and $v = x$. The general solution is, therefore,

$$y = C_1 \cos x + C_2 \sin x + \ln |\cos x| \cos x + x \sin x$$

Note that the method of undetermined coefficients could not be used to obtain y_p because $\sec x$ is not a solution of the homogeneous linear differential equation.

(ii) The corresponding homogeneous equation $y'' - y = 0$ has the general solution $y = C_1 e^x + C_2 e^{-x}$. Because of the nature of the driving function, y_p could not be found by the method of undetermined coefficients. In the method of variation of parameters, we put $y_1 = e^x$ and $y_2 = e^{-x}$ in the expressions for u' and v' to obtain

$$u' = \frac{\begin{vmatrix} 0 & e^{-x} \\ xe^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} \quad \text{and} \quad v' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & xe^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}$$

Therefore, $u' = x/2$ and $v' = \dfrac{-x\exp(2x)}{2}$ from which it readily follows that $u = x^2/4$ and $v = -(xe^{2x}/4) + (e^{2x}/8)$. The general solution is then

$$y = C_1 e^x + C_2 e^{-x} + \frac{1}{4} x^2 e^x - \frac{x}{4} e^x + \frac{1}{8} e^x$$

Finally, we note that $C_1 e^x$ and $\frac{1}{8} e^x$ can be combined as $(C_1 + \frac{1}{8}) e^x = Ce^x$ and the general solution may be written as

$$y = C_1 e^x + C_2 e^{-x} - \frac{1}{4} xe^x + \frac{1}{4} x^2 e^x$$

UNIT 3 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Structure

3.1 INTRODUCTION

In Unit 2, you have learnt how to solve second order ODEs with constant coefficients. The solutions of these equations are simple exponentials, trigonometric or hyperbolic functions known from calculus. But in many physical and engineering problems, we have to solve second order ordinary differential equations with variable coefficients. For example, we have to solve such equations to study the field distribution around a charged sphere or a cylinder, and energy production in a reactor. Similarly, when we wish to know how high a vertical column of uniform cross-section can be extended upward until it buckles under its own weight, we have to solve a second order ODE wit. variable coefficients. In such cases, simple algebraic or transcendental solutions do not exist and methods discussed in Unit 2 do not work. We, therefore; look for other methods.

One of the most elegant and efficient methods of solving such ODEs is the **power series method.** This is so particularly because it facilitates numerical computations. Even so, it has limited utility when coefficients of the given differential equation are not well defined at some point. In such cases, we use an extension of the power series method, called the **Frobenius' method,** You will learn these two methods in this unit. The properties of power series and certain other mathematical concepts are given in an Appendix at the end of this unit, It would be better if you study the appendix before studying this unit.

Objectives

After studying this unit, you should be able to

- define ordinary and singular points
- locate and classify the type of singularity
- use power series method to solve a second order ODE about an ordinary point
- use Frobenius' method to solve a second order ODE about a regular singular point.

3.2 SOME TERMINOLOGY

While studying first and second order ODEs with constant coefficients, you have learnt some basic terminology. You would come across some more common terms, which you do not know as yet, in reference to second order ODEs with variable coefficients. This section is intended to familiarise you with these concepts,

Analytic Functions

A function is said to be analytic at $x = x_0$ if it has a Finite value at $x = x_0$. Mathematically, it can be expressed as an infinite power series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots \tag{3.1}$$

where $a_0, a_1, \ldots$ are constants (also see Appendix **A).**

The term 'radius of convergence' is defined in the Appendix A at the end of this unit.

For the particular case of $x_0 = 0$, familiar examples of analytic functions are given in Table 3.1.

Table 3.1 : Power Series Expansion of Some Simple Functions Analytic at $x = 0$

Function	Power Series	Radius of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$	$\lvert x \rvert < 1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$	$\lvert x \rvert < \infty$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$	$\lvert x \rvert < \infty$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$	$\lvert x \rvert < \infty$

Ordinary and Singular Points

A second order ODE of the form

$$y'' + p(x)y' + q(x)y = 0 \tag{3.2}$$

has an ordinary point at $x = x_0$ if the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$. On the other hand, $x = x_0$ is a singular point of Eq. (3.2) if it is a singular point of either $p(x)$ or $q(x)$, ie., when either *function* is not analytic at $x = x_0$. For *example*, $x = 0$ is a singularity of $1/x$ as well as $\ln x$, and in the equation $x^2 y'' + xy' - 2y = 0$, $p(x)$ and $q(x)$ have a singular point at $x = 0$. Similarly, in the equation $(1 - x^2) y'' - 2xy' + 6y = 0$, $p(x)$ and $q(x)$ have singular points at $+1$ and -1.

Regular Singularity

A singular point $x = x_0$ of the differential equation (3.2) is said to be regular when $p(x)$ and/or $q(x)$ diverge but $\lim_{x \to x_0} (x - x_0)\, p(x)$ and $\lim_{x \to x_0} (x - x_0)^2 q(x)$ remain finite. If a singular point is not regular, it is called irregular. That is, when $x = x_0$ continues to be a singularity of $(x - x_0)\, p(x)$ or $(x - x_0)^2 q(x)$ it is irregular. These definitions hold for finite values of x_0. (The analysis for $x \to \infty$ leads to hypergeometric series.) The following example illustrates these concepts.

Example 1

Locate and classify the singular points of the equation

$$(1 - x^2)\, y'' - 2xy' + n(n+1)\, y(x) = 0$$

where n is a positive integer. This equation arises in many physical problems in electrostatics and quantum mechanics particularly in boundary value problems modelled in spherical polar coordinates, It is known as Legendre's equation.

Solution

On comparing the given equation with Eq. (3.2), you will note that

$$p(x) = -\frac{2x}{1 - x^2} \quad \text{and} \quad q(x) = \frac{n(n+1)}{1 - x^2}$$

Are these functions analytic at $x = \pm 1$? You will readily recognise that $x = +1$ and $x = -1$ are singular points of the given equation. To determine whether the singularities are regular or not, let us first compute

$$\lim_{x \to x_0} (x - x_0)\, p(x) \text{ and } \lim_{x \to x_0} (x - x_0)^2 q(x)$$

at $x = 1$. The result is

$$\lim_{x \to 1} (x-1)\, p(x) = -\frac{(x-1)\,2x}{(1-x^2)}$$

$$= \frac{2x}{1+x} = 1 \qquad \text{(i)}$$

and

$$\lim_{x \to 1} (x-1)^2 q(x) = \frac{(x-1)^2 n(n+1)}{1-x^2}$$

$$= -\frac{(x-1)\,n\,(n+1)}{1+x} \qquad \text{(ii)}$$

From (i) and (ii), you can say that $x = +1$ is a regular singularity. What about $x = -1$? It is also a regular singularity. However, you should convince yourself before proceeding further.

