **tool** (mathematical **technique**) cannot influence physics. We now illustrate this by solving Eq. (6.16) using both substitutions.

As before, let us assume a separable solution

$$f(x, y, t) = F(x, y)\, T(t)$$

where $F(x, y)$ is a function of space only and $T(t)$ is a function of time only. Subsfituting it in Eq. (6.16) we find that

$$F\ddot{T} = v^2 T\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y)$$

Dividing throughout by v^2FT, we get

$$\frac{\ddot{T}}{v^2 T} = \frac{1}{F}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y) \tag{6.18}$$

By comparing it with Eq. (6.6) you can say that the expression on **the** left-hand side depends only on t, whereas the expression on the right-hand side depends only on space variables. Following the arguments **used** for wave equation in two variables, we can say that both sides must be equal to a constant. We now know that only negative values of this constant will **lead to** a **nontrivial solution.** If we denote this constant by $-p^2$, we have

$$\frac{\ddot{T}}{v^2 T} = \frac{1}{F}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F(x, y) = -p^2 \tag{6.19}$$

This yields two differential **equations :**

$$\ddot{T} + p^2 v^2 T = 0 \tag{6.20}$$

and

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + p^2 F = 0 \tag{6.21}$$

You will note that **whereas** Eq. (6.20) is an ordinary differential equation, Eq. (6.21) still contains partial **derivatives in** x and y. That is, although we have separated the space and time variables, we have to separate space dependences. To do so, we assume that

$$F(x, y) = X(x)\, Y(y) \tag{6.22}$$

Substituting it in Eq. (6.21). we **obtain**

$$\frac{d^2X}{dx^2}Y = -X\left(\frac{d^2Y}{dy^2} + p^2 Y\right)$$

On dividing both sides by XY, we find that

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\left(\frac{d^2Y}{dy^2} + p^2 Y\right) \tag{6.23}$$

Note that the expression on **LHS** depends only on x, whereas the expression on RHS depends only on y. Therefore, both sides must be equal to a constant, which we take $-q^2$:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\left(\frac{d^2Y}{dy^2} + p^2Y\right) = -q^2 \tag{6.24}$$

This immediately leads to two ordinary equations:

$$\frac{d^2X}{dx^2} + q^2X = 0 \tag{6.25}$$

and

$$\frac{d^2Y}{dy^2} + \alpha^2Y = 0 \tag{6.26}$$

where $\alpha^2 = p^2 - q^2$.

We thus find that Eq. (6.16) which contained derivatives with respect to three independent variables has been reduced to three separate second-order **ODEs** (Eqs. **(6.20)**, (6.25) and **(6.26)**. Thus in the two-stage process of separation of variables, we separated the time dependence from the space dependence by clubbing them in one function, $F(x, y)$, which is subsequently **separated**.

Let us now split Eq. (6.16) by taking f (x, y, t) as a product of three functions as in Eq. **(6.17b)**. Then we can write

$$\frac{\partial^2 f}{\partial t^2} = X(x)\, Y(y)\, \ddot{T}(t)$$

$$\frac{\partial^2 f}{\partial x^2} = X''(x)\, Y(y)\, T(t)$$

and

$$\frac{\partial^2 f}{\partial y^2} = X(x)\, Y''(y)\, T(t)$$

On substituting these in Eq, (6.16) you will obtain

$$X(x)\, Y(y)\, \ddot{T}(t) = v^2\, [Y(y)\, T(t)\, X'' + X(x)\, Y''(y)\, T(t)]$$

On dividing throughout by $T(t)\, X(x)\, Y(y)$, this equation simplifies to

$$\frac{\ddot{T}(t)}{v^2 T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} \tag{6.27}$$

The left hand side of this identity is a function only of time and the right hand is a function only of the space variables. Therefore, we can write

$$\frac{1}{v^2}\frac{\ddot{T}}{T} = -k^2$$

or

$$\ddot{T} + k^2 v^2 T = 0 \tag{6.28a}$$

and

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -k^2$$

or

$$\frac{1}{X}\frac{d^2X}{dt^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} \tag{6.28b}$$

Here we have a function of x equated to a function of y. As before, we equate each side to another constant, $-m^2$. So we can split Eq. (6.28b) into two ODEs:

$$\frac{1}{X}\frac{d^2X}{dt^2} = -m^2 \qquad (6.29a)$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + m^2 = -n^2 \qquad (6.29b)$$

where we have introduced a new constant by $k^2 = m^2 + n^2$ to produce a symmetric set of equations. Thus we find that Eq. (6.16) has been replaced by three ODEs (Eqs. (6.28a), (6.29a) and (6.29b)).

If you identify p with k, m with q and n with α, Eqs. (6.28a), (6.29a) and (6.29b) will become identical to Eqs. (6.20) (6.25) and (6.26). respectively. We hope that now you have understood both the processes. To get a better grasp of these concepts you may like to work out an SAQ.

SAQ 2

Spend 15 minutes

The Helmholtz equation in Cartesian coordinates can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2 f(x, y, z) = 0$$

Reduce it to three ODEs using one step process.

