The change in T in one year (≃ 365 days) is 10×10^{-6} s, i.e. 10^{-5} s.

∴ The change in a day is $dT = \frac{10^{-5}\,s}{365} = 2.7 \times 10^{-8}$s.

Hence, the change in rotational R.E. will be

$$dE = -\frac{4\pi^2 \times (9.7 \times 10^{37}\,kg\,m^2) \times (2.7 \times 10^{-8}s)}{(86400\,s)^3}$$

$$= -1.6 \times 10^{17} kg\, m^2\, s^{-2}$$

So the rotational energy decreases by 1.6×10^{17} J per day.

UNIT 10 MOTION IN NON-INERTIAL FRAMES OF REFERENCE

Structure

10.1 INTRODUCTION

In the previous unit you have read about rigid body dynamics. The present unit will be the final one of our Elementary Mechanics course. We had introduced the concept of frame of reference in the very first unit of Block 1. In Unit 2 of Block 1 we introduced the idea of inertial and non-inertial observers. So far we have explained motion from the point of view of inertial observers. But as a matter of fact we live on a frame of reference (the earth) which is non-inertial. Moreover, we shall see that certain problems can be answered quite elegantly if we take the point of view of a non-inertial observer. So in this unit we shall study the description of motion relative to a non-inertial frame of reference. First we shall study what is meant by a non-inertial frame of reference.

You must have had the following experiences while travelling in a bus. You fall backward when the bus suddenly accelerates and forward when it decelerates. When the bus takes a turn you have sensation of an outward force. We shall explain these features by introducing the concept of inertial forces. Thereby we shall see how Newton's second law of motion gets modified in a non-inertial frame. This will be used to develop the concept of weightlessness.

Frames attached with rotating bodies like a merry-go-round, the earth and so on form the most interesting examples of non-inertial frames of reference. We shall derive the equation of motion of a body in such a frame of reference. Thereby we shall come across two inertial forces, namely, the centrifugal force and the Coriolis force. The former can be used to explain the action of a centrifuge. We will study a variety of applications of these forces in connection with the earth as a non-inertial frame of reference. Centrifugal force finds application in studying the variation of g with the latitude of a place.

Several natural phenomena like erosion of the banks of rivers, cyclones etc. can be explained using the concept of Coriolis force. Finally we shall study about Foucault's Pendulum experiment with a view to establishing the fact that the earth rotates about an axis passing through the poles.

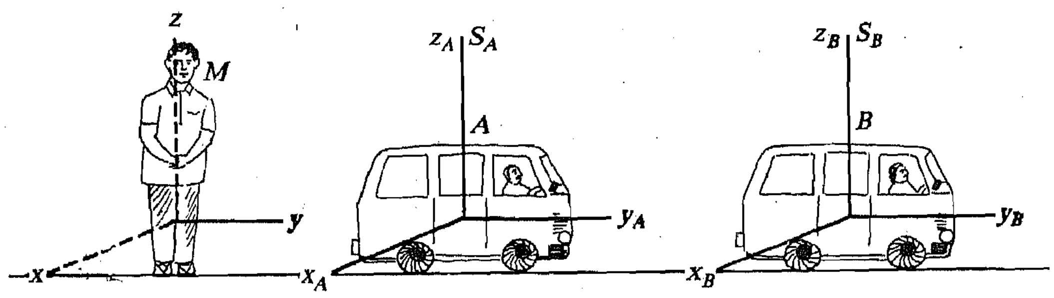
Objectives

After studying this unit you should be able to

- distinguish between an inertial and a non-inertial frame of reference
- write down the equation of motion of a body in a non-inertial frame of reference
- identify the inertial forces appearing in any non-inertial frame of reference
- solve problems on motion from the point of view of a non-inertial frame of reference.

10.2 NON-INERTIAL FRAME OF REFERENCE

In Sec. 2.2.1 of Block 1 we have discussed about inertial and non-inertial observers. You may recall that a car moving with a constant velocity and a man standing on the road are inertial with respect to each other. Let us now specify inertial and non-inertial *frames* of reference. Refer to Fig. 10.1.



Pig. 10.1: S and S_A are inertial with respect to each other. Sand S_B are non-inertial with respect to each other.

M is a person standing on the road. We take some point on the person of M as origin and define a three-dimensional Cartesian coordinate system S. Let Car A-move with a uniform velocity and Car B accelerate with respect to S. Let us now choose a point on each of Car A and B as origin and define the coordinate systems S_A and S_B.

The person will locate any object with reference to the coordinate system S. The drivers in the cars will locate objects with respect to S_A and S_B. They may choose a common zero on the time scale. Then you may recall from Sec. 1.2 of Block 1 that S, S_A and S_B are frames of reference. S and S_A are two inertial frames of reference with respect to each other. And S and S_B are two non-inertial frames of reference with respect to each other. In other words, the frames of *reference moving* with *uniform* velocity *with* respect to each other are inertial and those accelerating wirh *respect* to each other *are* called non-inertial. For the sake of convenience, from now onward we shall mostly use the word 'frame' in place of the phrase "frame of reference". Let us now discuss some examples of inertial and non-inertial frames of reference.

Consider a child sitting on a revolving merry-go-round in a park. A frame attached to'a fixed structure S in the park and the child are non-inertial with respect to each other because the merry-go-round has an acceleration due to rotation. Likewise the frame attached to a ball thrown up in the air by a child and S are non-inertial with respect to each other as the ball has an acceleration equal to g. The frame attached to some bench in the park and S are inertial with respect to each other as the bench is at rest with respect to the fixed structure. Similarly, the frame attached with a child walking leisurely (i.e. with a low uniform speed) and S are inertial with respect to each other.

You may now like to work out a simple SAQ to determine the nature of a frame, i.e. whether a frame is inertial or non-inertial with respect to any given frame.

SAQ 1

State giving reasons the nature of the frame attached

i) to a car moving along a curved path with a uniform speed with respect to a frame attached to a man standing on the road,

ii) to a falling rain drop during a drizzle (when it has attained a terminal velocity) with respect to a frame attached to the ground,

iii) to an electron moving in a uniform magnetic field produced by an electromagnet, with respect to a frame attached with a pole piece of the magnet.

So you have learnt how to identify inertial and non-inertial frames. Recall from what you have studied in Sec. 2.2.1 of Block 1 that for many purposes a frame fixed on the surface of earth can be considered as inertial. In all our previous units we had been analysing motion from the point of view of an inertial frame.

We shall see that certain problems of rotational dynamics become simpler when analysed from the point of view of a non-inertial frame. You may recall from Sec. 2.2.1 that Newton's first law of motion holds only in an inertial frame. You also know that the first law can be obtained from the second law. So we can say that the second law also holds only in an inertial frame. Let us now see how the second law will be modified for a non-inertial observer.

10.2.1 Motion Observed from a Nan-Inertial Frame

Let us take a simple example. Suppose you are standing on a road and observe a car about to start. We know that in order to start, a car has to accelerate. You would see that a person sitting inside the car gets pressed back against the seat by the acceleration. How would you explain this? Since you are an inertial observer with respect to another inertial observer, you will explain this as follows: This happens due to inertia of rest. The hips and the waist form part of the body of the man that is in direct contact with the seat of the car. The head and the torso are not in direct contact. This portion has a tendency to remain at rest. So as long as the car accelerates, the torso and the head tend to remain behind the waist and the hips. Thus, the person in the car gets pressed back against the seat.

Now, let us try to visualise the situation in a frame S' attached to the car. Due to the acceleration of the car, S' is non-inertial with respect to the person at rest. With respect to S' the portion of the person's body that is in direct contact with the seat of the car is at rest. The other portion falls back. How can this behaviour be explained from S'? We can say that in S' some force acts on the person in a direction opposite to the acceleration of the car. This force neutralises the accelerating force on the waist and hips and causes the other part to fall back.

But where does this force arise from? We have seen in Sec. 5.5 of Block 1 that forces occur either by way of contact (e.g. push, pull, friction) or due to some action at a distance (e.g. gravitational or electromagnetic field). But the force here does not have either of these as its origin. Moreover, such a force does not exist from the point of view of an inertial observer. However, this force is very much real from the point of view of S'. This is calledthe **inertial** force. From the example we have just now considered you can understand that the magnitude of this force is equal to the accelerating force and it is directed opposite to it. However, we shall quantify this force very soon in this section.

Continuing with the example, we find that in S' the man is held at rest by a force exerted on him by the back of the seat. If you were to remain at rest or in uniform motion with respect to an inertial frame of reference, no force would be needed. But in order to be at rest in a non-inertial frame of reference like that of the accelerating car, some force is required. This implies that the second law of motion will take a different form in a non-inertial frame. We shall now study that. In the process, we shall be able to quantify 'inertial force',

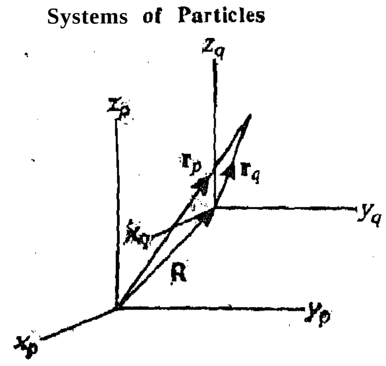


Fig. 10.2: The frames of reference of P and Q

The process of obtaining accelerations from the position vectors involves differentiation with respect to time, Incidentally, the time intervals are, strictly speaking, not the same in the two frames of P and Q. However the mathematical treatment corresponding to unequal time intervals will be very complicated. This issue will be resolved for two inertial and non-inenial frames by studying, respectively, the special arid general theories of relativity. For the sake of simplicity here we shall assume the time intervals to be equal.

10.2.2 Newton's Second Law and Inertial Forces

Suppose that two scientists P and Q decide to observe a series of events such as the position of a body of mass m as a function of time. Each has his own set of measuring devices and each works in his own laboratory. Let us suppose that P has confirmed by performing some experiments in his laboratory that the second law of motion holds there precisely. His frame of reference is, therefore, inertial. How can *P* find out whether *Q's* frame is inertial or not?