SAQ 1

Spend 10 min

To ensure that you have understood these concepts, we would like you to determine which of the following equations have regular singular point(s).

a) $x^2 y'' + 3xy' + y' = 0$

b) $x^2 y'' - y' + 2x^2 y = 0$

c) $2x^2 y'' - x y' + (x-5) y = 0$

Now you are familiar with the basic terminology. Let us, therefore, learn first to solve homogeneous ODEs of second order with variable coefficients by the so-called power series method. It gives solutions in the form of an infinite power series.

3.3 POWER SERIES METHOD

The basic idea of the power series method for solving ODEs with variable coefficients about an ordinary point is very simple. We will illustrate the procedure by considering specific examples. The steps used in this procedure are listed below.

The first step of the power series method is to assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad (3.3)$$

If $x = 0$ is an ordinary point, we obtain a power series in powers of x only :

$$y(x) = \sum_{n=0} a_n x^n$$

Next, we substitute Eq. (3.3) and its derivatives into the given differential equation. Then we obtain the values of a_n's by equating the coefficients of each power of x to zero. This technique is illustrated in the following example.

Example 2

Solve the differential equation $y'' + x^2 y = 0$ by assuming a power series solution about the point $x = 0$,

Solution

On comparing the given equation with Eq. (3.2), you will note that $p(x) = 0$ and $q(x) = x^2$ so that $x = 0$ is an ordinary point. So we assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots \tag{i}$$

and differentiate it with respect to x. This gives

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

In the summation notation, we can write

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

The lower limit on the summation index has been shifted from $n = 0$ to $n = 1$. This is because the term corresponding to $n = 0$ in (i) is constant and its derivative with respect to x is zero. Similarly, you can write

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots$$

or
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} \tag{ii}$$

Substituting for y and y'' in the given differential equation, we find that

$$(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \ldots) + (a_0 x^2 + a_1 x^3 + a_2 x^4 + \ldots) = 0$$

or
$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

Next, let us collect like powers of x. This gives

$$2a_2 + 6a_3 x + (12 a_4 + a_0) x^2 + (20 a_5 + a_1) x^3 + \ldots = 0$$

Since the RHS is identically zero, we equate the coefficient of every power of x on the LHS to zero.

Here 'coeff' is being used as an abbreviation of 'coefficient'.

Coeff. of x^0 : $2a_2 = 0 \Rightarrow a_2 = 0$

Coeff. of x^1 : $6a_3 = 0 \Rightarrow a_3 = 0$

Coeff. of x^2 : $12a_4 + a_0 = 0 \Rightarrow a_4 = -\dfrac{a_0}{12}$

Coeff. of x^3 : $20a_5 + a_1 = 0 \Rightarrow a_5 = -\dfrac{a_1}{20}$

Coeff. of x^4 : $30a_6 + a_2 = 0 \Rightarrow a_6 = 0$

Coeff. of x^5 : $42a_7 + a_3 = 0 \Rightarrow a_7 = 0$

Coeff. of x^{n+2} : $(n+3)(n+4)a_{n+4} + a_n = 0$

or
$$a_{n+4} = -\frac{1}{(n+3)(n+4)} a_n \quad \text{for } n \geq 0$$

With $a_2 = 0$, it follows that alternative even coefficients beyond a_4 (a_6, a_{10},) will be zero. Similarly, since $a_3 = 0$, it readily follows that $a_7 = a_{11} = \ldots = 0$. Hence, you can write

$$y(x) = a_0 + a_1 x + a_4 x^4 + a_5 x^5 + a_8 x^8 + a_9 x^9 + \ldots$$

$$= a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 + \frac{a_0}{672} x^8 + \frac{a_1}{1440} x^9 + \ldots$$

$$= a_0\left(1-\frac{x^4}{12}+\frac{x^8}{672}+\ldots\right)+a_1\left(x-\frac{x^5}{20}+\frac{x^9}{1440}+\ldots\right)$$

$$= a_0\, y_1(x) + a_1\, y_2(x)$$

Are these solutions linearly independent? You should check it by computing their Wronskian.

You may now ask : Can we not use the power series method to solve first or second order ODEs with constant coefficients? The answer is: Yes, we can use the above steps as outlined for second order equations with variable coefficients. You can easily convince yourself about this by solving the following SAQ.

SAQ 2

Spend 15 *min*

Use power series method to solve the following equations.

a) $y'' + \omega_0^2 y = 0$

b) $y' + xy = x^2 - 2x$

As pointed out earlier, the equation given in Example 1 is of particular interest in physics and nuclear engineering. It is known as **Legendre's equation.** Let us obtain its two linearly independent solutions using the power series method.

Example 3

Obtain series solution of the Legendre's equation given in Example 1

$$(1-x^2)\,y'' - 2xy' + m(m+1)\,y = 0$$

Solution

To solve Legendre's equation using the power series method, we first rewrite it as

$$y'' - \frac{2x}{1-x^2}\,y' + \frac{m(m+1)}{1-x^2}\,y = 0 \qquad \text{(i)}$$

From Example 1, you would recall that the functions $p(x) = -\frac{2x}{1-x^2}$ and $q(x) = \frac{m(m+1)}{1-x^2}$ have regular singularities at $x = \pm 1$. However, they are analytic at $x = 0$ and we can, therefore, use power series method to solve Legendre's equation in the range $-1 < x < 1$. Let us write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots \qquad \text{(ii)}$$

Substituting this and its derivatives

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

and

$$y''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots$$

in Legendre's equation, we get

$$(1-x^2)\,[2a_2 + 6a_3 x + 12a_4 x^2 + \ldots] - 2x\,[a_1 + 2a_2 x + 3a_3 x^2 + \ldots]$$

$$+ k\,[a_0 + a_1 x + a_2 x^2 + \ldots] = 0$$

or

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n\,a_n x^{n-1} + k\sum_{n=0}^{\infty} a_n x^n = 0$$

Where we have put $m(m+1) = k$,

You can rewrite it as

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\, a_n x^n - 2 \sum_{n=1}^{\infty} n\, a_n x^n + k \sum_{n=0}^{\infty} a_n x^n = 0 \qquad \text{(iii)}$$

In the expanded form, you can write

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) - (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + \dots)$$
$$- 2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots) + k(a_0 + a_1 x + a_2 x^2 + \dots) = 0$$

As before, let us collect the coefficients of each power of x. This gives

$$(2a_2 + ka_0) + (6a_3 - 2a_1 + ka_1)x + (12a_4 - 2a_2 - 4a_2 + ka_2)x^2 + \dots = 0$$

