UNIT 2 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Structure

2.1 INTRODUCTION

In Unit 1 you have learnt to solve first order ordinary differential equations. These equations provide a useful means of studying physical systems. For instance, radioactive decay, free fall of a body, fluid flow, current growth in electrical circuits, etc could be studied using these equations. But many systems require solutions of ordinary differential equations of order higher than one. From Unit 1 of **PHE-02** course on Oscillations and Waves, you would recall that the equation of motion of an undamped harmonic oscillator is a homogeneous second order differential equation with constant coefficients. Similarly, to determine the depression in horizontal beams, we have to solve a second order differential equation with constant coefficients. You will learn to solve such equations in Sec. 2.3. But can you study time-variation of charge in maintained *RLC* circuits or the phenomenon of resonance using a homogeneous second order differential equation? In such cases we have to solve nonhomogeneous second order differential equations with constant coefficients. But even this is not true in general. For instance, when we wish to study field distribution around a charged sphere or a cylinder, we have to solve second order differential equations with variable coefficients. Similar situations are also encountered in heat, optics, electromagnetic theory, energy production in a nuclear reactor, quantum mechanics, etc. In such cases we seek power series solutions or use Frobenius method. In the next unit you will learn these methods. But in this unit, we have discussed the basic techniques of solving second order differential equations with constant coeffidients. Some of their applications will be discussed in Unit 4.

Objectives

After studying this unit you should be able to

- compute the Wronskian of a given ODE
- obtain linearly independent solutions of homogeneous second order ordinary differential equations with constant coefficients
- use the method of undetermined coefficients and variation of parameters to obtain linearly independent solutions of non-homogeneous second order ordinary differential equations with constant coefficients.

2.2 SOME TERMINOLOGY

While studying first order ordinary differential equations (ODEs) in Unit 1 you have learnt some basic terminology. You would come across some common terms, which you do not know as yet, in the context of second order differential equations as well. It is important for you to be familiar with them. This section is intended for this purpose.

You know that a second order linear ordinary differential equation can be written as

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_0(x)y = g(x) \tag{2.1}$$

The function $g(x)$ is termed as the **forcing function**, and $p_1(x)$ and $p_0(x)$ are **coefficient** functions. These are continuous over the interval where the solution exists.

Linearly Independent Solutions and the Wronskian

From Unit 1 (Sec. 1.4.3) we recall that if y_1 and y_2 are linearly independent solutions of the homogeneous equation

$$y'' + p_1(x)y' + p_0(x)y = 0 \tag{2.2}$$

then their linear combination

$$y = C_1 y_1 + C_2 y_2 \tag{2.3}$$

where C_1 and C_2 are arbilrary constants, is a general solution of Eq. (2.2). For example, you know that $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$ are linearly independent solutions of the ODE for an undamped harmonic oscillator: $\frac{d^2y}{dt^2} + \omega^2 y = 0$. So the general solution of this equation is

$$y(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

You may ask: What do we mean by linearly independent solutions? How do we test linear independence? Will a linear combination of linearly independent solutions necessarily lead to a different solution? When does a set of solutions constitute the general solution of a linear differential equation? and so on. Let us now discover answers to these questions for ODEs of second order. We say that two solutions y_1 and y_2 are linearly independent on an interval if the identity

$$C_1 y_1 + C_2 y_2 = 0 \tag{2.4}$$

is satisfied only when $C_1 = C_2 = 0$. For, if C_1 and C_2 were non-zero constants, Eq. (2.4) would yield y_2/y_1 = constant, i.e., y_1 and y_2 would be proportional on sorns interval. Then, by definition, y_1 and y_2 would be linearly dependent functions on that interval. In other words, linear independence of y_1 and y_2 means that the ratio y_2/y_1 is not a constant. This implies that the differential of this ratio

$$\frac{y_2' y_1 - y_1' y_2}{y_1^2} \tag{2.5}$$

is not identically equal to zero. Therefore, we can write the condition of linear independence of two solutions y_1 and y_2 as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \tag{2.6}$$

The determinant $W(y_1, y_2)$ is called the **Wronski** determinant or the Wronskian of the given differential equation. We may, therefore, conclude that

> Two solutions y_1 and y_2 are linearly independent on an interval [a, b], if and only if, their Wronskian is non-zero for $a \le x \le b$.

For a harmonic oscillator, this means that

$$W(x) = \begin{vmatrix} \sin \omega t & \cos \omega t \\ \omega \cos \omega t & -\omega \sin \omega t \end{vmatrix} = -\omega$$

showing that sin wt and cos ωt are linearly independent. We also say that y_1 and y_2 are linearly dependent solutions on an interval **I**, if and only if their **Wronskian** is zero for some $x = x_0$ in I

Spend
5 min

SAQ 1

The solutions of the **equation**

$$y'' + 4y = 0$$

are given by $y_1 = \sin 2x$ and $y_2 = \cos 2x$. Are these **solutions** linearly independent?

Particular integral and complementary function

From **Sec.** 4.2 of the course Oscillations and Waves **(PHE-02),** you would recall that the equation of motion of a forced damped harmonic oscillator is

$$my'' + \gamma y' + ky = F_0 \cos \omega t$$

which is usually rewritten as

$$y'' + 2by' + \omega_0^2 y = f_0 \cos \omega t \qquad (2.7)$$

where $2b = \gamma/m$, $\omega_0^2 = k/m$ and $f_0 = F_0/m$.

Physically, the actual motion of this system is a sum of two oscillations: one of the frequency of damped oscillations and the other of the frequency of the driving force. Mathematically, we express it as

$$y(t) = y_1 + y_2 \qquad (2.8)$$

where y_1 is a solution of the homogeneous equation

$$y_1'' + 2by_1' + \omega_0^2 y_1 = 0 \qquad (2.9)$$

On substituting Eq. (2.8) in Eq. (2.7) and using Eq. (2.9) in the resultant expression, you will find that y_2 satisfies the equation

$$y_2'' + 2by_2' + \omega_0^2 y_2 = f_0 \cos \omega t$$

In the language of mathematics, y_1 is called the complementary function and y_2 is called the particular **integral**. We can write the general solution of a second order non-homogeneous linear differential equation with constant coefficients as the sum of a complementary **function** and the particular integral:

$$y(x) = y_c(x) + y_p(x) \qquad (2.10)$$

You know that the solution of a second order differential equation consists of only two **arbitrary** constants. This implies that the particular integral **will** not contain any arbitrary constant.

We **hope** that you are now **equipped** with all the necessary basic terminology. Let us now proceed to solve **homogeneous** linear **ODEs** of second order with constant coefficients.