So far we have considered a rectangular membrane fixed at its edges. The space variables in the PDE describing its vibrations were taken to be Cartesian. But you will readily recognise that for musical instruments like beating-drum and cymbal, use of spherical polar coordinates is desirable. You can mathematically model wave propagation in these instruments by considering the vibrations of a circular membrane, Let us now learn to separate wave equation in spherical coordinates.

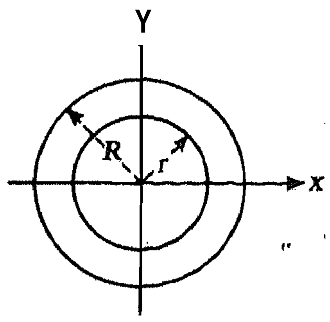

Fig. 6.3 : A circular membrane fixed at the perimeter

Circular Membrane

For a circular membrane held fixed at the perimeter, as shown in Fig. 6.3, the wave equation takes the form

$$\frac{\partial^2 f}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f(r, \theta, t) \qquad (6.30)$$

By assuming a solution in the separable form as

$$f(r, \theta, t) = F(r, \theta)\, T(t) \qquad (6.31)$$

you can readily show that Eq. (6.30) reduces to

$$\ddot{T} + \lambda^2 T = 0 \qquad (6.32a)$$

and

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F(r, \theta) = 0 \qquad (6.32b)$$

You would recall from PHE-02 course that the three dimensional wave equation is

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where ∇^2 is the Laplacian. You have studied its form in different coordinate systems in Unit 3 of the PHE-04 course.

where $\lambda = vk$; k being the separation constant. You will note that Eq. (6.32b) still contains two variables. We separate these as well and write

$$F(r, \theta) = R(r)\,\Theta(\theta) \qquad (6.33)$$

Substituting in Eq. (6.32b), we obtain

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + r^2k^2R\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = p^2$$

so that Eq. (6.32b) reduces to two ODEs; one involving R and the other one for Θ

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (r^2k^2 - p^2)\,R = 0 \tag{6.34a}$$

and

$$\frac{d^2\Theta}{d\theta^2} + p^2\,\Theta = 0 \tag{6.34b}$$

Spend 10 minutes

SAQ 3

By substituting

$$f(r, \theta, t) = R(r)\,\Theta(\theta)\,T(t)$$

in Eq. (6.30), show that it can be split into three ODEs.

To put Eq. (6.34a) in a more familiar form, let us introduce a change of variable by defining

$$s = kr$$

Then

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = k\frac{dR}{ds}$$

and

$$\frac{d^2R}{dr^2} = k^2\frac{d^2R}{ds^2}$$

Substituting these in Eq. (6.34a) you will get

$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + (s^2 - p^2)\,R = 0 \tag{6.35}$$

which is Bessel's equation of order p.

So far we have familiarised you with the basic technique of separating variables for reducing a PDE to a set of ODEs. You can solve these using the methods described in Block 1. However, in physical problems, we have to usually obtain unique solutions of PDEs, which correspond to certain initial and boundary conditions.

Before proceeding further to solve initial and boundary value problems in PDEs, let us stop for a while and summarise what we know about the method of separation of variables.

1) First of all, the unknown function of two (or more) variables is expressed as a product of two (or more) functions so that the dependence of one on an independent variable is in no way affected by the dependence of the other variable(s).

2) The assumed form of solution is inserted in the given differential equation. A second-order PDE in two variables splits into two ODEs. When the number of independent variables is more than two, we get ODEs equal in number to the independent variables.

3) You can solve the ODEs so obtained using methods known from Block-1. The solutions may be exponential functions, trigonometric functions, or power series.

4) The general solution of the given PDE is obtained by taking the product of the solutions of ODEs.

We hope that you can now use the method of separation of variables to reduce a PDE to a set of ODEs. (The number of ODEs equals the number of independent variables in the given PDE.) For the remainder of this unit, we shall confine ourselves to finding product solutions of wave equation, heat equation and Laplace's equation for different physical situations, under specific initial and boundary conditions.

6.3 SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS IN PHYSICS

When we solve a PDE, the number of solutions is, in general, very large. For example, if you consider Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

you can readily verify that each of the functions

$$f = x^2 - y^2, f = e^a \cos y \text{ and } f = \ln(x^2 + y^2)$$

satisfies the given equation. However, all these are completely different from each other. There are many other functions which would satisfy the above-said equation. Does this mean that we cannot obtain a general solution of a PDE? In physical problems, a general solution is seldom sought. Even if we can obtain a general solution, it involves too much arbitrariness. That is, it is not unique. You may ask: Why is it so? This is because a PDE with independent variables in space (x) and time (t), which is of second order in each'of these variables, requires two conditions at some x and two conditions at some t. (If the condition on x is specified at a boundary, we say that we are **specifying boundary** conditions. Usually the conditions on time are given at the instant we start making observations. These are referred to as initial conditions). So you may conclude that

Refer to IVP and BVP defined in Block I.

> To obtain a unique solution to a given PDE, we have to specify initial conditions (ICs) and boundary conditions (BCs) which correspond to the particular physical problem.

Let us consider certain IVPs and BVPs in PDEs that arise in physics.