Again the coefficient of each power of x is zero. Thus,

Coeff. of x^0 : $\quad 2a_2 + k a_0 = 0 \Rightarrow a_2 = -\frac{k}{2} a_0 \qquad$ (iv)

Coeff. of x^1 : $\quad 6a_3 + (k-2)a_1 = 0 \Rightarrow a_3 = -\frac{k-2}{6} a_1 \qquad$ (v)

Coeff. of x^2 : $\quad 12a_4 + (k-6)a_2 = 0 \Rightarrow a_4 = -\frac{k-6}{12} a_2 \qquad$ (vi)

In general, for the n th power of x, you can write

Coeff. of x^n : $\quad (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + ka_n = 0$

or

$$a_{n+2} = \frac{n(n-1) + 2n - k}{(n+1)(n+2)} a_n$$

$$= \frac{n(n+1) - k}{(n+1)(n+2)} a_n ; \quad n = 0, 1, 2, 3, \dots$$

You can readily verify by putting $k = m(m+1)$ that the numerator can be written as $(m-n)(n+m+1)$. Hence, this expression takes the form

$$a_{n+2} = -\frac{(m-n)(m+n+1)}{(n+1)(n+2)} a_n ; \quad n = 0, 1, 2, \dots \qquad \text{(vii)}$$

This equality enables us to determine each expansion coefficient in terms of the second one preceding it, except for a_0 and a_1 (which are arbitrary). Such a **relation** between the coefficients is called a recurrence **relation** or **recursion formula.**

The recurrence relation **(vii)** implies that coefficients with even subscripts can be expressed in terms of a_0 and those **with** odd subscripts in terms of a_1. **That** is,

$$a_2 = -\frac{m(m+1)}{2!} a_0 \qquad a_3 = -\frac{(m-1)(m+2)}{3!} a_1$$

$$a_4 = -\frac{(m-2)(m+3)}{12} a_2 \qquad a_5 = -\frac{(m-3)(m+4)}{20} a_3$$

$$= \frac{(m-2)m(m+1)(m+3)}{4!} a_0 \qquad = \frac{(m-3)(m-1)(m+2)(m+4)}{5!} a_1$$

and so on.

By inserting these values for the coefficients in (ii), you can write

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \qquad \text{(viii)}$$

where

$$y_1 = 1 - \frac{m(m+1)}{2!} x^2 + \frac{(m-2)m(m+1)(m+3)}{4!} x^4 - \dots + \dots$$

and

$$y_2 = 1 - \frac{(m-1)(m+2)}{3!} x^3 + \frac{(m-3)(m-1)(m+2)(m+4)}{5!} x^5 - \dots + \dots$$

are two solutions over the interval $-1 < x < 1$. You may now again ask: Are y_1 and y_2 linearly independent? To answer this question, we note that y_1 contains only even powers of x whereas y_2 contains only odd powers of x. As a result, the ratio y_1/y_2 will not be constant, implying that y_1 and y_2 are linearly independent solutions. And (viii) is a general solution of Legendre's equation.

Spend 15 min

SAQ 3

The equation

$$y'' - 2xy' + 2my = 0$$

plays a particularly important role in statistics. Its solutions are known as **Hermite** polynomials. We would like you to obtain the coefficients of the power series solution of this equation.

Before proceeding, let us summarise the steps you should follow to solve a second order ODE for which $x = 0$ is an ordinary point.

power Series Method

Step1 : Assume a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Step2 : Substitute the assumed solution and its derivatives into the given differential equation.

Step 3 : Equate the coefficients of each power of x to zero for a homogeneous equation. This results in a recursion relation, which helps us in determining successively the coefficients occurring in the power series in terms of two 'arbitrary constants.

Step 4 : Write the explicit form of the series solution which satisfies the given equation.

So far we have refrained from Mathematical rigour. It may, however, be pointed out here that the mathematical justification of power series method is contained in Fuchs' Theorem. We state it without giving proof:

If $x = x_0$ is an ordinary point of the equation

$$y'' + p(x)y' + q(x)y = 0$$

then there exists a unique function y (x) which is analytic and satisfies the given equation in the neighbourhood pf xo as well as the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$ where a_0 and a_1 are two arbitrary constants.

Several second order ODEs with variable coefficients appearing in many important physical problems have coefficients which are not'analytic functions. In particular, these equations may have a regular singularity at $x = x_0$. For such equations, power series solution of the form given by Eq. (3.2) is not physically acceptable and we use an extended power series which always provides at least one solution around a regular singular point:

$$y(x) = (x - x_0)^r \sum_{n=0}^{m} a_n (x - x_0)^n$$

$$= \sum_{n=0}^{m} a_n (x - x_0)^{n+r} \qquad (3.4)$$

Such a series is also called **Frobenius** series with index r. Here r may be any (real or complex) number so that $a_0 \neq 0$. You will note that the series given in Eq. (3.4) reduces to a power series (Eq. (3.3)) if r is a non-negative integer.

3.4 THE FROBENIUS' METHOD

We illustrate the use of **Frobenius'** method when $x = 0$ is a regular singular **point** of **Eq.** (3.2):

$$y'' + p(x)y' + q(x)y = 0$$

Since $p(x)$ and $q(x)$ are **not** analytic at $x = 0$ **but** $x = 0$ is a regular singularity, let us rewrite this equation in **terms** of $b(x) = x\,p(x)$ and $c(x) = x^2 q(x)$, **which** will be analytic at $x = 0$. So, we multiply throughout by x^2 to obtain

$$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$$

or

$$x^2y'' + x\,b(x)y' + c(x)y = 0 \tag{3.5}$$

Since $b(x)$ and $c(x)$ are analytic at $x = 0$, we can write

$$b(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots = \sum_{n=0}^{\infty} b_n x^n \tag{3.6a}$$

and

$$c(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots = \sum_{n=0}^{\infty} c_n x^n \tag{3.6b}$$

You will note that if it happens **that** $b_0 = c_0 = c_1 = 0$, then $x = 0$ defines an ordinary point rather than a **regular** singular point.