As per convention let the frames be defined by two Cartesian coordinate systems (Fig. 10.2) with identical scale units. In general, the coordinate systems do not coincide. We shall assume that none of the frames is executing a rotation and that they are executing relative motion with their corresponding axes always parallel to each other. Let the position vectors of *m* be $\mathbf{r}_p$ and $\mathbf{r}_q$ with respect to *P* and Q, respectively. If the origins of the two frames are displaced by a vector R, then we have from Fig. 10.2

$$\mathbf{r}_q = \mathbf{r}_p - \mathbf{R}. \qquad (10.1)$$

If P sees m accelerating at a rate $\mathbf{a}_p = \ddot{\mathbf{r}}_p$ he concludes from the second law that there is a force on *m* given by

$$\mathbf{F}_p = m\mathbf{a}_p.$$

Q observes m to be accelerating at a rate $\mathbf{a}_q = \ddot{\mathbf{r}}_q$ as if it were experiencing a force

$$\mathbf{F}_q = m\mathbf{a}_q$$

Let us now find out how $\mathbf{F}_q$ is related to the force $\mathbf{F}_p$. We know from Sec. 1.5 of Block 1 that if *Q* be moving with a uniform velocity relative to P, i.e. if *Q* is also inertial, then $\mathbf{a}_q = \mathbf{a}_p$ and

$$\mathbf{F}_q = m\mathbf{a}_q = m\mathbf{a}_p = \mathbf{F}_p.$$

So we find that the force is same in both the frames. In other words, the equations of motion have the same form in both the frames. So all inertial frames are equivalent. ***There** is no dynamical experiment that leads us to prefer one inertial frame from another.*

Let us now see what happens if Q were accelerating with respect to P. How about working out the relation between $\mathbf{F}_p$ and $\mathbf{F}_q$ in this case?

SAQ 2

Find the relation between $\mathbf{F}_p$ and $\mathbf{F}_q$ when the acceleration of Q with respect to P is a?

Now that you have solved SAQ 2, we can express the relation between $\mathbf{F}_q$ and $\mathbf{F}_p$ as

$$\mathbf{F}_q = \mathbf{F}_p + \mathbf{F}' = m\mathbf{a}_q \qquad (10.2a)$$

where $$\mathbf{F}' = -m\mathbf{a}. \qquad (10.2b)$$

So we are able to preserve the relationship between the net force on the object and its, acceleration. But the net force in the Q-frame is now made up of two parts: a force $\mathbf{F}_p$ and another force $\mathbf{F}'$ equal to $-m\mathbf{a}$. The latter originates from the fact that the frame Q has an acceleration a with respect to P. This force F' is called the ***inertial force.*** Its expression is given by Eq. 10.2b. Its magnitude is equal to the product of the mass of the body and the acceleration of the non-inertial frame. It is directed opposite to the acceleration of the frames. An important special case of Eq. 10.2a is that in which the force $\mathbf{F}_p$ is zero. *In* such a case the body as observed in Q, moves under the action of the inertial force alone. The situation of the torso and the head of the man in the car is very much like that. Let us now work out an example to understand the meaning of inertial force better.

Example 1

A small ball of mass m hangs from a string in a car (Fig. 10.3a) which accelerates at a rate a. What angle does the string make with the vertical and what is the value of tension in it?

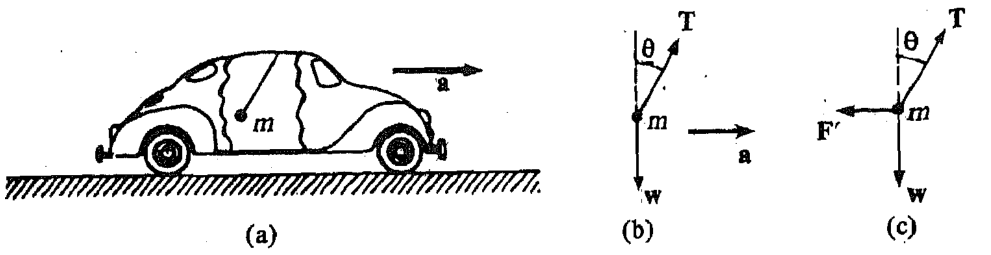


Fig. 10.3: (a) A car accelerating at the rate a; (b) force diagram with respect to an inertial frame; (c) force diagram with respect to a frame accelerating with the Car.

We shall analyse the problem both with respect to an inertial frame and in a frame accelerating with the car. Let the tension in the string be T and let it make an angle 0 with the vertical.

Motion in inertial frame

Refer to **Fig. 10.3b**. With respect to an inertial frame the mass moves in the direction of motion of the car with an acceleration a $(\mathbf{a} = a\hat{\mathbf{i}})$. This is caused by the tension T and the weight mg $(\mathbf{g} = -g\hat{\mathbf{j}})$. There is no motion in the y-direction.

$$\therefore\ T\cos\theta\,\hat{\mathbf{j}} + mg\,(-\hat{\mathbf{j}}) = 0 \quad \text{or} \quad T\cos\theta = mg. \qquad (10.3a)$$

Equation of motion in the x-direction is given by

$$T\sin\theta\,\hat{\mathbf{i}} = ma\,\hat{\mathbf{i}} \ \text{or}\ T\sin\theta = ma. \qquad (10.3b)$$

From Eqs. 10.3a and b, we get

$$\tan\theta = \frac{a}{g} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{a}{g}\right) \qquad (10.3c)$$

$$\text{and } T = \sqrt{(T\cos\theta)^2 + (T\sin\theta)^2}$$

$$\therefore\ T = m\sqrt{g^2 + a^2}. \qquad (10.3d)$$

Motion in the frame accelerating with the car

Refer to **Fig 10.3c.** In this frame apart from the forces T and mg there is an inertial force **F′** arising out of the acceleration of the frame. With respect to this frame the mass is at rest, i.e. it is in equilibrium under the influence of T, mg **and F′**:

$$\therefore\ \mathbf{T} + m\mathbf{g} + \mathbf{F}' = 0$$

$$\text{or}\quad T\cos\theta\,\hat{\mathbf{j}} + T\sin\theta\,\hat{\mathbf{i}} + mg\,(-\hat{\mathbf{j}}) + F'\,(-\hat{\mathbf{i}}) = 0$$

$$\text{or}\quad (T\cos\theta - mg)\,\hat{\mathbf{j}} + (T\sin\theta - F')\,\hat{\mathbf{i}} = 0$$

$$\therefore\ T\cos\theta - mg = 0, \quad \text{i.e. } T\cos\theta = mg \qquad (10.3a')$$

$$\text{and}\qquad T\sin\theta - F' = 0 \quad \text{or} \quad T\sin\theta = F'.$$

F' is the magnitude of F and it is equal to ma. So we gqt

$$T\sin 0 = ma. \qquad (10.3b')$$

From Eqs, 10.3a' and 10.3b' we get as in the previous case

$$\theta = \tan^{-1}\left(\frac{a}{g}\right) \qquad (10.3c')$$

$$\text{and}\qquad T = m\sqrt{g^2 + a^2} \qquad (10.3d')$$

which are identical with the values of θ and T obtained in Eqs. 10.3c and 10.3d. In fact Eq. 10.3a′ is identical with Eq. 10.3a and 10.3b′ is same as 10.3b. But there is an element of difference. Eqs. 10.3a and 10.3a′ both occur as conditions of equilibrium. But 10.3b occurs as an equation of motion whereas 10.3b′ arises out of a condition of equilibrium.

Sometimes the inertial force is called 'fictitious force' or 'pseudo-force' (pseudo means false) as it does not arise from any basic interaction. But these names are misleading as the force actually exists for a non-inertial observer.

Moreover, you must remember that the inertial force does not exist for the inertial observer. This is because inertial forces experienced in an accelerating frame of reference do not arise from physical interactions. They originate in the acceleration of the frame of reference. So for a non-inertial observer such forces are present. For example, suppose we wish to keep an object at rest in a non-inertial frame by tying it down with springs. Then these springs would be observed to elongate or contract in such a way as to provide an opposing force to balance the inertial force.

You may now like to work out an SAQ on the above concept.

SAQ 3

a) **A** glass half filled with water is kept on a horizontal table in a train. Will the free surface of water remain horizontal as the train starts?

b) A man of mass m is standing in a lift which is accelerating upwards at a rate **f**. Write down the expression for the inertial force acting on the man. Hence prove that he feels heavier than usual.

Now that you have worked out SAQ 3(b), you will be able to appreciate the concept of weightlessness.

10.2.3 Weightlessness

Suppose that the lift was accelerating downwards at the rate f (Fig. 10.4a). Then the net force acting on the man in the frame attached with the lift is given by

$$\mathbf{F} = m\mathbf{g} - m\,\mathbf{f}$$
$$= m\,(g - f)\,\hat{\mathbf{j}},$$ where $\hat{\mathbf{j}}$ is the unit vector in the vertically downward direction.

Now if the lift were falling freely, i.e. f = g, then F = 0. Thus, the force acting on the man is zero. You know that the weight of an object is defined as the force needed to keep it at rest. So in the lift's frame, the reaction of **F** is the weight of the man, since it is the force required to keep the man at rest. Since **F** is zero in a freely falling lift, the man feels weightless. Likewise, every freely falling object is weightless in a frame attached with itself

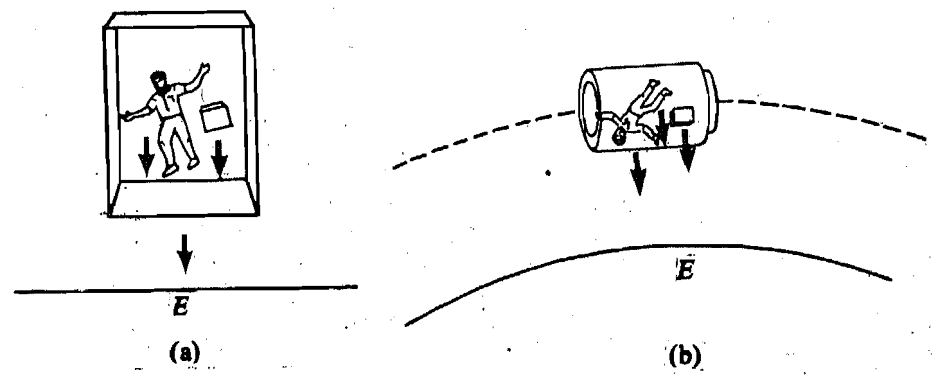


Fig. 10.4: Objects feel weightless in a freely falling frame of reference as they experience the same acceleration as the frame: a) A freely falling elevator near the earth's surface; b) a spacecraft orbiting the earth E. The person, book and the elevator or spaceship all have the same acceleration towards the earth.