One-dimensional wave equation

Let us consider a wave propagating on a string. We put equidistant marks to identify particles of the string. We wish to determine instantaneous displacement of a particle at any of these marked positions. Mathematically speaking, we wish to determine a function $f(x, t)$, which depends on two independent variables. We have to supplement the PDE describing this phenomenon by BCs. The boundary conditions will involve f, or some of its derivatives, or both, on the curve (boundary) enclosing the region (of independent variables) over which a solution is being sought.

Proceeding further, we note, that we have to solve a one-dimensional equation for a string of length L such as a guitar, an Ektara or a violin string illustrated in Fig; 6.1 :

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f(x, t)}{\partial x^2}$$

where v is wave speed.

Since the string is fixed at x = 0 and x = L for all times, the boundary conditions may be written as

$$f(0, t) = 0$$

and

$$f(L, t) = 0 \text{ for all } t > 0 \qquad (6.36)$$

Since the solution of the wave equation depends on t as well, we must also know as to what happens at t = 0. That is, we have to specify initial conditions on displacement and velocity. Since the string is released from rest, the initial velocity is zero. In mathematical terms, we seek a function f (x, t) which satisfies the initial conditions

$$f(x, 0) = h(x) \qquad 0 < x < L$$

and

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{t=0} = 0 \tag{6.37}$$

Separating variables in wave equation, you will obtain

$$X'' + \mu^2 X(x) = 0$$

and

$$\ddot{T}(t) + \mu^2 v^2 T(t) = 0.$$

whose solutions are given by Eq. (6.11):

$$X(x) = A \cos \mu x + B \sin \mu x$$

and

$$T(t) = C \cos \mu v t + D \sin \mu v t$$

Now since $f(0, t) = X(0)\, T(t) = 0$ and $f(L, t) = X(L)\, T(t) = 0$, we must have $X(0) = 0$ and $X(L) = 0$. Using the first of these conditions, we find that $A = 0$. Therefore

$$X(x) = B \sin \mu x$$

The second condition now implies that

$$X(L) = B \sin \mu L = 0$$

This equality will be **satisfied** if $B = 0$ or $\sin \mu L = 0$. If $B = 0$, then $X = 0$ so that $f = 0$, which is a trivial solution. Hence, we must have $B \neq 0$ and the only option is $\sin \mu L = 0$. This implies that $\mu L = n\pi$ or $\mu = n\pi/L$ for $n = 0, 1, 2, 3, \ldots$. The solution for $n = 0$ is a trivial solution. For any arbitrary value of B, we obtain infinite solutions of the form

$$X(x) \equiv X_n(x) = B_n \sin\left(\frac{n\pi}{L}\right)x \qquad n = 1, 2, 3, \ldots \tag{6.38}$$

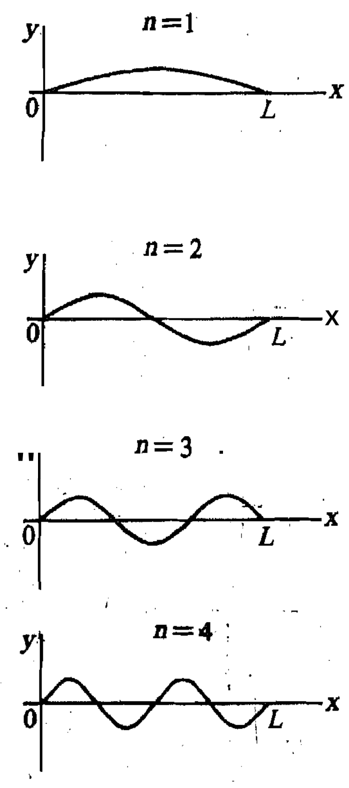

Fig. 6.4: A plot of Eq. (6.38) for the first few modes

The values of $\mu = n\pi/L$ for $n = 1, 2, 3, \ldots$ are called **eigenvalues** of **Eq.** (6.1). With B = 1, Eq. (6.38) is depicted in Fig. 6.4 for $n = 1, 2, 3$ and 4. **Hence**, the solution of Eq. (6.2) which **satisfies** the given boundary conditions **can now** be written as

$$\begin{aligned} f_n(x,t) &= \left[C \cos\left(\frac{n\pi v t}{L}\right) + D \sin\left(\frac{n\pi v t}{L}\right)\right] B_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \left[a_n \cos\left(\frac{n\pi v t}{L}\right) + b_n \sin\left(\frac{n\pi v t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{6.39}$$

where we have put $CB_n = a_n$ and $DB_n = b_n$ since each value of n may require different **constants.** You would note that the subscript n has been added to $f(x, t)$. Do you know why? This is just to allow for a different function for each value of n. In the present case, each value of n defines harmonic motion of the string with frequency $(n\, v/2L)$ **Hz.** Whereas $n = 1$ defines the **fundamental** mode, $n > 1$ **characterises overtones.**

You would agree that $f_n(x, t)$ is not a solution of the given problem since initial conditions have not yet been imposed. Moreover, since the wave equation is linear and homogeneous, we expect that the **most** general solution, which satisfies the given boundary **conditions,** is given by the superposition principle:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi v t}{L}\right) + b_n \sin\left(\frac{n\pi v t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \tag{6.40}$$

To match the initial conditions, we set $t=0$ in the above **equation.** This gives

$$f(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = h(x) \tag{6.41}$$

Now the **question** arises: How to evaluate a_n? To **determine** the **constants** a_n, we must **know** the form of the function $h(x)$. Let us take $h(x) = \xi_0 \sin \frac{\pi x}{L}$. Then by comparison, we have

$$a_1 = \xi_0$$

and

$$a_2 = a_3 = \ldots = 0 \qquad (6.42)$$

For any general form of $h(x)$ you would require Fourier series, which you will learn in the next two units.