Differentiating the series expansion given in **Eq.** (3.4) for $x_0 = 0$ term by **term,** you will get

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^{n+r-2}$$

Substituting for $y(x)$, $y'(x)$, $y''(x)$, $b(x)$ and $c(x)$ in **Eq.(3.5),** we get

$$x^r \sum_{n=0}^{\infty} (n+r)(n+r-1)\,a_n x^n + x^r \left[\sum_{n=0}^{\infty} (n+r)\,a_n x^n\right]\left[\sum_{m=0}^{\infty} b_m x^m\right]$$

$$x^r\left[\sum_{m=0}^{\infty} c_m x^m\right]\left[\sum_{n=0}^{\infty} a_n x^n\right] = 0$$

As before, let us equate the sum of the coefficients of each power of x to zero. This yields a system of **equations** involving the unknown coefficients a_n**'s**. The coefficient of x^r, which is the lowest power of x, is obtained **from** the $n = 0$ term. Equating this coefficient to zero, you will obtain

Coefficient of x^r: $\quad [\,r(r-1) + r\,b_0 + c_0\,]\,a_0 = 0$

Since $a_0 \neq 0$, this equality will be satisfied if

$$r^2 + (b_0 - 1)\,r + c_0 = 0 \tag{3.7}$$

Eq.(3.7) is called the **indicial** equation **corresponding** to the given differential equation. Since the **indicial** equation is quadratic, **it will** have two roots. This means that there should be two **Frobenius series solutions.** Will these solutions always be linearly independent? Not necessarily. In fact, the roots of the indicial equation give us some idea of the nature of solutions of the **ODE** of interest. In practice, the desired solutions are obtained using the steps outlined for **the** power series method. However, before plunging into these details, we would **like** you to go through the following example.

Example 4

Determine the roots of the indicial equation around the origin for the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$$

Solution

To be able to compare the given equation with Eq. (3.2), we divide throughout by x^2. This gives

$$y'' + \frac{1}{x}y' + \left(\frac{x^2 - 1/9}{x^2}\right)y = 0 \qquad \text{(i)}$$

You would readily recognise that this equation has a singularity at $x = 0$. Is it regular or irregular? To discover this, compare (i) with Eq. (3.2). You will note that $p(x) = \frac{1}{x}$ and $q(x) = \frac{x^2 - \frac{1}{9}}{x^2}$ so that $xp(x) = 1$ and $x^2 q(x) = x^2 - (1/9)$. In the limit $x \to 0$, $xp(x) = 1$ and $x^2 q(0) = -(1/9)$. That is, $b(x)$ and $c(x)$ are analytic at $x = 0$. Thus, $x = 0$ is a regular singular point.

We can, therefore, assume a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

Differentiating it with respect to x, we get

$$y'(x) = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1}$$

and

$$y''(x) = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$$

On substituting these in the given equation, we get

$$\sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \frac{1}{9}\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

To arrive at the indicial equation, we equate the coefficients of the lowest power of x, i.e., x^r to zero. This gives

$$a_0\left[r(r-1) + r - \frac{1}{9}\right] = 0$$

For $a_0 \neq 0$, the indicial equation takes the form

$$r^2 - \frac{1}{9} = 0$$

which has roots $r = \pm\frac{1}{3}$. That is, the roots of the indicial equation are distinct. Moreover, they do not differ by an integer.

Let us now pause for a minute and ask: Will the roots always be distinct? Certainly not. Moreover, even when they are distinct, they may differ by an integer. In fact, there are three possibilities. The indicial equation may have (i) distinct roots not differing by an Integer, (ii) double roots, and (iii) distinct roots differing by an integer. To discover these, you should solve the following **SAQ**.

SAQ 4

Spend 10 min

Determine the roots of the indicial equations corresponding to the following ODEs about $x = 0$.

(a) $x(x-1)y'' + (3x-1)y' + y = 0$

(b) $x^2(x^2-1)y'' - (x^2+1)xy' + (x^2+1)y = 0$

You now know that there are **three** possible cases. But, only one of these, **viz.** when the roots of the indicial equation are distinct and do not differ by an **integer** is important **from** the view-point of physics. So we shall consider it in detail. For repeated roots or for distinct roots differing by an integer, we shall just quote the results and you will not be **tested** for, these (in **TMA/CMA/term-end** examination). Let us learn to obtain solutions for this case.

***Case* 1: Distinct Roots of the Indicial Equation not Differing by an Integer**

When the two roots (r_1, r_2) of the indicial equation are distinct and do not differ by an integer, we expect two linearly independent Frobenius series **solutions** of the form

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \tag{3.8a}$$

and

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} d_n x^n \tag{3.8b}$$

As such, this is the simplest case. We have illustrated the procedure of obtaining the **two** linearly independent **solutions** using **Frobenius'** method in the following example.

Example 5

Determine the **two** Frobenius series **solutions** around $x = 0$ for the **ODE**

$$x^2 y'' + \left(x^2 + \frac{5}{36}\right) y = 0$$

Solution

Let us **first rewrite** the given **ODE** in the standard form on dividing **by** x^2:

$$y'' + \left(\frac{x^2 + \frac{5}{36}}{x^2}\right) y = 0 \tag{i}$$

Here $p(x) = 0$ and $q(x) = \dfrac{x^2 + (5/36)}{x^2}$. You will **easily recognise** that the **point** $x = 0$ is a **singularity.** Further, since $\lim_{x \to 0} x^2 q(x) = 5/36$, you may logically **conclude that** $x = 0$ is a **regular singular** point of the given differential equation.

By assuming a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

and substituting it in (i), you **will** obtain

$$\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} + \frac{5}{36} \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

To obtain **the** indicial equation, we equate the sum of the **coefficients** of x^r to zero. **This** yields

$$a_0 \left[r(r-1) + \frac{5}{36} \right] = 0$$

For $a_0 \neq 0$, we find that

$$r(r-1) + \frac{5}{36} = 0$$

is the required **indicial** equation. You can readily verify that the two distinct roots are 5/6 and 1/6. Moreover, these do not differ by an integer.

To determine a_n corresponding to r = 5/6, let us equate the sum of the coefficients of $x^{r+1}, x^{r+2}, \ldots x^{n+r}$ to zero:

Coeff. of x^{r+1}: $\quad a_1\left[r(r+1)+\frac{5}{36}\right] = 0$

Since the bracketed term does not vanish for $r = r_1 = 5/6$, this equality will hold only if we choose $a_1 = 0$. Similarly

Coeff. of x^{r+2}: $\quad a_2(r+1)(r+2)+a_0+\frac{5}{36}a_2 = 0$

or

$$a_2 = -\frac{a_0}{(r+1)(r+2)+\frac{5}{36}}$$

Coeff. of x^{m+r}: $\quad a_m(m+r)(m+r-1)+a_{m-2}+\frac{5}{36}a_m = 0$

or

$$a_m\left[(m+r)(m+r-1)+\frac{5}{36}\right] = -a_{m-2}$$

On substituting $r = r_1 = 5/6$, you will find that

$$a_m\left[\left(m+\frac{5}{6}\right)\left(m-\frac{1}{6}\right)+\frac{5}{36}\right] = -a_{m-2}$$

or

$$a_m = -\frac{a_{m-2}}{m\left(m+\frac{2}{3}\right)} \qquad \text{(ii)}$$

Since $a_1 = 0$, this recurrence relation implies that all odd coefficients will vanish.

