On adding these results and collecting the coefficients of, $\hat{e}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ we get

$$\nabla \times \mathbf{F} = \frac{\hat{\mathbf{e}}_1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(F_3 h_3) - \frac{\partial}{\partial u_3}(F_2 h_2)\right] + \frac{\hat{\mathbf{e}}_2}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(F_1 h_1) - \frac{\partial}{\partial u_1}(F_3 h_3)\right] + \frac{\hat{\mathbf{e}}_3}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(F_2 h_2) - \frac{\partial}{\partial u_2}(F_1 h_1)\right]$$

This can be written as

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

4. i) In spherical polar coordinates

$\mathbf{r} = r\,\hat{\mathbf{e}}_r$

and

$\boldsymbol{\omega} = \omega\,(\cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta)$

Hence

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$
$$= \omega\, r\,(\cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta) \times \hat{\mathbf{e}}_r$$
$$= -r\,\omega \sin\theta\,(\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r)\,.$$
$$= \omega\, r \sin\theta\,\hat{\mathbf{e}}_\phi$$

ii)
$$\nabla \times \mathbf{v} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \omega\, r^2 \sin^2\theta \end{vmatrix}$$

$$= \frac{1}{r^2 \sin 8}\left[\hat{\mathbf{e}}_r\, 2\,0\, r^2 \text{ sine } \cos\theta - \hat{\mathbf{e}}_\theta\, 2\omega\, r^2 \sin^2\theta\right]$$
$$= 2\,\hat{\mathbf{e}}_r\,\omega\cos\theta - 2\omega\,\hat{\mathbf{e}}_\theta \sin\theta$$
$$= 2\omega\,(\hat{\mathbf{e}}_r \cos\theta - \hat{\mathbf{e}}_\theta \sin\theta) = 2\omega\,\hat{\mathbf{k}} = 2\,\boldsymbol{\omega}$$

5. $\mathbf{F} = \hat{\mathbf{e}}_r \dfrac{2\, r_0 \cos\theta}{r^3} + \hat{\mathbf{e}}_\theta \dfrac{r_0}{r^3}\sin\theta$

Hence

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & \hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{2\, r_0 \cos\theta}{r^3} & \frac{r_0}{r^3}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta}\left[r\sin\theta\,\hat{\mathbf{e}}_\phi\left(-\frac{3r_0}{r^4}\sin\theta + \frac{2r_0}{r^3}\sin\theta\right]\right.$$
$$= \hat{\mathbf{e}}_\phi \frac{\sin\theta}{r^4}\left(2r_0 - 3\frac{r_0}{r}\right)$$
$$= \hat{\mathbf{e}}_\phi \sin\theta\left(\frac{2r\, r_0 - 3r_0}{r^5}\right).$$

UNIT 4 INTEGRATION OF SCALAR AND VECTOR FIELDS

Structure

4.1 INTRODUCTION'

In the previous unit you have studied about different coordinate systems. In the process you have learnt to represent the gradient of a scalar field and the divergence and curl of a vector field in several coordinate systems. In Unit 2 you have studied about differentiation of vectors, In this unit you will learn about integration of scalar and vector fields.

These integrals find many applications in physics. For example, the work done by a force or the magnetic field due to a current-carrying conductor can be expressed as a line integral. The flux of a magnetic field can be represented as a surface integral. Line, surface and volume integrals find application in determining the potential due to a continuous distribution of matter or charges. We shall discuss some of these examples here. These integrals form the foundations of many important equations in physics like the Maxwell's equation of Electromagnetism, the equation of continuity and so on.

In this unit, first we shall discuss the integration of a vector with respect to a scalar. You will find that this will basically be an extension of the idea'of 'ordinary integral. This finds application in determining the trajectory of a particle if its equation of motion is known. We shall also discuss the integration of scalar and vector products of vectors with respect to a scalar.

Next we shall discuss the integration of scalar and vector fields with respect to coordinates. As you know a scalar or a vector field may be a function of one or more coordinates. Their integrals which we shall come across will necessarily not be in terms of single variable. So we shall learn about double and triple integrals. You will see that the three kinds of field integrals, i.e. the line, surface and volume integrals are respectively the extensions of ordinary, double and triple integrals.

Finally, we shall discuss how one kind of a field integral can be transformed into another type. In the process you will learn to apply the vector integral theorems — namely the Gauss' divergence theorem, Stokes' theorem and Green's theorem, Here, we shall only state these theorems without proof, However, if you are interested in knowing the proofs you may *go* through the Appendix. These theorems provide us with very elegant methods for arriving at several fundamental equations of physics.

In the next block, we shall take up Statistics and Probability and their applications in physics.

Objectives

After going through this unit you should be able to:

- solve problems based on integration of a vector with respect to a scalar
- evaluate double and triple integrals
- solve problems based on line, surface and volume integration of scalar and vector fields
- solve problems based on Gauss', Stokes' and Green's theorems and apply them to relevant physical situations.

4.2 INTEGRATION OF A VECTOR WITH RESPECT TO A SCALAR

Many a time we have to obtain the description of motion of a system from the knowledge of the forces acting on it. In order to solve such a problem we need to integrate the velocity and position vectors with respect to time, a scalar. Let us see how we do it.

In Unit 2, you have learnt to differentiate a vector with respect to a scalar. You know that integration is the reverse process of differentiation. This holds also for integration of vectors with respect to scalars. Let us consider a vector **a** which is a function of a scalar t i.e. $\mathbf{a} = \mathbf{a}(t)$. Now, let

$$\frac{d\,\mathbf{a}(t)}{dt} = \mathbf{b}(t) \tag{4.1}$$

Then the (indefinite) integral of $\mathbf{b}(t)$ with respect to t is equal to $\mathbf{a}(t) + \mathbf{c}$, where **c** is an arbitrary constant vector. In physics we deal with quantities that generally have dimensions. Therefore, **c** is a vector whose dimension is the same as that of **a**. Symbolically, we write

$$\int \mathbf{b}(t)\,dt = \mathbf{a}(t) + \mathbf{c} \tag{4.2}$$

In actual problems, **c** is determined by using suitable initial conditions.

To illustrate Eq. (4.2) let us take the example of determining the trajectory of a particle when the force acting on it is known as a function of time. Let the force be $\mathbf{F}(t)$. Then its acceleration is given by $\mathbf{f}(t) = \frac{\mathbf{F}(t)}{m}$, where m is the mass of the particle. The relation between acceleration, **f** and velocity, **v** is quite well-known. It is given by

$$\mathbf{f}(t) = \frac{d}{dt}\mathbf{v}(t)$$

and

$$\mathbf{v}(t) = \int \mathbf{f}(t)\,dt + \mathbf{c}_1, \tag{4.3}$$

where $\mathbf{c}_1$ has the dimensions of velocity. Again velocity **v** is related to the position vector **r** as

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t)$$

$$\therefore\ \mathbf{r}(t) = \int \mathbf{v}(t)\,dt + \mathbf{c}_2 \tag{4.4}$$

where $\mathbf{c}_2$ is a constant vector having the dimension of length.

In order to evaluate the integrals in Eqs. (4.3) and (4.4), we first express the concerned vector in the component form. This means that for evaluating the integral $\int \mathbf{a}(t)\,dt$, we express $\mathbf{a}(t)$ as

$\mathbf{a}(t) = a_1(t)\,\hat{\mathbf{i}} + a_2(t)\,\hat{\mathbf{j}} + a_3(t)\,\hat{\mathbf{k}}$, where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are respectively the x, y, and z components of $\mathbf{a}(t)$. Then, we have

$$\int \mathbf{a}(t)\,dt = \hat{\mathbf{i}}\int a_1(t)\,dt + \hat{\mathbf{j}}\int a_2(t)\,dt + \hat{\mathbf{k}}\int a_3(t)\,dt \tag{4.5}$$

So, in effect we have to perform integration of a scalar with respect to a scalar with which you are familiar, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ come outside the integration signs as they do not depend on t.

The following linearity property holds for integrals of vectors. For p and q being two constants,

$$\int [p\mathbf{V}_1(t) + q\mathbf{V}_2(t)]\,dt = p\int \mathbf{V}_1(t)\,dt + q\int \mathbf{V}_2(t)\,dt. \qquad (4.6)$$

Let us work out an example based on what we have discussed so far.

Example 1

The force acting on a particle of constant mass m is given in terms of *t* by

$$\mathbf{F} = b(\cos\omega t\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}})$$

If the particle is initially at rest at the origin, compute its velocity as a function of *t*.

Solution

From Newton's second law, the acceleration of the particle is

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(t) = \frac{\mathbf{F}}{m} = \frac{b}{m}\cos\omega t\,\hat{\mathbf{i}} + \frac{b}{m}\sin\omega t\,\hat{\mathbf{j}}$$

Using Eqs. (4.1) and (4.6), we get

$$\mathbf{v}(t) = \frac{b}{m}\left[\hat{\mathbf{i}}\int\cos\omega t\,dt + \hat{\mathbf{j}}\int\sin\omega t\,dt\right]$$

These are ordinary integrals and can be readily evaluated to give

$$\mathbf{v}(t) = \frac{b}{m}\left[\hat{\mathbf{i}}\frac{\sin\omega t}{\omega} + \hat{\mathbf{j}}\left(-\frac{\cos\omega t}{\omega}\right)\right] + \mathbf{c}_1$$

We now have to determine $\mathbf{c}_1$.

Since the particle is at rest initially, we have $\mathbf{v}(0) = \mathbf{0}$. So,

$$\mathbf{0} = \frac{b}{m\omega}(-\hat{\mathbf{j}}) + \mathbf{c}_1$$

$$\therefore\quad \mathbf{c}_1 = \frac{b}{m\omega}\hat{\mathbf{j}}$$

Hence $\qquad \mathbf{v}(t) = \dfrac{b}{m\omega}[\sin\omega t\,\hat{\mathbf{i}} + (1 - \cos\omega t)\,\hat{\mathbf{j}}]$

You should now determine the position of the particle as a function of t yourself.

SAQ 1

Spend 2 min

Find the position vector of the particle of Example 1 as a function of t. (Remember that the particle was initially at the origin.)

Now that you have gone through Example 1 and worked out **SAQ 1**, you must have realised that while integrating a vector with respect to a scalar we follow the technique of ordinary integration. Does the same apply to scalar and vector products ? We will now attempt to answer this question.

4.2.1 Integrals Involving Scalar and Vector Products of Vectors

Let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ be two vector functions of a scalar *t*. Then for evaluating the integrals $I_1 = \int[\mathbf{a}(t)\cdot\mathbf{b}(t)]\,dt$ and $I_2 = \int[\mathbf{a}(t)\times\mathbf{b}(t)]\,dt$, we first compute the scalar and vector products in the integrands. You may then recall from Sec. 1.4 that I_1 will reduce to an integral of a scalar function of t with respect to t, and I_2 will be an integral of a vector function of t with respect to t. Let us take a specific example to discuss the evaluation of I_1. After that you will work out an **SAQ** to evaluate I_2.

Example 2

In free space a transverse electromagnetic (EM) wave propagating in the x-direction has an electric field $\mathbf{E} = E_0\cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{j}}$ and a magnetic induction field $\mathbf{B} = B_0\cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{k}}$. Here c and λ are respectively the velocity and the wavelength of the EM-wave and $E_0 = B_0 c$.

The energy flowing through a volume V per unit time is given by $U = \frac{V}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$, where $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. Here ε_0 and μ_0 are the permittivity and the magnetic permeability, respectively of free space and c can be expressed as $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Compute the total energy flowing through V during one complete cycle of EM oscillation if its time period is T.

Solution

The energy flow during time dt will be given by U dt. So the total energy will be the definite integral of U from $t = 0$ to $t = T$, i.e.

$$U_0 = \int_0^T U\, dt$$

$$= \frac{V}{2}\int_0^T (\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})\, dt = \frac{V}{2}(I_E + I_B)$$

where $I_E = \int_0^T \mathbf{E}\cdot\mathbf{D}\, dt$ and $I_B = \int_0^T \mathbf{B}\cdot\mathbf{H}\, dt.$

Both I_E and I_B are integrals of the type I,. So we shall first evaluate the scalar products.

$$\mathbf{E} = E_0 \cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{j}}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} = \varepsilon_0 E_0 \cos\frac{2\pi}{\lambda}(ct - x)\,\hat{\mathbf{j}}$$

$$\therefore \quad \mathbf{E}\cdot\mathbf{D} = \varepsilon_0 E_0^2 \cos^2\frac{2\pi}{\lambda}(ct - x)$$

Similarly, you can show that

$$\mathbf{B}\cdot\mathbf{H} = \frac{B_0^2}{\mu_0}\cos^2\frac{2\pi}{\lambda}(ct - x)$$

$$\therefore \quad U_0 = \frac{V}{2}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right) I$$

where $I = \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)\, dt = \frac{T}{2}$

$$\therefore \quad U_0 = \frac{VT}{4}\left(\varepsilon_0 E_0^2 + \frac{B_0^2}{\mu_0}\right)$$

Again $B_0^2 = \frac{E_0^2}{c^2} = \varepsilon_0 \mu_0 E_0^2 \quad \left(\because c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}\right)$

$$\therefore \quad \frac{B_0^2}{\mu_0} = \varepsilon_0 E_0^2.$$

Hence $U_0 = \frac{VT}{2}\varepsilon_0 E_0^2$

$$\frac{2\pi c}{\lambda} = \frac{2\pi}{T} \quad (\because \lambda = cT)$$

$$\cos^2\frac{2\pi}{\lambda}(ct - x)$$

$$= \cos^2\left(\frac{2\pi t}{T} - kx\right),$$

where $k = \frac{2\pi}{\lambda}$

$$= \frac{1}{2}\left\{\cos\left[2\left(\frac{2\pi t}{T} - kx\right)\right] + 1\right\}$$

$$\therefore \int_0^T \cos^2\frac{2\pi}{\lambda}(ct - x)\,dt$$

$$= \frac{1}{2}\int_0^T \cos\left(\frac{4\pi t}{T} - 2kx\right)dt$$

$$+ \frac{1}{2}\int_0^T dt$$

$$= \frac{1}{2}\frac{T}{4\pi}\left|\sin\left(\frac{4\pi t}{T} - 2kx\right)\right|_0^T$$

$$+ \frac{T}{2}$$

$$= \frac{T}{8\pi}[\sin(4\pi - 2kx)$$

$$- \sin(-2kx)] + \frac{T}{2}$$

$$= \frac{T}{8\pi}(-\sin 2kx + \sin 2kx) + \frac{T}{2}$$

$$= \frac{T}{2}$$

SAQ 2

The mass of the bob of the simple pendulum shown in Fig. 4.1 is m. It executes small oscillations (i.e. its angular amplitude does not exceed 4°) in the xy plane. Using the result $\tau = \frac{d\mathbf{L}}{dt}$, where τ and L are respectively the torque and angular momentum of the bob about the origin, show that

$\mathbf{L} = m\sqrt{l^3 g}\,\theta_0 \sin\left(\sqrt{\frac{g}{l}}\,t\right)\hat{\mathbf{k}} + \mathbf{L}_0$, where $\mathbf{L} = \mathbf{L}_0$ at $t = 0$. It is given that the time-variation of

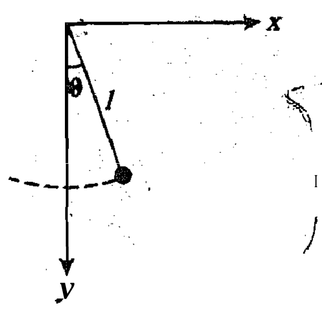


Fig. 4.1 : A simple pendulum of effective length l.

the angular displacement of the bob can be expressed as $\theta = \theta_0 \cos \frac{2\pi t}{T}$, where $T = 2\pi\sqrt{\frac{l}{g}}$

(Hint: Use the result $\tau = \mathbf{r} \times \mathbf{F}$)

Spend 5 min

Now that you have worked out Example 2 and SAQ 2, you understand that the process of integration of the scalar and vector products of two vectors with respect to a scalar essentially reduces to determining the integral of a scalar with respect to a scalar. Thus we find that the overall aspect of integration of vector and scalar function of a scalar can be managed by evaluating ordinary integrals. However, in physics we come across many applications involving the integration of scalar and vector fields (about which you have read in Unit 2) with respect to scalars and vectors. Such applications include the determination of work done by a force field, flux of a vector field and so on. The knowledge of ordinary integrals is not sufficient for carrying out integrations. For this we have to learn to evaluate multiple integrals.