To determine b_n, we first differentiate Eq. *(6.40)* with respect to t and then set $t = 0$. The result is

$$\frac{\partial f}{\partial t} = \sum_{n=1}^{\infty} \left[- a_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi vt}{L}\right) + b_n \left(\frac{n\pi v}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

so that

$$\left.\frac{\partial f}{\partial t}\right|_{t=0} = 0 = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

You will readily cdnclude by looking at this expression that

$$b_n = 0\,; \; n = 1, 2, \ldots \qquad (6.43)$$

Hence the unique solution of the one-dimensional wave equation on a string tied at both ends corresponding to the given initial and boundary conditions is given by

$$f(x, t) = \xi_0 \cos \omega t \sin \left(\frac{\pi x}{L}\right) \qquad (6.44)$$

where $\omega = \frac{\pi v}{L}$ is angular frequency and ξ_0 is amplitude.

SAQ 4

Spend 2 *minutes*

Determine the constants a_n's occurring in Eq. *(6.40)* when

$$h(x) = \xi_0 \left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

The boundary value problems considered in SAQ 4 refer to Cartesian geometry. You know of many physical problems which involve spherical and cylindrical coordinates. You have studied these in Unit 3 of Block 1, PHE-04 course entitled Mathematical Methods in Physics-I. In particular, we may mention wave propagation on the membrane of a tabla or a beating drum, electric field around a long current carry wire, energy produced in a nuclear reactor, etc. In the following examples, we have considered physical problems involving spherical polar and cylinderical coordinates.

Example 1 : Circular Membrane

The radial part of wave equation for a circular membrane of radius r_0 fixed at its circumference is

$$\frac{\partial^2 f}{\partial t^2} = v^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right)$$

Specify the boundary conditions and obtain a unique solution.

Solution

The boundary conditions in this case are

$$f(r, t) = 0 \qquad \text{for } r = r_0 \text{ at all } t$$

$$= 0 \qquad \text{for } t = 0 \text{ at all } r$$

The wave equation describing the motion of circular membrane fixed at its perimeter can be reduced to the following ODEs :

$$\ddot{T} + \lambda^2 T = 0 \text{ where } \lambda = vk \tag{i}$$

and

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2 R = 0 \tag{ii}$$

By introducing a chance of variable through the relation

$$s = kr$$

we find that

$$\frac{dR}{ds} = \frac{dR}{dr}\frac{dr}{ds} = \frac{1}{k}\frac{dR}{dr}$$

and

$$\frac{d^2R}{ds^2} = \frac{1}{k}\frac{d}{dr}\left(\frac{dR}{dr}\right)\frac{dr}{ds}$$

$$= \frac{1}{k^2}\frac{d^2R}{dr^2}$$

so that we can rewrite (ii) as

$$k^2\frac{d^2R}{ds^2} + \frac{k^2}{s}\frac{dR}{ds} + k^2 R = 0$$

or
$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} + R = 0 \tag{iii}$$

Do you recognise this equation? This is zeroth order Bessel equation (see Unit 3 of Block 1 of this course). Its general solution is

$$R(s) = c_1 J_0(s) + c_2 Y_0(s)$$

In terms of r, we can write

$$R(r) = c_1 J_0(kr) + c_2 Y_0(kr) \tag{iv}$$

where J_0 and Y_0 are zeroth order Bessel's functions of the first and the second kind, respectively.

The Bessel's function of the second kind is known to tend to $-\infty$ as $r \to 0$. Therefore, we must choose $c_2 = 0$. Then (iv) reduces to

$$R(r) = c_1 J_0(kr) \tag{v}$$

Since the membrane is fixed at the perimeter ($r = r_0$), the boundary condition $f(r, t) = 0$ implies that

$$R(r) = 0 \text{ for } r = r_0$$

Obviously if we choose $c_1 = 0$ we will obtain a trivial solution. Therefore, we must choose $J_0(kr_0) = 0$. If we denote the zeros of $J_0(kr_0)$ by $\alpha_1, \alpha_2, \ldots$, we get

$$kr_0 = \alpha_m$$

or $$k \equiv k_m = \frac{\alpha_m}{r_0} \qquad \text{(vi)}$$

Hence, (v) takes the form

$$R(r) = c_1 J_0\left(\frac{\alpha_m}{r_0}\right)$$

This solution is depicted in Fig. 6.5

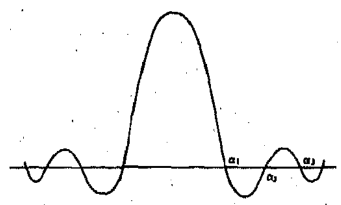

Fig. 6.5 : Plot of radial solution of wave equation

Example 2

Consider a solid cylindrical cooling tower as shown in Fig. 6.6. The steady-state temperature distribution is described by Laplace's equation $\nabla^2 T = 0$

i) Specify the boundary conditions, and

ii) determine the steady-state temperature.