To evaluate even coefficients, let us introduce the change $m = 2p$. This means that

$$a_{2p} = -\frac{a_{2p-2}}{2p\left(2p+\frac{2}{3}\right)} = -\frac{3}{4}\frac{a_{2p-2}}{p(3p+1)}$$

Hence, with

$$p = 1: \quad a_2 = -\frac{3}{4}\left(\frac{a_0}{1\times 4}\right)$$

$$p = 2: \quad a_4 = -\frac{3}{4}\left(\frac{a_2}{2\times 7}\right) = \left(\frac{3}{4}\right)^2\frac{a_0}{1\times 2\times 4\times 7} = \left(\frac{3}{4}\right)^2\frac{a_0}{2!\,4\times 7}$$

Similarly, you can write

$$p = 3: \quad a_6 = -\left(\frac{3}{4}\right)^3\frac{a_0}{3!\,4\times 7\times 10}$$

$$\vdots$$

$$a_{2p} = (-1)^p\left(\frac{3}{4}\right)^p\frac{a_0}{p!\,1\times 4\times 7\times 10\ldots..(3p+1)}$$

Hence, one of the solutions of the given equation is

$$y_1(x) = a_0x^{5/6}\left[1-\frac{3}{16}x^2+\frac{9}{16}\cdot\frac{x^4}{2\times 4\times 7}-\ldots+(-1)^p\left(\frac{3}{4}\right)^p\frac{x^{2p}}{p!\,1\times 4\times 7\ldots(3p+1)}\right]$$

This can be expressed in a compact form as

$$y_1(x) = a_0 x^{5/6}\left[1+\sum_{p=1}^{\infty}(-1)^p\left(\frac{3}{4}\right)^p \frac{x^{2p}}{p!\,1\times4\times7\,....\,(3p+1)}\right]$$

For r = r_2 = 1/6 also, all odd coefficients will vanish and we leave it as an exercise **for** you to verify that

$$y_2(x) = d_0 x^{1/6} + \sum_{p=1}^{\infty} d_{2p} x^{2p+(1/6)}$$

where

$$d_{2p} = (-1)^p\left(\frac{3}{4}\right)^p \frac{d_0}{p!\,2\times5\times8\,...\,(3p-1)}$$

Case 2 : Double Root of the **Indicial Equation**

When roots of the indicial equation are equal, we cannot obtain two linearly independent solutions using the procedure outlined above. In fact, there can be only one Frobenius **series** solution. To determine this solution, we first find r from Eq. (3.7):

$$r = \frac{-(b_0-1)\pm\sqrt{(b_0-1)^2-4c_0}}{2}$$

The condition for equal roots of a quadratic equation $ax^2+bx+c=0$ is $b^2-4ac=0$. This means that the term under the radical sign will vanish.

Using the condition for equal roots, you will find that the common root is $-\frac{b_0-1}{2} = \frac{1-b_0}{2}$.

Hence, it readily follows from Eq. **(3.4)** that one of the solutions will be of the **form**

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= x^{(1-b_0)/2} \sum_{n=0}^{\infty} a_n x^n \tag{3.9}$$

where a_n**'s** are unknown constants.

The second linearly independent **solution** is

$$y_2(x) = y_1(x)\ln x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)\left(\sum_{m=1}^{\infty} a_m x^m\right)$$

$$= y_1(x)\ln x + x^r \sum_{i=1}^{\infty} A_i x^i$$

where A_i is some other constant.

Case 3 : Roots of the **Indicial** Equation **Differing** by an **Integer**

When the roots (r_1, r_2) of the **indicial** equation are distinct but differ by an integer, we can always determine the first Frobenius series solution as before. If r_1 (= r) and r_2 (= $r - p$, where p is a **positive** integer) are the two **roots**, then

$$y_1(x) = x^{r_1} \sum_{n=0}^{m} a_n x^n$$

The other linearly independent Frobenius solution is

$$y_2(x) = k_p\, y_1(x)\ln x + x^{r_2} \sum_{m=0}^{\infty} D_m x^m$$

Let us now summarise what you have studied in this unit.

3.5 SUMMARY

- A second order ODE with variable coefficients

$$y'' + p(x)y' + q(x)y = 0$$

is said to have an ordinary point at $x = x_0$ if $p(x)$ and $q(x)$ are analytic at $x = x_0$. Otherwise, the point is said to be singular. The singularity is regular if $\lim_{x \to x_0} (x - x_0) p(x)$ and $\lim_{x \to x_0} (x - x_0)^2 q(x)$ are finite at $x = x_0$.

- We can solve a second order ODE with variable coefficients around an ordinary point at $x = 0$ by assuming a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$. The constants a_n's are determined using the recursion relation.

- The solution around a regular singularity at $x = 0$ is obtained using Frobenius method by assuming a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

- The indicial equation is obtained by equating the sum of the coefficients of the lowest power of x to zero.

- The roots of the indicial equation give us an idea of the nature of solutions of the ODE. The roots of the indicial equation may be distinct, repeated or differ by an integer:

- For distinct roots, two linearly independent solutions are

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

and

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} d_n x^n$$

3.6 TERMINAL QUESTIONS

1. Like Legendre's equation, another ODE that arises in advanced studies in physics and applied mathematics is the Bessel's equation of order m:

$$x^2 y'' + xy' + (x^2 - m^2) y = 0$$

In Example 4, we solved Bessel's equation of order 1/3. Use Frobenius' method to solve this equation.

2. Solve the ODE

$$y'' + y = e^x$$

around the point $x = 0$.

3.7 SOLUTIONS AND ANSWERS

SAQs

1(a) $x^2 y'' + 3xy' + y = 0$

To compare it with the standard form ($y'' + p(x)y' + q(x)y = 0$), you should divide it by x^2. This gives

$$y'' + \frac{3}{x} y' + \frac{1}{x^2} y = 0$$

so that $p(x) = \frac{3}{x}$ and $q(x) = \frac{1}{x^2}$. Both functions diverge at $x = 0$. To determine whether the singularity is regular or not, let us compute $\lim_{x \to 0} xp(x)$ and $\lim_{x \to 0} x^2 q(x)$. These are respectively equal to 3 and 1. So we can say that $x = 0$ is a regular **singularity.**

(b) $$x^2 y'' - y' + 2x^2 y = 0$$

Dividing throughout by x^2, we get

$$y'' - \frac{1}{x^2} y' + 2y = 0$$

which shows that $p(x) = -\frac{1}{x^2}$ is not analytic at $x = 0$. In fact, $\lim_{x \to 0} x\,p(x) = -\frac{1}{x}$ and the function diverges at $x = 0$. So $x = 0$ is an irregular singularity of the given equation.