4.3 MULTIPLE INTEGRALS

So far we have performed integration with respect to a single variable. There will be cases, such as the determination of moment of inertia, product of inertia, the coordinates of centre of mass of continuous matter where we shall have to do so with respect to more than one variable. These are called multiple integrals. Here, we shall discuss douhle and triple integrals.

In Unit 2 you learnt to differentiate a function of several variables. The method was just an extension of the differentiation of a function of a single variable. Also we know that integration is the reverse process of differentiation. So you will find that the method of evaluation of multiple integrals is an extension of that of the ordinary integral. In the case of a definite integral

$$\int_a^b f(x)dx$$

the integrand is a function $f(x)$ that exists for all x in an interval $a < x < b$. Since x is the only variable of integration we call this a single integral. This integral can be expressed as the limit of the sum

$$\sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

as n tends to infinity. Here Δx_i is an infinitesimal segment on the x-axis in the neighbourhood of the point $x = x_i$. You know to evaluate such an integral. So we will not go into details. Instead, we proceed to introduce the double integral.

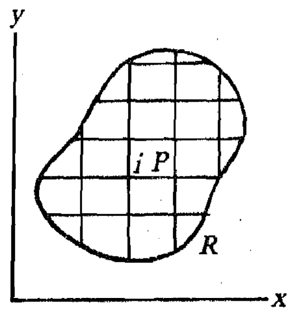


Fig. 4.2

4.3.1 Double Integral

In the case of a double integral, the integrand say $\phi(x, y)$ is given for all (x, y) in a closed bounded region R of the xy-plane (Fig. 4.2). The double integral can be defined in a manner analogous to that used for a single integral. We subdivide the region R by drawing lines parallel to x and y-axes. We number the rectangles that are within R from 1 to n. Now, let us consider a point P, in the ith rectangle. Let the value of the function at that point be ϕ_i. Now we form the sum $S_n = \sum_{i=1}^{n} \phi_i \Delta A_i$ where ΔA_i is the area of the ith rectangle and n is a positive integer. We take this sum for larger and larger values of n. Then the rectangles become smaller. And as n goes to infinity, the length of the maximum diagonal of the rectangles approaches zero and the value of ϕ_i may be considered as constant at all points within the ith rectangle. If $\phi(x, y)$ is continuous in R, the limit of S_n, as n tends to infinity, is finite and its value is independent of the choice of the sub-divisions. This limit is called the **double integral** of $\phi(x, y)$ over the region R. Thus

As you know a closed interval on the x-axis, say between $x = a$ and $x = b$ is expressed as $a \le x \le b$. Similarly for a two dimensional region; being 'closed', its boundary must be a part of the region. Moreover, if the region can be enclosed in a circle of sufficiently large radius, then it is a "closed bounded region".

$$\lim_{n \to \infty} \sum_{i=1}^{n} \phi_i \Delta A_i = \iint_R \phi(x, y)\, dx\, dy = \iint_R \phi(x, y)\, dA \qquad (4.7)$$

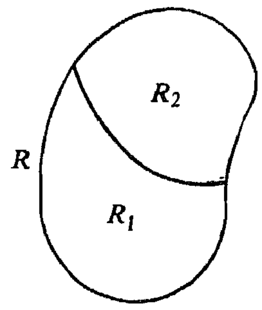


Fig. 4.3

where $dA = dxdy$.

From the definition it is quite evident that the double integral has certain properties which are quite similar to that of single integrals. We shall now state these properties.

Let ϕ_1 and ϕ_2 be the functions of x and y, defined and continuous in a region R. Then

$$\iint_R k\,\phi_1\,dxdy = k\iint_R \phi_1\,dxdy \tag{4.8a}$$

where k is a constant,

$$\iint_R k\,(\phi_1+\phi_2)\,dxdy = k\iint_R \phi_1\,dxdy + k\iint_R \phi_2\,dxdy \tag{4.8b}$$

and

$$\iint_R \phi_1\,dxdy = \iint_{R_1} \phi_1\,dxdy + \iint_{R_2} \phi_1\,dxdy \tag{4.8c}$$

where R has been subdivided into two regions R_1 and R_2 (Fig.4.3).

Evaluation **of Double** Integrals

Double integrals over a region R may be calculated by way of evaluating two successive integrations. The first integration is performed with respect to one of the variables, keeping the other variable constant. Then the double integral reduces to an ordinary definite integral which can be easily evaluated. We shall illustrate this with the help of an example.

Example **3**

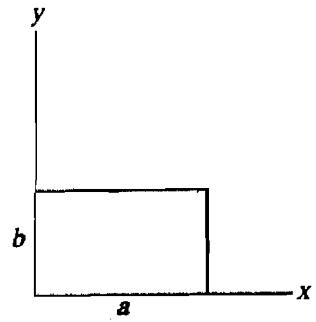


Fig. 4.4

The product of inertia of a lamina in the xy-plane about the x and y-axes is given by

$$I_{xy} = I_{yx} = \iint_R \sigma xy\,dxdy$$

where R is the region of space covered by the lamina. And σ is the mass per unit area of the lamina. Determine I_{xy} for the rectangle shown in Fig. 4.4.

Solution

$$I_{xy} = \int_0^a\int_0^b \frac{m}{ab}\,xy\,dxdy$$

where m is the mass of the rectangle.

$$\therefore\ I_{xy} = \frac{m}{ab}\int_0^a\left(\int_0^b xydy\right)dx$$

Now for evaluating the inner integral we have to treat x as constant.

$$\therefore\qquad \int_0^b xydy = \left|\frac{xy^2}{2}\right|_0^b = \frac{xb^2}{2}$$

Hence,

$$I_{xy} = \frac{m}{ab}\int_0^a x\,\frac{b^2}{2}\,dx = \frac{mb}{2a}\int_0^a x\,dx$$

$$= \frac{mb}{2a}\left|\frac{x^2}{2}\right|_0^a = \frac{mb}{2a}\,\frac{a^2}{2} = \frac{mab}{4}$$

We shall now illustrate the evaluation of double integral using spherical polar coordinates, which you studied in Unit 3.

The moment of inertia of a body like a disc, a spherical shell or a rectangular lamina, about an axis, is given by

$$I = \int r^2\,dm$$

where the integral extends over the entire body. Here dm is the mass of an infinitesimal element of the body and r is its distance from the axis. You may find some examples of determination of I in Block-2 of PHE-01 (Elementary Mechanics) course. In this block we have mentioned the expression for moment of inertia of a spherical shell about a diameter. For a shell of mass M and radius R it is $\frac{2}{3}MR^2$. We shall illustrate the method to evaluate it in the following example.

Example 4

Write down the expression for an element of area of the spherical shell of Fig. 4.5 in spherical polar coordinates and calculate its moment of inertia about any diameter. The mass of the shell is M and its radius R.

Solution

Let us take the z-axis along the diameter AB about which we intend to calculate I. We take an element of area, da, on the shell included between the polar angles θ and $\theta + d\theta$ and the azimuthal angles ϕ and $\phi + d\phi$. You know from Eq. (3.18) that

$$da = (Rd\theta)\,(R\sin\theta\, d\phi)$$

Now $dm = \sigma d\alpha$, where σ, the mass per unit area is $\frac{M}{4\pi R^2}$. Hence, you can write

$$dm = \frac{M}{4\pi R^2} R^2 \sin\theta d\theta d\phi = \frac{M}{4\pi}\sin\theta d\theta d\phi$$

From Fig. 4.5 it is evident that $r = R\sin\theta$ so that

$$I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (R\sin\theta)^2 \frac{M}{4\pi}\sin\theta d\theta d\phi$$

You may now complete the remaining part of the example.

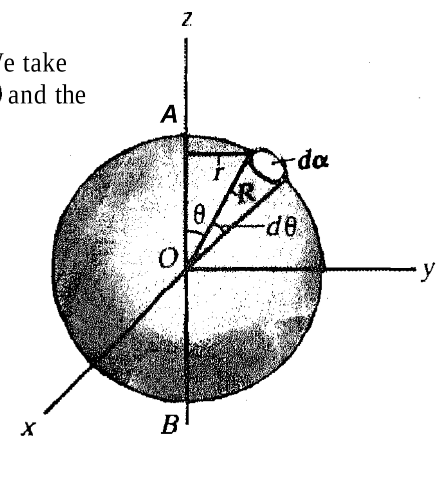


Fig. 4.5

SAQ 3

Spend 2 min

Evaluate the double integral given above and show that

$$I = \frac{2}{3}MR^2$$

While going through Examples 3 and 4, you must have realised that the differential term in each integral refers to an area. In other words, we have performed integration over areas. In physics, we will be required to perform integrations over volumes as well. Such is the case of determining the moment of inertia of a solid body or the quantity of fluid flowing out of a system. So we will now discuss triple integrals as a first step towards understanding integration over volumes.

4.3.2 Triple Integrals

For defining the triple integral we shall follow an approach similar to that for defining a double integral. Let us consider a function $\psi(x, y, z)$ defined in a bounded closed region T in space. In this connection you may recall that in sub-section 4.3.1, we had considered a region R in a plane. But now T is a region in 3-dimensional space, for example, a solid cube, a football or the region between two concentric spheres. We subdivide this three-dimensional region T by planes parallel to the three coordinate planes. Then we get many rectangular parallelepipeds inside that space which we number from 1 to n. Let us consider a point, in the ith parallelepiped. Let the value of the function be ψ_i. Then we form the sum

Refer to the margin remark corresponding to the beginning of Sec. 4.3.1 for the meaning of the term 'Closed'. However, here 'bounded' means that the region can be enclosed in a sphere of sufficiently large radius.

$$S_n = \sum_{i=1}^{n} \psi_i \Delta V_i$$

where ΔV_i is the volume of the ith parallelepiped.. We take this sum for larger and larger positive integers n, so that the parallelepipeds become smaller and smaller and lengths of the edges of the largest parallelepiped of subdivision approach zero as n goes to infinity. In this case, the value of the function ψ_i may be considered to be same at all points with the parallelepiped. If $\psi(x, y, z)$ is continuous in the region of space T, the limit of S_n as n tends to infinity is finite and its value is independent of the subdivisions. This limit is called the **triple integral** of $\psi(x, y, z)$ over the region T. Thus,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \psi_i \, \Delta V_i = \iiint_T \psi(x,y,z)\, dxdydz = \iiint_T \psi(x,y,z)\, dV \qquad (4.9)$$

where $dV = dxdydz$.

Triple integrals can be evaluated by three successive integrations. This is similar to the evaluation of double integrals by two successive integrations. The method of evaluation is illustrated in the following example.

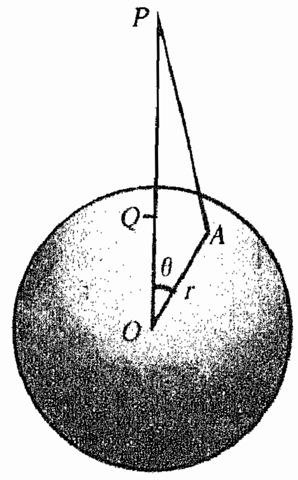


Fig. 4.6

Example 5

Determine the gravitational potential due to a solid sphere of mass M and radius a at a point (i) outside the sphere and (ii) inside the sphere.

Solution

Refer to Fig.4.6. O is the centre of the sphere. P and Q are, respectively, an external and an internal point. Here O, P and Q are collinear. Since a sphere is symmetrical about its centre, the potential depends only on the distance of the observation point from the centre of the sphere. It is independent of its position. In other words, the potential due to the sphere at all external points which lie on a sphere of radius OP and centre at O will be equal to that at P. Likewise, the potential at all internal points lying on a sphere of radius OQ and centre at O will be equal to that at Q.

Now let us consider an infinitesimal volume element dV of the sphere at A, where $OA = r$ and $\angle AOP = \theta$. Here OP is taken to be the polar axis and the volume element dV is included between r and $r + dr$, θ and $\theta + d\theta$, ϕ and $\phi + d\phi$. In other words

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

We have solved the problem given in Example 5 in unit 5 of Block-I of PHE-01 course. The method adopted there has too many steps. First you have to determine the potential due to a spherical shell at an inside and on outside point and then extend it for the case of a solid sphere. The process is particularly tedious for the case of the inside point. Here we shall solve it much more elegantly using the idea of triple integrals.

The point where the potential has to be determined is called the field point.

Now, let r' and R be the distances of the field point from A and O, respectively. Hence in case (i), $r' = AP$, $R = OP$ and in case (ii), $r' = AQ$, $R = OQ$.

You will note that in case (i) $R > a$ and in case (ii) $R < a$.