2.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A second order homogeneous linear ordinary differential **equation** with constant coefficients **finds** wide applications in engineering, biological and physical systems. In particular, you know its use in mechanical and **electrical** vibrations. Many techniques of solving such equations have been developed. These include the reduction of order **technique,** which you learnt **in** Unit 1, and the method of exponential functions, which we will discuss now.

A homogeneous second order **ordinary** differential **equation with constant coefficients can** be expressed in the form

$$ay'' + by' + cy = 0 \qquad (2.11)$$

where a, b and c are real constants.

From Unit 1, you would recall that the solution of the first order homogeneous linear ordinary differential equation ($y' + y = 0$) is an exponential function of the form

$$y = A\exp(-kx)$$

Let us, therefore, seek a solution of Eq. (2.11) of the form

$$y = A\exp(mx) \qquad (2.12)$$

where dimensions of m are inverse of those of x. This ensures that the power of exponential is **dimensionless.**

Substituting this and its derivatives

$$y' = A\,m\exp(mx)$$

and

$$y'' = A\,m^2\exp(mx)$$

in Eq. **(2.11)**, you will obtain

$$(am^2 + bm + c)A\exp(mx) = 0$$

Since $A\exp(mx)$ is finite, this equation will be satisfied **only** if

$$am^2 + bm + c = 0 \qquad (2.13)$$

This quadratic equation is called the characteristic equation (or auxiliary equation). Its roots are

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

For example, the auxiliary equation for

$$y'' + 5y' - 7y = 0 \quad \text{is} \quad m^2 + 5m - 7 = 0$$

So you will agree that

$$y_1(x) = A\exp(m_1 x) \qquad (2.14a)$$

and

$$y_2(x) = A\exp(m_2 x) \qquad (2.14b)$$

are solutions of Eq. **(2.11)**. Using the principle of superposition, you can write its most general solution as

$$y(x) = C_1\exp(m_1 x) + C_2\exp(m_2 x) \qquad (2.14c)$$

for a suitable choice of constants C_1 and C_2 determined by initial or boundary conditions.

The **Wronskian** of these **solutions** is

$$W(x) = \begin{vmatrix} A\exp(m_1 x) & A\exp(m_2 x) \\ m_1 A\exp(m_1 x) & m_2 A\exp(m_2 x) \end{vmatrix}$$

$$= (m_2 - m_1)B\exp[(m_1 + m_2)x] \qquad (2.15)$$

where B is a constant. This shows that for $m_1 \neq m_2$, the **solutions** will be linearly independent:

From Eq. (2.11), **you would note** that y, y' and y'' are linearly dependent. This demands that y be an exponential function. For, the derivative **of any** order of an exponential function is linearly dependent with itself, **i.e.,** some **multiple of** the original exponential.

You must have noticed that the process of solving a homogeneous linear second order ordinary differential equation with **constant** coefficients using an **exponential function** as a **solution** reduces to finding the **roots** of a quadratic equation. The roots of this equation can be

1. real and distinct for $b^2 - 4ac > 0$ or $b^2 > 4ac$
2. real **and** equal when $b^2 - 4ac = 0$, or $b^2 = 4ac$ and
3. **complex conjugate for** $b^2 - 4ac < 0$ **or** $b^2 < 4ac$

Let us now discover solutions corresponding to these roots.

Distinct Real Roots

For distinct real roots, $\exp(m_1 x)$, and $\exp(m_2 x)$ are linearly independent and the general solution is given by

$$
\begin{aligned}
y &= C_1 \exp(m_1 x) + C_2 \exp(m_2 x) \\
&= \exp\left[-\left(\frac{bx}{2a}\right)\right][C_1 \exp(\alpha x) + C_2 \exp(-\alpha x)] \qquad (2.16)
\end{aligned}
$$

where $\alpha = \dfrac{\sqrt{b^2 - 4ac}}{2a}$.

The constants C_1 and C_2 can be determined by using given initial and boundary conditions. We now illustrate this method with the following example.

Example 1

Solve the equation

$$y'' + 3y' + 2y = 0$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = 2$.

Solution

In this case, the auxiliary equation is

$$m^2 + 3m + 2 = 0$$

which has roots $m = -1$ and $m = -2$. Therefore, the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} \qquad \text{(i)}$$

To determine C_1 and C_2, we first use the condition that at $x = 0$, $y = 1$. **This** gives

$$1 = C_1 + C_2 \qquad \text{(ii)}$$

Further, since

$$y' = -C_1 e^{-x} - 2C_2 e^{-2x}$$

we find **that**

$$y'(0) = 2 = -C_1 - 2C_2 \qquad \text{(iii)}$$

You **can** readily solve (ii) and (iii) for C_1 and C_2 to obtain $C_1 = 4$ and $C_2 = -3$. Hence, the desired particular solution is

$$y = 4e^{-x} - 3e^{-2x}$$

When two roots are equal ($m_1 = m_2$), $W(x) = 0$. This means that $e^{m_1 x}$ and $e^{m_2 x}$ are linearly dependent. What docs this imply? It implies that (i) Eq. (2.14) does not hold and (ii) our, starting assumpdon is false. You may now ask: **How** can we obtain two linearly independent functions when auxiliary equation of a second order differential equation has two equal roots? In such a situation, we use **the** method of reduction of order to construct a second linearly independent solution. This is illustrated **below.**

Repeated **Real** Roots

When a **second** order **differential** equation has two equal roots, we obtain the correct form of the **second** solution by **assuming** that

$$y_2 = u(x) \exp(mx) \qquad \textbf{(2.17)}$$

where m **is** a root **of the** auxiliary equation (Eq. (2.13)). **Differentiating** Eq. (2.17) with **respect to** x, we get

$$y_2' = u' \exp(mx) + mu \exp(mx)$$

and

$$y_2'' = u'' e^{mx} + 2m\, u' e^{mx} + m^2 u\, e^{mx}$$

Substituting these in Eq. (2.11), we have

$$(a m^2 + b m + c)\, u(x)\, e^{mx} + (2ma + b)\, e^{mx} u' + a\, e^{mx} u'' = 0$$