(c) $$2x^2 y'' - xy' + (x-5)y = 0$$

Dividing by $2x^2$ throughout, you will obtain

$$y'' - \frac{1}{2x} y' + \frac{x-5}{2x^2} y = 0$$

On comparing with the standard form, you will recognise that $p(x) = -\frac{1}{2x}$ and $q(x) = \frac{x-5}{2x^2}$ which diverges at $x = 0$. However, $\lim_{x \to 0} xp(x) = -\frac{1}{2}$ and $\lim_{x \to 0} x^2 q(x) = -\frac{5}{2}$ so that $x = 0$ is a **regular singularity.**

2(a) $$y'' + \omega_0^2 y(x) = 0$$

Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and differentiate it with respect to x. The result is

$$y' = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these in the given equation, we find that

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \omega_0^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

In the expanded form, you can write

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \ldots) + \omega_0^2 (a_0 + a_1 x + a_2 x^2 + \ldots) = 0$$

Collecting the coefficients of like powers of x, we find that

$$(2a_2 + \omega_0^2 a_0) + (6a_3 + \omega_0^2 a_1) x + (12a_4 + \omega_0^2 a_2) x^2 + \ldots = 0$$

Equating the coefficient of each power of x to zero, we have

Coeff. of x^0: $2a_2 + \omega_0^2 a_0 = 0 \Rightarrow a_2 = -\frac{\omega_0^2}{2!} a_0$

Coeff. of x^1: $6a_3 + \omega_0^2 a_1 = 0 \Rightarrow a_3 = -\frac{\omega_0^2}{6} a_1 = -\frac{\omega_0^2}{3!} a_1$

Coeff. of x^2: $12a_4 + \omega_0^2 a_2 = 0 \Rightarrow a_4 = -\frac{\omega_0^2}{12} a_2 = \frac{\omega_0^4}{24} a_0 = \frac{\omega_0^4}{4!} a_0$

That is, $a_2, a_4, \ldots$ can be expressed in terms of ao and a_3, as, ... etc., can be expressed in terms of a_1. Hence

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= a_0 + a_1 x - \frac{\omega_0^2}{2!} a_0 x^2 - \frac{\omega_0^2}{3!} a_1 x^3 + \frac{\omega_0^4}{4!} a_0 x^4 + \ldots$$

Collecting the coefficients of a_0 and a_1, we find that

$$y(x) = a_0 \left(1 - \frac{\omega_0^2}{2!} x^2 + \frac{\omega_0^4}{4!} x^4 - \ldots \right) + a_1 \left(x - \frac{\omega_0^2}{3!} x^3 + \frac{\omega_0^5}{5!} x^5 - \ldots \right)$$

You would recognise that the coefficient of a_0 defines $\cos \omega_0 x$ whereas the coefficient of a_1 defines $\sin \omega_0 x$. Hence, the general solution can be written as

$$y(x) = a_0 \cos \omega_0 x + a_1 \sin \omega_0 x$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

From Unit 2, you may recall that $\cos \omega_0 x$ and $\sin \omega_0 x$ are linearly independent .solutions.

(b) The given differential equation has an ordinary point at $x = 0$. So there exists a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $a_0 \neq 0$.

Substituting the power series for y and y' in the given equation, we obtain

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = x^2 - 2x$$

or

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = x^2 - 2x$$

In the expanded form, we can write

$$(a_1 + 2a_2 x + 3a_3 x^2 + \ldots) + (a_0 x + a_1 x^2 + a_2 x^3 + \ldots) = x^2 - 2x$$

Collecting the coefficients of each power of x on LHS, we get

$$a_1 + (a_0 + 2a_2)x + (a_1 + 3a_3)x^2 + \ldots = x^2 - 2x$$

The coefficient a_0 is arbitrary. To find the other coefficients, we proceed by equating coefficients of like powers of x on both sides of this equation:

Coeff. of x^0 : $a_1 = 0$

Coeff. of x^1 : $2a_2 + a_0 = -2 \Rightarrow a_2 = -\frac{a_0 + 2}{2}$

Coeff. of x^2 : $3a_3 + a_1 = 1 \Rightarrow a_3 = \frac{1-a_1}{3} = \frac{1}{3}$

Coeff. of x^3 : $4a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4} = \frac{a_0+2}{8}$

Coeff. of x^{n-1} : $na_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{1}{n} a_{n-2}$ for n a 4

Iteration of this formula for $n \geq 4$ yields

$$a_4 = -\frac{1}{4} a_2 = -\frac{1}{4}\left(-\frac{a_0+2}{2}\right) = \frac{a_0+2}{4 \times 2}$$

$$a_5 = -\frac{1}{5} a_3 = -\left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = -\frac{1}{5 \times 3}$$

$$-\frac{1}{6} a_4 = -\frac{a_0+2}{6 \times 4 \times 2}$$

$$a_7 = -\frac{1}{7} a_5 = -\frac{a_0+2}{7 \times 5 \times 3}$$

$$a_8 = -\frac{1}{8} a_6 = \frac{a_0+2}{8 \times 6 \times 4 \times 2}$$

$$a_9 = -\frac{1}{9} a_7 = -\frac{1}{9 \times 7 \times 5 \times 3}$$

The solution can then be written as

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 - \frac{a_0+2}{2} x^2 + \frac{1}{3} x^3 + \frac{a_0+2}{4 \times 2} x^4 - \frac{1}{5 \times 3} x^5 - \frac{a_0+2}{6 \times 4 \times 2} x^6 + \frac{1}{7 \times 5 \times 3} x^7 + \ldots$$

$$= a_0 - (a_0+2)\left(\frac{1}{2} x^2 - \frac{1}{4 \times 2} x^4 + \frac{1}{6 \times 4 \times 2} x^6 + \ldots\right)$$

$$+ \left(\frac{1}{3} x^3 - \frac{1}{5 \times 3} x^5 + \frac{1}{7 \times 5 \times 3} x^7 - \ldots\right)$$

3. You would readily recognise that $x = 0$ is an ordinary point of the given equation. So we assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these in the given equation, we find that

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} na_n x^n + 2m\sum_{n=0}^{\infty} a_n x^n = 0$$

In the expanded form,

$$(2a_2 + 6a_3 x + 12a_4 x^2 + ... + n(n-1)a_n x^{n-2} + ...)$$

$$-2(a_1 x + 2a_2 x^2 + 3a_3 x^3 + ... + na_n x^n + ...)$$

$$+ 2m(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...) = 0$$

Collecting the coefficients of each power of x, we get

$$(2a_2 + 2m\, a_0) + (6a_3 - 2a_1 + 2m\, a_1)x$$

$$+ (12a_4 - 4a_2 + 2m\, a_2)x^2 + ...$$

$$+ [(n+2)(n+1)a_{n+2} - 2n\, a_n + 2m\, a_n]x^n + ... = 0$$

Next we equate the coefficient of each power of x to zero. The result is

Coeff. of x^0: $\quad 2a_2 + 2m\, a_0 = 0 \Rightarrow a_2 = -\dfrac{2m\, a_0}{2\times 1}$

