UNIT 9 RIGID BODY DYNAMICS

Structure

9.1 INTRODUCTION

In the previous unit you have studied the phenomenon of scattering. We had treated the projectile there as a point mass. In Units 6 and 7 you have studied about the motion of planets around sun by treating them as point masses. As a matter of fact so far in this course, we have been concerned primarily with the motion of point masses. In nature, however, we hardly come across an ideal point mass. We have to deal with motion of bodies which have finite dimensions. So we need to develop a technique for studying the motion of such bodies.

A special class of such bodies is known as ***rigid bodies.*** In this unit you will first learn what a rigid body is. You will see that the definition of a rigid body provides a model for studying the motion of various kinds of physical bodies. You will then study about the different kinds of motion of a rigid body. A rigid body can execute both translational and rotational motion. We shall see that the general motion of a rigid body is a combination of both translation and rotation.

You will find that the translational motion of a rigid body can be described in terms of the motion of its centre-of-mass. So, we shall be able to apply the dynamics of point masses for description of translational motion. Hence, our chief concern will be the study of dynamics of rotational motion of rigid bodies.

In Unit 4 of Block I you have studied the dynamics of rotational motion of a particle. You already know the concepts of angular displacement, angular velocity, angular acceleration, moment of inertia, kinetic energy, torque and angular momentum for a particle. In this unit we shall extend these concepts to the case of rigid bodies. This will enable us to study about a variety of applications such as the rotation of flywheels, despinning of satellites, motion of rolling objects and so on.

Finally, in this unit we shall revisit the important principle of conservation of angular momentum. We shall see that the principle holds for rigid and other extended bodies. We shall apply the principle to explain the acrobatics performed by a diver or a ballerina. Finally we shall discuss very briefly about precessional motion.

In this unit we shall very often refer to the contents of Unit 4 of Block 1. So it is suggested that you go through that unit once again before you start this unit.

In the next unit we shall aim to study the analysis of motion from the point of view of a non-inertial observer.

Objectives

After studying this unit you should be able to

- identify a rigid body
- distinguish between the features of translational and rotational motion of a rigid body
- outline the features of the general motion of a rigid body
- explain the significance of moment of inertia of a rigid body about a certain axis
- solve problems based on the concept of rotational dynamics of rigid bodies.

9.2 A RIGID BODY AND ITS MOTION

Let us consider the motion of a Yo-Yo (Fig.9.1). It runs up and down as the spool winds and unwinds. The Yo-Yo rotates about an axis passing through its centre and perpendicular to the plane of this paper. You can see that this axis does not remain fixed in space. It moves vertically downward or upward with the Yo-Yo. In principle we can use Newton's laws of motion to analyse such a motion as each particle of the Yo-Yo obeys them. But obtaining a description on a particle-by-particle basis will be an uphill task as the number of particles is very large. *So* we would like to find a simple method for analysing the general motion of an extended body like a Yo-Yo. We can find such a method by using the model of a rigid-body. So let us first learn what a rigid body is.

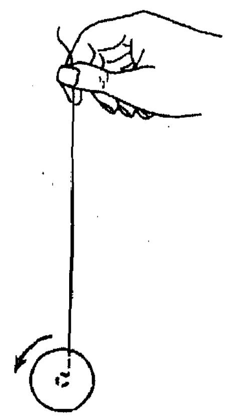


Fig. 9.1: A Yo-Yo

9.2.1 What is a Rigid Body ?

You must have seen a wheel rotating about its axle. Let us consider any two points on the wheel. We find that the relative separation between them does not change when the wheel is in motion. But if we take the example of the diver of Fig. 7.12 we find that the relative separation between two different parts of her body does change. The former is an example of a rigid body whereas the latter is not.

Technically speaking, a rigid body is defined as an aggregate of point masses such that the relative separation between any two of these always remains invariant, i.e. for any position of the body r_{ik} = a constant (Fig. 9.2). So a rigid body is one which has a definite shape. It does not change even when a deforming force is applied. In nature there is no perfectly rigid body as all real bodies experience some deformation when forces are exerted. So a perfectly rigid body can only be idealised. But we shall see that this model is quite useful in cases where such deformations can be ignored. For example, the deformation of a cricket ball as it bounces off the ground can be ignored. You know that if a heavy block is dragged along a plane, frictional force acts on it (see Sec. 2.2.2 of Block 1). But its deformation due to the frictional force can be neglected. However, you cannot neglect the deformation of a railway track due to the weight of the train. Likewise, the deformation of the fibre glass pole used by a pole-vaulter can also not be neglected. So in the last two cases we cannot apply the rigid body model.