The first term in this expression vanishes in view of Eq. (2.13). The coefficient of u' is zero since $m = -b/2a$ in this case. Hence, the above expression simplifies to

$$\exp(mx)\, a\, u'' = 0$$

Multiplying by $\exp(-mx)$ and integrating, you will get

$$u' = K$$

where K is an arbitrary constant of integration.
Integrating again, you will get

$$u = Kx + C$$

Hence, the desired solution is

$$y_2 = x\, e^{mx} = x\, e^{-bx/2a} \tag{2.18}$$

Alternatively, you can also arrive at Eq. (2.18) by constructing the following linear combination of $e^{m_1 x}$ and $e^{m_2 x}$

$$\frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$$

In the limit $m_1 \to m_2$ $(= m)$, this takes the form

$$\lim_{h \to 0} \frac{e^{(m+h)x} - e^{mx}}{h}; \quad h = m_1 - m_2$$

$$= \frac{d}{dm} e^{mx}$$

$$= x\, e^{mx}$$

where the arbitrary constants K and C have been dropped (since we are seeking only a second linearly independent solution). Hence, the general solution of a second order differential equation, when auxiliary equation has repeated real roots, is

$$y(x) = C_1 e^{-bx/2a} + C_2\, x\, e^{-bx/2a}$$

$$= (C_1 + C_2 x) \exp\left(-\frac{bx}{2a}\right) \tag{2.19}$$

To test that $e^{-bx/2a}$ and $x\, e^{-bx/2a}$ are linearly independent, you can compute their Wronskian

$$W(x) = \begin{vmatrix} \exp\left(-\frac{bx}{2a}\right) & x \exp\left(-\frac{bx}{2a}\right) \\ -\frac{b}{2a} \exp\left(-\frac{bx}{2a}\right) & -\frac{b}{2a} x \exp\left(-\frac{bx}{2a}\right) + \exp\left(-\frac{bx}{2a}\right) \end{vmatrix}$$

$$= e^{-(bx/a)} > 0 \quad \text{for} \quad a \le x \le b \tag{2.20}$$

It implies that $e^{-(bx/a)}$ and $x\, e^{-bx/2a}$ are acceptable solutions. The arbitrary constants C_1 and C_2 occurring in Eq. (2.19) can be determined using specified initial or boundary conditions.

You may, therefore, conclude as follows:

> When the auxiliary equation for a second order ODE with constant coefficients has repeated real roots ($m_1 = m_2 = m$), the general solution is given by
>
> $$y = (C_1 + C_2 x) \exp(mx)$$
>
> where C_1 and C_2 are arbitrary constants.

You may now like to solve an SAQ to be sure that you have grasped this method.

SAQ 2

Spend 10 min

Solve the initial value problem

$$y'' + 6y' + 9y = 0; \quad y(0) = 2 \text{ and } y'(0) = 1$$

Complex Roots

Let us now consider the case for which the characteristic equation has complex roots of the form $m = \alpha + i\beta$, where α and β are real. From your school mathematics, you know that complex roots of a real polynomial equation always occur in conjugate pairs. That is, if

$$m_1 = \alpha + i\beta$$

is one of the roots, then

$$m_2 = \alpha - i\beta$$

is also a root.

As before, we can obtain the general solution as a linear combination of two linearly independent solutions as

$$y = A \exp(m_1 x) + B \exp(m_2 x)$$

$$= A e^{(\alpha + i\beta)x} + B e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x}(A e^{i\beta x} + B e^{-i\beta x}) \quad (2.21)$$

You will note that this solution is complex. Can you express it as a real solution? To do so, we use **Euler's formula:**

$$e^{\pm i\theta} = \cos\theta \pm i \sin\theta \quad (2.22)$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Put $x = i\theta$ in this expansion. The result is

$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} \ldots + \ldots$

$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right)$

$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$

$= \cos\theta + i \sin\theta$

This gives

$$y = e^{\alpha x}[A(\cos\beta x + i \sin\beta x) + B(\cos\beta x - i\sin\beta x)]$$

$$= e^{\alpha x}[(A + B)\cos\beta x + (A - B)\, i \sin\beta x]$$

By letting $C_1 = A + B$ and $C_2 = (A - B)\, i$, you can write

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) \quad (2.23a)$$

Putting $C_1 = C\cos\phi$ and $C_2 = C\sin\phi$, you can rewrite Eq. (2.23a) as

$$y = C e^{\alpha x}\cos(\beta x - \phi) \quad (2.23b)$$

where C and ϕ are arbitrary constants. These are related to C_1 and C_2 by

$$C = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan\phi = \frac{C_2}{C_1}$$

You will note that when the roots of characteristic equation are complex, they generate solutions of the form of a product of an exponential and a trigonometric function. Therefore, you may conclude as follows:

> If the characteristic equation of a second order ODE has complex roots of the form $m = \alpha \pm i\beta$, the general solution is of the form
> $$y = C e^{\alpha x}\cos(\beta x - \phi)$$

You know that the differential equation governing the motion of an undamped spring-mass system is $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$ where $\omega_0^2 = k/m$. You would readily note that the characteristic equation for this case is

$$m^2 + \omega_0^2 = 0$$

which has roots

$$m_1 = i\omega_0 \quad \text{and} \quad m_2 = -i\omega_0$$

Hence, the general solution is

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

The oscillatory motion described by this example should continue indefinitely. In practice, however, several factors lead to loss of energy of the system. As a result, oscillations tend to decrease. In the following example, we illustrate the method discussed in this section to study the motion of a damped harmonic oscillator.

Example 2

Consider a spring-mass system which is damped by a viscous force (Fig.2.1), which can be modelled so that it is linearly proportional to velocity. What differential equation describes its motion and what are its acceptable solutions?

Solution

We know that viscous force opposes motion. Hence, the differential equation describing the motion of a damped spring-mass system is

$$M\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+kx = 0$$

or
$$\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega_0^2 x = 0 \qquad (2.24)$$

where $2b = \gamma/M$ and $\omega_0^2 = k/M$.

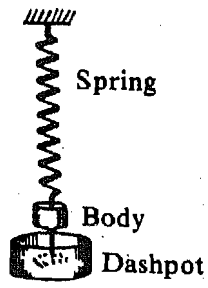

Fig. 2.1: A damped spring-mass system.

On comparing it with Eq. (2.11); you can write the characteristic equation as

$$m^2+2bm+\omega_0^2 = 0$$

which has roots

$$m_1 = -b+\sqrt{b^2-\omega_0^2} \quad \text{and} \quad m_2 = -b-\sqrt{b^2-\omega_0^2}$$

These roots depend on damping and determine the motion of the oscillator. Depending on the value of $(b^2-\omega_0^2)^{1/2}$, we have three possibilities.

Case 1: If $b > \omega_0$, $\sqrt{b^2-\omega_0^2}$ is positive and we have two real distinct roots.

Case 2: If $b = \omega_0$, $b^2-\omega_0^2 = 0$ and we have real repeated roots.

Case 3: If $b < \omega_0$, $b^2-\omega_0^2$ is negative and $\sqrt{b^2-\omega_0^2}$ is imaginary, i.e., we have a complex conjugate pair of roots.

Let us now discuss these cases.

Case 1

When the roots are real and distinct, the system is said to be heavily damped and the general solution of Eq. (2.24) is given by

$$x(t) = \exp(-bt)\,[C_1\exp(\beta t)+C_2\exp(-\beta t)] \qquad (2.25)$$

where $\beta = \sqrt{b^2-\omega_0^2}$.