Coeff. of x^1: $\quad 6a_3 + (2m-2)a_1 = 0 \Rightarrow a_3 = \dfrac{2(1-m)}{3\times 1}a_1$

Coeff. of x^2: $12a_4 + (2m-4)a_2 = 0 \Rightarrow a_4 = \dfrac{(2-m)}{3\times 2}a_2 = -\dfrac{2m(2-m)}{4\times 3}a_0$

Coeff. of x^n: $(n+2)(n+1)a_{n+2} - 2(n-m)a_n = 0$

or $\quad a_{n+2} = \dfrac{2(n-m)}{(n+2)(n+1)}a_n$

4(a) $\quad x(x-1)y'' + (3x-1)y' + y = 0$

Let us first rewrite this equation in the standard form by dividing throughout by $x(x-1)$:

$$y'' + \frac{(3x-1)}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$$

You can readily identify that

$$p(x) = \frac{(3x-1)}{x(x-1)} \quad \text{and } q(x) = \frac{1}{x(x-1)}$$

Both functions diverge at $x = 0$ as well as $x = 1$. But $\lim\limits_{x\to 0} xp(x) = 1, \lim\limits_{x\to 0} x^2 q(x) = 0$, $\lim\limits_{x\to 1}(x-1)p(x) = 2$ and $\lim\limits_{x\to 1}(x-1)^2 q(x) = 0$ so that $x = 0$ and $x = 1$ are regular singularities. For $x = 0$, let us, therefore, assume a solution of the form

$$y(x) = \sum_{n=0}^{m} a_n x^{n+r}$$

so that $\quad y'(x) = \sum\limits_{n=0}^{\infty}(n+r)a_n x^{n+r}$

and $\quad y''(x) = \sum\limits_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

On substituting for $y(x)$, $y'(x)$ and $y''(x)$ in the given equation, we find that

$$x(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$+(3x-1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}$$

$$+3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

You will note that the lowest power of x is x^{r-1}. By equating the sum of its coefficients to zero, we have

$$[-r(r-1)-r]a_0=0$$

Since $a_0 \bullet 0$, we must have $r^2=0$

Hence, this indicial equation has a double root: $r=0$.

(b) $$(x^2-1)x^2y''-(x^2+1)xy'+(x^2+1)y=0$$

us divide throughout by $(x^2-1)x^2$ to put it in the standard form ((Eq. (3.2)):

$$y''-\frac{x^2+1}{x(x^2-1)}y'+\frac{x^2+1}{x^2(x^2-1)}y=0$$

You will readily recognise that $x=0$ is a regular singular p i n t of the given ODE. Let us, therefore, assume a solution of the form

$$y(x)=\sum_{n=0}^{\infty}a_n x^{n+r}$$

so that $$y'(x)=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

and

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Substituting these in the given equation, we find that .

$$(x^2-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}-(x^2+1)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+(x^2+1)\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Performing the multiplication, we find that

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+2}-\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+2}-\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}a_n x^{n+r+2}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Combining the first, third and fifth series together, and second and last **series** together, **we find that**

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)-(n+r)+1]\,a_n x^{n+r+2}$$

$$-\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-1]\,a_n x^{n+r} = 0$$

On simplification, we find that

$$\sum_{n=0}^{\infty}(n+r-1)^2 a_n x^{n+r+2} - \sum_{n=0}^{\infty}(n+r-1)(n+r+1)\,a_n x^{n+r} = 0$$

The lowest power of x in this equation is x^r. By equating its power to zero, you will obtain the required indicial equation:

$$(r+1)(r-1) = 0$$

whose roots are $r_1 = 1$ and $r_2 = -1$. You will readily **recognise** that these roots differ by an integer.

Terminal Questions

1. The family of ODEs

$$x^2y'' + xy' + (x^2 - m^2)\,y = 0$$

is known as **Bessel's equations.** The parameter m is real and non-negative. You would readily note that $x = 0$ is a regular singular point of the equation. So we assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad \text{(i)}$$

Substitute $y(x)$ and its derivatives in the given equation. This yields

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} m^2 a_n x^{n+r} = 0$$

Changing the first summation so that the exponent on x is $n+r$ and collecting other series, we have

$$\sum_{n=0}^{m}[(n+r)(n+r-1)+(n+r)-m^2]\,a_n x^{n+r} + \sum_{n=2}^{m} a_{n-2} x^{n+r} = 0 \qquad \text{(ii)}$$

The smallest power of x is x^r ($n = 0$). Equating the coefficient of x^r to zero, we get

$$[r(r-1)+r-m^2]\,a_0 = 0$$

Since $a_0 \neq 0$, we have the indicial equation

$$r^2 - m^2 = 0 \qquad \text{(iii)}$$

which has roots $r_1 = m$ and $r_2 = -m$.

Depending on the value of m, the solutions can differ vastly:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+m} \qquad \text{(iv)}$$

and

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-m} \qquad \text{(v)}$$

To find y_1, let us write **(ii)** in the expanded form

$$x^r[(r^2-m^2)\,a_0 + \{(r+1)^2 - m^2\}\,a_1 x$$
$$+\{(r+2)^2 - m^2\}\,a_2 x^2 + \ldots + \{(n+r)^2 - m^2\}\,a_n x^n + \ldots]$$
$$+\,x^r[a_0 x^2 + a_1 x^3 + a_2 x^4 + \ldots + a_{n-2} x^n + \ldots] = 0$$

Equating the coefficient of each power of x to zero, we get

Coeff. of x^r : $(r^2 - m^2) a_0 = 0$

Coeff. of x^{r+1} : $[(r+1)^2 - m^2] a_1 = 0$

Coeff. of x^{r+2} : $[(r+2)^2 - m^2) a_2 - a_0 = 0 \Rightarrow a_2 = -\frac{1}{(r+2)^2 - m^2} a_0$

Coeff. of x^{r+n} : $[(n+r)^2 - m^2] a_n + a_{n-2} = 0 \Rightarrow a_n = -\frac{1}{(n+r)^2 - m^2} a_{n-2}$

For $r = m$, we find that $a_1 = 0$ since the bracketed quantity does not vanish. Then recursion relation implies that $a_3 = a_5 = a_7 = 0 = ...$ That is, all odd subscripted coefficients vanish. For even subscripted coefficients, we find that

$$a_2 = -\frac{a_0}{(m+2)^2 - m^2}$$

$$= -\frac{a_0}{(m+2-m)(m+2+m)}$$

$$= -\frac{a_0}{2^2(m+1)}$$

Similarly,

$$a_4 = -\frac{a_2}{2^2 \times 2(m+2)} = \frac{a_0}{2^4 \times 2(m+1)(m+2)}$$

$$a_6 = -\frac{a_4}{2^2 \times 3(m+3)} = -\frac{a_0}{2^6 \times 3 \times 2(m+1)(m+2)(m+3)}$$

In general,

$$a_{2n} = \frac{(-)^n a_0}{2^{2n} n!(m+1)(m+2)\dots(m+n)} \qquad n = 0, 1, 2, \dots$$

The solution corresponding to $r_2 = -m$ is found by simply replacing m by $-m$ provided m is not an integer.