You may have seen Squadron Leader Rakesh Sharma floating in the spaceship. In fact, he could lift his fellow astronaut on the tip of his finger. How could this happen?

This is because weightlessness occurs in any orbiting spaceship (Fig. 10.4b), as it is always in a state of free fall. You must remember that weight depends on the frame of reference. The astronaut is weightless only in the freely falling frame of the spaceship. So weightlessness does not imply absence of gravitational force.

Let us now consider the same situation in a frame at rest with respect to the earth. In this frame the net force acting on the astronaut is mg. Therefore, both the spaceship and the astronaut have weight with respect to this frame. The astronaut can float because he is falling towards the earth at the same rate as that of the spaceship.

So far we have not considered the rotation of frames with respect to one another. We know that a rotating body has an acceleration. So a frame attached with such a body rotates and is non-inertial. Our interest in rotating frames of references arises mainly because we live on one such frame, the earth. Another example of a rotating frame is the one attached to a merry-go-round. We shall be able to explain several natural phenomena by considering rotating frames. For example, the occurrence of weather disturbances, the variation of g with latitude and many other phenomena can be explained if we regard the earth as a rotating frame. So let us now analyse motion from the point of view of a rotating frame of reference.

10.3 ROTATING FRAME OF REFERENCE

In Sec. 10.2.2 we have seen how the second law of motion transforms from an inertial frame to a translating non-inertial frame. We shall now see how the second law transforms when one goes from an inertial frame to a rotating frame of reference. As in the previous case the transformed version of the second law will contain the inertial force. We shall see that in a rotating frame more than one inertial force will occur. Our aim will be to determine these inertial forces.

Let us consider a particle of mass m which is accelerating at a rate $\mathbf{a}_{in}$ with respect to an inertial frame. Then its equation of motion in that frame is

$$\mathbf{F} = m\mathbf{a}_{in}.$$

Again let its acceleration with respect to a rotating frame be $\mathbf{a}_{rot}$.

Then its equation of motion in that frame would be

$$\mathbf{F}_{rot} = m\mathbf{a}_{rot}.$$

Let the relative acceleration of the inertial frame with respect to the rotating frame be $\mathbf{a}'$. Then we have

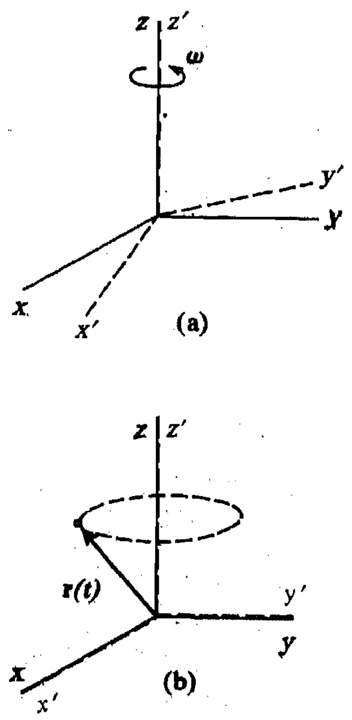
$$\mathbf{a}_{in} = \mathbf{a}_{rot} + \mathbf{a}'$$

$$\text{or} \quad \mathbf{F}_{rot} = m(\mathbf{a}_{in} - \mathbf{a}') = \mathbf{F} + \mathbf{F}', \qquad (10.4)$$

where $\mathbf{F}'$ is the inertial force given by $\mathbf{F}' = -m\,\mathbf{a}'$. Our task now is to determine $\mathbf{a}'$ for a rotating frame. We know that acceleration is the time-derivative of velocity which again is the time-derivative of displacement. So we shall first relate the infinitesimal displacements of a particle as measured from an inertial and a rotating frame of reference. We shall take the time-derivative of this relation to obtain the relation between the velocities of the particle measured in these frames. Then the time derivative of the relation between the velocities will give the desired expression of the accelerations. So effectively, we shall now study, the relations between the time-derivatives of different kinematical variables in inertial and rotating frames of reference.

10.3.1 Time Derivatives in Inertial and Rotating Frames

Let the motion of a particle of mass m be observed by an inertial and a rotating observer. Let the inertial observer O have a Cartesian coordinate system (x, y, z) as its frame of reference (Fig. 10.5a). The frame of reference of another observer O', who is rotating, is given by another Cartesian coordinate system (x', y', z'). In practice we will be dealing with situations where a frame rotates uniformly about an inertial frame. So here we shall assume that the set of axes (x', y', z') rotates about (x, y, z) with a uniform angular velocity. We are interested in pure rotation, i.e. O' has no translational motion with respect to O. So we have taken the origin of the coordinate systems to be coincident. Also let us suppose that the (x', y', z') system is so rotating that the z and z'-axes always coincide. Thus, the constant angular velocity $\boldsymbol{\omega}$ of the rotating system, lies along the z-axis. Further, let the x and x'-axes coincide at an instant of time t.



Fig, 10.5: (a) The inertial frame (x, y, z) and the rotating frame (x', y', z') (b) A position vector $\mathbf{r}(t)$ in the xz (and $x'z'$) plane.

Imagine now that the particle has a position vector **r** (t) in the xz -plane (and $x'z'$ -plane) at time t (Fig. 10.5b). At time t + At, the position vector is **r** (t + At) and from Fig. 10.6a the displacement of the particle in the inertial frame is given by

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t). \qquad (10.5a)$$

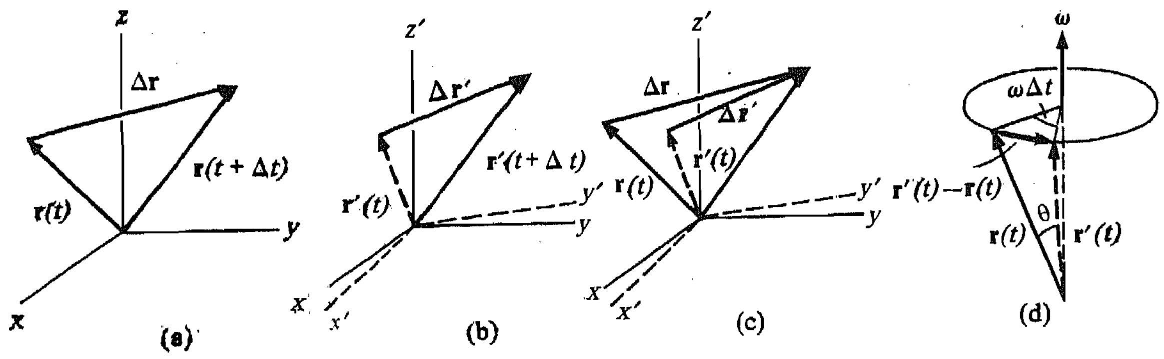


Fig. 10.6: (a) The change **Ar** in the position vector in the inertial frame; (b) the change Ar' in the position vector in the rotating frame; (c) illustrating that **Ar** and **Ar'** are not the same; (d) diagram for obtaining the relation between { **r'**(t) – **r**(t) } and **ω**.

The situation is different for the rotating observer. He also notes the same final position vector **r** (t + At) but in obtaining the displacement he ensures that the initial position vector **r'**(t) in his coordinate system (Fig. 10.6b) was in the $x'z'$-plane. So he measures the displacement as

$$\text{Ar}' = \text{r}'(t + \text{At}) - \text{r}'(\text{t}). \qquad (10.5b)$$

It can be seen from Fig. 10.6c that the $x'z'$ - plane is now rotated away from its previous position. So Ar and Ar' are not the same. From Eqs. 10.5a and b we get

$$\Delta \mathbf{r} = \text{Ar}' + \mathbf{r}'(\text{t}) - \mathbf{r}(t). \qquad (10.6)$$

We shall now express {**r'** (t) – **r** (t)} in terms of **ω** and At. For this let us refer to Fig. 10.6d. It can be seen that

$$|\,\text{r}'(\text{t}) - \text{r}(\text{t})\,| = (\text{r} \sin\theta)(\omega\, \text{At})$$

$$= \omega r \sin\theta\, \Delta t = |\,\boldsymbol{\omega} \times \mathbf{r}\,|\, \Delta t$$

where r and **r** stand for $r(t)$ and **r** (t), respectively. Again from the right hand rule for determining the direction of vector product, we see that (r' (t) – r (t) } is along (**ω** × **r**). So the vector quantity (**ω** × **r**) At represents {**r'**(t) – **r** (t) } in magnitude as well as directioh. Thus

$$\text{r}'(\text{t}) - \mathbf{r}(t) = (\boldsymbol{\omega} \times \mathbf{r})\, \Delta t.$$

Hence, from Eq. 10.6, we get

$$\frac{\text{Ar}}{At} = \frac{\text{Ar}'}{\text{At}} + \text{O} \times \sim .$$

Now taking limits on both sides of above as At → 0 we get

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \boldsymbol{\omega} \times \mathbf{r}.$$

Now $\frac{d\mathbf{r}}{dt} = \mathbf{v}_{in}$ = velocity of particle in the inertial frame,

and $\frac{d\mathbf{r}'}{dt} = \mathbf{v}_{rot}$ = velocity of the particle in the rotating frame. Thus

$$\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\omega} \times \text{r}. \qquad (10.7)$$

You must have noted that in the above proof we did not use the special arrangement of axes of our choice. So the result given by Eq. 10.7 is a general one.

An alternative way of expressing **Eq.** 10.7 is as follows.

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.8)$$

For obtaining **Eq.** 10.8 we have only used the geometric properties of **r**. So it can be generalised for any vector **A.** Thus we have the general result

$$\left(\frac{d\mathbf{A}}{dt}\right)_{in} = \left(\frac{d\mathbf{A}}{dt}\right)_{rot} + (\boldsymbol{\omega} \times \mathbf{A}). \qquad (10.9)$$

We shall now use **Eq.** 10.8 to determine $\mathbf{a}'$ $(= \mathbf{a}_{in} - \mathbf{a}_{rot})$.