This represents non-oscillatory behaviour. The system is said to be heavily damped and such a motion is called dead beat.

The actual displacement of any such system will be determined by the initial conditions. To compute this, we would like you to solve the following SAQ.

SAQ 3

Spend 5 min

A heavily damped oscillator in its equilibrium position is suddenly kicked so that at $t = 0, x = 0$ and $\frac{dx}{dt} = v_0$. Compute the particular solution and interpret the resulting expression for displacement.

On working out this SAQ, you will find that the displacement of a heavily damped oscillator is determined by the interplay of an increasing hyperbolic function and a decaying exponential. As a result, the displacement increases initially, attains a maximum and thereafter decays.

The hyperbolic functions are so named because $x = a\cosh\theta$, and $y = a\sinh\theta$ define a rectangular hyperbola $x^2 - y^2 = a^2$. Compare it with the parametric equation of the circle $x^2 + y^2 = a^2$ which is defined by $x = a\cos\theta$ and $y = a\sin\theta$.

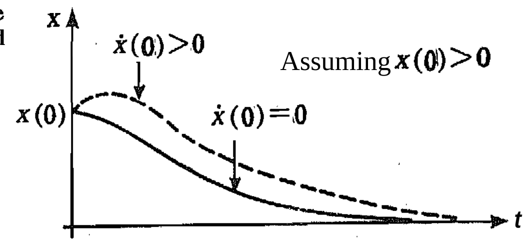

Fig. 2.2: Displacement-time graph for an overdamped spring-mw system

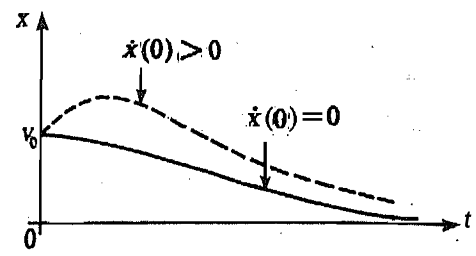

Fig 23: Displacement-time graph for a critically damped spring-mass system

Fig 2.2 shows two typical over-damped motions for $\frac{dx(0)}{dt} = 0$ and $\frac{dx(0)}{dt} > 0$

Case 2

When we have repeated real roots, the general solution of Eq. *(2.24)* is given by

$$x(t) = (C_1 + C_2\, t)\exp(-bt) \tag{2.26}$$

Note that here C_1 has dimensions of length and C_2 those of velocity. As before, these constants can be determined by specifying initial conditions. You can easily verify that for initial conditions given in SAQ 3, $C_1 = 0$ and $C_2 = v_0$ so that the complete solution is

$$x(t) = v_0\, t\exp(-bt) \tag{2.27}$$

Such a system is said to be **critically damped.** Typical graph of a critically damped system for $\frac{dx(0)}{dt} = 0$ and $\frac{dx(0)}{dt} > 0$ is shown in Fig. 2.3.

Case 3

When the roots are imaginary, let us write

$$\sqrt{b^2 - \omega_0^2} = \sqrt{-1}\,(\omega_0^2 - b^2)^{1/2} = i\,\omega_d$$

where $i = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - b^2}$ is a real positive quantity. Hence, the displacement is given by

$$x(t) = \exp(-bt)\,[C_1\exp(i\,\omega_d t) + C_2\exp(-i\,\omega_d t)]$$

$$= C\exp(-bt)\cos(\omega_d t + \phi) \tag{2.28}$$

where $C = \sqrt{C_1^2 + C_2^2}$ and $\phi = \cos^{-1}\left(\frac{C_1 + C_2}{2\sqrt{C_1 C_2}}\right)$.

You will note that Eq. *(2.28)* represents oscillatory motion whose amplitude decreases **exponentially** at a rate governed by ***b***. Such a system is said to be **weakly damped.** The displacement of a weakly damped system is depicted in Fig. *2.4.*

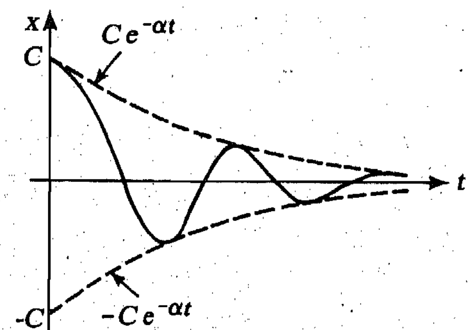

Fig. 2.4 : Oscillations of a weakly damped spring-mass system

Let us sum up what you have learnt so far:

1. We can solve a second order homogeneous linear ordinary differential equation with constant coefficients using exponential functions. The form of the solution depends on the roots of the characteristic equation.
2. When we have distinct real roots, there exist two linearly independent functions of the form $\exp(m_1 x)$ and $\exp(m_2 x)$ and the general solution is given by

$$y(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$

3. When we have real equal roots, the two linearly independent functions in the general solution are of the form $\exp(mx)$ and $x \exp(mx)$, i.e.,

$$y(x) = (C_1 + C_2 x) \exp(mx)$$

4. When we have a complex conjugate pair of roots, the two linearly independent solutions are of the form $\exp(\alpha x) \sin \beta x$ and $\exp(\alpha x) \cos \beta x$, and the general solution can be written as

$$y(x) = \exp(\alpha x)(C_1 \sin \beta x + C_2 \cos \beta x)$$
$$= C \exp(\alpha x) \cos(\beta x - \phi)$$

So far we have considered homogeneous linear equations with constant coefficients. These equations do not satisfactorily model forced mechanical and electrical systems. In fact, such systems can be fairly accurately represented by nonhomogeneous second order linear equations. We now wish to obtain solutions of such equations. Can you use the method discussed in the preceding section to solve non-homogeneous equations? No, we have to look for new methods. Let us learn some of these now.

2.4 NONHOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

From Sec. 2.2, you would recall that the general solution of a second order nonhomogeneous linear ordinary differential equation with constant coefficients Is composed of the particular integral (PI) and the complementary function (CF). You can easily verify this by substituting $y = y_p + y_c$ in the equation

$$a y'' + b y' + C y = g(x)$$

You can obtain $y_c(x)$ by using the method of the preceding section. For instance, CF for the equation

$$y'' - 2y' - 3y = \sin x$$

is given by

$$y_c = C_1 e^{-3x} + C_2 e^{-x}$$

(You can check this by direct substitution.)

This means that finding the particular integral is at the heart of the method of solving a non-homogeneous equation. But how to get y_p? One systematic approach to find y_p is based on the method of reduction of order, which you have learnt in Unit 1. The other commonly used methods are the method of undetermined **coefficients** and **the variation of parameters.** Let us pow learn these.