The gravitational potential due to the volume element dV at any point at a distance r' from it is given by

$$d\phi = -\frac{G\rho \, dV}{r'}$$

where ρ is the density of the material of the sphere and is given by

$$\rho = \frac{M}{\frac{4}{3}\pi a^3} = \frac{3M}{4\pi a^3}$$

Since $r' = \sqrt{r^2 + R^2 - 2rR\cos\theta}$, you can write

$$d\phi = -\frac{G\rho\, r^2 \sin\theta \, dr\, d\theta\, d\phi}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}$$

so that

$$\phi = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} -\frac{G\rho\, r^2 \sin\theta\, dr\, d\theta\, d\phi}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}$$

We now separate out the triple integral into three integrals with respect to ϕ, θ and r, i.e.,

$$\phi = -G\rho \int_{r=0}^{a} r^2 dr \int_{\theta=0}^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \int_{\phi=0}^{2\pi} d\phi$$

The ϕ-integral is quite simple and its value is 2π. The result of the θ-integral is a function of r, say $f(r)$. Then you can write

$$\phi = -2\pi G\rho \int_{0}^{a} r^2 f(r)\, dr,$$

where

$$f(r) = \int_{\theta=0}^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}.$$

So far as the evaluation of $f(r)$ is concerned, you have to remember that θ is the only variable quantity. Any thing other than θ should be considered as constant. We now introduce a change of variable by writing

$$\sqrt{r^2 + R^2 - 2rR\cos\theta} = u$$

or $$r^2 + R^2 - 2rR\cos\theta = u^2$$

Taking differentials on both sides, we get

$$2Rr\sin\theta d\theta = 2udu \quad \text{or} \quad \frac{\sin\theta \, d\theta}{u} = \frac{du}{Rr}$$

When $\theta = \pi$, $\cos\theta = -1$, and $u^2 = (r + R)^2$, i.e. $u = R + r$

When $\theta = 0$, $\cos\theta = 1$ and $u^2 = (r - R)^2$

Thus $\theta = 0$ will correspond to $u = R - r$ for $R > r$ and $r - R$ for $R < r$. Now, as r ranges from 0 to a for case (i), $R > r$ always, so that $u = R - r$ for $\theta = 0$. But case (ii) is a little tricky.

$$\left.\begin{array}{l} \text{For } 0 < r < R, \quad R > r \quad \text{and } u = R - r \\ \text{and for } R < r < a, \quad R < r \quad \text{and} \quad u = r - R \quad \text{for } \theta = 0. \end{array}\right\}$$

Now let us take up calculation of ϕ for case (i)

$$f(r) = \int_{R-r}^{R+r} \frac{du}{Rr} = \frac{1}{Rr}\,[(R + r) - (R - r)] = \frac{2}{R}$$

$$\therefore \ \phi = -2\pi G\rho \int_0^a \frac{2r^2}{R}\, dr$$

$$= -\frac{4\pi G\rho}{R}\int_0^a r^2\, dr = -\frac{4\pi G\rho}{R}\frac{a^3}{3}$$

$$\therefore \ \phi = -\frac{GM}{R} \quad \left(\because \ M = \frac{4\pi a^3}{3}\rho\right)$$

You should now work out case (ii) yourself.

SAQ 4

Spend 5 min.

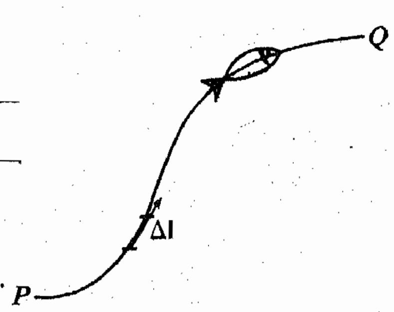
Complete the solution of Example 5 by working out ϕ for an inside point.

Now that you have learnt about the double and triple integrals, you can go over to learn the integration of scalar and vector fields. You have studied about vector fields in Sec. 2.5.1. Scalar fields can be expressed as scalar functions of position coordinates. Vector fields can be represented as vector functions of position coordinates. We shall now learn about the integrals of these functions. There can be three kinds of integrals called line integral, surface integral and volume integral. We shall first try to understand how and where these integrals arise. Let us begin with the line integral.

Planktons are minute plant and animal beings found in water.

4.4 LINE INTEGRAL OF A FIELD

Imagine a fish swimming through the sea and gathering planktons as it moves. The total mass of planktons gathered by the fish during a given time interval depends upon the path (Fig. 4.7a) which it follows and the density of plankton at each point it swims through. In an extremely short interval of time, the mass gathered by the fish is approximately equal to $|f\Delta\mathbf{l}|$, where f is the linear density of the planktons (i.e. number of planktons per unit length) at any point through which the fish passes, and $\Delta\mathbf{l}$ is the displacement of the fish during that short time interval. In general, the path of a fish is along an arc. For a short time



(a)

Fig. 4.7a

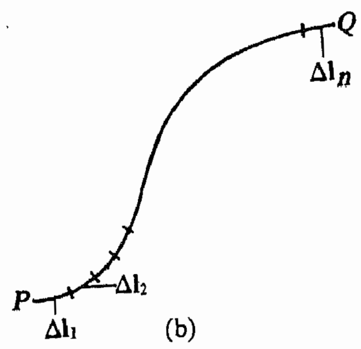


Fig. 4.7b

interval, an extremely small arc can be regarded as coincident to a straight line. So the magnitude of $\Delta \mathbf{l}$ is equal to the length of this arc or straight line and it points along the tangent to the arc (Fig. 4.7a) at any point on it. Its sense is along the direction of motion of the fish.

Now, the mass gathered during this short interval of time Δt can be obtained by dividing the entire path into such small intervals (Fig. 4.7b) and adding up the terms $|f\Delta \mathbf{l}|$ for these intervals. In general, f varies along the path. Let $f_1, f_2 ..., f_n$ be the values of the linear density function f at the positions of the segments $\Delta \mathbf{l}_1$, $\Delta \mathbf{l}_2$,..., $\Delta \mathbf{l}_n$. So the mass m gathered by the fish can be expressed as the limit of the sum

$$S = |f_1 \Delta \mathbf{l}_1| + |f_2 \Delta \mathbf{l}_2| + \ldots + |f_n \Delta \mathbf{l}_n|$$

as the largest of the Δl s tends to a null vector. So m will be the magnitude of the line integral of the linear density function over the line of motion of the fish, i.e.

$$m = \left| \int_P^Q f(l)\, d\mathbf{l} \right| \tag{4.10}$$

where P and Q indicate the end points of the path (Fig. 4.7b). The segment, included between the points P and Q of the curve along which the fish moves is called the **path of integration** for Eq. (4.10).

Thus you can understand that the line integral is a generalisation of the single integral $\int_a^b f(x)dx$ in the sense that the path of integration can be any line, straight or curved. Before going into further discussions on line integrals we mention some important features about the path of integration. We shall require them for the evaluation of the line integrals.

4.4.1 Path of Integration

As mentioned earlier the path of integration is in general a curve. Let us refer to that by the letter C. As you have studied in Sec.2.2.1. C is the locus of a point which can be represented by an equation

$$\mathbf{r} = \mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \tag{4.11}$$

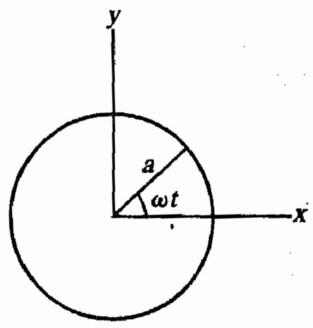


Fig. 4.8

where t is a parameter and $\mathbf{r}(t)$ is a vector function of t. A representation of the form (4.11) is useful in many applications, where t may be time. For example, suppose a particle executes a uniform circular motion along the circumference of a circle of radius a with an angular speed ω (Fig. 4.8). Then the equation of its path is given by

$$\mathbf{r} = a \cos\omega t\,\hat{\mathbf{i}} + a \sin\omega t\,\hat{\mathbf{j}}$$

where t ranges from 0 to $2\pi/\omega$.

Similarly, the parabola $y = x^2$ (Fig. 4.9) can be expressed in the parametric form as

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t^2\hat{\mathbf{j}}, \quad -\infty < t < \infty$$

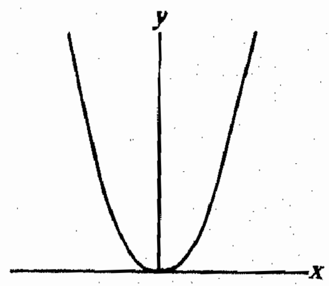


Fig. 4.9

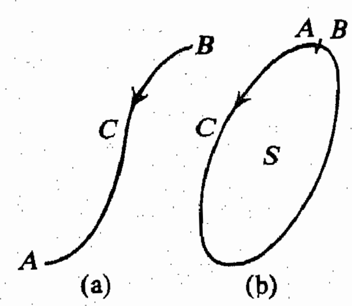


Fig. 4.10

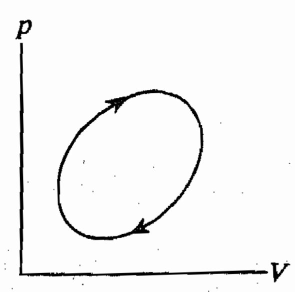


Fig. 4.11

Now, refer to Fig.4.10a. We call C a smooth curve if it has a representation of the form (4.11), with $a \le t \le b$, where $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are the position vectors of the initial and terminal points A and B. In addition, $\mathbf{r}(t)$ must have a continuous derivative $\frac{d\mathbf{r}}{dt}$ whose magnitude is always non-zero. The points A and B may coincide as in Fig. 4.10b. The curve C is called a **closed curve.** For example, in thermodynamics a cyclic process is represented by a closed curve on a pressure-volume $(p - V)$ diagram (Fig. 4.11).

The geometrical significance of C is that there exists a unique tangent at each of its points whose direction varies continuously as we traverse C. The direction of the tangent is along the unit tangent vector $\hat{\mathbf{u}} \left(= \frac{d\mathbf{r}}{dt} \Big/ \left| \frac{d\mathbf{r}}{dt} \right| \right)$. C is said to be oriented when a definite sense on C is associated with $\hat{\mathbf{u}}$. Conventionally, the positive orientation corresponds to the direction of increasing t.

A somewhat different convention is used for closed curves. Let S be a region of the plane bounded by a single **simple** (i.e. non-intersecting) closed curve C (Fig. 4.10b). By convention, we choose the positive orientation of C in such a way that as we move along the boundary of the region S, the region always lies to our left. This implies that the anticlockwise sense is positive. The symbol of integration along any path C is $\int_C$. If the path is closed we use $\oint_C$ where the arrow indicates the sense of traversing C. You may now like to work out a simple SAQ on what you have learnt in this subsection.

A machine which intercepts the flow of heat and converts some of it into useful work is called a heat engine. To be useful such an engine must be able to operate continuously. This means that the processes which take place inside the engine must not lead to any permanent change. Hence, the working of an engine involves a series of operations, called strokes so that the state of the system at the beginning and end of such operation are identical. In other words the initial and final states are represented by identical pair of $p - V$ values. So such an operation is an example of a cyclic process.

SAQ 5

Spend 5 min

a) Refer to Fig. 4.12. Represent the curves shown here in parametric form. Mention the corresponding ranges of the parameter.

b) What is the sign of the sense of traversing the closed curve of Fig. 4.11 ?

c) Refer to Fig. 4.13. Are the curves $ABCD$ and PQR smooth?

4.4.2 Types of Line Integrals

We encounter three kinds of line integrals of fields

i) $\int_C \phi \, d\mathbf{r}$

ii) $\int_C \mathbf{A} \cdot d\mathbf{r}$

and

iii) $\int_C \mathbf{A} \times d\mathbf{r}$

where ϕ and $\mathbf{A}$ represent a scalar and vector field, respectively. While (i) and (iii) give vectors, (ii) defines a scalar.

The general properties of line integrals are as follows :

$$\int_C kI = k \int_C I \qquad (4.12a)$$

$$\int_C (I_1 + I_2) = \int_C I_1 + \int_C I_2 \qquad (4.12b)$$

and (See Fig. 4.14)

$$\int_C I = \int_{C_1} I + \int_{C_2} I \qquad (4.12c)$$

where I, I_1, I_2 may represent integrands of any type among (i), (ii), (iii).

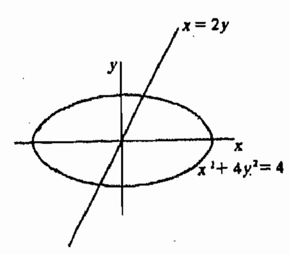


Fig. 4.12

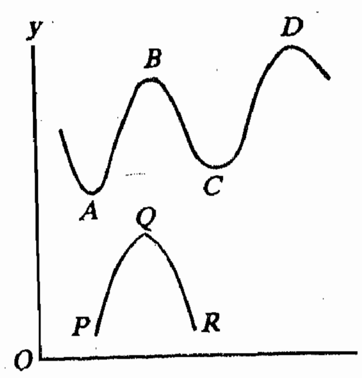


Fig. 4.13

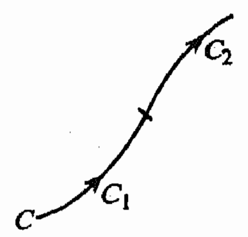


Fig. 4.14

We shall now take up the evaluation of line integrals. We have already considered an example of (i) while introducing the line integral. So we shall concentrate on (ii) and (iii) in course of discussing the methods of evaluation of line integrals.

4.4.3 Evaluation of Line Integrals

For evaluating the integral $\int_C \mathbf{A} \cdot d\mathbf{r}$, we shall take an example where $\mathbf{A}$ will be a variable force $\mathbf{F}$. From Sec. 1.4, we know that $\mathbf{F} \cdot d\mathbf{r}$ is the work done by the force $\mathbf{F}$ when displacement is $d\mathbf{r}$. In other words, the line integral of a force over a certain path is the work done by the force over that path. Let us now take up a specific case.

Example 6

A two-dimensional force field is defined as

$$\mathbf{F} = \frac{k(x\hat{\mathbf{j}} - y\hat{\mathbf{i}})}{x^2 + y^2},$$

where k is a constant. Compute the work done by this force in taking a particle from point $P(1,0)$ to $Q\ (0,1)$ shown in Fig. 4.15 along the straight line PQ.

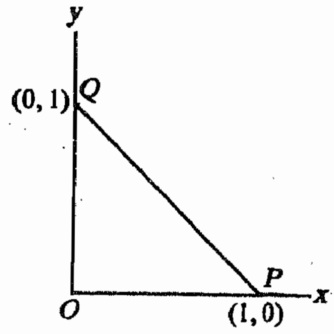


Fig. 4.15

Solution

In order to evaluate the integral we have to express $\mathbf{F}$ and $d\mathbf{r}$ as a function of the same parameter, say t. The equation of PQ is

$$x + y = 1.$$

In going from P to Q, x changes from 1 to 0 and y changes from 0 to 1. This can be expressed in the parametric form as $x = t$, $y = 1 - t$, where t goes from 1 to 0.