We know that $\mathbf{a}_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in}$, i.e. the time derivative of $\mathbf{v}_{in}$ in the inertial frame.

and $\mathbf{a}_{rot} = \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot}$, i.e. the time derivative of $\mathbf{v}_{rot}$ in the rotating frame.

On applying **Eq.** 10.9 for $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\mathbf{a}_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{in}$$

On using **Eq.** 10.7 we get

$$\mathbf{a}_{in} = \frac{d}{dt}(\mathbf{v}_{rot} + \boldsymbol{\omega} \times \mathbf{r})_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Since $\boldsymbol{\omega}$ is constant, we get

$$\mathbf{a}_{in} = \mathbf{a}_{rot} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

or $$\mathbf{a}_{in} = \mathbf{a}_{rot} + 2\boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\therefore \quad \mathbf{a}' = 2\boldsymbol{\omega} \times \mathbf{v}_{rot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (10.10)$$

Thus the inertial force is given by

$$\mathbf{F}' = -m\mathbf{a}' = -2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.11)$$

In **Eq.** 10.11 we have written $\mathbf{v}'$ in place of $\mathbf{v}_{rot}$ for the sake of convenience.

Hence, from **Eq.** 10.4 we can write

$$\mathbf{F}_{rot} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}') - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \qquad (10.12)$$

Eq. 10.12 shows that the dynamics of motion as observed from a uniformly rotating frame of reference may be analysed in terms of the following three categories of forces:

i) F: This is the sum of all forces on the particle, arising out of physical interactions or due to contact. They may be tensions in strings and forces due to fundamental interactions. Only these forces are present in an inertial frame.

ii) $-2m(\boldsymbol{\omega} \times \mathbf{v}')$: This is called the Coriolis force. It acts at right angles to the plane containing $\boldsymbol{\omega}$ and $\mathbf{v}'$ and points in the direction of advancement of the screwhead when the screw is rotated from $\mathbf{v}'$ towards $\boldsymbol{\omega}$. This force is absent when the particle has no velocity with respect to the rotating frame.

iii) $-m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$: This is called the **Centrifugal** force. It always acts radially outward. The two observers in the inertial and rotating frame do agree on the position vector of a particle at a given instant. Hence r may be replaced by r', provided their origins coincide.

Coriolis force is named after the French engineer and mathematician Gustave Gaspard Coriolis (1792-1843). He was the first man to provide a description of the force. The term centrifugal comes from 'centre' and 'fugal'. The latter means to fly off.

We shall now study some examples of these forces. Let us begin with the centrifugal force.

10.3.2 Centrifugal Force

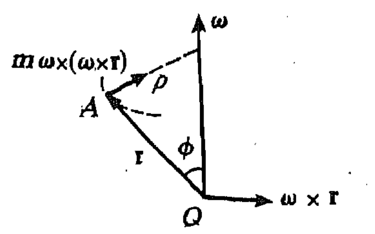


Fig. 10.7: The centrifugal force. Mathematically $\omega \times (\omega \times r)$ acts at O. But physically $-m\omega \times (\omega \times \mathbf{r})$ is a force acting on the body. Thus F_{cent} acts at A and is a vector antiparallel to $\omega \times (\omega \times r)$

Let us first determine the magnitude and direction of the centrifugal force $F_{cent} = -m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. See Fig. **10.7.** $\boldsymbol{\omega} \times \mathbf{r}$ is perpendicular to the plane of $\boldsymbol{\omega}$ and $\mathbf{r}$. Let the angle between $\boldsymbol{\omega}$ and r be ϕ. Then the magnitude of $\boldsymbol{\omega} \times \mathbf{r}$ is $\omega r \sin\phi = \omega\rho$, where $\rho = r \sin\phi$ is the perpendicular distance from the axis of rotation to the head of r. Hence $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector with magnitude $\omega^2 \rho$, since the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \mathbf{r}$ is 90". From the right-hand rule this vector is directed radially inward towards the axis of rotation. Therefore, $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector of magnitude $m\omega^2\rho$. It points radially outward from the axis of rotation to the head of $\mathbf{r}$. So we can also write

$$\mathbf{F}_{cent} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\omega^2 \rho\, \hat{\boldsymbol{\rho}} = m\,\omega^2 r \sin\phi\, \hat{\boldsymbol{\rho}}, \qquad (10.13a)$$

where $\hat{\boldsymbol{\rho}}$ is the unit vector along the direction from the axis of rotation to the head of r. If the body's position vector $\mathbf{r}$ were measured from the centre of the circle in which it is rotating, then $\phi = 90^\circ$ and

$$\mathbf{F}_{cent} = m\omega^2 r\, \hat{\mathbf{r}}. \qquad (10.13b)$$

The centrifugal force is familiar to us in our daily life. If we tie an object to a string and whirl it around it seems to pull on us. This effect can be explained in terms of the centrifugal force. Let's see how.

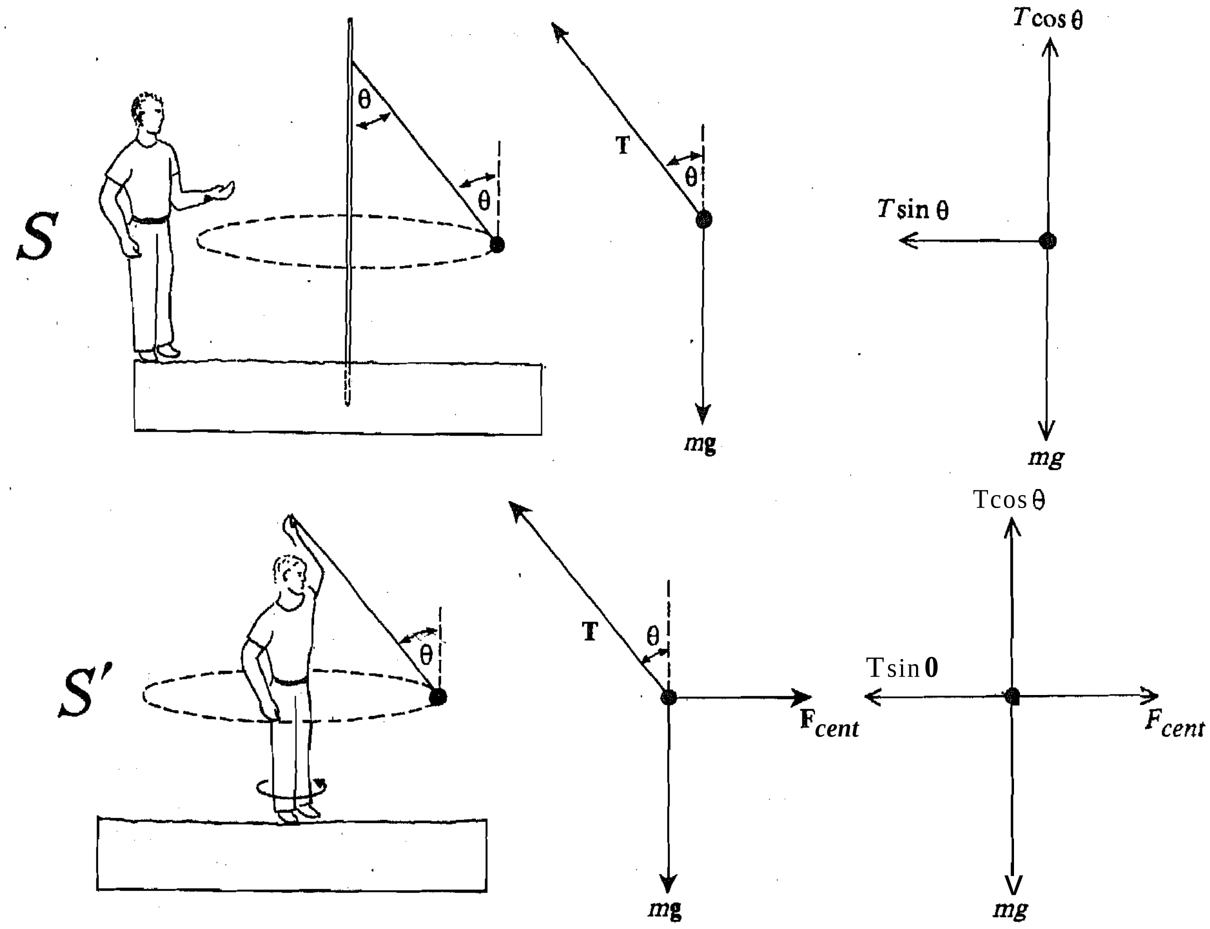


Fig. 10.8: In the frame S the forces acting are the tension in the string and the weight of the object. Tcos θ balances mg and Tsin θ provides the necessary centripetal force. In the frame S' apart from the tension and weight we have the centrifugal force. These forces are in equilibrium.

Suppose that a ball is being whirled around in horizontal circular motion (Fig. 10.8) with constant angular speed ω. Let us analyse the motion of the ball from two frames of reference. **A** stationary (inertial) frame S, and a rotating (non-inertial) frame S' that rotates with the same angular speed as the ball. So the angular speed of S' with respect to S is also ω. Look at the force diagrams in the S frame and S' frame.

In the S frame the ball has a centripetal acceleration $(-\omega^2 r\, \hat{\mathbf{r}})$. The force responsible for this acceleration is provided by the tension in the cord. On resolving the force T into its components Re get

$$T \cos \theta = mg,$$

$$T \sin \theta = m\omega^2 r$$

In the S' frame, the ball is at rest. This is because in this frame alongwith T and mg a centrifugal force $\mathbf{F}_{cent}$ also acts on it. Resolution of forces gives

$$T \cos \theta = mg,$$

$$F_{cent} = T \sin \theta = m\omega^2 r$$

We have taken this example also to caution you against the misuse of the term centrifugal force. Sometimes you may come across statements like 'The Moon does not fall down as it moves around the earth because the centrifugal force balances the force of gravitation and hence there is no neb force to make it fall.'

Any such statement goes against Newton's first law. Why? Because if no net force were acting on a body, it would *move* in a *straight line*. Any body moving on a curved path must have an unbalanced force on it. Now in the inertial frame the moon (or the ball) is seen to *move in a circular path*. Thus, an unbalanced centripetal force given by the force of gravitation (or the tension in the string) acts on the moon or the ball.