2.4.1 The Method of Undetermined Multipliers

The basic idea of this method is to first construct the general form of the particular integral From the forcing function. Then we determine coefficients for y_p that allow it to satisfy the given differential equation. In the following example, we have illustrated this concept.

Example 3

Determine the particular integral of $y'' + 3y' + 2y = \exp(2x)$.

Solution

Since the driving function is $\exp(2x)$, we assume the PI to be of the form $y_p = A\exp(2x)$. We substitute y_p and its derivatives

$$y' = 2A\exp(2x)$$

and

$$y'' = 4A\exp(2x)$$

in the given equation and solve for A. This gives

$$A = \frac{1}{12}.$$

Thus

$$y_p = \frac{1}{12}\exp(2x)$$

is a PI of the given equation.

You may now ask: Will this method work always? No, we have to refine it when the forcing function in the differential equation is a **polynomial** of order n. For example, consider the differential equation $y'' + 3y' + 2y = 5x^2$. Following the method outlined above, you may choose

$$y_p = Ax^2$$

On substituting y_p and its derivatives in the differential equation of interest, you will get

$$2A + 6Ax + 2Ax^2 = 5x^2$$

It is impossible to solve it for A. This means that the method illustrated in Example 3 fails to give a particular solution. Then, how to find $y_p(x)$? We tabulate below the form of $y_p(x)$ depending on the form of $g(x)$.

Table 2.1

Form of forcing function	Nature of the root of auxiliary equation	Form of particular integral
$A e^{kx}$	When k is not a root k is single root k is double root	$C e^{kx}$ $C x e^{kx}$ $C x^2 e^{kx}$
Polynomial $A x^n (n = 0, 1, \ldots)$	k = 0 is not a root k = 0 is single root k = 0 is double root	$C_0 + C_1 x + C_2 x^2 + \ldots$ $x(C_0 + C_1 x + \ldots)$ $x^2(C_0 + C_1 x + C_2 x^2 + ..)$
$A \cos kx$ $A \sin kx$	ik is not a root ik is a single root	$C \cos kx + D \sin kx$ $x(C \cos kx + D \sin kx)$

From the table you will note that if $g(x)$ is of the form given in column 1, the corresponding PI will be of the form given in column 3. The form of PI will also be determined by the nature of the root of the auxiliary equation as given in column 2. Note also that if a term in $g(x)$ is a solution of the homogeneous equation corresponding to the given non-homogeneous ODE, the form of y_p is modified as follows: y_p is multiplied by x or x^2 depending on whether the root of the auxiliary equation is a single or a double root. This is termed the **modification** rule.

Further, if $g(x)$ is a sum of the functions in column 1, we choose y_p as the sum of the functions in the corresponding rows.

SAQ 4

Spend 10 min

Find the PI of the equation

$$y'' - y = x + \frac{x^2}{2}$$

Let us now consider an example involving a trigonometric forcing function.

Example 4

Find the general solution of the differential equation

$$y'' + 4y = 2\sin 2x$$

Solution

The solution of the homogeneous equation is

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$$

Since the forcing function is itself a solution of the corresponding homogeneous equation, using the modification rule, we assume a solution of the form

$$y_p(x) = Ax \cos 2x + Bx \sin 2x$$

Substituting this into the original differential equation, we get

$$-2A \sin 2x + 2B \cos 2x - 2A \sin 2x + 2B \cos 2x - 4A\, x \cos 2x$$
$$- 4Bx \sin 2x + 4Ax \cos 2x + 4Bx \sin 2x = 2 \sin 2x$$

Equating coefficients of different trigonometric functions, we get

coefficient of $\sin 2x$: $-2A - 2A = 2$

coefficient of $\cos 2x$: $2B + 2B = 0$

coefficient of $x \sin 2x$: $-4B + 4B = 0$

coefficient of $x \cos 2x$: $-4A + 4A = 0.$

These equations require that $A = -\frac{1}{2}$ and $B = 0$ Thus,

$$y_p(x) = -\frac{1}{2} x \cos 2x$$

A general solution is then

$$y(x) = y_c(x) + y_p(x)$$
$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2} x \cos 2x$$

We now hope that you have understood the method of solving second order non-homogeneous ordinary differential equations with constant coefficients using the method of undetermined multipliers.

Let us now summarise:

> In the method of undetermined multipliers, the particular integral is constructed from the forcing function. The arbitrary constant(s) is (are) determined by solving equations obtained on comparing coefficients of like terms on the two sides of the given equation.

To ensure that your progress is satisfactory, we would like you to solve the following SAQ.

SAQ 5

Spend 20 min

Find the general solutions of the following differential equations:

(i) $y'' + y = x^2$

(ii) $y'' + 4y = 3\cos x$

(iii) $y'' + y' + 2y = 4e^x + 2x^2$

From Unit 4 of PHE-02 course, you would recall that non-homogeneous linear differential equations find an immediate application to damped spring-mass systems acted upon by external forces. Suppose that the applied force causes the weight in a spring-mass system to move up and down in some prescribed manner. Denoting the external applied force by $F(t)$, the differential equation describing one-dimensional motion of such a system is

$\ddot{x} = \frac{d^2x}{dt^2}$

$\dot{x} = \frac{dx}{dt}$

$$m\ddot{x} + \gamma\dot{x} + kx = F(t)$$

Suppose the forcing function is given by $F(t) = F_0 \cos \omega t$, where F_0 is the constant amplitude and ω is the angular frequency. Then, you can write

$$m\ddot{x} + \gamma\dot{x} + kx = F_0 \cos \omega t \tag{2.29}$$

Since the solution of this equation depends on the damping force, we have to consider separately the cases $\gamma = 0$ (undamped) and $\gamma > 0$ (damped). Let us consider the first case now.

Undamped Forced Vibrntions

If there is no damping force, the differential equation describing the motion of the spring-mass system becomes

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{2.30}$$

Let us assume the weight to be initially at rest and that $\omega \neq \omega_0 = \sqrt{k/m}$. By the method of **undetermined coefficients**, you can show that the general solution of this equation is

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

Then, using $\dot{x}(0) = x(0) = 0$, we get

$$C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad C_2 = 0$$

The desired solution is then

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

If we use the identity $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$, this expression can be rewritten in the form

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2}\right) t \sin\left(\frac{\omega_0 - \omega}{2}\right) t \tag{2.31}$$

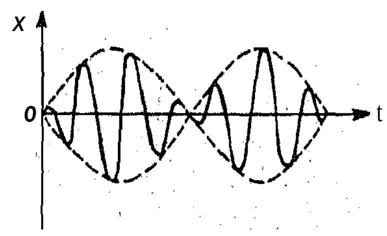

Fig. 2.5 : Displacement time plot of undamped forced vibrations

Since the two sine functions are of different frequencies, there will be occasions, especially when ω is close to ω_0, when their amplitudes will either magnify or cancel one another (see Fig. 2.5). This magnification and cancellation occurs at regular intervals and is called beat. In acoustics, these fluctuations can be heard when two tuning forks of slightly different frequencies are set into vibration simultaneously. The same phenomenon occurs in electronics, where it is called **amplitude modulation.**

You must have noticed that the method of **undetermined coefficients** is limited to those **equations** whose driving **functions** are **of very** special **form.** Let us now discuss the so-called variation of parameters method that is applicable to all linear differential equations with constant coefficients.