Now, $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} = t\hat{\mathbf{i}} + (1-t)\hat{\mathbf{j}}$

$$\therefore\ d\mathbf{r} = dt\,\hat{\mathbf{i}} - dt\,\hat{\mathbf{j}} = (\hat{\mathbf{i}} - \hat{\mathbf{j}})dt$$

and

$$\mathbf{F} = k\frac{t\hat{\mathbf{j}} + (t-1)\hat{\mathbf{i}}}{t^2 + (1-t)^2}$$

$$\therefore\ \mathbf{F}\,.\,d\mathbf{r} = \frac{k[(t-1)\hat{\mathbf{i}} + t\hat{\mathbf{j}}]\,.\,(dt\,\hat{\mathbf{i}} - dt\,\hat{\mathbf{j}})}{t^2 + (1-t)^2}$$

$$= k\frac{(t-1)\,dt - t\,dt}{2t^2 - 2t + 1} = -\frac{k\,dt}{2t^2 - 2t + 1}$$

So the work done is given by

$$W = \int_1^0 -\frac{k\,dt}{2t^2 - 2t + 1} = k\int_0^1 \frac{dt}{2t^2 - 2t + 1} = \frac{k\pi}{2}$$

$$\int_0^1 \frac{dt}{2t^2 - 2t + 1}$$
$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + 1/2}$$
$$= \frac{1}{2}\int_0^1 \frac{dt}{(t - 1/2)^2 + (1/2)^2}$$
$$= \frac{1}{2}\,.\,2\left|\tan^{-1}\left(\frac{t - 1/2}{1/2}\right)\right|_0^1$$
$$= |\tan^{-1}(2t-1)|_0^1$$
$$= \tan^{-1} 1 - \tan^{-1}(-1)$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

You can also solve it in an alternative way. To illustrate it we compute the scalar product $\mathbf{F}\,.\,d\mathbf{r}$:

$$\mathbf{F}\,.\,d\mathbf{r} = \frac{k}{x^2 + y^2}(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}})\,.\,(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}})$$

or

$$\mathbf{F}\,.\,d\mathbf{r} = \frac{k}{x^2 + y^2}(xdy - ydx) \qquad (4.13)$$

(i) Now the equation of the straight line PQ is $x + y = 1$

$$\therefore\ dx + dy = 0,\ \text{i.e.}\ dy = -dx,\ \text{and}$$

$$x^2 + y^2 = x^2 + (1-x)^2$$

$$= 2x^2 - 2x + 1$$

So from Eq. (4.13), we get

$$\int_P^Q \mathbf{F}\,.\,d\mathbf{r} = k\int_P^Q \frac{-xdx + (x-1)\,dx}{2x^2 - 2x + 1} = -k\int_P^Q \frac{dx}{2x^2 - 2x + 1}$$

$$\therefore\ \mathbf{F}\,.\,d\mathbf{r} = k\int_0^1 \frac{dx}{2x^2 - 2x + 1}$$

$$= \frac{k\pi}{2}$$

You may now work out the value of work done along another path joining P and Q.

SAQ 6

Spend 5 min

For the force field of Example 6, calculate the work done when the particle is taken from P to Q in Fig. 4.15 along the combination of the paths PO and OQ.

Now that you have gone through Example 6 and worked out SAQ 6, you would realise that the work done by the given force field along two different paths joining the same initial and final points are not the same. However in physics, we come across a particular kind of force field for which the work done along different paths joining the same pair of points is always the same. Such a force field is said to be conservative. We shall study briefly about **it now**.

You must have noted that in the alternative method, we get

$$\int_P^Q \mathbf{F} \cdot d\mathbf{r} = \int_P^Q F(x, y)dx + \int_P^Q F(x, y)dy$$

The two integrals on the right hand side are line integrals but F is a function of two variables. While integrating over x, we replace y in $F(x, y)$ by $f(x)$ where $y = f(x)$ is the equation of the path of integration connecting P and Q. Similarly for integrating over y, we replace x in $F(x, y)$ where $x = g(y)$ is the equation of the path of integration connecting P and Q.

Conservative Force Field

If the work done by a force field in taking a particle from one point to another depends only on its initial and final positions and is independent of the intermediate path, then the force field is **conservative.** Otherwise it is called **non-conservative.** The gravitational force between any two bodies is conservative. So is the force between two charges. To understand more about a conservative force field, refer to Fig. 4.16. A and B are two points in the field of a conservative force $\mathbf{F}$. So, according to the definition of a conservative force we must have

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} = \int_{ADB} \mathbf{F} \cdot d\mathbf{r} = \int_{AEB} \mathbf{F} \cdot d\mathbf{r} \qquad (4.14)$$

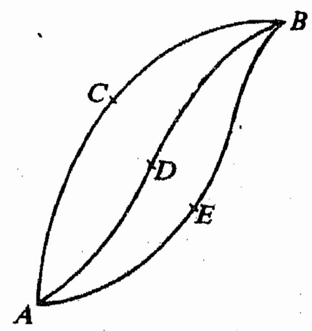


Fig. 4.16

This means that $\mathbf{F} \cdot d\mathbf{r}$ may be expressed as the differential of a scalar function of position coordinates only. So, we may write that

$$\mathbf{F} \cdot d\mathbf{r} = -dU. \qquad (4.15)$$

where $U = U(x,y,z)$. We have brought the negative sign on the right-hand side so that U may be identified with the potential energy (P.E.) of the particle in the given force field. You may find detailed discussion about this in Sec 3.3 of Block-1 of PHE-01 (Elementary Mechanics) course. You must note that P.E. is a characteristic of a conservative force field only. You may now work out an SAQ. Its first part deals with the relation between the force and the P.E. The second part refers to the relation between the line integral and the change in kinetic energy (K.E.) of the particle.

SAQ 7

Spend 10 min.

a) Using Eq. 4.15, prove that $\mathbf{F} = -\nabla U$. (**Hint :** Express $-dU$ in terms of dx, dy and dz and $\mathbf{F} \cdot d\mathbf{r}$ in component form. Then equate them.)

b) Prove that

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = T_B - T_A,$$

where T_A and T_B are the K.Es of the particle at A and B, respectively. (**Hint :** Write $\mathbf{F} = m\dfrac{d\mathbf{v}}{dt}$, where $\mathbf{v}$ is the velocity of the particle at time t and m its mass. Then express $d\mathbf{r}$ in terms of dt and evaluate the integral.)

Having worked out SAQ 7, you would realise that this result is true for any force. However, if $\mathbf{F}$ is conservative we may use Eq. (4.15) for expressing $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ in an alternative form. It is very simple and is given by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = -\int_A^B dU = -(U_B - U_A) = U_A - U_B$$

Hence, from the result of SAQ 7b, we get

$$U_A - U_B = T_B - T_A$$

or $$U_A + T_A = U_B + T_B = \text{a constant.}$$

This means that the sum of P.E. and K.E. is always a constant. So the total mechanical energy is conserved for such a force field and that is why it is said to be conservative.

So far we have discussed the evaluation of line integral of type (ii). We shall now illustrate the evaluation of the line integral of type (iii) with particular reference to a physical situation.

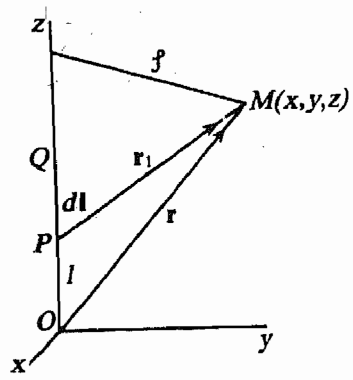


Fig. 4.17

Example 7

Refer to Fig. 4.17. An electric current I flows upwards along an infinitely long wire. Let us choose the z-axis along the length of the wire. Calculate the magnetic induction **B** due to the current in the wire at the point M (x, y, z)

Solution

The magnetic induction due to an element $d\mathbf{l}$, carrying current I, at a point whose position vector is **r** with respect to the element, is given by

$$d\mathbf{B} = \frac{C\,I\,d\mathbf{l} \times \mathbf{r}}{r^3} \qquad (4.16)$$

where C is a constant depending on the nature of the medium in which the wire is placed.

Now, let $OP = l$. We consider infinitesimally small element **PQ** (= $d\mathbf{l}$) of the wire. According to Eq. (4.16) the induction at M due to this element is

$$d\mathbf{B} = \frac{CI\,d\mathbf{l} \times \mathbf{r}_1}{r_1^3}$$

Since the wire is infinitely long, we can integrate this expression from $-\infty$ to ∞. This gives

$$\mathbf{B} = CI \int_{-\infty}^{+\infty} \frac{d\mathbf{l} \times \mathbf{r}_1}{r_1^3} = -CI \int_{-\infty}^{+\infty} \left(\frac{\mathbf{r}_1}{r_1^3}\right) \times d\mathbf{l}$$

Can you identify this integral ? It is the line integral of type (iii) . To evaluate this, we note that

$$\mathbf{r}_1 = \mathbf{OM} - \mathbf{OP}$$

Now, $\mathbf{OM} = x\,\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{OP} = l\hat{\mathbf{k}}$ so that

$$\mathbf{r}_1 = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - l)\hat{\mathbf{k}}$$

and $$r_1^3 = [\sqrt{x^2 + y^2 + (z-l)^2}\,]^3 = [\rho^2 + (z-l)^2]^{3/2}$$

where $\rho (= \sqrt{x^2 + y^2})$ is the perpendicular distance of M from the wire.

You should try the remaining part of the solution yourself.

Spend 10 min

SAQ 8

Prove that $\mathbf{B} = 2CI\,(-\frac{y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{x^2 + y^2})$

Now that you have worked out SAQ 8, you can visualise that a pole of a bar magnet experiences a force of the type given in Example 6 for which the work done is path-dependent. This implies that such a force is non-conservative.

So far we have dealt with the line integral of scalar and vector fields. As you have seen in Sec. 2.5, the flux of a vector field is a very important concept in physics. To understand it we need to study surface integrals of vector field. It is also found useful in determining the number of nucleons escaping into the environment from the core of a nuclear reactor. So let us now introduce surface integrals of scalar and vector fields.

4.5 SURFACE INTEGRAL OF A FIELD

We have seen that the line integral is a generalisation of a single integral. Likewise a surface integral is a generalisation of double integral.

If we move a bar magnet M towards a circular coil C (Fig. 4.18), we know that an electromotive force is induced in the coil. This happens because the number of lines of force linked with the coil changes with time. But the question arises : How to know the number of lines of force (called magnetic flux) linked with the coil at a particular position ? We definitely do not count them.

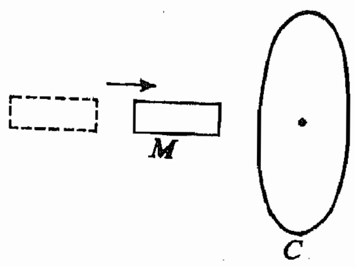


Fig. 4.18

From Sec 1.4 you would recall that an area can be visualised as a vector. Since the plane of the coil is an area at every point of which the intensity of the magnetic field produced by the bar magnet has a different value, the integral of this field over the area of the coil gives a measure of the number of lines of force. This indicates that the flux of the magnetic field through the coil is given by the surface integral of the magnetic field vector over the area of the coil. We shall now learn evaluation of surface integration. In order to do that we must **know how to represent the surface of integration. We do that now.**

4.5.1 Surface of Integration

A surface may or may not lie in a plane. If we have a surface S enclosed by a closed curve C (Fig. 4.19) in a plane, then it is an **open surface** lying on that plane. This surface is then a vector quantity whose magnitude is equal to the area of the surface, and its direction is along the normal to this surface. The choice of positive direction of the normal depends on the sense in which the perimeter of this surface is traversed. If the right hand fingers are placed in the sense of travel around the perimeter, the positive normal points in the direction indicated by the thumb of the right hand shown in Fig. 4.19. The surface shown there, when traversed in the sense $+x \rightarrow +y \rightarrow -x \rightarrow -y \rightarrow +x$ will have its positive normal parallel to the positive direction of the z-axis. Here we have taken the example of a surface for which the normals at all the points are parallel to one another. We may come across surfaces where they may not be parallel as we shall see now.

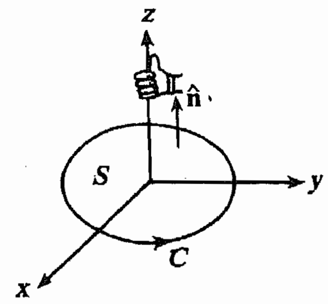


Fig. 4.19

If a volume is enclosed by a curved surface, like the shell of a whole egg, it is called a closed surface (Fig. 4.20a). For such a surface the direction of the normal varies from point to point. However, at any point, it is the convention to take the direction of the outward drawn normal as positive.

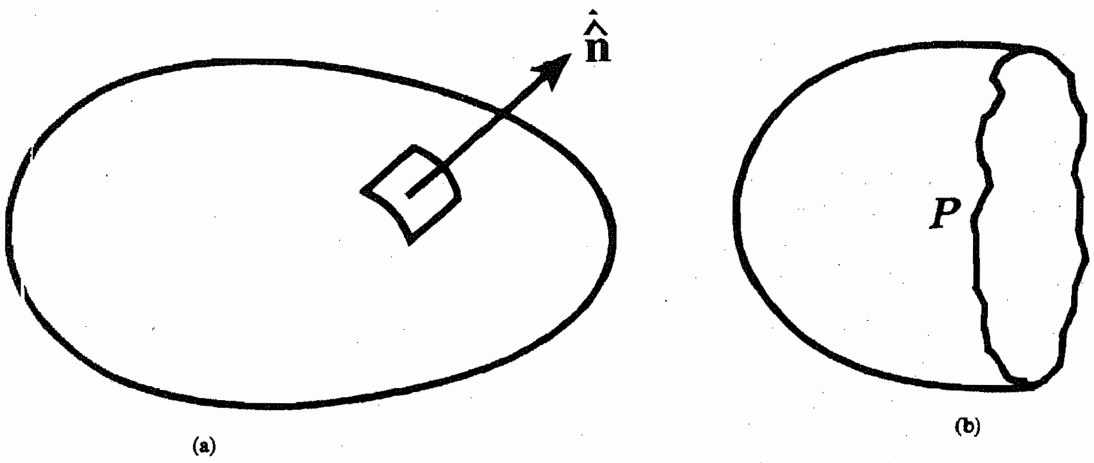


Fig. 4.20

We may sometime come across curved surfaces such as the shell of a cracked egg (Fig. 4.20b). It is an open surface as it has a periphery P. In this case one side of the surface is chosen arbitrarily as outside and at any point the outward normal is taken to be positive. However, in actual applications we come across open and closed surfaces of the types shown in Figs. 4.19 and 4.20a, respectively.