However, in the rotating frame of reference moving at the same angular speed, these objects would be seen to be at *rest*. Only in such frames would the centrifugal force balance the gravitational force on the moon (or the horizontal component of the tension in the string). *So* remember centrifugal forces arise only in rotating frames of reference. If we analyse a rotating object's motion from a non-rotating frame there is no such thing as centrifugal force. Of course, either frame is valid for analysing the problems. But never use inertial forces in inertial frames. They arise only in non-inertial frames.

Let us round off this section with an example of centrifugal force.

Example **2:** Centrifuge

An interesting application of the centrifugal force is a device called a centrifuge. It has uses, such as for separating heavy particles suspended in a liquid, for separating chemicals etc. You may like to know how it works.

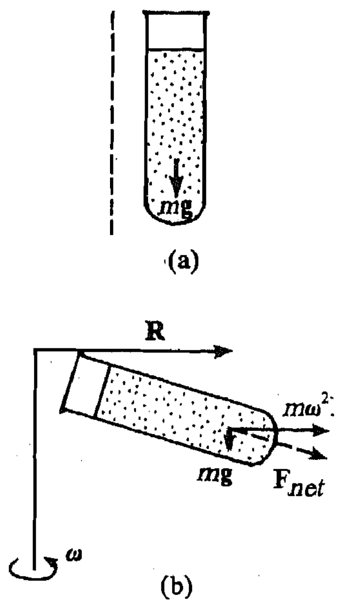


Fig. 10.9: (a) A test-lube in a centrifuge. The dotted line is the axis of the centrifuge: (b) when the centrifuge rotates, the centrifugal force makes the free end of the test tube swing out.

Suppose we have a test tube containing small particles suspended in a liquid. If the particles are heavier than the liquid, they will settle to the bottom, but if the particles are extremely small, this will take a long time. To speed up the process, we attach the test tube to a centrifuge. It is a mechanical device whose operation depends on centrifugal force.

For a rigorous analysis of the situation we need to account for the buoyant forces on the suspended particles and the viscous force acting on the mobile particles. Since these forces are small compared to the force of gravity and the centrifugal force, we shall ignore them.

Initially the tube hangs vertically, as in Fig. 10.9a. The centrifuge is carefully balanced with other tubes (not shown in the figure). When the centrifuge is spun about its central vertical axis, the tubes feel a centrifugal force (in the frame rotating with the centrifuge) pointing in the horizontal direction. The resultant of the force of gravity and centrifugal force acts like an *effective force of gravity*. At high values of angular speed $\mathbf{F}_{cent}$ is much greater than mg. So this effective force is much stronger and points almost horizontally (Fig. 10.9b). The Lube rises until it is oriented along the direction of the net force $\mathbf{F}_{net}$ on it. The surface of the liquid orients itself normal to the net force it feels. **A** particle suspended in the liquid moves in the direction of the net force it feels. This is essentially towards the bottom of the tube. Since $\mathbf{F}_{net}$ is much greater than mg for high values of ω, the suspended particles settle to the bottom of the tube much more rapidly than they would otherwise.

You may now like to work out an SAQ to consolidate your understanding of centrifugal forces.

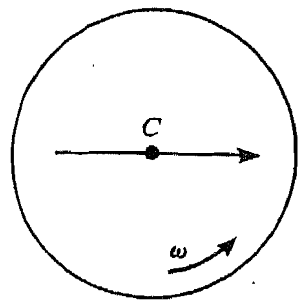
SAQ 4

(a) When we drive a car too fast around a curve, it skids outward. To us it seems as if it is pushed by a centrifugal force. If you were standing by tlie roadside watching this happen, how would you explain the car's motion?

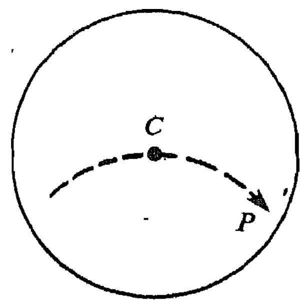
(b) A tiny virus particle of mass 6×10^{-19} kg is in a water suspension in an ultracentrifuge which is essentially a centrifuge where extremely high angular speed can be generated. It is 4 cm from the vertical axis of rotation. The angular speed of rotation is $2\pi \times 10^3$ rad s^{-1}.

(i) What is the effective value of 'g' relative to the frame rotating with the ultracentrifuge?

(ii) What is the net centrifugal force acting on the particle'?



(a)



(b)

Fig. 10.10: Motion of a frictionless ball passing over the rotation axis at C, as seen from above in (a) an inertial frame (solid line) and (b) the rotating frame (dashed line).

10.3.3 Coriolis Force

Let us consider a particle which moves with a velocity v_{rot} with respect to a rotating frame. The effect of Coriolis force is relatively easy to visualize at the axis of rotation, where tlie centrifugal force is negligible. So let us begin with that case.

A rotating horizontal disc is shown in Fig. 10.10a. The axis of rotation is perpendicular to the plane of this paper at point C which is the centre of the disc. Let us now consider a ball passing through C. If friction can be ignored, tlie ball is free of horizontal forces. Therefore, it moves in a straight line (the solid line of Fig. 10.10a) with constant velocity v relative to the inertial frame. As seen from this frame, the rotating disc turns, say, counterclockwise with angular speed ω. But as seen from a frame fixed in the disc, it is the inertial frame that rotates, with the same angular speed in the opposite sense, clockwise. So in the rotating frame the ball's trajectory also turns clockwise, following the curved path indicated by the dashed line in Fig. 10.10b. Thus, there must be an inertial force in the rotating frame to provide the curvature that was not present in the inertial frame. It is indeed the Coriolis force.

The magnitude of the Coriolis force can be appreciable on a turntable or merry-go-round. For example, if ω is 1 rad s^{-1} and v_{rot} is 5 $m\,s^{-1}$ the Coriolis acceleration $2\omega v_{rot}$ is 10 ms^{-2}, equal to the acceleration due to gravity.

The Coriolis force associated with the earth's rotation is much weaker than the effect considered above because the earth rotates only once per day, corresponding to an angular speed $\omega = 2\pi \times 10^{-5}$ rad s^{-1}. Even at projectile velocities of 10^3 $m\,s^{-1}$, the Coriolis acceleration $2\omega v_{rot}$ is only of the order of 10^{-2} ms^{-2} which is far less than g. That is why the Coriolis force is not intuitively familiar. However, when the Coriolis force associated with the earth's rotation acts over a sufficient period of time. say for several days, it can have striking effects. The centrifugal and Coriolis forces associated with the earth's rotation are responsible for many a natural phenomena. For example, the variation in g with latitude. the deflection of a moving body, wind patterns in the two hemispheres can be explained using the concepts of centrifugal or Coriolis force arising on a rotating earth. So let us now study the earth as a rotating frame.

10.4 THE EARTH AS A ROTATING FRAME OF REFERENCE

A number of important phenomena are driven by the inertial forces acting in a rotating frame of reference attached to the earth's surface. Let us study some of these phenomena.

10.4.1 The Variation of g with Latitude

You may know that a person weighs more at the poles than at the equator. This effect arises clue to the rotation of the earth. In fact we have already stated this result giving the variation of g with latitude (recall Eq. 5.44 of Unit 5. Block 1). Let us now prove the result.

Let a particle P be at rest with respect to tlie earth at latitude λ near the earth's surface. Then in the earth's frame it is subjected to the force of gravity F_g (= mg) and the centrifugal force

$\mathbf{F}_{cent}$ shown in Fig. 10.11a. The Coriolis force is zero for this particle, since it is at rest in the rotating frame. The magnitude of $\mathbf{F}_{cent}$ is given from Eq. 10.13a as

$$F_{cent} = m\omega^2 R \sin\phi = m\omega^2 R \cos A. \ [\because \ \lambda = \frac{\pi}{2} - \phi\ ]$$

where R is the earth's radius. Let the resultant of $\mathbf{F}_g$ and $\mathbf{F}_{cent}$ be $\mathbf{F}_g^*$. Let us resolve these three forces along the radial and transverse directions. Note that on the earth, the radial direction corresponds to the vertical (opposite to $\mathbf{F}_g$) and the transverse to the horizontal. Let g_v^* and g_h^* represent the vertical and horizontal components of **g'**, respectively (Fig. 10.11b). So we have

$$m\, g_v^* = F_g - F_{cent} \cos\lambda = mg - m\,\omega^2 R \cos^2\lambda$$

$$\text{or } \ g_v^* = g - \omega^2 R \cos^2\lambda \tag{10.14a}$$

$$\text{and } \ m g_h^* = F_{cent} \sin\lambda = m\omega^2 R \cos A \ \sin\lambda$$

$$\text{or } \ g_h^* = \omega^2 R \cos\lambda \sin A \tag{10.14b}$$

Now, the maximum magnitude of the centrifugal acceleration, (F_{cent}/m), is $\omega^2 R$. Let us calculate its value.

$$\omega^2 R = \left(\frac{2\pi \text{ rad}}{24 \times 60 \times 60\text{s}}\right)^2 \times (6.37 \times 10^6 \text{m}) = 3.4 \times 10^{-2}\ \text{ms}^{-2}.$$

Thus $\omega^2 R \ll g$ and $g_v^* \approx g$, i.e. the angle between $\mathbf{g}_v'$ (the apparent vertical) and **g** (the real vertical) is very small. Let us compute its value. From Fig. 10.11b

$$\tan\alpha \approx \alpha = \frac{g_h^*}{g_v^*} = \frac{\omega^2 R \cos \text{L} \sin\lambda}{R} = \frac{\omega^2 R \sin 2\lambda}{2g}$$

It has a maximum value at $\lambda = 45°$ which is

$$\alpha_{max} = \frac{(3.4 \times 10^{-2}\ \text{ms}^{-2})}{2 \times 9.8\ \text{ms}^{-2}} = 0.0017 \text{ rad} = 0°\, 6'.$$

So effectively $g_h^* = 0$ and $g_v^* = g^*$.

From Eq. 10.14a, we get

$$g^* = g - \omega^2 R \cos^2\lambda, \tag{10.15}$$

At the poles $\lambda = 90°$ and $g_v^* = g$, i.e. $g_{pole} = g$.