2.4.2 The Method of Variation of Parameters

Let y_1 and y_2 be any linearly independent solutions of the homogeneous equation corresponding to the given non-homogeneous equation. To find the particular integral, let us assume that

$$y_p = u y_1 + v y_2 \tag{2.32}$$

where u and v are unknown functions of x. To determine these, we differentiate Eq. (2.32) with respect to x. This gives

$$y_p' = u' y_1 + u y_1' + v' y_2 + v y_2' \tag{2.33}$$

We seek a solution such that

$$u' y_1 + v' y_2 = 0 \tag{2.34}$$

Using this condition in Eq. (2.33), you will get

$$y_p' = u y_1' + v y_2' \tag{2.35}$$

Differentiate this expression again. The result is

$$y_p'' = u y_1'' + v y_2'' + u' y_1' + v' y_2' \tag{2.36}$$

On substituting y_p, y_p' and y_p'' for y, y' and y'', respectively in the equation

$$a y'' + b y' + c y = g(x)$$

you will obtain

$$a (u y_1'' + v y_2'' + u' y_1' + v' y_2') + b (u y_1' + v y_2') + c (u y_1 + v y_2) = g(x)$$

This can be rearranged as

$$u (a y_1'' + b y_1' + c y_1) + v (a y_2'' + b y_2' + c y_2) + a (u' y_1' + v' y_2') = g(x) \tag{2.37}$$

Since y_1 and y_2 are solutions of the homogeneous equation

$$a y'' + b y' + c y = 0$$

the terms in both parentheses drop out. Hence, Eq.(2.37) reduces to

$$a (u' y_1' + v' y_2') = g(x) \tag{2.38}$$

This means that u' and v' satisfy the system of Eqs. (2.34) and (2.38). These can be solved by Cramer's rule. Thus

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{g(x)}{a} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 g(x)}{aW}$$

and

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{g(x)}{a} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 g(x)}{aW} \tag{2.39}$$

You will note that the denominator in this equation is the Wronskian of two linear independent functions y_1 and y_2 and is non-zero.

These equations can be integrated to obtain

For two simultaneous linear equations of the form $a_{11} x_1 + a_{12} x_2 = b_1$ and $a_{21} x_1 + a_{22} x_2 = b_2$ Cramer's rule tells us that the solutions for x_1 and x_2 are

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D}$$

and

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D}$$

where

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is a non-zero determinat.

(2.40)

The particular solution is then

$$y_p(x) = -y_1 \int \frac{y_2\, g(x)}{aW}\, dx + y_2 \int \frac{y_1\, g(x)}{aW}\, dx \tag{2.41}$$

You would agree that the integration required to obtain u and v can make this method difficult to use. So whenever possible, you should use the method of undetermined coefficients.

In SAQ 5, you obtained the PI of the equation

$$y'' + y = x^2$$

using the **method of undetermined** multipliers. Now let us obtain a particular **solution** of this equation using the method of variation of parameters.

Since the two independent solutions are

$$y_1(x) = \sin x \quad \text{and} \quad y_2(x) = \cos x$$

the Wronskian is given by

$$W(x) = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}$$

$$= -\sin^2 x - \cos^2 x = -1$$

A particular **solution** is then given by

$$y_p(x) = \sin x \int x^2 \cos x \& - \cos x \int x^2 \sin x\, dx$$

Integrating by parts, we get

$$y_p(x) = x^2 - 2$$

This solution is, however, not unique. There can be other solutions as well.

To sum up this section, we say that

> In the **method of variation of parameters,** we construct particular integral by a linear combination of **independent** solutions of the **homogeneous** equation corresponding to the **given** non-homogeneous **equation** subject to the condition
>
> $$u' y_1 + v' y_2 = 0$$
>
> where u and v are given by
>
> $$u = -\int \frac{y_2\, g(x)}{aW}\, dx \quad \text{and} \quad v = \int \frac{y_1\, g(x)}{aW}\, dx$$
>
> and W is the **Wronskian** of y_1 **and** y_2.

You may now ask: Can we use each of these methods for every second order ODE? The applicability of a method essentially depends on the differential equation under consideration. Yet the method of variation of parameters is the most general one and the method of **undetermined** multipliers is probably the most popular.

Let us now summarise **this** unit **.**

2.5 SUMMARY

- If y_1 and y_2 are solutions of a second order ODE, then they are linearly independent, if and only if, their Wronskian, defined as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is non-zero.

o A second order homogeneous ODE with constant coefficients can be solved using exponential functions. The form of the solution depends on the roots of the auxiliary equation.

For distinct real roots, there exist two linearly independent functions of the form $\exp(m_1 x)$ and $\exp(m_2 x)$, and the general solution is given by

$$y(x) = C_1 \exp(m_1 x) + C_2 \exp(m_2 x)$$

When roots are equal ($m_1 = m_2 = m$), the general solution is given by

$$y(x) = (C_1 + C_2 x) \exp(mx)$$

For a complex conjugate pair of roots, the two lincarly independent solutions are of the form $\exp(\alpha x) \sin \beta x$ and $\exp(\alpha x) \cos \beta x$, and the general solution can be written as

$$y(x) = \exp(\alpha x)(C_1 \sin \beta x + C_2 \cos \beta x)$$

$$= C \exp(\alpha x) \cos(\beta x - \phi)$$

- The solution of a non-homogeneous second order ODE with constant coefficients is a sum of the particular integral and the complementary function.
- The complementary function is the general solution of the homogeneous equation corresponding to the given non-homogeneous equation.
- In the method of undetermined multipliers, the particular integral is constructed from the forcing function. Depending on the form of the forcing function, y_p is given as follows:

Form of forcing function	Nautre of the root of auxiliary equation	Form of particular integral
$A e^{kx}$	When k is not a root k is single root k is double root	$C e^{kx}$ $C x e^{kx}$ $C x^2 e^{kx}$
Polynomial $A x^n (n = 0, 1, \ldots)$	$k = 0$ is not a root $k = 0$ is single root $k = 0$ is double root	$C_0 + C_1 x + C_2 x^2 + \ldots$ $x(C_0 + C_1 x + \ldots.)$ $x^2(C_0 + C_1 x + C_2 x^2 + ..)$
$A \cos kx$ $A \sin kx$	ik is not a root ik is a single root	$C \cos kx + D \sin kx$ $x(C \cos kx + D \sin kx)$

The arbitrary constants are determined by solving equations obtained on comparing coefficients of like terms on the two sides of the given lion-homogeneous ODE.