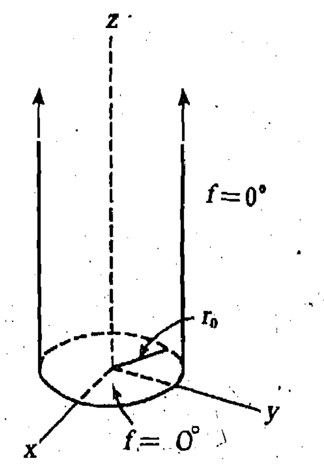

Fig. 6.6 : A cylindrical cooling tower

Solution

i) By looking at the figure, we can express the boundary conditions as,

$$T(2, z) = 0, \qquad 0 < z < 4,$$

$$T(\rho, 0) = 0, \qquad T(\rho, 4) = T_0,\ 0 < \rho < 2 \qquad \text{(i)}$$

ii) To determine the steady-state temperature we have to express the Laplacian in cylindrical polar coordinates. Since the given geometry exhibits circular symmetry we can write

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}$$

so that

$$\frac{\partial^2 T}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial T}{\partial\rho} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad 0 < \rho < 2;\ 0 < z < 4$$

Putting $T = R(\rho)\,Z(z)$ and separating variables, you will obtain

$$\frac{R'' + \frac{1}{\rho}R'}{R} = -\frac{Z''}{Z} = -\lambda^2$$

which reduces to

$$\rho R'' + R' + \lambda^2 \rho R = 0 \qquad \text{(ii)}$$

and $$Z'' - \lambda^2 Z = 0 \qquad \text{(iii)}$$

The negative separation constant is used since there is no reason to expect the solution to be periodic in z.

Eq. (ii) is the zeroth order parametric Bessel's equation. Its general solution is

$$R = c_1 J_0(\lambda\rho) + c_2 Y_0(\lambda\rho)$$

where J_0 and Y_0 are Bessel's functions of the first and the second kind of order zero.

Since the solution of (iii) is defined on the finite interval (0, 4), we write

$$Z = c_3 \cosh\lambda z + c_4 \sinh\lambda z.$$

In order to have a bounded temperature $T(\rho, z)$ at $\rho = 0$, we must define $c_2 = 0$. The condition $T(2, z) = 0$ implies $R(2) = 0$ or

$$J_0(2\lambda) = 0 \qquad \text{(iv)}$$

This equation will hold for $\lambda_1 = \alpha_1/2,\ \lambda_2 = \alpha_2/2, \ldots, \lambda_n = \alpha_n/2$ where $\alpha_1, \alpha_2, \ldots \alpha_n$ are zeros of the Bessel function. Lastly $Z(0) = 0$ implies $c_3 = 0$. Hence we have $R = c_1 J_0(\lambda_n \rho)$, $Z = c_4 \sinh \lambda_n z$, and

$$u_n = A_n \sinh \lambda_n z \; J_0(\lambda_n \rho)$$

The general solution is therefore of the form

$$u(r, z) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n z \; J_0(\lambda_n \rho)$$

Example 3

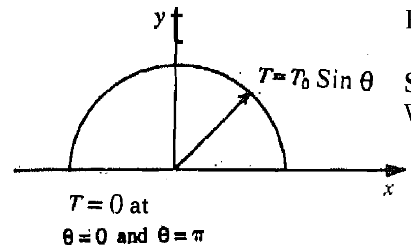

Fig. 6.7 : A semi circular plate

Find the steady-state temperature $T(r, \theta)$ in the semi-circular plate shown in Fig. 6.7.

Solution

We must solve

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad 0 < \theta < \pi; \quad 0 < r < a_0$$

subject to the boundary conditions

$$T(a_0, \theta) = T_0 \sin \theta \qquad 0 < \theta < \pi$$

$$T(r, 0) = 0,\ T(r, \pi) = 0 \qquad 0 < r < a_0$$

If we define $T = R(r)\,\Theta(\theta)$, then separation of variables gives

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

so that original **PDE** reduces to

$$r^2 R'' + rR' - \lambda^2 R = 0 \qquad \text{(i)}$$

and

$$\Theta'' + \lambda^2 \Theta = 0 \qquad \text{(ii)}$$

Applying the boundary conditions $\Theta(0) = 0$ and $\Theta(\pi) = 0$ to the solution $\Theta = c_1 \cos \lambda\theta + c_2 \sin \lambda\theta$ of (ii) yields $c_1 = 0$ and $\lambda = n$; $n = 1, 2, 3, \ldots$ Hence $\Theta = c_2 \sin n\theta$. For $\lambda = n$, (i) is Cauchy-Euler equation. You can solve it using methods discussed in Unit 3. It has the solution

$$R = c_3 r^n + c_4 r^{-n}$$

In order that the solution $T(r, \theta)$ is finite as $r \to 0$ we must demand that $c_4 = 0$. Therefore

$$T_n = A_n r^n \sin n\theta$$

and

$$T(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

The boundary condition at $r = a_0$ gives

$$T_0 \sin \theta = \sum_{n=1}^{\infty} A_n a_0^n \sin n\theta$$

so that $A_1 = \dfrac{T_0}{a_0}$ and $A_2 = A_3 = 0 = \ldots = A_n$

Hence

$$T(r, \theta) = \frac{T_0}{a_0} r \sin \theta$$

Let us now sum up the unit.