At the equator $\lambda = 0$, so that

$$g_h^* = 0, \ g_v^* = g - \omega^2 R.$$

$$\therefore \ g_e = g - \omega^2 R, \text{ where } g_e \text{ is the value of } g \text{ at equator.}$$

Now using Eq. 10.15, we may write,

$$g^* = g - \omega^2 R\,(1 - \sin^2\lambda) = (g - \omega^2 R) + \omega^2 R \sin^2\lambda$$

$$\text{or } \ g^* = g_e + \omega^2 R \sin^2\lambda,$$

which is same as Eq. 5.44 of Block 1.

So the value of acceleration due to gravity at the poles will be greater by 3.4 $\times 10^{-2}$ ms^{-2} than its value at the equator if we take earth's rotation into account. However the measured difference is 5.2 $\times 10^{-2}$ ms^{-2}. This discrepancy arises because the earth is not a perfect sphere. It is flattened at the poles and bulging at the equator. Due to the centrifugal force arising from earth's rotation a plumb line does not point exactly towards the centre of the earth. Instead it swings through a small angle. You may now like to work out an SAQ on the above concept.

Fig. 10.11 Variation of g with λ. (a) Resultant of $\mathbf{F}_g$ and $\mathbf{F}_{cent}$. The dotted line E represents the equator. PV is the vertical direction at P (b) g^*, g_h^* and g_v^*.

SAQ 5

a) What must be the angular speed of the earth so that the centrifugal force makes objects fly off its surface? (Take $g = 10\ ms^{-2}$).

b) If the angular Speed is just enough to make this happen, from which part of the earth would the objects fly off?

In the above discussion we have considered the body to be at rest with respect to the earth. What can you say about a body moving with respect to the earth's surface? We will now have to take into account the Coriolis force also. Let us analyse this motion.

10.4.2 Motion on the Rotating Earth

Let us consider a particle of mass m moving with velocity $\mathbf{v}$ at latitude λ on the surface of the spherical earth. So $\mathbf{v}$ is tangential to the sphere. Let the earth's angular velocity be $\boldsymbol{\omega}$. Then in the earth's frame of reference, the force on m is given from Eq. 10.12 as

$$\mathbf{F} = m\mathbf{g} - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Let us analyse the additional term due to Coriolis force. Refer to Fig. 10.12a. Let us decompose $\boldsymbol{\omega}$ into a vertical part $\boldsymbol{\omega}_V$ and horizontal part $\boldsymbol{\omega}_H$. Then the Coriolis force is given by

$$\mathbf{F}_{cor} = -2m\,(\boldsymbol{\omega} \times \mathbf{v})$$

$$= -2m\,(\boldsymbol{\omega}_V \times \mathbf{v}) - 2m\,(\boldsymbol{\omega}_H \times \mathbf{v}).$$

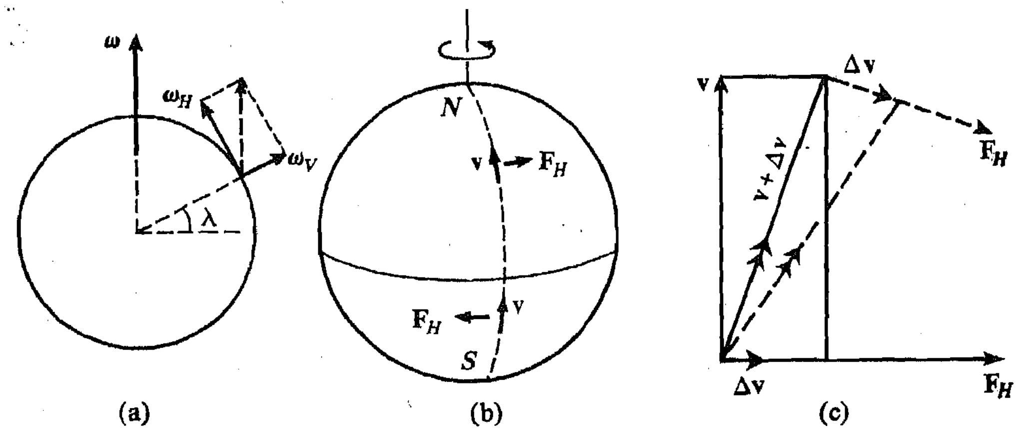


Fig. 10.12: Deflection of a moving particle due lo Coriolis force. (a) Breaking $\boldsymbol{\omega}$ into components $\boldsymbol{\omega}_H$ and $\boldsymbol{\omega}_V$; (b) directions of $\mathbf{F}_H$ in N and S-hemisphere; (c) clockwise turning of v in N-hemisphere.

Now $\boldsymbol{\omega}_H$ and $\mathbf{v}$ are horizontal, so $\boldsymbol{\omega}_H \times \mathbf{v}$ is vertical. And $\boldsymbol{\omega}_V \times \mathbf{v}$ alone gives rise to the horizontal component $\mathbf{F}_H$ of the Coriolis force. $\boldsymbol{\omega}_V$ is perpendicular to $\mathbf{v}$. So $\boldsymbol{\omega}_V \times \mathbf{v}$ has magnitude $\omega_V v$. Now let $\hat{\mathbf{r}}$ be a vector perpendicular to the surface at latitude λ, i.e. $\hat{\mathbf{r}}$ is along $\boldsymbol{\omega}_V$. Then we have that

$$\boldsymbol{\omega}_V = \omega_V\,\hat{\mathbf{r}} = \omega \cos\left(\frac{\pi}{2} - \lambda\right)\hat{\mathbf{r}} = \omega \sin\lambda\,\hat{\mathbf{r}}$$

and $\mathbf{F}_H = -2m\,(\boldsymbol{\omega}_V \times \mathbf{v}) = -2m\,\omega \sin\lambda\,(\hat{\mathbf{r}} \times \mathbf{v})$

The magnitude of $\mathbf{F}_H$ is $2mv\omega \sin\lambda$. $\mathbf{F}_H$ is a force perpendicular to $\mathbf{v}$ (Fig. 10.12b). So its effect is to produce circular motion. Let us see how.

The effect of $\mathbf{F}_H$ will be to produce a deflection towards the right in the northern hemisphere. $\mathbf{F}_H$ produces a change in the direction of $\mathbf{v}$. Let the change in $\mathbf{v}$ be $\Delta\mathbf{v}$ in an infinitesimal interval of time Δt. From Fig. 10.12c you can see that the resultant velocity vector moves towards the right. $\mathbf{F}_H$ is now perpendicular to $\mathbf{v} + \Delta\mathbf{v}$. So in the next such time interval Δt, the velocity vector will further turn towards right. So the effect of $\mathbf{F}_H$ in the northern hemisphere is to produce a clockwise rotation of the velocity vector. In the southern hemisphere this will be anticlockwise.

So you can see that this effect of Coriolis force is that it turns straight line motion into circular motion. Ttiis result has a number of interesting consequences. For example, rivers flowing in the northern hemisphere wash out their right banks, and those in the southern hemisphere their left banks. Again in the northern hemisphere the right hand rails of the rail tracks are worn out faster if it is a double-track railway. This is because on each track the train always goes in one direction. Due to $\mathbf{F}_H$ its motion has a component to the right from the direction of motion. Similarly, the left hand rail is worn out faster in the southern hemisphere.

Air flow patterns in the atmosphere can also be explained by this result. Imagine that temperature difference in the various layers of air has given rise to a low pressure region in the atmosphere (Fig. 10.13a). The closed curves in the figure represent lines of constant pressure, called isobars. The pressure gradient gives rise to a force on each element of air. In the absence of other forces winds would blow inward and the pressure in the region would become uniform.

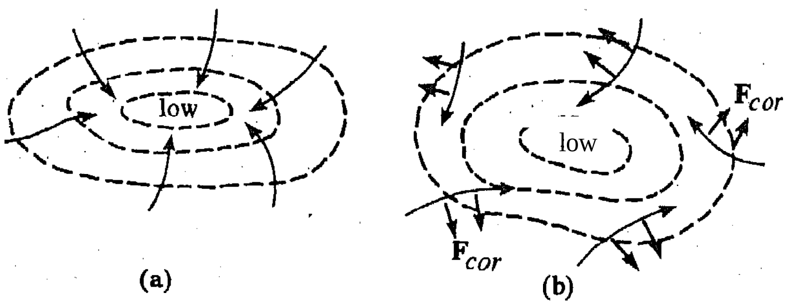


Fig. 10.13: Air-tlow patterns: (a) Dotted lines represent the isobars: (b) right deflection of the air particles.

However, the pressure of Coriolis force considerably changes the air flow pattern. Let us consider this event in the northern hemisphere. As the air flows inward towards the low pressure region it is deflected toward the right as shown in Fig. 10.13b. The result is that wind rotates anticlockwise about the regions of low pressure. This effect causes most cyclones to be anticlockwise in the northern hemisphere and clockwise in the southern hemisphere. This effect can be seenquite clearly in the INSAT pictures of clouds taken during a cyclonic storm.

So far we have discussed some natural phenomena which arise due to rotation of the earth. We can also demonstrate rotation of the earth in a laboratory using the Foucault's pendulum.

10.4.3 Foucault's Pendulum

In 1851, J.B.L. Foucault for the first time demonstrated the rotation of the earth. He suspended a heavy metal sphere of 28 kg on a wire almost 70m long, The suspension point of the pendulum was free to rotate in any direction. The motion of the pendulum was observed from a point above. With successive swings of the pendulum it seemed that the plane of its motion rotated. In 1h the plane of the swing changed by 11°. A full circuit was completed in about 32h.

Why does the plane of motion of the pendulum rotate?

To understand this, we shall visualise this experiment at the North Pole (Fig. 10.14a). In an inertial frame the only forces acting on the pendulum are the force of gravity and the tension of the wire. Both these forces act in the plane of oscillation. So they cannot rotate it. Therefore, with respect to an inertial frame the plane of the oscillation of the pendulum would remain fixed. The earth would, of course, rotate from west to east under the pendulum once in every 24h. The rotation of the earth is anticlockwise as seen from the North Pole. So to an observer standing at the North Pole, the plane of the oscillation would seem to rotate clockwise (east to west} (Fig. 10.14b). It can also be explained for other latitudes but we are not going into those details here.