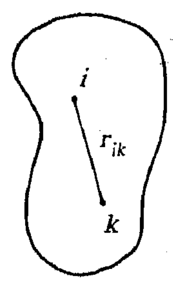


Fig. 9.2: For any position of a **rigid body,** *r_{ik} = a constant*

You may now like to identify the objects that can be approximated by the rigid body model.

SAQ 1

Which of the following can be considered as rigid bodies ?

a) **A** top b) A rubber band c) A bullet d) **A** balloon e) The earth.

Let us now study the motion of a rigid body. **A** rigid body can execute both translational and rotational motion. Let us discuss their basic features.

9.2.2 Translational Motion of a Rigid Body

Suppose you are travelling in a train. Then during a certain interval of time your displacement will be exactly equal to that of your co-passenger provided both of you do not move with respect to the train. This will also be true for any two objects attached to the body of the train, say a bulb and a switch. This is the characteristic of translational motion. A rigid body is said to execute pure translational motion if each particle in it undergoes the same displacement as every other particle in any given interval of time. Translational motion of a rigid body is shown schematically in Fig. 9.3.

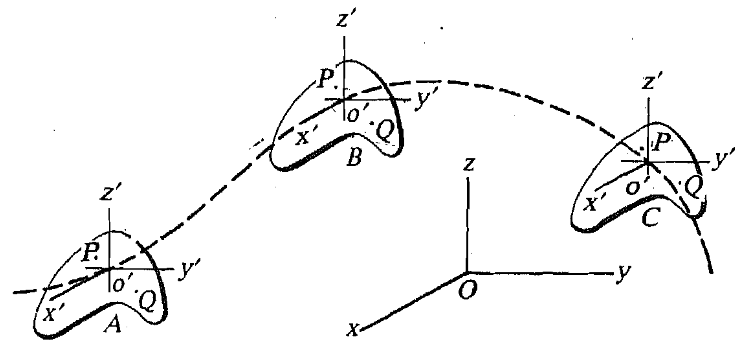


Fig. 9.3: Translational motion of a rigid body

You must have noted that the path taken is not necessarily a straight line. Now let us measure the magnitudes of the displacements of the points P, O', Q as the body moves from the position *A* to B. Each is equal to 3.9 cm and the lines joining these positions are parallel to each other. So they undergo the same displacement. You may verify the same for the motion of the body between positions B and C.

SAQ 2

(a) Measure the magnitudes of the displacements of P, O' and Q between positions B and C and verify that they are equal.

(b) Give two examples of a pure translational motion.

Now that you have worked out SAQ **2,** you can see that if we are able to describe the motion of a single particle in the body, we can describe the motion of the body as a whole. We have done this exercise a number of times before. However, you may like to consolidate your understanding by working out the following SAQ.

SAQ **3**

A rigid body of mass M is executing a translational motion under the influence of an external force F,. Suggest a suitable differential equation of motion of the body.

What does the answer to SAQ 3 signify? We know that the relative separation between any two points of a rigid body does not change, i.e.

$$\frac{dr_{ik}}{dt} = 0. \qquad (9.1)$$

So all the points follow the same trajectory as the c.m. Hence, for studying translational motion, the body may be treated as a particle of pass M located at its c.m. You may recall that we had treated the sun and a planet as particles in Unit *6.* They were treated as particles as their sizes are negligible compared to the distances between them and also because the shapes of these bodies were insignificant. But here we are considering a rigid body as a particle for another reason as explained above.

Thus we can represent the translational motion of a rigid body as a whole in terms of the motion of its c.m. It becomes easier to describe the translational motion in this way. In the previous units we have dealt with cases like a body falling down an inclined plane, a cricket ball hit by a bat, etc. There we had applied the above idea. So before we go over to the next sub-section it would be worthwhile to know about the position of c.m. of a rigid body.