- In the method of variation of parameters, the particular integral is given by

$$y_p = u y_1 + v y_2$$

where y_1 and y_2 are linearly independent solutions of the homogeneous part of the given differential equation. The functions u and v can be computed using the relations

$$u = -\int \frac{y_2 g(x)}{aW} dx \quad \text{and} \quad v = \int \frac{y_1 g(x)}{aW} dx$$

where W is the Wronskian of y_1 and y_2.

2.6 TERMINAL QUESTIONS

1. The motion of a damped harmonic oscillator is described by the equation

$$m \frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + kx = F_0 \cos \omega t$$

Obtain the particular integral.

2. Solve the differential equations

i) $\frac{d^2y}{dx^2} + y = \sec x$

ii) $\frac{d^2y}{dx^2} - y = x e^x$

2.7 SOLUTIONS AND ANSWERS

SAQs

1. The Wronskiap for these functions is

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2\sin^2 2x - 2\cos^2 2x$$

$$= -2$$

Since $W(x) \neq 0$ for all x, the functions $\sin 2x$ and $\cos 2x$ are linearly independent.

2. The auxiliary equation corresponding to the given ODE is

$$m^2 + 6m + 9 = 0$$

which has a double root $m = -3$. Hence, the general solution is

$$y(x) = (C_1 + C_2 x) e^{-3x} \qquad \text{(i)}$$

The condition $y(0) = 2$ gives

$$2 = C_1 \qquad \text{(ii)}$$

Differentiate (i) with respect to x. This gives

$$\frac{dy}{dx} = C_2 e^{-3x} - 3(C_1 + C_2 x) e^{-3x}$$

Using the condition $\frac{dy(0)}{dx} = 1$, we get

$$1 = C_2 - 3C_1$$

or

$$C_2 = 1 + 3C_1 = 1 + 6 = 7 \qquad \text{(iii)}$$

Hence, the desired solution is

$$y(x) = (2 + 7x) e^{-3x}$$

3. $$x(t) = \exp(-bt)\,[C_1 \exp(\beta t) + C_2 \exp(-\beta t)] \qquad \text{(i)}$$

At $t = 0, x = 0$. This gives

$$0 = C_1 + C_2$$

or

$$C_1 = -C_2$$

Differentiate the given expression with respect to time. The result

$$\frac{dx}{dt} = -b\exp(-bt)\,[C_1 \exp(\beta t) + C_2 \exp(-\beta t)]$$

$$+ \exp(-bt)\,[\beta C_1 \exp(\beta t) - \beta C_2 \exp(-\beta t)]$$

Using the condition $\frac{dx(0)}{dt} = v_0$, we find that

$$v_0 = -b\,(C_1 + C_2) + \beta\,(C_1 - C_2)$$

or

$$C_1 = \frac{v_0}{2\beta} = -C_2$$

Hence

$$x(t) = \frac{v_o}{2\,\beta}\exp(-bt)\,[\,\exp(\beta t) - \exp(-\beta t)\,]$$

$$= \frac{v_0}{\beta}\exp(-bt)\sinh\beta t$$

This shows that the resultant motion of a heavily damped oscillator is determined by the interplay of a decaying exponential and a hyperbolic function,

4. Assume that the PI is of the form

$$y_p = C_0 + C_1 x + C_2 x^2 \qquad \text{(i)}$$

Substituting it and its second derivative

$$\frac{d^2 y_p}{dx^2} = 2C_2$$

in the given **ODE,** you would get

$$2C_2 - (C_0 + C_1 x + C_2 x^2) = x + \frac{x^2}{2} \qquad \text{(ii)}$$

For (i) to be an acceptable solution, (ii) should be an identity. This gives

$$C_0 = -1$$

$$C_1 = -1$$

and

$$C_2 = -\frac{1}{2}$$

Therefore

$$y_p(x) = -\frac{x^2}{2} - x - 1$$

5. (i) The solution of the homogeneous equation

$$\frac{d^2 y}{dx^2} + y = 0$$

is found to be

$$y_c(x) = C_1 \cos x + C_2 \sin x$$

A particular integral is assumed to have the form

$$y_p(x) = Ax^2 + Bx + C$$

This is substituted into the original differential equation to give

$$2A + Ax^2 + Bx + C = x^2$$

Equating coefficients of the various gapers of x, we have

Coefficient of x^0: $2A + C = 0$

Coefficient of x^1: $B = 0$

Coefficient of x^2: $A = 1$

These equations are solved simultaneously to give the particular solution

$$y_p(x) = x^2 - 2$$

Finally, the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1 \cos x + C_2 \sin x + x^2 - 2$$

(ii) The solution of the corresponding homogeneous equation $y'' + 4y = 0$ which has roots $\pm 2i$, is

$$y_c = C_1 \sin 2x + C_2 \cos 2x$$

To find a particular solution of the given equation, we assume that the general form of y_p is

$$y_p = A \sin x + B \cos x$$

This is the correct expression since neither of these functions is in y_c. Substituting y_p and y_p'' into the given differential equation, we obtain

$$(-A \sin x - B \cos x) + 4 (A \sin x + B \cos x) = 3 \cos x$$

Expanding and collecting like terms yields

$$3A = 0, \text{ and } 3B = 3$$

which has the solution $A = 0$ and $B = 1$. Hence, $y_p = \cos x$ and the general solution is

$$y = C_1 \sin 2x + C_2 \cos 2x + \cos x$$

(iii) Assume the particular solution to have the form

$$y_p(x) = Ae^x + Bx^2 + Cx + D$$

Substitute this into the given differential equation, The result is

$$Ae^x + 2B + Ae^x + 2Bx + C + 2Ae^x + 2Bx^2 + 2Cx + 2D = 4e^x + 2x^2$$

Equating the various coefficients, you would obtain

Coefficient of e^x : $A + A + 2A = 4$

Coefficient of x^0 : $2B + C + 2D = 0$

Coefficient of x^1 : $2B + 2C = 0$

Coefficient of x^2 : $2B = 2$

From the above equations, we find that $A = 1$, $B = 1$, $C = -1$ and $D = -\frac{1}{2}$. Thus,

$$y_p(x) = e^x + x^2 - x - \tfrac{1}{2}$$

Terminal Questions

1. The given differential equation is

$$m\ddot{x} + \gamma \dot{x} + kx = F_0 \cos\omega t \qquad \text{(i)}$$

where dot over x denotes derivative with respect to time. As such, you have learnt to solve it in your **PHE-02** course. But we repeat the solution for the sake of completeness. The corresponding homogeneous equation is

$$m\ddot{x} + \gamma \dot{x} + kx = 0$$

which is the same as the equation describing damped vibration without a forcing function. Its solution depends upon the sign of $\gamma^2 - 4mk$. Thus