With these ideas we can say that an element of area on any surface is a vector. It is denoted as $d\mathbf{S}$ and is given by

$$d\mathbf{S} = \hat{\mathbf{n}}\, dS, \qquad (4.17)$$

$\hat{\mathbf{n}}$ being the unit vector along the positive normal, as given by the right-hand rule.

For example, the element of surface included between the angles θ and $\theta + d\theta$ as well as ϕ and $\phi + d\phi$ of the sphere of radius r (Fig. 4.21) is given by (see Sec. 3.3.2) as

$$d\mathbf{S} = (r\, d\theta)\,(r \sin\theta\, d\phi)\, \hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}} \qquad (4.18)$$

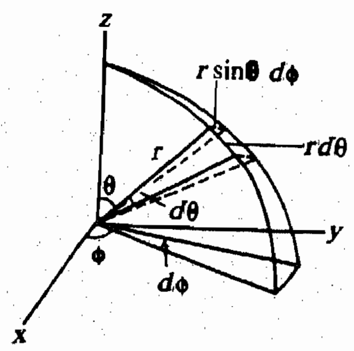


Fig. 4.21

You may now like to work out a simple SAQ on this idea.

Spend 5 min

SAQ 9

Refer to Fig. 4.22. Represent vectorially the elementary surface included between z and $z + dz$, ϕ and $\phi + d\phi$ of the cylinder having radius a.

Fig. 4.22

The symbol for a surface integral over an open surface is $\iint_S$ whereas a surface integral over a closed surface is represented by $\oiint_S$

Let us now see what the different types of surfaces integrals are.

4.5.2 Types of Surface Integrals

Analogous to line integrals, surface integrals may appear in the forms

i) $\iint_S \phi \, d\mathbf{S}$

ii) $\iint_S \mathbf{A} \cdot d\mathbf{S}$

iii) $\iint_S \mathbf{A} \times d\mathbf{S}$

where ϕ is a scalar field and $\mathbf{A}$ is a vector field.

Type (ii) is the most commonly encountered form. We shall discuss only this here. It is referred to as the flux of the vector **A** through the surface S. Thus for the example cited at the beginning of this section the flux through the coil is the integral $\iint_S \mathbf{B} \cdot d\mathbf{S}$, where **B** is the magnetic induction vector due to the magnet at the position of the element $d\mathbf{S}$ of the area of the coil. Here S indicates the overall area of the coil.

$\mathbf{J} = ne\mathbf{v}$.
where n is the number of electrons per unit volume, e is the charge on an electron and $\mathbf{v}$ is the average drift velocity of an electron.

The current I flowing through a wire is the flux of the current density (**J**) vector across a cross-section of the wire. i.e.

$$I = \iint_S \mathbf{J} \cdot d\mathbf{S} \tag{4.19}$$

where $d\mathbf{S}$ is an element of a cross-section of the wire. Similarly, the mass m of a fluid flowing out of a volume V is given by the flux of the vector $\rho\mathbf{v}$ across the closed surface S enclosing V, where ρ is the density of the fluid and **v** its average flow velocity. Mathematically, we can write

$$m = \oiint_S \rho \, \mathbf{v} \cdot d\mathbf{S} \tag{4.20}$$

We shall now discuss the method of evaluation of surface integral with reference to form (ii).

4.5.3 Evaluation of Surface Integrals

You may recall that for evaluating the line integral $\mathbf{A} \cdot d\mathbf{r}$, we had to express **A** and $d\mathbf{r}$ in terms of the same single parameter. For evaluating the surface integral $\iint_S \mathbf{A} \cdot d\mathbf{S}$, we have to express **A** and $d\mathbf{S}$ in terms of the **same pair of parameters.** Then this integral will reduce to an ordinary double integral. Let us consider an example.

Example 8

An under-sea electric cable of total length l has a leakage current of $k(l - z)$ amperes per unit area normal to the surface, where k is a constant and z is the length of cable measured from the shore. The cable is circular in cross-section with radius a. What is the total leakage current through the surface of the cable ?

Solution

Here the leakage current density is given by

$$\mathbf{J} = k(l - z)\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is along the normal to the surface of the cable. An element of the surface of the cable may be expressed as

$$d\mathbf{S} = \hat{\mathbf{n}}\, dS$$

Since a cable is cylindrical it will be convenient to use the cylindrical polar coordinates (see Fig. 4.23). We have $dS = ad\phi dz$, so that $d\mathbf{S} = ad\phi dz\hat{\mathbf{n}}$. Hence from Eq (4.19), we get

$$I = \iint_S \mathbf{J} \cdot d\mathbf{S} = \iint_S k(l-z)\, \hat{\mathbf{n}}.ad\phi dz\hat{\mathbf{n}}$$

$$= \int_{z=0}^{l} \int_{\phi=0}^{2\pi} k(l-z)\, adzd\phi$$

$$= ak\int_0^l 2\pi\,(l-z)\, dz$$

$$= 2\pi ak \left| lz - \frac{z^2}{2} \right|_0^l = \pi akl^2$$

So the current is πakl^2 amperes.

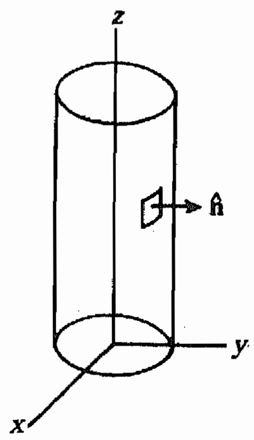


Fig. 4.23

We shall now consider another example of surface integral.

Example 9

Evaluate $\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S}$ where S is the surface of a sphere of radius R.

Solution

Refer to Fig. 4.24. Since $d\mathbf{S}$ points along the outward drawn normal, it points along $\hat{\mathbf{r}}$ at every point on the sphere so that $d\mathbf{S} = \hat{\mathbf{r}}\, dS$

$$\therefore\ \hat{\mathbf{r}} \cdot d\mathbf{S} = \hat{\mathbf{r}} \cdot dS\hat{\mathbf{r}} = dS(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) = dS \qquad (\because\ \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1)$$

Hence $\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S} = \oiint_S dS = S$, which is the surface area of the sphere.

Thus

$$\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S} = 4\pi R^2 \qquad (4.21)$$

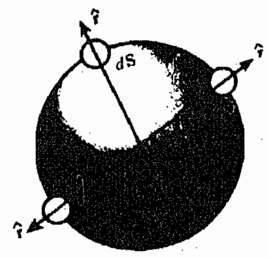


Fig. 4.24

This surface integral finds use in defining solid angle, which is used quite often in physics. We will study briefly about that now.

Solid angle

Refer to Fig. 4.25a. We know that an arc of length Δl subtends an angle $\Delta\theta = \frac{\Delta l}{r}$ at P which is at a distance r from it. This angle lies on a plane and is formed by joining point P to the two ends of the arc. Now if instead of an arc we have an area ΔS (Fig. 4.25b), then the lines joining all the peripheral points of the area with some point Q give a solid angle $\Delta\Omega$. You may like to know : How do we measure $\Delta\Omega$?

Refer to Fig. 4.25c. The solid angle subtended by a sphere at its centre is the sum of the total angles lying above and below any plane passing through the diameter. i.e. $2\pi + 2\pi = 4\pi$.

Now, we find from Eq. (4.21) that

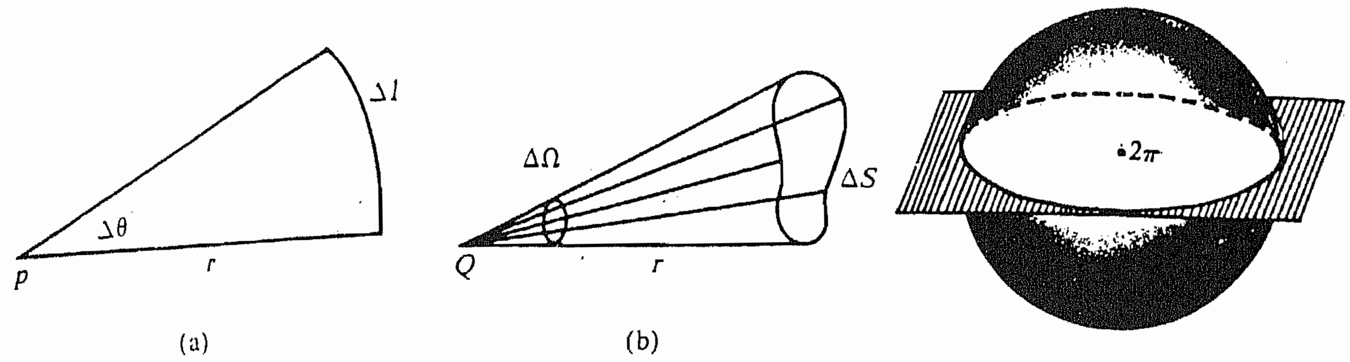
$$\frac{\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S}}{R^2} = 4\pi = \text{the solid angle subtended at the centre (i.e. a point at a distance } R$$

from surface of the sphere) by the whole sphere.

We may generalise this result to calculate the solid angle subtended by an area ΔS at a distance r from it (Fig. 4.25b). The surface integral in the numerator is taken over ΔS, and R

in the denominator is obviously replaced by r. So we have

$$\Delta\Omega = \frac{\iint_{\Delta S} \hat{\mathbf{r}} \,.\, d\mathbf{S}}{r^2} \qquad (4.22)$$



(a) (b) (c)

Fig. 4.25 : (a) The angle subtended at P by the arc Δl is given by $\theta = \frac{\Delta l}{r}$; (b) the solid angle $\Delta\Omega$ subtended at Q by the surface area ΔS; (c) the total solid angle subtended at the centre of a sphere by its upper hemisphere is 2π.

Using Eq. (4.22) we may obtain the expression for the solid angle subtended at the centre of a sphere by an area ΔS on its surface, which is intercepted between two directions defined by θ and $\theta + d\theta$ (see Fig. 4.21)

From Eq. (4.18), we have $d\mathbf{S} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}$

$$\therefore \; \hat{\mathbf{r}} \,.\, d\mathbf{S} = r^2 \sin\theta d\theta d\phi \qquad (\because \; \hat{\mathbf{r}} \,.\, \hat{\mathbf{r}} = 1)$$

Now, as ΔS is infinitesimal, the integral

$$\iint_{\Delta S} \hat{\mathbf{r}} \,.\, d\mathbf{S} = \hat{\mathbf{r}} \,.\, d\mathbf{S} = r^2 \sin\theta d\theta d\phi$$

Hence, from Eq. (4.22), we get

$$\Delta\Omega = \sin\theta d\theta d\phi \qquad (4.23)$$

Eq. (4.23) finds applications in many problems of scattering, kinetic theory of gases and electrodynamics.

Before we proceed we would like you to work out an SAQ on the evaluation of surface integral.

SAQ 10

Spend 10 min

The electric field due to a point charge q, at a point whose position vector with respect to the location of q is $\mathbf{r}$, is given by

$$\mathbf{E} = \frac{kq}{r^3} \mathbf{r} \qquad (r \neq 0)$$

where k is a constant dependent on the nature of the medium.

Now determine the flux of $\mathbf{E}$ through a sphere of radius a (Fig. 4.26) whose centre is at the position of the charge q.

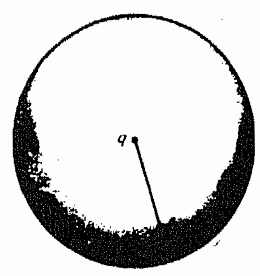


Fig. 4.26

We have discussed only about the surface integral of type (ii) i.e. $\iint_S \mathbf{A} \,.\, d\mathbf{S}$. As mentioned earlier, the other two forms do not occur often in physics. However, all the three forms find use in having the integral definitions of gradient, divergence and curl, which you will read in the Appendix. It is given as a prelude to the study of the vector integral theorems. However, you will learn to apply these theorems very shortly. But prior to that you will need to learn about the volume integral.

4.6 VOLUME INTEGRAL OF A FIELD

A volume integral is a generalisation of the triple integral. It is represented symbolically as $\iiint\limits_V$. A volume element dV is a scalar. Hence, we have volume integrals appearing in two forms

i) $\iiint\limits_V \mathbf{A}dV$

ii) $\iiint\limits_V \phi dV$

Suppose a body made up of a continuous distribution of matter, whose density ρ' varies from point to point, moves with a velocity $\mathbf{v}$. Its momentum then will be given by an integral of the form (i) as $\iiint\limits_V \rho' \mathbf{v} dV$ where V is the total volume of the body. The electrostatic potential at an external point due to a distribution of charge over a volume V in free space (Fig. 4.27) is given by

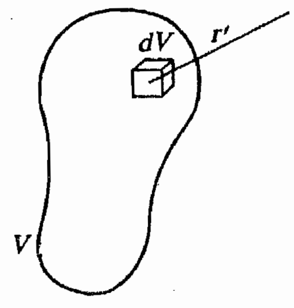


Fig. 4.27

$$U_e = \frac{1}{4\pi\varepsilon_0} \iiint\limits_V \frac{\rho\, dV}{r'}$$

where ρ is the charge per unit volume, and ε_0 is the permittivity of free space. This integral is of type (ii).

The evaluation of an integral of type (i) is done as follows

$$\iiint\limits_V \mathbf{A}dV = \iiint\limits_V (A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}})\, dV$$

$$= \hat{\mathbf{i}}\iiint\limits_V A_x dV + \hat{\mathbf{j}}\iiint\limits_V A_y dV + \hat{\mathbf{k}}\iiint\limits_V A_z dV$$

So the integral of type (i) reduces to a combination of integrals of form (ii). So we shall discuss the evaluation of integrals of type (ii) in the following subsection.

Evaluation of Volume Integrals

For evaluating a volume integral of the type $\iiint\limits_V \phi dV$ we have to express dV in terms of the same three parameters with respect to which ϕ is expressed. Let us consider a simple example.