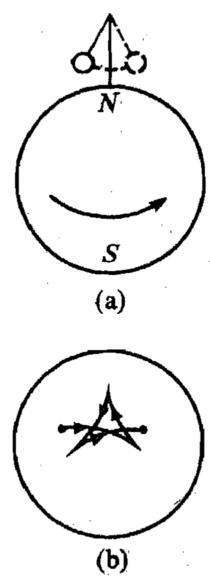


Fig. 10.14: Foucault's pendulum. (a) The pendulum on the N-Pole. The arrow indicates the direction of rotation of the earth; (b) rotation of the plane of oscillation.

Let us now summarise what we have studied in this unit.

10.5 SUMMARY

- The frames of reference accelerating with respect to each other are called non-inertial frames.
- The net force acting on any object in the non-inertial frame S' having an acceleration a with respect to an inertial frame S is made up of two parts: the force F, acting on the object in the S frame and an inertial force equal to $-m$ a. Inertial forces arise only in non-inertial frames.
- The equation of motion of an object in a rotating frame of reference is given as

$$\mathbf{F}_{rot} = \mathbf{F} - 2m\,(\boldsymbol{\omega} \times \mathbf{v}') - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where F is the sum of all forces acting on the object as seen from the inertial frame. The second and the third terms are the Coriolis and the centrifugal forces, respectively.

- Any frame of reference attached to the earth is a non-inertial frame of reference. Rotation of the earth is responsible for many a natural phenomena, such as variation of g with latitude, deflection of moving bodies, etc. The earth's rotation can be demonstrated with the help of Foucault's pendulum.

10.6 TERMINAL QUESTIONS

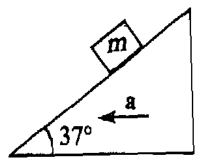


Fig. 10.15: Diagram for TQ 1

1. An inclined plane (Fig. 10.15) is accelerated horizontally to the left. The magnitude of the acceleration is gradually increased until a block of mass m, originally at rest with respect to the plane, just starts to slip up the plane. The coefficient of static friction between the plane and the block is 0.8. (It is given that sin 37° = 3/5, $g = 10\,\text{ms}^{-2}$).
   a) Draw diagrams showing the forces acting on the block, just before it slips (i) in an inertial frame fixed to the floor and (ii) in the non-inertial frame moving along with the block.
   b) Find the acceleration at which the block begins to slip using both the force diagrams (i) and (ii) of part (a).

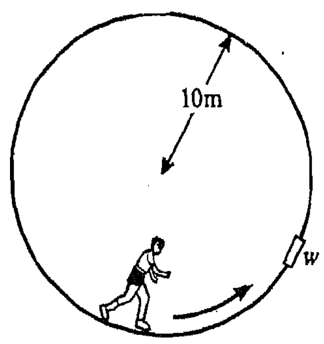


Fig. 10.16: Diagram for TQ 2a.

2. a) A space station of radius 10m Spins so that a person inside it (Fig. 10.16) has a sensation of 'artificial gravity' when afloat in space. The rate of spin is chosen to attain $g = 10\,\text{ms}^{-2}$. Find the length of the 'day' as seen in the spacecraft through a window W.
   b) A 4.0×10^5 kg train runs south at a speed of 30 ms^{-1} at a latitude of 60° N. What is the horizontal force on the tracks? What is the direction of this force?
3. Your weight is measured to be equal to W when you are at rest with respect to the earth. Will your weight be different from W when you are in motion with respect to the earth?

10.7 ANSWERS

SAQs

1. i) Since the car is moving along a curved path its velocity vector is continually changing its direction. So it has a non-zero acceleration with respect to the man standing on the road. So the frame attached to it is non-inertial with respect to the man.

   ii) Since the raindrop has attained a terminal velocity it is falling with a constant velocity with respect to the ground. So the frame attached to it is inertial with respect to the ground.

   iii) An electron moving in a uniform magnetic field experiences a force. So it will be accelerating with respect to a pole piece. Hence, the frame attached to the electron is non-inertial with respect to the pole piece.

2. Differentiating Eq. 10.1 twice with respect to time, we get

$$\ddot{\mathbf{r}}_q = \ddot{\mathbf{r}}_p - \ddot{\mathbf{R}}$$

or $$\mathbf{a}_q = \mathbf{a}_p - \mathbf{a}$$

∴ $$\mathbf{F}_q = m\mathbf{a}_q = m\mathbf{a}_p - m\mathbf{a}$$

or $$\mathbf{F}_q = \mathbf{F}_p - m\mathbf{a}.$$

3. a) In order to start, the train has to accelerate. Let this acceleration be a, and directed along the x-axis. Now, following Eqs. 10.2a and 10.2b, we can write the total force acting on the water in the frame of reference of the train as (see Fig. 10.17)

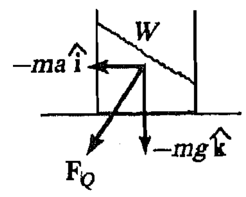


Fig. 10.17

$$\mathbf{F}_Q = m\mathbf{g} + (-m\mathbf{a}),$$

where m is the total mass of the water and the glass.

$$\mathbf{g} = -g\,\hat{\mathbf{k}} \quad \text{and} \quad \mathbf{a} = a\,\hat{\mathbf{i}}\,.$$

The surface of water takes up a position normal to the force $\mathbf{F}_Q$ as shown in Fig. 10.17.

b) Let the lift be accelerating in the z-direction (Fig. 10.18). The inertial force acting on the man is given by

$$\mathbf{F}' = -m\mathbf{f},$$

where m is the mass of the man. So the total force on the man is given by

$$\mathbf{F} = m\mathbf{g} + \mathbf{F}' = m\mathbf{g} - m\mathbf{f}.$$

But $\mathbf{g} = -g\hat{\mathbf{k}},\ \mathbf{f} = f\hat{\mathbf{k}}.$

Hence, $\mathbf{F} = -m\,(g + f)\,\hat{\mathbf{k}}.$

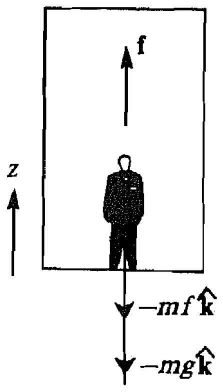


Fig. 10.18

So the magnitude of the force on the man is greater than mg. Hence, he feels heavier than usual.

a) The observer on the roadside will analyse the situation as follows: A centripetal force ($= mv^2/r$) where m is the mass of the car, v its speed and r the radius of curvature of the bend, is required by the car to move along the curve. You may recall from Sec. 4.3.1 of Block 1 that this is normally provided by way of the banking on the road an4 the friction between the tyres and the road, Let the contribution due to banking and friction be F_1, F_2, respectively. Then the equation of motion of the car will be

$$F_1 + F_2 = mv^2/r \quad \text{or} \quad \frac{F_1 + F_2}{m} = \frac{v^2}{r}$$

Now, the left hand side is a fixed quantity depending on m. So if v is large, r should be large in order to make the above equation hold. In other words, the car has to move more outward to have a large r, when it is moving very fast.

b. i) For this problem

$$\omega^2 r = (2\pi \times 10^3\ \mathrm{s}^{-1})^2 \times (0.04\mathrm{m}) = 1.6 \times 10^6\ \mathrm{m\,s}^{-2}.$$

Since this is much larger than the usual value of 'g' the effective value of 'g' can be considered to be equal to $1.6 \times 10^6\ \mathrm{m\,s}^{-2}$.

ii) The net centrifugal force $= m\omega^2 r$, where $m = 6 \times 10^{-19}$ kg. So its value is $(6 \times 10^{-19}\ \mathrm{kg}) \times (1.6 \times 10^6\ \mathrm{m\,s}^{-2}) = 9.6 \times 10^{-13}$ N.

5. a) The required angular speed will correspond to $g^* = 0$. We know from Eq. 10.15 that $g^* = g - \omega^2 R \cos^2\lambda$. So the required condition is

$$\omega = \sqrt{\frac{g}{R\cos^2\lambda}}$$

So the minimum value of ω corresponds to the maximum value of $\cos^2$ h, i.e. 1 for $\lambda = 0^\circ$. This happens at the equator. And the required angular speed of earth is given by

$$\omega_{min} = \sqrt{\frac{g}{R}},$$

where R is the equatorial radius of the earth = 6.37 X 10^6 m.

$$\therefore \omega_{min} = \sqrt{\frac{10\ \text{ms}^{-2}}{6.37 \times 10^6\ \text{m}}} = 1.3 \times 10^{-3}\ \text{rad s}^{-1}.$$

b) At equator as explained in the answer to part (a).

Terminal Questions

1. a) Refer to Figs. 10.19 (a and b) for parts (i) and (ii), respectively.

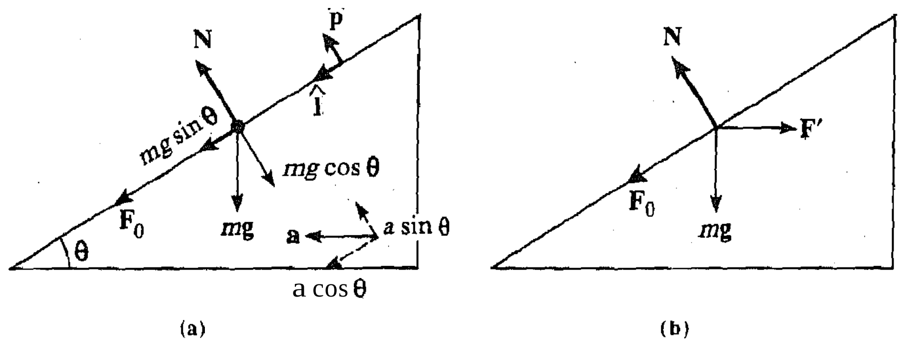


Fig. 10.19: F_0 is the force of friction, N is the normal reaction and mg is the weight of the block, a) The resultant of three forces F_0, N and mg is equal to ma. Components of a along $\hat{p}$ and $\hat{i}$ have also been shown; (b) In addition to F_0, N and mg, we have F′, the inertial force (= −ma).The farces F_0, N, mg and F′ are in equilibrium.

b) Using the force diagram for part (i), i.e. Fig. 10.19a, we have the equation of motion

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_0 = m\mathbf{a}. \qquad (10.16)$$

Now, let the unit vectors along $\mathbf{F}_0$ and N be $\hat{i}$ and $\hat{p}$, respectively. So we have

$$mg\cos\theta\,(-\hat{p}) + mg\sin\theta\,(\hat{i}) + N\,(\hat{p}) + F_0\,(\hat{i}) = ma\cos\theta\,(\hat{i}) + ma\sin\theta\,(\hat{p}).$$

Thus,

$$(F_0 + mg\sin\theta - ma\cos\theta)\,\hat{i} + (N - mg\cos\theta - ma\sin\theta)\,\hat{p} = 0.$$

$$\left.\begin{aligned} \therefore\ F_0 + mg\sin\theta - ma\cos\theta &= 0 \\ \text{and } N - mg\cos\theta - ma\sin 8 &= 0 \end{aligned}\right\} \qquad (10.17)$$

Now if a be the magnitude of acceleration at which the block just begins to slip up we have $F_0 = \mu N$ where $\mu = 0.8$.