6.4 SUMMARY

A linear partial differential equation in two variables can be solved by assuming a solution in the form of a product $f = XY$ where X is a function of x only, and Y is a function of y only. This **method of separation of variables** leads to two ordinary differential equations.

A boundary value problem consists of finding a function that satisfies a partial differential equation as well as conditions consisting of boundary conditions and initial conditions.

- The method of solving a boundary value problem using separation of variables consists of the following basic steps.
  - i) Write the function f as a product of two (or more) functions involving independent variables, i.e. $f(x, y) = X(x)\, Y(y)$ and insert it in the given PDE. You will obtain a set of ODEs. A PDE in two variables splits into two ODEs. If the number of independent variables is more, we get ODEs whose number is equal to the number of variables.
  - ii) Solve the separated ordinary differential equations. The solutions may be exponential functions, trignometric functions or a power series.
  - iii) Substitute the solutions so obtained in the above product.
  - iv) Use boundary and/or the initial condition(s), and solve for the coefficients'in the series.

6.5 TERMINAL QUESTIONS

1. i) The electrostatic potential in the exterior and interior of a spherical shell is calculated by using Laplace's equation : $\nabla^2 f = 0$. Use the method of separation of variables to split it into three ODEs.
(Hint : Express ∇^2 in spherical polar coordinates.)

   ii) The wave propagation in space is described by 3-*D* equation

$$\nabla^2 f(\mathbf{r}, t) = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} (\mathbf{r}, t)$$

Working in Cartesian coordinates show that it can be reduced to four ODEs:

$$\ddot{T} + \omega^2 T = 0$$

$$X'' + l^2 X = 0$$

$$Y'' + m^2 Y = 0$$

and

$$Z'' + n^2 Z = 0$$

where $w = vk$ and I, m, n are separation constants,

2. The one dimensional wave equation for em wave propagation in free space is given by (for $E \parallel \hat{y}$)

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0$$

Solve,this equation and obtain the eigen frequencies of the cavity if $E_y = 0$ at $x = 0$ and $x = L$.

3. Consider a rod whose ends are kept at a constant temperature and the lateral surface is insulated. The heat flow is described by one-dimensional heat equation subject to the conditions

$$f(0, t) = 0, f(L, t) = 0 \qquad \text{for } t > 0$$

and

$$f(x, t) = f(x) \qquad \text{for } 0 < x < L$$

Obtain a unique solution.'

6.6 SOLUTIONS AND ANSWERS

SAQs

1. i) Letustake

$$T(r, z) = R(r)\, Z(z) \qquad \text{(i)}$$

Then,

$$\frac{\partial T}{\partial r} = \frac{dR}{dr} Z$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{d^2 R}{dr^2} Z$$

and

$$\frac{\partial^2 T}{\partial z^2} = R \frac{d^2 Z}{dz^2} \qquad \text{(ii)}$$

Substituting these in the given PDE, we obtain

$$Z \frac{d^2 R}{dr^2} + \frac{Z}{r} \frac{dR}{dr} + R \frac{d^2 Z}{dz^2} = 0$$

On dividing throughout by ZR, we get

$$\frac{1}{R}\left[\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right] = -\frac{1}{Z}\frac{d^2 Z}{dz^2} \qquad \text{(iii)}$$

The LHS of this equality involves functions which depend only on r, whereas the expression on RHS is a function of z only. So both sides must be equal to a constant, k. Hence the given equation splits into the following two ODEs:

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - kR = 0 \qquad \text{(iv)}$$

and

$$\frac{d^2 Z}{dz^2} + kZ = 0 \qquad \text{(v)}$$

ii) The given PDE can be rewritten as

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r} + \cot\theta \frac{\partial V}{\partial \theta} + \frac{\partial^2 V}{\partial \theta^2} = 0 \qquad \text{(i)}$$

Let us now write

$$V(r, \theta) = R(r)\, \Theta(\theta) \qquad \text{(ii)}$$

Then, differentiation with respect to r gives

$$\frac{\partial V}{\partial r} = \frac{dR}{dr}\Theta$$

and

$$\frac{\partial^2 V}{\partial r^2} = \frac{d^2R}{dr^2}\Theta \qquad \text{(iii)}$$

Similarly, differentiation with respect to θ gives

$$\frac{\partial V}{\partial \theta} = R\frac{d\Theta}{d\theta}$$

and

$$\frac{\partial^2 V}{\partial \theta^2} = R\frac{d^2\Theta}{d\theta^2} \qquad \text{(iv)}$$