Example 10

Evaluate $\iiint\limits_V \nabla\cdot\hat{\mathbf{r}}\, dV$, where V is the volume of a sphere of radius R. (Fig. 4.24)

Solution

For evaluating $\nabla\cdot\hat{\mathbf{r}}$, we shall use the expression of $\nabla\cdot\mathbf{A}$ in spherical polar coordinates [Sec. 3.4]. As $A_\theta = A_\phi = 0$, the calculation will be simplified considerably. We have $A_r = 1$

$$\therefore\ \nabla\cdot\hat{\mathbf{r}} = \frac{1}{r^2}\frac{d}{dr}(r^2) = \frac{1}{r^2}2r = \frac{2}{r}$$

Now an element of the volume of the sphere included between r, $r + dr$, θ, $\theta + d\theta$ and ϕ, $\phi + d\phi$ is given by $dV = r^2 \sin\theta\, d\theta\, d\phi\, dr$, where $0 \le r \le R$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

$$\therefore\ \iiint\limits_V \nabla\cdot\hat{\mathbf{r}}\, dV = \int_{r=0}^{R} dr \int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} d\phi\ \frac{2}{r} r^2\sin\theta$$

$$= \left(\int_0^R 2r dr\right)\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = R^2 \times 2 \times 2\pi = 4\pi R^2$$

Now, if you look back at Example 9, you will find that

$$\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S} = 4\pi R^2$$

where S signifies the surface of the sphere. So we find that

$$\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \hat{\mathbf{r}} dV,$$

where V is enclosed by S.

Is the above result true only for the vector $\hat{\mathbf{r}}$ or can it be generalised for any vector **A** ? Were it true for any vector, it will be possible for us to convert a volume integral into a surface integral. We shall examine this possibility now. In the process we shall arrive at some very useful theorems known as the vector integral theorems.

4.7 VECTOR INTEGRAL THEOREMS

If you are interested in the proofs of the vector integral theorems you may go through the Appendix.

Vector integral theorems may be used for converting volume integral into surface integrals or surface integrals into line integrals. That is, by using these theorems, the mathematical calculations involved in obtaining vector integrals may be simplified. They find wide applications. They can be used to study various aspects of conservative force fields. They are very often applied in the study of electrodynamics, fluid mechanics and so on. We shall study some of the applications here. As mentioned earlier, we will simply state them and discuss their significance and applications. Let us start with the **Gauss' divergence theorem.**

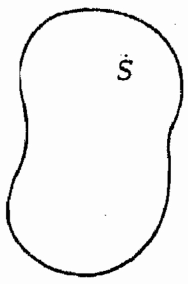


Fig. 4.28

4.7.1 Gauss' Divergence Theorem

It states that : '**The integral of the divergence of a vector field over a volume V is equal to the surface integral of the vector over the closed surface bounding V.**

It is expressed mathematically as

$$\oiint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \text{div}\mathbf{A}\, dV \tag{4.24}$$

where V is enclosed by S (Fig. 4.28)

We shall now illustrate this theorem with an application. You have obtained the result

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi\, kq$$

in SAQ 10, where S is the surface of a sphere that encloses a charge q. It can be shown that the above result is true for any charge distribution. Suppose that a closed surface enclosing a volume V has a continuous distribution of charge. If the charge per unit volume is σ, then $q = \iiint_V \sigma dV$. An example of such a distribution is a charged sphere. For this distribution, we have

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi\, k \iiint_V \sigma\, dV \tag{4.25}$$

But using Eq. (4.24), we have

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{E}\, dV \tag{4.26}$$

From Eqs. (4.25) and (4.26), we get

$$\iiint_V \nabla \cdot \mathbf{E}\, dV = 4\pi\, k \iiint_V \sigma\, dV$$

or

$$\iiint_V (\nabla \cdot \mathbf{E} - 4\pi\, k\sigma)\, dV = 0$$

Since dV is an arbitrary infinitesimal volume element, the integrand in above expression must be zero:

$$\nabla \cdot \mathbf{E} = 4\pi k\sigma \tag{4.27}$$

Eq. (4.27) signifies that the divergence of the electric field vector due to a continuous distribution of charge is independent of the extent of the distribution. It depends only the charge per unit volume. In charge free space $\sigma = 0$ so that

$$\nabla \cdot \mathbf{E} = 0. \tag{4.28}$$

The advantage of the Gauss' divergence theorem is that it enables us to convert a volume integral to a surface integral. You must note that in direct calculation of a volume integral, we take contributions from all parts of the volume. But in evaluating surface integrals we need to take contributions from the surface only. This makes the mathematical steps simpler. This theorem also provides us with an alternative method of obtaining the divergence of a vector.

At times it may be necessary to convert a surface integral to a volume integral. Even then we have to take help of this theorem. You can see that for yourself in the following SAQ.

SAQ 11

Spend 5 min

Show that

$$\frac{1}{3}\oiint_S \hat{\mathbf{r}} \cdot d\mathbf{S} = V.$$

where V is the volume enclosed by the closed surface S.

Thus, you have learnt a theorem that permits us to convert surface integrals to volume integrals and vice versa. You will now learn a theorem that enables us to convert a surface integral to a line integral and vice versa.

4.7.2 Stokes' Theorem

This theorem states that : **The integral of the curl of a vector over a surface S is equal to the line integral of the vector over the closed path bounding S.**

It is expressed mathematically as

$$\boxed{\oint_C \mathbf{A} \cdot d\mathbf{l} = \iint_S \text{curl}\,\mathbf{A} \cdot d\mathbf{S}} \tag{4.29}$$

where the surface S is bounded by the closed path C (Figs. 4.29a and 4.29b).

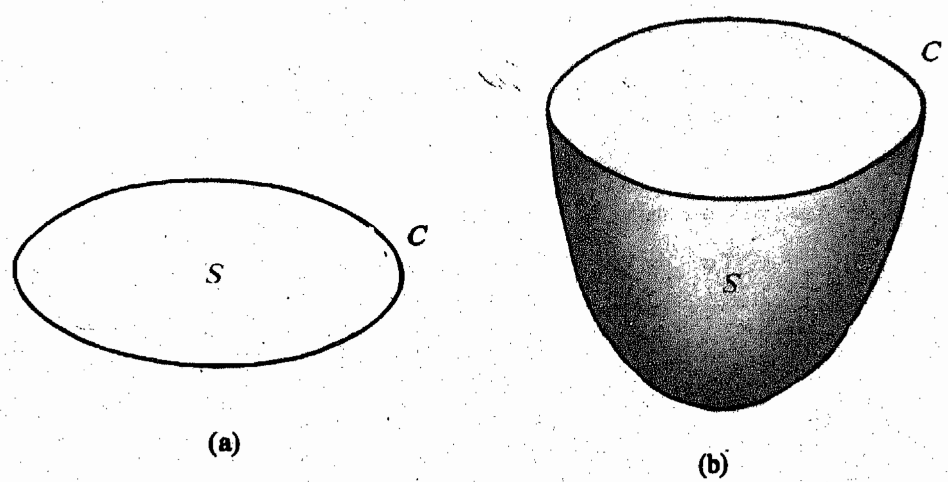


Fig. 4.29 : (a), (b) Two different cases of a surface S and its boundary C.

We shall now discuss an application of this theorem. The direct evaluation of curl **B**, where **B** is magnetic induction due to a current carrying conductor is quite tedious. To obtain curl **B** we shall use Stokes' theorem and the circuital form of Ampere's law according to which

The circuital form of Ampere's law relates the magnetic induction due to a current-carrying conductor with the current. It is thus a fundamental law in electrodynamics.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 i \tag{4.30}$$

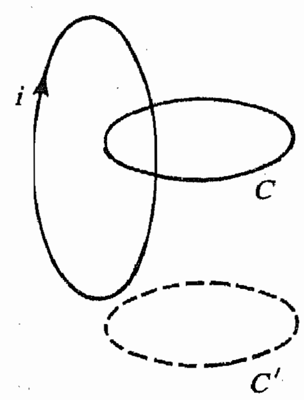


Fig. 4.30

where C is any closed path that is linked with the current (Fig. 4.30). For a path like C', which is not linked with the current, we have

$$\oint_{C'} \mathbf{B} \cdot d\mathbf{l} = 0$$

Now, our task is to calculate curl **B**. Using Eq. (4.29), we get

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \iint_S \text{curl } \mathbf{B} \cdot d\mathbf{S} \qquad (4.31)$$

where S is enclosed by C.

Again using the idea of Eq (4.19), we may write

$$i = \iint_S \mathbf{J} \cdot d\mathbf{S}$$

where **J** is the surface current density.

Hence, from Eqs. (4.30 and (4.31), we get

$$\iint_S \text{curl } \mathbf{B} \cdot d\mathbf{S} = \iint_S \mu_0 \mathbf{J} \cdot d\mathbf{S}$$

or

$$\iint_S (\text{curl } \mathbf{B} - \mu_0 \mathbf{J}) \cdot d\mathbf{S} = 0$$

But as $d\mathbf{S}$ is arbitrary, the integrand must be zero. Therefore,

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}$$

Thus we see that **B** has a non-vanishing curl. For a conservative force field, the curl vanishes, as you will see yourself in the following SAQ.

SAQ 12

Spend 10 min

Using Eq. (4.14) and Stokes' theorem, prove that curl of a conservative force field is zero everywhere.

There are several possibilities for the extension of the Gauss' divergence theorem and the Stokes' theorem. The one which is used very often is the Green's theorem. We shall discuss it now.

4.7.3 Green's Theorem

This theorem appears in two forms. The first form follows from Gauss' divergence theorem. To obtain it we substitute

$$\mathbf{A} = \phi \nabla \psi - \psi \nabla \phi$$

in Eq. (4.24) where ϕ and ψ are two scalar fields. Then we get

$$\iiint_V \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi)\, dV = \oiint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S}$$

$$\therefore \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \oiint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S} \qquad (4.32)$$

We know from Eq. (2.26c), that
$\nabla \cdot (s\mathbf{A}) = s\nabla \cdot \mathbf{A} + \nabla \cdot \nabla s$
$\therefore \nabla \cdot (\phi \Delta \psi) = \phi \nabla \cdot \nabla \psi + \nabla \psi \cdot \nabla \phi$
$= \phi \nabla^2 \psi + \nabla \psi \cdot \nabla \phi$
and $\nabla \cdot (\psi \nabla \phi)$
$= \psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi$
$\therefore \nabla \cdot (\phi \nabla \psi - \nabla \phi \cdot \nabla \phi)$
$= \phi \nabla^2 \psi - \psi \nabla^2 \phi$

Eq. (4.32) is the mathematical representation of one of the forms of Green's theorem. This theorem finds wide application in solving problems on electrostatic potential. We will not discuss these applications here.

The other form of Green's theorem follows from Stokes' theorem. For obtaining that, we substitute in Eq. (4.29), **A** equal to a two-dimensional vector field :

$$\mathbf{A} = P(x,y)\hat{\mathbf{i}} + Q(x,y)\hat{\mathbf{j}}$$

Now let S be a surface on the xy-plane so that

$d\mathbf{l} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$ and $d\mathbf{S} = dxdy\hat{\mathbf{k}}$

Thus $\mathbf{A}.d\mathbf{l} = Pdx + Qdy$ and

$$\text{curl } \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

Since P and Q are independent of z, we have

$$\text{curl } \mathbf{A} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

and

$$\text{curl } \mathbf{A} \cdot d\mathbf{S} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{dy}\right) dxdy \qquad (\because \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1)$$

So from Eq. (4.29), we get

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \qquad (4.33)$$

Eq. (4.33) is referred to as the Green's theorem in plane. We illustrate it with a simple example.

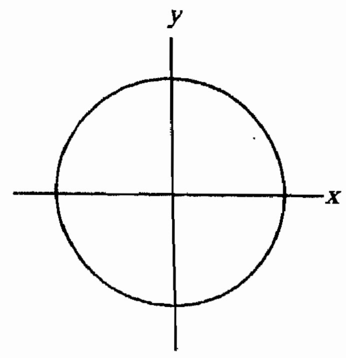


Fig. 4.31

Example 11

Using Eq. (4.33), evaluate $\int_C (-ydx + xdy)$, if C is the circumference of the circle $x^2 + y^2 = 1$ (Fig. 4.31).

Solution

From Eq. (4.33), we get

$$\oint_C (-ydx + xdy) = \iint_S \left[\frac{\partial x}{\partial x} - \frac{\partial}{\partial y}(-y)\right] dxdy = \iint_S 2dxdy, \text{ where } S \text{ is the area of the circle}$$

$$= 2S = 2\pi \qquad (\because S = \pi.1^2 = \pi)$$

Let us now summarise this unit.

4.8 SUMMARY

- If $\mathbf{a}(t)$ is a vector function of a scalar variable t, then

  $$\frac{d}{dt}\mathbf{a}(t) = \mathbf{b}(t)$$

  and

  $$\int \mathbf{b}(t)\, dt = \mathbf{a}(t) + \mathbf{c}$$

  where $\mathbf{c}$ is a constant vector
- For two vectors $\mathbf{V}_1(t)$ and $\mathbf{V}_2(t)$ of the scalar t, we have

  $$\int [p\mathbf{V}_1(t) \pm q\mathbf{V}_2(t)]\, dt = p\int \mathbf{V}_1(t)\, dt \pm q\int \mathbf{V}_2(t) dt$$

  where p and q are constants.
- An integral of the type $\int_a^b f(x)dx$ is called a single integral.

  If ϕ is a function of two variables x and y, then integral of ϕ over these variables in a planar region R, expressed as

  $$\iint_R \phi(x,y)\, dx\, dy$$

  is called a double integral.

If ψ is a function of three variables x, y and z then integral of ψ over these variables over a volume V, expressed as

$$\iiint_V \psi\,(x,y,z)\,dx\,dy\,dz$$

is called a triple integral.

- A line integral of a scalar or a vector field is a generalisation of the single integral where the path of integration may be any curve in space. It can occur in three forms:

$$\int_C \phi\,d\mathbf{r},\ \int_C \mathbf{A}\,.\,d\mathbf{r} \text{ and } \int_C \mathbf{A}\times d\mathbf{r}$$

- A surface integral of a scalar or a vector field is the generalisation of the double integral where the region of integration may be any surface. It can occur in three forms:

$$\iint_S \phi\,d\mathbf{S},\ \iint_S \mathbf{A}\,.\,d\mathbf{S} \text{ and } \iint_S \mathbf{A}\times d\mathbf{S}$$

- A volume integral of a scalar or a vector field is the generalisation of the triple integral where the integration is carried out over any volume. It can occur in two forms:

$$\iiint_V \phi\,dV \text{ and } \iiint_V \mathbf{A}\,dV$$

- Gauss' divergence theorem is mathematically expressed as

$$\oiint_S \mathbf{A}\,.\,d\mathbf{S} = \iiint_V \text{div}\mathbf{A}\,dV$$

where S is a closed surface enclosing the volume V.