2. Letting $y = \sum_{n=0}^{\infty} a_n x^n$, we find that

$$y' = \sum_{n=0}^{\infty} a_n x^{n-1}$$

and

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

From Table 3.1, you would recall that series expansion for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Substituting in the given differential equation, we obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Rearranging the terms, you will obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \left(a_n - \frac{1}{n!}\right) x^n = 0$$

Since the right-hand side of the equality is identically zero, we equate each coefficient of every power of x on the left-hand side to zero. To evaluate a_n, the exponents in the two series must be made the same by shifting the index. Here, we choose to replace n with $n-2$ in the second series. Thus,

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} + \sum_{n=2}^{\infty}\left(a_{n-2} - \frac{1}{(n-2)!}\right)x^{n-2} = 0$$

We combine the two series to obtain

$$\sum_{n=2}^{\infty} x^{n-2}\left[n(n-1)\,a_n + a_{n-2} - \frac{1}{(n-2)!}\right] = 0$$

The coefficients of x^{n-2} for $n \geq 2$ must be equal to zero. Thus,

$$n(n-1)\,a_n + a_{n-2} - \frac{1}{(n-2)!} = 0$$

Solving for a_n, we obtain the recursion formula

$$a_n = \frac{1}{n!} - \frac{1}{n(n-1)}a_{n-2}, \qquad n = 2, 3, 4 \ldots$$

iteration of this formula yields

$$a_2 = \frac{1}{2!} - \left(\frac{1}{2\times 1}\right)a_0 = \frac{1}{2!} - \frac{1}{2!}a_0$$

$$a_3 = \frac{1}{2!} - \left(\frac{1}{3\times 2}\right)a_1 = \frac{1}{3!} - \frac{1}{3!}a_1$$

$$a_4 = \frac{1}{4!} - \left(\frac{1}{4\times 3}\right)a_2 = \frac{1}{4!}a_0$$

$$a_5 = \frac{1}{5!} - \left(\frac{1}{5\times 4}\right)a_3 = \frac{1}{5!}a_1$$

and so on

Since $y(x) = \sum_{n=0}^{\infty} a_n x^n$, and a_0 and a_1 are arbitrary, we get

$$y(x) = a_0 + a_1 x + \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \left(\frac{1}{2!} - \frac{1}{2!}a_0\right)x^2 + \left(\frac{1}{3!} - \frac{1}{3!}a_1\right)x^3 + \frac{1}{4!}a_0 x^4 + \frac{1}{5!}a_1 x^5 + \ldots$$

$$= a_0\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots\right) + a_1\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + ..\right)$$

$$+ \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$

$$= a_0 y_1(x) + a_1 y_2(x) + y_p$$

where y_1 and y_2 are, respectively, infinite series of even and odd terms, and

$$y_p = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \ldots$$

Since this differential equation is **linear** and of the second order, the form of the solution is $y = y_c + y_p$, where y_c consists of a linear combination of two linearly independent functions and y_p is a particular solution to the given nonhomogeneous **equation.**

APPENDIX A: POWER SERIES

A power series about a point x_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{A.1}$$

where the numbers $a_0, a_1, a_2, \ldots, a_n, \ldots$ are called the coefficients of the power series. A power series does not include terms with negative or fractional powers.

We say that the power series **converges** if

$$\lim_{n \to \infty} \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{A.2}$$

exists. The value of the limit is called the sum of the power series at the point $x = x_0$. If the limit does not exist, the power series is said to **diverge.** The interval **of** values of x for which a power series converges is called the interval **of convergence** and is denoted as $|x - x_0| < R$, where x_0 is called the centre of the power series and R is called the **radius of convergence.** If R = 0, the series converges only at x_0; if R = ∞, the series converges for all values of x.

Within a common interval of convergence, two power series may be added term by term. That is,

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \tag{A.3}$$

Further, within the interval of convergence, the power series represents a function whose derivative and integral may be found from term-by-term differentiation and integration. That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

then

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

and

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n+1}$$

A power series expansion of $f(x)$ around $x = 0$ is called a **Maclaurin series.** The Maclaurin series of f (x) is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}$$

where f $^{(n)}(0)$ means the value of the nth derivative of f at $x = 0$. Recall that $f^{(0)} =$ f and $0! = 1$

UNIT 4 SOME APPLICATIONS OF ODEs IN PHYSICS

Structure

4.1 INTRODUCTION

In Units 1 to 3 you have learnt various techniques of solving first and second order ODEs. Recall that we have studied these techniques so that we are able to answer real-life questions such as the following: Does the quantity of fuel burned by a rocket affect its velocity? How long does it take a polluted gulf to return to its natural state, once man-made pollution is stopped? What is the response of an LCR circuit to an applied signal? You are now on the verge of being able to answer such questions. This unit on applications of ODEs will further help you in this respect.

This unit is primarily concerned not with ***how to do*** mathematics but with how ***to use*** it. Here we shall be applying the techniques you have learnt so far to solve a variety of real problems involving differential equations. These problems have both their origin and solutions outside mathematics. Doing the mathematics is only a part of the process called mathematical modelling. Today, as scientists seek to further our understanding of nature, the technique of representing our "real- world" in mathematical terms has become an invaluable tool. Indeed, ***theprocess of mimicking reality by using the language of mathematics is known as mathematical modelling.***

In this unit, you will first learn what is involved in the process of mathematical modelling, especially with ODEs. As the unit progresses, you will realise that mathematical modelling with ODEs finds immense use in various areas of physics. We hope that having studied the unit, you will be able to answer the kind of questions posed above and many others you are likely to come across as you delve deeper in physics.

In this block you have studied various methods of solving first and second order ordinary differential equations and their applications in physics. In the next block, we shall discuss partial differential equations (PDEs). You will learn how to model physical systems with PDEs and various methods of salving them.

Objectives

After studying this unit you should be able to