So from Eqs. 10.17 we get

$$\mu N = m(a\cos\theta - g\sin 8)$$

$$\text{or } \mu m\,(g\cos\theta + a\sin\theta) = m\,(a\cos\theta - g\sin\theta)$$

$$\therefore g\,(\mu\cos\theta + \sin\theta) = a\,(\cos\theta - \mu\sin\theta)$$

$$\text{or } a = g\left(\frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin 0}\right)$$

Since, $\sin\theta = 0.6$, $\cos\theta = 0.8$

$$\therefore a = (10\ \text{ms}^{-2})\left(\frac{0.8 \times 0.8 + 0.6}{0.8 - 0.8 \times 0.6}\right) = 39\ \text{ms}^{-2}.$$

Using the force diagram for part (ii), i.e. Fig. 10.19b, we have,

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_0 + \mathbf{F}' = \mathbf{0}.$$

Since $\mathbf{F}' = -m\mathbf{a}$, we get

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_0 = m\mathbf{a}.$$

This is same as Eq. 10.16. So the succeeding analysis will follow as in the previous case and we shall get a = 39 ms^{-2}. You must have noted that we come across an equation of motion in the inertial frame, but a condition of equilibrium'in the non-inertial frame.

2. a) Let the required rate of spin be ω, Then the corresponding length of day is given by

$$T = \frac{2\pi}{\omega}.$$

Since the person inside has a sensation of artificial gravity, we have

$\omega^2 r = g$, where r = 10m.

$$\therefore \frac{4\pi^2}{T^2} r = g$$

$$\text{or } T = 2\pi\sqrt{\frac{r}{g}}$$

$$= 2\pi\sqrt{\frac{10\text{m}}{10\ \text{ms}^{-2}}} = 6.3\text{s}.$$

b) Refer to Fig. 10.20. NPM and *QPR* are, respectivcly, the longitude and latitude through *P*, the position of the train. *AB* is the equator. The horizontal force is due to the Coriolis force given by

$$\mathbf{F}_{cor} = -2m(\boldsymbol{\omega} \times \mathbf{v}).$$

Since the angle between $\boldsymbol{\omega}$ and v is (180" – k) (see figure caption), the magnitude of the horizontal force is 2 $m\omega v$ sink,

where m = 4.0 x 10^5 kg, v = 30 ms^{-1}, $\omega = \dfrac{2\pi}{24 \times 60 \times 60}$ rad s^{-1} and λ = 60".

So the magnitude of the horizontal force on the tracks is

$$2 \times (4 \times 10^5\ \text{kg}) \times (30\ \text{ms}^{-1}) \times \left(\frac{2\pi}{24 \times 60 \times 60}\ \text{s}^{-1}\right) \times \sin 60^\circ = 1.5 \times 10^2\ \text{N}.$$

The direction is opposite to ($\boldsymbol{\omega} \times$ v) . Now, (ω x v) points tangentially to the latitude *QPR* in the sense *Q* to P. So $\mathbf{F}_{cor}$ will be tangential to *QPR* in the sense *P* to Q, i.e. towards west.

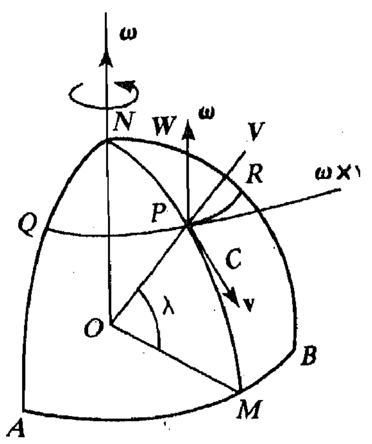


Fig 10.20 : Diagram for terminal question 2b. You must note that $\angle NOP = 90^\circ - \lambda$. It is equal to the corresponding angle ($\angle WPV$). And $\angle VPC = 90^\circ$. So $\angle WPC = 90^\circ - \lambda + 90^\circ = 180^\circ - \lambda$

3. The weight of your body is given by

$$\mathbf{F} = m\mathbf{g} - \mathbf{F}_{cent} - \mathbf{F}_{cor},$$

where m is your mass. If you are at rest with respect to the earth $\mathbf{F}_{cor} = \mathbf{0}$. But if you are moving $\mathbf{F}_{cor} \neq \mathbf{0}$. So your weight will be different from *W* when you are in motion with respect to the earth.

FURTHER READING

1. Berkeley Physics Course – Volume I, Mechanics; C. Kittel, W.D. Knight, M.A. Ruderman, A.C. Helmholz, B. J. Moyer; Asian Student Edition, McGraw-Hill International Book Company, 1981.
2. An Introduction to Mechanics; D. Kleppner, R.J. Kolenkow; International Student Edition, McGraw-Hill International Book Company, 1984.
3. Introduction to Classical Mechanics; A.P. French, M.G. Ebison; Van Nostrand Reinhold (UK) Co. Ltd., 1984.
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5. The Mechanical Universe, Mechanics and Heat Advanced Edition; S.C. Frautschi, R.P. Olenick, T.M. Apostol, D.L. Goodstein; Cambridge University Press, 1986.
6. Physics Volume I; R. Wolfson, J.M. Pasachoff; Little, Brown and Company, 1987.

APPENDIX A

CONIC SECTIONS

The curves obtained by slicing a cone with a plane not passing through its vertex are called *conic* sections or simply conics. If the cutting plane is parallel to the side of the cone, as in Fig. A.1a, the conic is aparabola. Otherwise the intersection is called-an ellipse or a *hyperbola*, according as the plane cuts just one or both nappes (portion of the cone) as shown in Figs. A.1b and A.1c. Circle is the special case of ellipse when the intersecting plane is parallel to the base of the cone (Fig. A.1d).

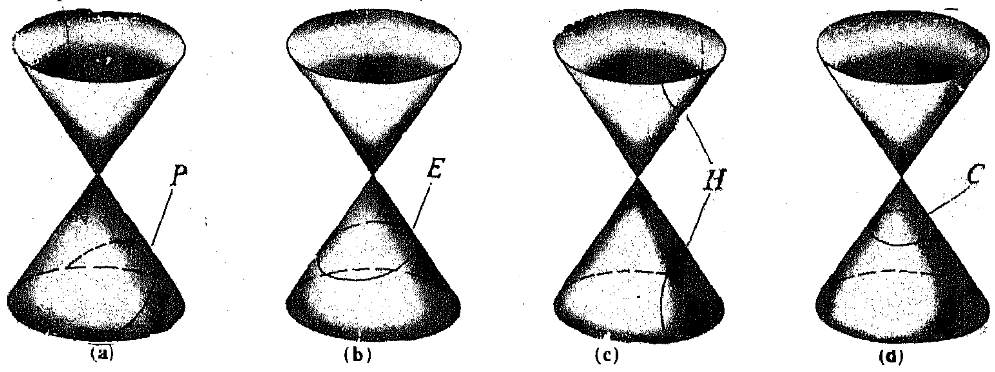

Fig.A.l: Conic sections: *P* - Parabola, E - Ellipse, *H* - Hyperbola, C - Circle

We shall now present a unified treatment for all conics. For this we shall define a term called 'eccentricity'.

A.1 Eccentricity and Polar Equation of a Conic

Refer to Fig. A.2. A conic section may be defined as a curve traced out by a point moving in a plane such that the ratio of its distance from a fixed point *F* (a focus) and a fixed line *AB* (a directrix) is constant. This constant ratio is called the eccentricity. It is denoted by e.

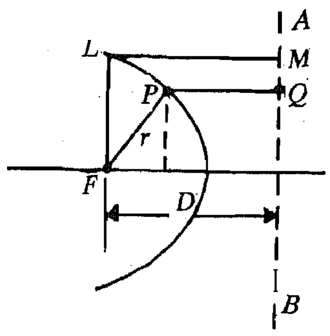

Fig. A.2: Polar equation of a conic

If $0 < e < 1$, the conic is an ellipse. If $e = 1$ it is a *parabola* and if $e > 1$, it is a hyperbola.

In the Fig. A.2, let P be any point on the conic. *PQ* is perpendicular on *AB* from P. Then according to the definition,

$$e = \frac{FP}{PQ}. \tag{A.1}$$

Using Eq. A.1 we shall obtain the polar equation of a conic, when the pole (i.e. the origin of the plane polar coordinates) is inside the curve. Let the pole be at F. The polar axis *Fx* is so *chosen* that it is perpendicular to the directrix. L is a point on the conic such that *FL* is perpendicular to *Fx*. *FL* is called the semi-latus rectum of the conic. Let *FL* = p. *LM* is again the perpendicular from *L* on *AB*. Let *LM* = D.

Now, we have

$$FP = r \quad \text{and} \quad QP = D - r\cos\theta.$$

So from Eq. A.1, we get

$$r = e\,(D - r\cos\theta)$$

$$\text{or} \quad r = \frac{eD}{1 + e\cos\theta} \tag{A.2}$$

But from the definition of e, we find that

$$\frac{FL}{LM} = e, \quad \text{i.e.}\ \frac{p}{D} = e \quad \text{or}\ p = eD.$$

So, we get from Eq. A.2 that

$$r = \frac{p}{1 + e\cos\theta}. \tag{A.3}$$

Eq. A.3 is the polar equation of a conic with pole inside.

The three types of conics have been shown in Fig. A.3. Because $\cos(-\theta) = \cos\theta$, all the