- Stokes' theorem states that if $\mathbf{A}$ is a vector field, then

$$\oint_C \mathbf{A}\,.\,d\mathbf{l} = \iint_S \text{curl}\,\mathbf{A}\,.\,d\mathbf{S}$$

where C is a closed curve enclosing the surface S.

- Green's theorem, for two scalar fields ϕ and ψ states that

$$\oiint_S (\phi\,\nabla\psi - \psi\,\nabla\phi)\,.\,d\mathbf{S} = \iiint_V (\phi\,\nabla^2\psi - \psi\nabla^2\phi)\,dV$$

where S is a closed surface enclosing V.

- Green's theorem in plane, for two scalar functions $P\,(x,y)$ and $Q\,(x,y)$ states that

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where R is a region on the xy-plane bounded by the closed curve C.

4.9 TERMINAL QUESTIONS

Spend 40 min

1) A particle of mass m moves under the influence of a force field given by $\mathbf{F} = b\,(\sin\omega t\,\hat{\mathbf{i}} + \cos\omega t\,\hat{\mathbf{j}})$. If the particle is initially at rest, prove that the work done on the particle up to time t is $\dfrac{b^2}{m\omega^2}(1 - \cos\omega t)$.

2) Express an infinitesimal volume element of a solid right circular cylinder in cylindrical polar coordinates and hence determine its moment of inertia about its axis of symmetry. The mass of the cylinder is m and its radius is R.

3) Verify Stokes' theorem for the vector $\mathbf{A} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$, where C is the path shown in the yz plane (Fig. 4.32).

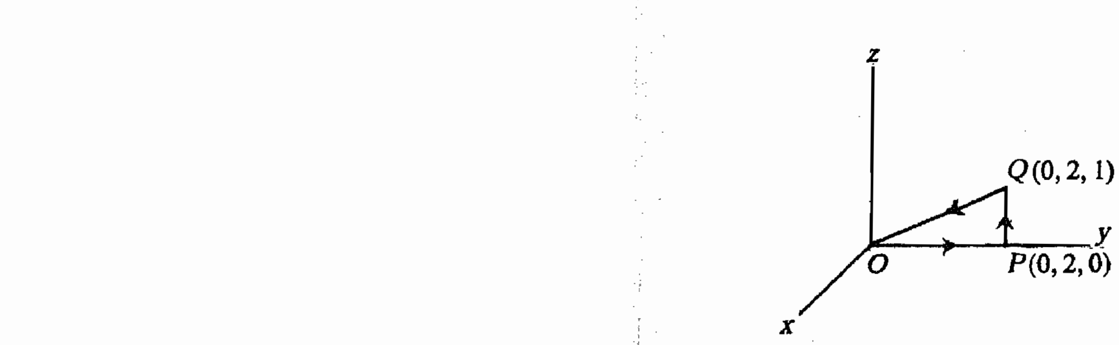


Fig. 4.32

4.10 SOLUTIONS AND ANSWERS

SAQs

1) Using Eqs. (4.4), (4.5) and (4.6), we get

$$\mathbf{r}(t) = \frac{b}{m\omega}\left[\hat{\mathbf{i}}\int \sin\omega t\;dt + \hat{\mathbf{j}}\int (1 - \cos\omega t)dt\right]$$

$$= \frac{b}{m\omega}\left[-\frac{\cos\omega t}{\omega}\hat{\mathbf{i}} + \left(t - \frac{\sin\omega t}{\omega}\right)\hat{\mathbf{j}}\right] + \mathbf{c}_2$$

Since the particle is initially at origin, we have $\mathbf{r}(0) = \mathbf{0}$.

So $\mathbf{0} = -\frac{b}{m\omega^2}\hat{\mathbf{i}} + \mathbf{c}_2$

$\therefore \quad \mathbf{c}_2 = \frac{b}{m\omega^2}\hat{\mathbf{i}}$

and $\mathbf{r}(t) = \frac{b}{m\omega^2}[(1 - \cos\omega t)\,\hat{\mathbf{i}} + (\omega t - \sin\omega t)\,\hat{\mathbf{j}}]$

2) Refer to Fig. 4.33

$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{OP} \times mg\hat{\mathbf{j}}$

The component of **OP** along y-direction is $l\cos\theta$ and that along x-direction is $l\sin\theta$.
So, $\mathbf{OP} = l\sin\theta\hat{\mathbf{i}} + l\cos\theta\hat{\mathbf{j}}$

$\therefore \quad \tau = (l\sin\theta\hat{\mathbf{i}} + l\cos\theta\hat{\mathbf{j}}) \times mg\,\hat{\mathbf{j}}$

$= mgl\sin\theta\hat{\mathbf{k}} \qquad (\because\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \text{ and } \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \mathbf{0})$

If θ is small, $\sin\theta \approx \theta$. Hence, $\tau = mgl\theta\hat{\mathbf{k}}$

or

$$\tau = \frac{d\mathbf{L}}{dt} = mgl\theta_0 \cos\frac{2\pi}{T}t\hat{\mathbf{k}}$$

Hence

$$\mathbf{L} = \hat{\mathbf{k}}(mgl\theta_0)\int \cos\frac{2\pi t}{T}\,dt$$

$$= \hat{\mathbf{k}}\frac{T}{2\pi}mgl\theta_0\sin\frac{2\pi t}{T} + \mathbf{C},$$

where **C** is constant vector of integration.

It is given that $\mathbf{L} = \mathbf{L}_0$ at $t = 0$. On applying this condition, we get

$$\mathbf{L}_0 = \mathbf{0} + \mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \mathbf{L}_0$$

Again, as $T = 2\pi\sqrt{\frac{l}{g}}$, we have $\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$

$$\therefore \quad \mathbf{L} = \left[m\sqrt{l^3 g}\ \theta_0 \sin\left(\sqrt{\frac{g}{l}}\,t\right)\right]\hat{\mathbf{k}} + \mathbf{L}_0$$

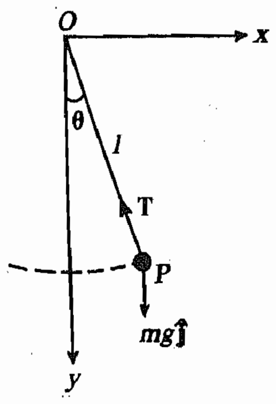


Fig. 4.33

3) $I = \frac{MR^2}{4\pi}\int_0^{2\pi}\left(\int_0^{\pi}\sin^3\theta\,d\theta\right)d\phi$

Now,

$$\int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}$$

$$\int_0^{\pi}\sin^3\theta\,d\theta = \int_0^{\pi}\sin^2\theta\,\sin\theta\,d\theta$$

Let $u = \cos\theta \therefore du = -\sin\theta\,d\theta$
when $\theta = 0, u = 1$
$\theta = \pi, u = -1$

$$\therefore \int_0^{\pi}\sin^3\theta\,d\theta = -\int_1^{-1}(1 - u^2)\,du$$

$$= \int_{-1}^{1}(1 - u^2)\,du$$

$$= 2\int_0^{1}(1 - u^2)\,du$$

$$= 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

$$\therefore \quad I = \frac{MR^2}{4\pi}\frac{4}{3}\int_0^{2\pi}d\phi = \frac{MR^2}{3\pi}2\pi = \frac{2}{3}MR^2$$

4) As shown in example 5, $f(r) = \frac{2}{R}$ for $R > r$ and $f(r) = \int_{r-R}^{r+R}\frac{du}{Rr}$ for $R < r$

i.e. $f(r) = \frac{1}{Rr}[(r + R) - (r - R)] = \frac{2}{r}$ for $R < r$

Now, $\phi = -2\pi G\rho\int_0^{a} r^2 f(r)\,dr$

$$= -2\pi G\rho\left(\int_0^{R} r^2\frac{2}{R}\,dr + \int_R^{a} r^2\frac{2}{r}\,dr\right)$$

$$= -4\pi G\rho\left[\frac{1}{R}\frac{R^3}{3}+\left(\frac{a^2}{2}-\frac{R^2}{2}\right)\right]$$

$$= -2\pi G\rho\left(a^2-\frac{R^2}{3}\right)$$

On putting $\rho = \dfrac{3M}{4\pi a^3}$, we get

$\phi = -\dfrac{GM}{2a^3}(3a^2 - R^2)$ when the field point is inside the sphere.

5 a) For $x = 2y$, $\quad \mathbf{r} = t\,\hat{\mathbf{i}} + 2t\,\hat{\mathbf{j}} \quad (-\infty < t < \infty)$

For $x^2 + 4y^2 = 4$, $\quad \mathbf{r} = 2\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} \quad (0 < \theta < 2\pi)$

b) Negative

c) *ABCD* is smooth. *PQR* is not smooth as a unique tangent cannot be drawn at *Q*.

6) Here, $\int_P^Q \mathbf{F}\,.\,d\mathbf{r} = \int_P^O \mathbf{F}\,.\,d\mathbf{r} + \int_O^Q \mathbf{F}\,.\,d\mathbf{r}$

Equation of *PO* is $y = 0 \quad \therefore\ dy = 0$

So from Eq. (4.13), we get $\mathbf{F}.d\mathbf{r} = 0$ and $\int_P^O \mathbf{F}\,.\,d\mathbf{r} = 0$

Again, the equation of *OQ* is $x = 0 \quad \therefore\ dx = 0$

So from Eq. (4.13), we again get $\mathbf{F}.d\mathbf{r} = 0$ and $\int_O^Q \mathbf{F}\,.\,d\mathbf{r} = 0$

Thus, for the given path $\int_P^Q \mathbf{F}\,.\,d\mathbf{r} = 0$

7 a) $U = U(x, y, z)$

$$\therefore\ dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

Again as $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$ and $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, we have

$$\mathbf{F}\,.\,d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz$$

$\therefore$ From Eq. (4.15), we get

$$F_x\,dx + F_y\,dy + F_z\,dz = -dU = -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right)$$

or $\left(F_x + \dfrac{\partial U}{\partial x}\right)dx + \left(F_y + \dfrac{\partial U}{\partial y}\right)dy + \left(F_z + \dfrac{\partial U}{\partial z}\right)dz = 0$

Since dx, dy, dz are independent, we must have

$$F_x + \frac{\partial U}{\partial x} = F_y + \frac{\partial U}{\partial y} = F_z + \frac{\partial U}{\partial z} = 0$$

or

$$F_x = -\frac{\partial U}{\partial x},\ F_y = -\frac{\partial U}{\partial y},\ \text{and}\ F_z = -\frac{\partial U}{\partial z}$$

$$\therefore\ \mathbf{F} = -\frac{\partial U}{\partial x}\hat{\mathbf{i}} - \frac{\partial U}{\partial y}\hat{\mathbf{j}} - \frac{\partial U}{\partial z}\hat{\mathbf{k}} = -\left(\frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}\right)$$

or $\mathbf{F} = -\nabla U$.

b) $\mathbf{F} = m\dfrac{d\mathbf{v}}{dt}$, $d\mathbf{r} = \dfrac{d\mathbf{r}}{dt}dt = \mathbf{v}\,dt$

$$\therefore\ \mathbf{F}\,.\,d\mathbf{r} = m\frac{d\mathbf{v}}{dt}\,.\,\mathbf{v}\,dt$$

We know from Eq. (2.8a) that $\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{v^2}{2}\right)$

$$\therefore \quad \mathbf{F} \cdot d\mathbf{r} = m\frac{d}{dt}\left(\frac{v^2}{2}\right) dt = d\left(\frac{mv^2}{2}\right)$$

$$\therefore \quad \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B d\left(\frac{1}{2}mv^2\right) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2.$$

where v_A and v_B are the magnitudes of the velocity of the particle at A and B, respectively. Now as K.E. $= \frac{1}{2}mv^2$, we get

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = T_B - T_A$$

8) The integrand can be expressed as $\frac{1}{r_1^3}(\mathbf{r}_1 \times d\mathbf{l})$, since r_1^3 *is a scalar*

Now,

$$\mathbf{r}_1 \times d\mathbf{l} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z-l \\ 0 & 0 & dl \end{vmatrix} = y\,dl\,\hat{\mathbf{i}} - x\,dl\,\hat{\mathbf{j}}$$

$$\therefore \text{ The integrand} = \frac{(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}})\,dl}{[\rho^2 + (z-l)^2]^{3/2}}$$

Now, you must remember that x, y and z are independent of dl as they are coordinates of an arbitrary point M.

$$\therefore \quad \mathbf{B} = CI(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \int_{-\infty}^{+\infty} \frac{dl}{[\rho^2 + (z-l)^2]^{3/2}}$$

We now make the substitution

$l - z = \rho\tan\theta$ so that $dl = \rho\sec^2\theta\, d\theta$ and $[\rho^2 + (z-l)^2]^{3/2} = (\rho^2\sec^2\theta)^{3/2} = \rho^3\sec^3\theta$

When $l \to \infty$, $\tan\theta \to \infty$ i.e., $\theta \to \pi/2$

and when $l \to -\infty$, $\tan\theta \to -\infty$, i.e., $\theta \to -\pi/2$

$$\therefore \quad \mathbf{B} = CI(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \int_{-\pi/2}^{+\pi/2} \frac{\rho\sec^2\theta\, d\theta}{\rho^3\sec^3\theta}$$

$$= \frac{CI}{\rho^2}(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}) \int_{-\pi/2}^{+\pi/2} \cos\theta\, d\theta$$

Thus we have expressed $d\mathbf{l}$ and $\frac{\mathbf{r}_1}{r_1^3}$ in terms of the same parameter θ. Finally, we have,

$$\mathbf{B} = 2CI\left(\frac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2 + y^2}\right) \qquad \left(\because \int_{-\pi/2}^{+\pi/2} \cos\theta\, d\theta = 2 \text{ and } \rho^2 = x^2 + y^2\right)$$

9) $d\mathbf{S} = (a\,d\phi)(dz)\,\hat{\boldsymbol{\rho}}$ [See Eq. (3.16a)]

$= a\,d\phi\,dz\,\hat{\boldsymbol{\rho}}$

10) Refer to Fig. 4.34. The required surface integral is $\oint\!\!\oint_S \mathbf{E} \cdot d\mathbf{S}$, where S is the surface of the sphere of radius a. Here

$$\mathbf{E} = \frac{kq}{r^3}\mathbf{r} = \frac{kq}{r^3}r\hat{\mathbf{r}} = \frac{kq}{r^2}\hat{\mathbf{r}}$$