Substituting these results in the given equation, we obtain

$$r^2\Theta\frac{d^2R}{dr^2} + 2r\Theta\frac{dR}{dr} + \cot\theta\, R\frac{d\Theta}{d\theta} + R\frac{d^2\Theta}{d\theta^2} = 0$$

As before, on dividing throughout by R 8 , we get

$$\frac{1}{R}\left[r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right] = -\frac{1}{\Theta}\left[\cot\theta\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2}\right] \qquad \text{(v)}$$

The LHS is a function of r only whereas RHS is a function of θ only. Hence, putting them equal to a constant, k, we get

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - kR = 0$$

or

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - kR = 0 \qquad \text{(vi)}$$

and

$$\cot\theta\frac{d\Theta}{d\theta} + \frac{d^2\Theta}{d\theta^2} + k\Theta = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + k\Theta = 0 \qquad \text{(vii)}$$

iii) The given equation is

$$\frac{\partial^2\psi}{\partial x^2} + \alpha\frac{\partial\psi}{\partial t} = 0$$

Express $\psi(x, t)$ as a product of two separable functions:

$$\psi(x, t) = X(x)\,T(t)$$

Substituting it in the given **PDE,** you will obtain

$$X''(x)\,T(t) + \alpha X(x)\,\dot{T}(t) = 0$$

Dividing throughout by $X(x)\,T(t)$, we find that

$$\frac{X''(x)}{X(x)} = -\alpha\frac{\dot{T}(t)}{T(t)} = -k^2$$

so that the Schrbdinger equation splits into following equations:

$$X''(x) + k^2 X = 0$$

and

$$\dot{T}(t) - \lambda^2 T(t) = 0$$

where $\lambda^2 = k^2/\alpha$.

2. The Helmholtz equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x, y, z) + k^2 f = 0 \qquad \text{(i)}$$

Let us write

$$f(x, y, z) = X(x)\, Y(y)\, Z(z) \qquad \text{(ii)}$$

Substituting it in the given PDE, we get

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} + k^2 XYZ = 0$$

Dividing throughout by X Y Z and rearranging terms, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad \text{(iii)}$$

The LHS is a function of **x** alone, whereas the RHS depends only on **y** and **z**. Let us choose

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2 \qquad \text{(iv)}$$

Then you can write

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad \text{(v)}$$

Here we have a function of y equated to a function of z. We now set

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2 \qquad \text{(vi)}$$

so that

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2 \qquad \text{(vii)}$$

where we have put $k^2 = l^2 + m^2 + n^2$. Eqs. (iv), (vi) and (vii) are the three ODEs into which Helmholtz equation splits.

3. We know that

$$\frac{\partial^2 f}{\partial t^2} = v^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) f(r, \theta, t) \qquad \text{(i)}$$

By substituting

$$f(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t) \qquad \text{(ii)}$$

you can write

$$\frac{\partial^2 f}{\partial t^2} = \ddot{T} R \Theta$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{d^2R}{dr^2}\Theta T$$

and

$$\frac{\partial^2 f}{\partial \theta^2} = RT \frac{d^2 \Theta}{d\theta^2} \qquad \text{(iii)}$$

Using these results in (i) you will find that

$$R \Theta \ddot{T} = v^2 \left(\Theta T \frac{d^2 R}{dr^2} + \frac{\Theta T}{r} \frac{dR}{dr} + \frac{RT}{r^2} \frac{d^2 \Theta}{d\theta^2} \right)$$

All the partial derivatives have now become ordinary derivatives. Dividing throughout by R Θ T, we get

$$\frac{1}{v^2} \frac{\ddot{T}}{T} = \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} \qquad \text{(iv)}$$

A function of t on the left equals a function of r and θ. Let us choose

$$\frac{1}{v^2} \frac{\ddot{T}}{T} = -l^2 \qquad \text{(v)}$$

so that

$$\ddot{T} + v^2 l^2 T = 0$$

or $\ddot{T} + \lambda^2 T = 0$ (vi)

where $\lambda = vl$.

Then (iv) can be rewritten as

$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -l^2$$

or

$$-\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = r^2 l^2 + \frac{r}{R} \frac{dR}{dr} + \frac{r^2}{R} \frac{dR}{dr}$$

We may set the left hand side to m^2 so that

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \qquad \text{(vii)}$$

and

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (r^2 l^2 - m^2) R = 0 \qquad \text{(viii)}$$

Eqs. (vi), (vii) and (viii) are identical to Eqs. (6.32a), (6.34b) and (6.34a) respectively, if we identify l with k and m with p.