If $\gamma^2 - 4mk > 0$

$$x_c(t) = C_1 e^{-(\alpha-\beta)t} + C_2 e^{-(\alpha+\beta)t}$$

If $\gamma^2 - 4mk = 0$

$$x_c(t) = e^{-\alpha t}(C_1 t + C_2)$$

If $\gamma^2 - 4mk < 0$

$$x_c(t) = e^{-\alpha t}(C_1 \cos \omega' t + C_2 \sin \omega' t)$$

$$= Ce^{-\alpha t} \cos (\omega' t - \delta)$$

where $\alpha = \gamma/2m$, $\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}$, $\omega' = \frac{1}{2m}\sqrt{4mk - \gamma^2}$, $C = \sqrt{C_1^2 + C_2^2}$

and $\tan \delta = C_2/C_1$.

Since no constant multiple of the driving function $F_0 \cos \omega t$ is a term of x_c (t), the particular solution is of the form

$$x_p(t) = A \cos \omega t + B \sin \omega t$$

Differentiating twice with respect to time, we get

$$\dot{x}_p(t) = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$\ddot{x}_p(t) = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

Substituting these in (i) and collecting the coefficient of sine and cosine terms, we have

$$[(k - m\omega^2)A + \omega\gamma B] \cos \omega t + [-\omega\gamma A + (k - m\omega^2) B] \sin \omega t = F_0 \cos \omega t$$

Equating the coefficients of the sine and cosine terms on both sides of this equality, we get

$$-\omega \gamma A + (k - m\omega^2)B = 0$$

and $\quad (k - m\omega^2)A + \omega \gamma B = F_0$

Solving these for A and B, you will get

$$A = \frac{F_0 (k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2\gamma^2} \quad \text{and} \quad B = \frac{\gamma\omega F_0}{(k - m\omega^2)^2 + \omega^2\gamma^2}$$

Recalling that $\sqrt{k/m} = \omega_0$, we can write A and B as

$$A = \frac{F_0 m (\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \quad \text{and} \quad B = \frac{\gamma\omega F_0}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

We choose to write $x_p(t)$ in the form

$$x_p(t) = C \cos (\omega t - \delta)$$

where $C = F_0 \Big/ \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$ and $\tan \delta = \omega\gamma / m(\omega_0^2 - \omega^2)$.

For large values of t, the motion is essentially described by $x_p(t)$. For this reason, $x_p(t)$ is called the **steady-state solution.**

2.(i) Since two linearly independent solutions of the corresponding homogeneous equation are $\cos x$ and $\sin x$, the general solution of the given equation is

$$y = C_1 \cos x + C_2 \sin x + y_p$$

where $y_p = u \cos x + v \sin x$ and u' and v' are, respectively given by

$$\frac{du}{dx} = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} \quad \text{and} \quad \frac{dv}{dx} = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}}$$

Therefore, $\frac{du}{dx} = -\tan x$ and $\frac{dv}{dx} = 1$, from which it readily follows that $u = \ln|\cos x|$

and $v = x$. The general solution is, therefore,

$$y = C_1 \cos x + C_2 \sin x + \ln|\cos x| \cos x + x \sin x$$

Note that the method of undetermined coefficients could not be used to obtain y_p because sec x is not a solution of the homogeneous linear differential equation.

(ii) The corresponding homogeneous equation $y'' - y = 0$ has the general solution $y = C_1 e^x + C_2 e^{-x}$. Because of the nature of the driving function, y_p could not be found by the method of undetermined coefficients. In the method of variation of parameters, we put $y_1 = e^x$ and $y_2 = e^{-x}$ in the expressions for u' and v' to obtain

$$u' = \frac{\begin{vmatrix} 0 & e^{-x} \\ xe^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} \quad \text{and} \quad v' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & xe^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}$$

Therefore, $u' = x/2$ and $v' = \frac{-x \exp(2x)}{2}$ from which it readily follows that $u = x^2/4$ and $v = -(xe^{2x}/4) + (e^{2x}/8)$. The general solution is then

$$y = C_1 e^x + C_2 e^{-x} + \frac{1}{4} x^2 e^x - \frac{x}{4} e^x + \frac{1}{8} e^x$$

Finally, we note that $C_1 e^x$ and $\frac{1}{8} e^x$ can be combined as $(C_1 + \frac{1}{8}) e^x = Ce^x$ and the general solution may be written as

$$y = C_1 e^x + C_2 e^{-x} - \frac{1}{4} xe^x + \frac{1}{4} x^2 e^x$$

UNIT 3 SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Structure

3.1 INTRODUCTION

In Unit 2, you have learnt how to solve second order ODEs with constant coefficients. The solutions of these equations are simple exponentials, trigonometric or hyperbolic functions known from calculus. But in many physical and engineering problems, we have to solve second order ordinary differential equations with variable coefficients. For example, we have to solve such equations to study the field distribution around a charged sphere or a cylinder, and energy production in a reactor. Similarly, when we wish to know how high a vertical column of uniform cross-section can be extended upward until it buckles under its own weight, we have to solve a second order ODE with variable coefficients. In such cases, simple algebraic or transcendental solutions do not exist and methods discussed in Unit 2 do not work. We, therefore; look for other methods.

One of the most elegant and efficient methods of solving such ODEs is the **power series** method. This is so particularly because it facilitates numerical computations. Even so, it has limited utility when coefficients of the given differential equation are not well defined at some point. In such cases, we use an extension of the power series method, called the **Frobenius' method.** You will learn these two methods in this unit. The properties of power series and certain other mathematical concepts are given in an Appendix at the end of this unit. It would be better if you study the appendix before studying this unit.

Objectives

After studying this unit, you should be able to

- define ordinary and singular points
- locate and classify the type of singularity
- use power series method to solve a second order ODE about an ordinary point
- use Frobenius' method to solve a second order ODE about a regular singular point

3.2 SOME TERMINOLOGY

While studying first and second order ODEs with constant coefficients, you have learnt some basic terminology. You would come across some more common terms, which you do not know as yet, in reference to second order ODEs with variable coefficients. This section is intended to familiarise you with these concepts,