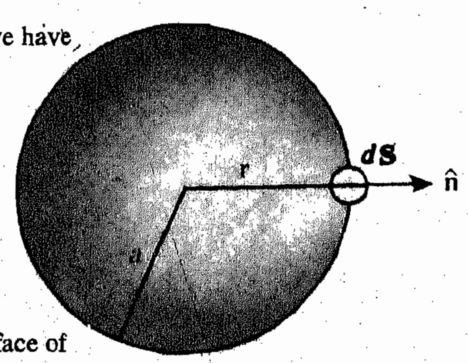


Fig. 4.34

Contributions to a surface integral come from the surface only. So we have to know E on the surface of the sphere, which is $\frac{kq}{a^2}\hat{\mathbf{r}}$

Again, we know from Eq. (4.17) that $d\mathbf{S} = dS\,\hat{\mathbf{n}}$

Now, at every point on the sphere $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ $\quad \therefore d\mathbf{S} = dS\,\hat{\mathbf{r}}$

Hence, the required flux $= \oiint_S \frac{kq}{a^2}\hat{\mathbf{r}} \cdot dS\,\hat{\mathbf{r}}$

$$= \oiint_S \frac{kq}{a^2}\, dS \qquad (\because \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1)$$

$$= \frac{kq}{a^2}\oiint_S dS \qquad (\because k, q \text{ and } a \text{ are constants})$$

Now, $\oiint_S dS$ is the surface area of the sphere $= 4\pi a^2$. So flux $= \frac{kq}{a^2} 4\pi a^2 = 4\pi kq$

11) We know from Gauss' divergence theorem, that

$$\oiint_S \mathbf{r} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{r}\, dV$$

where V is the volume enclosed by the surface S

$$\text{Now } \nabla \cdot \mathbf{r} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\therefore \oiint_S \mathbf{r} \cdot d\mathbf{S} = 3\iiint_V dV = 3V$$

or

$$\frac{1}{3}\oiint_S \mathbf{r} \cdot d\mathbf{S} = V$$

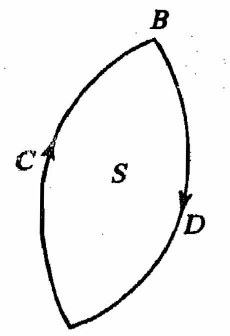


Fig. 4.35

12) Refer to Fig. 4.35. We have from Eq. (4.14)

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} = \int_{ADB} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}$ is a conservative force field.

$$\therefore \int_{ACB} \mathbf{F} \cdot d\mathbf{r} = -\int_{BDA} \mathbf{F} \cdot d\mathbf{r}$$

or

$$\int_{ACB} \mathbf{F} \cdot d\mathbf{r} + \int_{BDA} \mathbf{F} \cdot d\mathbf{r} = 0$$

i.e. $\oint_{ACBDA} \mathbf{F} \cdot d\mathbf{r} = 0$

From Stokes' theorem, we know that

$$\oint_{ACBDA} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$$

So,

$$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = 0$$

But $d\mathbf{S}$ is arbitrary. Hence the integrand is zero. Moreover, since the path $ACBDA$ has been chosen anywhere in the field, we can write

$$\text{curl}\,\mathbf{F} = \mathbf{0} \text{ everywhere in the field.}$$

Terminal Questions

1) $\mathbf{v} = \int \mathbf{f}\, dt = \int \frac{\mathbf{F}}{m}\, dt$

$$= \frac{b}{m}\int (\sin\omega t\,\hat{\mathbf{i}} + \cos\omega t\,\hat{\mathbf{j}})\, dt$$

$$= \frac{b}{m}\left(-\frac{\cos\omega t}{\omega}\,\hat{\mathbf{i}} + \frac{\sin\omega t}{\omega}\,\hat{\mathbf{j}}\right) + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector.

We know that $\mathbf{v} = \mathbf{0}$ at $t = 0$

$$\therefore\ \mathbf{0} = \frac{b}{m}\left(-\frac{1}{\omega}\,\hat{\mathbf{i}}\right) + \mathbf{C} \text{ or } \mathbf{C} = \frac{b}{m\omega}\,\hat{\mathbf{i}}$$

Hence

$$\mathbf{v} = \frac{b}{m\omega}\,[1 - \cos\omega t)\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}}]$$

We know from the result of SAQ 7b that the work done in time t is equal to the change in K.E. during that time. Since the particle is at rest $t = 0$, the initial K.E. is zero. So the change in K.E. = K.E at time t

$$\therefore \text{ The required work done} = \frac{1}{2} m v^2 = \frac{1}{2} m\,\mathbf{v}\cdot\mathbf{v}$$

$$= \frac{m}{2}\left(\frac{b}{m\omega}\right)^2 [(1 - \cos\omega t)^2 + \sin^2\omega t)$$

$$= \frac{b^2}{2m\omega^2}(1 - 2\cos\omega t + \cos^2\omega t + \sin^2\omega t)$$

$$= \frac{b^2}{2m\omega^2}\,2(1 - \cos\omega t) = \frac{b^2}{m\omega^2}(1 - \cos\omega t)$$

2) $I = \int r^2\, dm$

Now, $dm = \rho_0\, dV$ where ρ_0 is the density of the material of the cylinder. So,

$$I = \rho_0 \int_V r^2\, dV$$

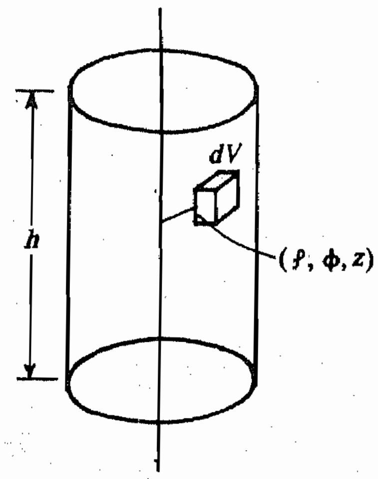


Fig. 4.36

Now, refer to Fig. 4.36. For this problem $r = \rho$ and we know from Sec. 3.3 that $dV = (d\rho)\,(\rho d\phi)\, dz = \rho d\rho d\phi dz$, where $0 < \rho < R$, $0 < \phi < 2\pi$, and $0 < z < h$.

$$\therefore\ I = \rho_0 \int_{\rho=0}^{R} \rho^3\, d\rho \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{h} dz$$

$$= \rho_0\, h \left(\int_{\rho=0}^{R} \rho^3\, d\rho\right)\left(\int_{\phi=0}^{2\pi} d\phi\right) = 2\pi\rho_0\, h \int_{\rho=0}^{R} \rho^3\, d\rho$$

$$\therefore\ I = 2\pi\,\rho_0\, h\,\frac{R^4}{4} = (\pi R^2\, h\,\rho_0)\,\frac{R^2}{2} = \frac{m}{2} R^2 \quad (\because \pi R^2 h\rho_0 \text{ is the mass of the cylinder})$$

3) First we shall calculate $\oint_C \mathbf{A}\cdot d\mathbf{l}$, where C is shown in Fig. 4.37. Here

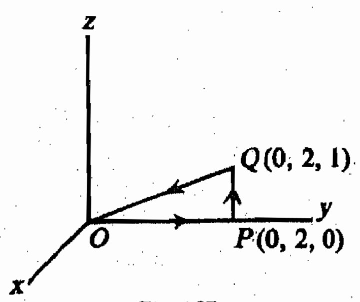


Fig. 4.37

$$\mathbf{A} = z^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}},\ d\mathbf{l} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$$

$$\therefore\ \mathbf{A}\cdot d\mathbf{l} = z^2\, dy + yz\, dz$$

$$\text{Now} \oint_C \mathbf{A}\cdot d\mathbf{l} = \int_O^P \mathbf{A}\cdot d\mathbf{l} + \int_P^Q \mathbf{A}\cdot d\mathbf{l} + \int_Q^O \mathbf{A}\cdot d\mathbf{l}$$

For O to P, $x = 0$, y goes from 0 to 2, $z = 0$. Hence $\int_O^P \mathbf{A}\cdot d\mathbf{l} = 0$

For P to Q, $x = 0, y = 2, z$ goes from 0 to 1, Hence $dy = 0$ and

$$\int_P^Q \mathbf{A} \cdot d\mathbf{l} = \int_0^1 2z\, dz = 1$$

And for Q to O, $x = 0$, $y = 2z$, z goes from 1 to 0 , Hence $dy = 2dz$ and

$$\int_Q^O \mathbf{A} \cdot d\mathbf{l} = \int_1^0 4z^2 dz = -\frac{4}{3}$$

$$\therefore \oint_C \mathbf{A} \cdot d\mathbf{l} = 0 + 1 - \frac{4}{3} = -\frac{1}{3}$$

Now,

$$\text{curl } \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z^2 & yz \end{vmatrix} = -z\hat{\mathbf{i}}$$

Since, the path C is traversed anticlockwise, we have $d\mathbf{S} = dS\, \hat{\mathbf{i}}$.

Moreover, as S lies on the yz-plane, $dS = dydz$ $\therefore$ $d\mathbf{S} = dy\, dz\, \hat{\mathbf{i}}$

$\therefore$ curl $\mathbf{A} \cdot d\mathbf{S} = -z\hat{\mathbf{i}} \cdot dy\, dz\, \hat{\mathbf{i}} = -z\, dy\, dz$

or

$$\iint_S \text{curl } \mathbf{A} \cdot d\mathbf{S} = -\iint_S z\, dy\, dz = -\int_0^2 dy \left(\int_0^{y/2} z\, dz \right)$$

$$= -\int_0^2 \frac{y^2}{8} dy = -\frac{1}{8} \left| \frac{y^3}{3} \right|_0^2 = -\frac{1}{3}$$

$$\therefore \iint_S \text{curl } \mathbf{A} \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{l},$$

which is Stokes' theorem.

Further Reading

1) *Advanced Engineering Mathematics*, E. Kreyszig, Wiley Eastern, New Delhi, 1987

2) *Vector Analysis; Theory and Problems*, M.R. Speigel, Schaum's series, Mc Graw Hill, 1974

3) *Vector Analysis,* M.L. Krashov, A.I. Kiselev, G.I. Makarenko, Mir Publishers Moscow, 1983

4) *Physics, A Textbook for Classes XI-XII,* Ed. by D.D. Pant and S.K. Joshi, NCERT, 1988

APPENDIX : PROOFS OF THE VECTOR INTEGRAL THEOREMS

Here we shall provide proofs of the vector integral theorems which you have learnt to apply in Sec. 4.7. Before we prove these theorems, we shall give integral definitions of gradient, divergence and curl.

A.1 Integral Definitions of Gradient, Divergence and Curl

Let us evaluate $\oint\!\!\oint_{\Delta S} \phi d\mathbf{S}$, where ϕ is the value of a scalar field at the point $P(x, y, z)$ and ΔS is the surface area of an infinitesimal parallelepiped having edges Δx, Δy, Δz, centred around the point P (Fig. A.1).

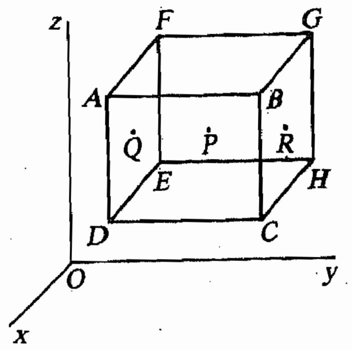


Fig. A.1

Now the closed surface ΔS has six faces; two perpendicular to each axis. Each of them will contribute to $\oint\!\!\oint_{\Delta S} \phi d\mathbf{S}$. Let us first find out the contributions of the faces perpendicular to the y-axis, i.e. $AFED$ and $CBGH$. For this we need to evaluate ϕ at Q and R, the mid-points of $AFED$ and $CBGH$ respectively, Since ϕ is known at P and it varies with position, we shall use Taylor series expansion to obtain ϕ_Q and ϕ_R. In the expansion we shall retain terms only up to the first order in Δx, Δy and Δz as these are extremely small.

Now, $QP = PR = \dfrac{\Delta y}{2}$

$$\therefore \quad \phi_Q = \phi - \frac{\partial \phi}{\partial y}\frac{\Delta y}{2}$$

$$= \phi - \frac{1}{2}\frac{\partial \phi}{\partial y}\Delta y$$

(The negative sign appears because as we go from P to Q, ϕ varies only with y and the y-coordinate decreases by $\Delta y/2$.)

Taylor's series :
$f(x+\Delta x)$
$= f(x) - f'(x)\,\Delta x$
$+ \dfrac{f''(x)}{2!}(\Delta x)^2 + ...$

Since the area $AFED$ is infinitesimal, the value of ϕ may be considered to be equal to ϕ_Q everywhere on that area. So while evaluating the integral $\iint_{AFED} \phi d\mathbf{S}$, the term $\left(\phi - \frac{1}{2}\frac{\partial \phi}{\partial y}\Delta y\right)$ may be considered constant.

$$\therefore \iint_{AFED} \phi d\mathbf{S} = \left(\phi - \frac{\partial \phi}{\partial y}\frac{\Delta y}{2}\right)\iint_{AFED} d\mathbf{S}$$

Now $d\mathbf{S} = -\Delta y \Delta z\,\hat{\mathbf{j}}$, as the direction of $d\mathbf{S}$ must be along the outward normal, i.e., along the negative y- axis.

$$\therefore \iint_{AFED} \phi\, d\mathbf{S} = -\left(\phi - \frac{1}{2}\frac{\partial \phi}{\partial y}\Delta y\right)\Delta y\, \Delta z\, \hat{\mathbf{j}} \qquad \text{(A.1a)}$$

Now, $\phi_R = \phi + \frac{1}{2}\frac{\partial \phi}{\partial y}\Delta y$, as on going from P to R the y-coordinate increases by $\dfrac{\Delta y}{2}$ and $\iint_{CBGH} d\mathbf{S} = \Delta y \Delta z\,\hat{\mathbf{j}}$, as the surface normal in this case is along positive y-axis.

$$\therefore \iint_{CBGH} \phi d\mathbf{S} = \left(\phi + \frac{1}{2}\frac{\partial \phi}{\partial y}\Delta y\right)\Delta y \Delta z\,\hat{\mathbf{j}} \qquad \text{(A.1b)}$$

Thus, the overall contribution from these faces is the sum of the right hand sides of Eqs. (A.1a) and (A.1b), i.e.

$$\frac{\partial \phi}{\partial y}\Delta x\, \Delta y\, \Delta z\,\hat{\mathbf{j}} \qquad \text{(A.2a)}$$

Similarly, the contributions to $\iint_{\Delta S} \phi\, d\mathbf{S}$ from the two faces perpendicular to the x-axis and the two faces perpendicular to the z-axis are , respectively, given by

$$\left(\frac{\partial \phi}{\partial x}\Delta x\, \Delta y\, \Delta z\,\hat{\mathbf{i}}\right) \qquad \text{(A.2b)}$$