4. The solution of 1-D wave equation is given by Eq. (6.40). To determine the unknown constants, substitute the given form of $h(x)$ in Eq. (6.41). This gives

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \xi_0 \left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]$$

On comparison of like terms on two sides of this expression, we find that

$$a_1 = \xi_0 = a_2$$

and

$$a_3 = a_4 = \ldots\ldots a_n = 0$$

Terminal Questions

1. i) In spherical polar coordinates, the Laplace's equation can be written as

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right]=0$$

In the **method** of separation of variables, we write

$$f(r, \theta, \phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

By substituting back into the given equation and dividing by R Θ Φ, we have

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=0$$

On multiplying throughout by $r^2\sin^2\theta$, we can isolate ϕ dependent term:

$$-\frac{1}{R}\sin^2\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

As before, you will note that this equation relates a function of ϕ alone to a function of r and θ. Since r, θ, ϕ are **Independent** variables, we can equate each side to a constant. Let us choose it to be $-m^2$. Then

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2 \quad \text{(i)}$$

and

$$\frac{1}{R}\sin^2\theta\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=m^2$$

or

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{m^2}{\sin^2\theta}-\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

Again equating each side to a constant, we get the required result:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-nR=0 \quad \text{(ii)}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\Theta+n\Theta=0 \quad \text{(iii)}$$

where n is separation constant.

ii) The 3-*D* wave equation is

$$\nabla^2 f(\mathbf{r}, t)=\frac{1}{v^2}\frac{\partial^2 f(\mathbf{r}, t)}{\partial t^2} \quad \text{(i)}$$

In Cartesian coordinates, the **Laplacian** can be written as

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad \text{(ii)}$$

so that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(x, y, z, t)=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{(iii)}$$

The function f depends on four variables. Let us write it as

$$f(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t)$$

On substituting back in the given equation and dividing by $XYZT$, we **find** that

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} \qquad \text{(iv)}$$

The LHS is a function of space variables whereas **RHS** is a function of time alone. Let us therefore choose

$$\frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2$$

so that

$$\frac{d^2T}{dt^2} + \omega_0^2 T = 0 \qquad \text{(v)}$$

where $\omega_0 = kv$.

Then, (iv) reduces to

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad \text{(vi)}$$

Again LHS is a function of x only, whereas the RHS depends only on y and z. We, therefore, choose

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2$$

so that

$$\frac{d^2X}{dx^2} + l^2X = 0 \qquad \text{(vii)}$$

and

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = l^2 - k^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}$$

Proceeding along the same lines you can show that

$$\frac{d^2Y}{dy^2} + m^2Y = 0$$

and

$$\frac{d^2Z}{dz^2} + n^2Z = 0$$

where $n^2 = k^2 - l^2 - m^2$.

The given equation describes **e.m.** wave **propagation** in free space:

$$\frac{\partial^2E_y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2E_y}{\partial t^2} = 0$$

Let us **make the** substitution

$$E_y = X(x)\, T(t)$$

so that

$$X''\, T(t) - \frac{1}{c^2}X(x)\, \ddot{T} = 0$$

Dividing by $X(x)\,T(t)$, you will get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -k^2$$

so that

$$X'' + k^2 X = 0$$

and

$$\ddot{T} + \omega_0^2 T = 0$$

where $\omega_0 = ck$.

The solutions of these equations are of the form

$$X = A\cos kx + B\sin kx$$

and

$$T = C\cos\omega_0 t + D\sin\omega_0 t$$

The condition

$$X(x)\,T(t) = E_y = 0 \text{ at } x = 0 \text{ and at } x = L$$

implies that for all $t > 0$, $X(x) = 0$ at $x = 0$ and $x = L$. This leads to

$$X(L) = B\sin kL = 0$$

For a non-trivial solution, the eigenvalues are

$$k_n L = n\pi$$

or

$$k_n = (n\pi/L)$$

with corresponding eigen functions

$$X_n(x) = B\sin\left(\frac{n\pi}{L}\right)x$$

3. We have to solve 1-D heat equation :

$$\frac{1}{v}\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

subject to the conditions

$$f(0, t) = 0, f(L, t) = 0$$

and

$$f(x, 0) = f(x) \qquad 0 < x < L \tag{ii}$$

We first write

$$f(x, t) = X(x)\,T(t)$$

and substitute it back in the given equation. Then dividing by $X(x)\,T(t)$, we get

$$\frac{1}{v}\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

The LHS depends only on t, whereas the **RHS** depends on x only. Therefore, we choose

$$\frac{1}{v}\frac{1}{T}\frac{dT}{dt}=\frac{1}{X}\frac{d^2X}{dx^2}=-k^2 \qquad \text{(iii)}$$

so that

$$\frac{dT}{dt}+k^2 v T=0 \qquad \text{(iv)}$$

and

$$\frac{d^2X}{dx^2}+k^2 X=0 \qquad \text{(v)}$$

which has solutions of the form

$$T(t)=Ae^{-k^2 v t}$$

and

$$X(x)=B\cos kx+C\sin kx$$

so that

$$f(x,t)=X(x)\,T(t)=(P\cos kx+Q\sin kx)\,e^{-k^2 v t}$$

The condition $f(0,t)=0$ implies that $P=0$ and $f(L,T)=0$ demands that $k=n\pi/L$. Hence, the desired solution is given by

$$f(x,t)=\sum_{n=0}^{\infty} Q_n \sin\left(\frac{n\pi x}{L}\right)\exp\left[-v\,t\left(\frac{n\pi}{L}\right)^2\right] \qquad \text{(vi)}$$