The problem of locating the c.m. of a rigid body is complicated when its shape is asymmetrical. However, we shall deal mostly with bodies having a symmetrical shape. Positions of c.m.s (c) of several symmetrical bodies have been shown in Fig. 9.4.

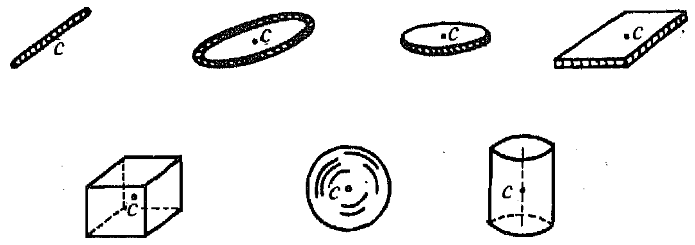


Fig. 9.4: Centres-of-mass of symmetrical rigid bodies

Let us now discuss the rotational motion of a rigid body.

9.2.3 Rotational Motion of a Rigid Body.

Let us consider the motion of the earth. Every point on it moves in a circle (the corresponding latitude), the centres of which lie on the polar axis. Such a motion is an example of a rotational motion. A rigid body is said to execute rotational motion if all the particles in it move in circles, the centres of which lie on a straight line called the axis of rotation. Fig. 9.5 shows the rotational motion of a rigid body about the z-axis. When a rigid body rotates about an axis every particle in it remains at a fixed distance from the axis. So each point in the body, such as P, desaribes a circle about this axis. You must have realised that perpendiculars drawn from any point in the body to the axis will sweep through the same angle as any other such line in any given interval of time.

Fig. 9.5: An example of rotational motion of a rigid body.

We shall now study about the general motion of a rigid body.

9.2.4 General Motion of a Rigid Body

Thegeneral motion of a rigid body is a combination of translation and rotation. This can be understood by considering a simple example shown in Fig. 9.6.

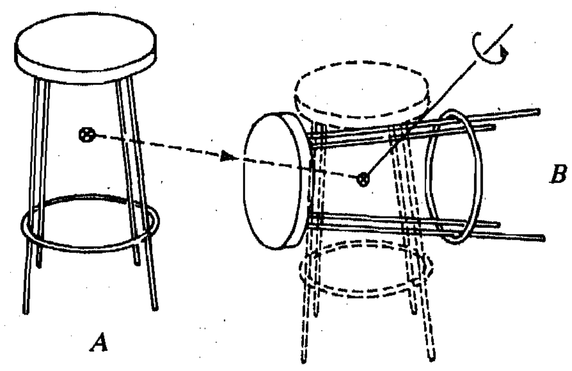


Fig. 9.6: To bnng the body from position A to some new position *B*, first translate it so that the centre-of-mass coincides with the new centre-of-mass. and then rotate it around the appropriate axis through the centre-of-mass until the body is in the desired position.

You may now perform an activity for the sake of better understanding of the general motion of a rigid body.

Activity

Take any book lying on your table and keep it in the bookshelf in its erect posture.

Here you first shift the c.m. of the book to a new position, Then you turn the book about a suitable axis through the c.m. to make it stand erect on the shelf. So you can see that the whole motion of the book is-a-combination of translation and rotation. Now, study the following figure carefully.

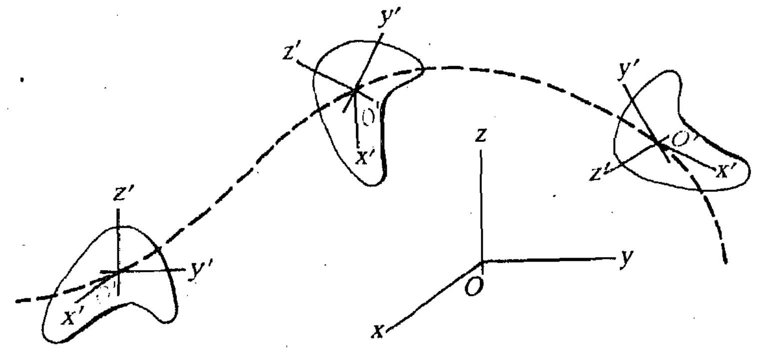


Fig. 9.7: A rigid body moving in combined translational and rotational motion as seen from reference frame (x, y, z). Notice that the reference frame fixed on the body (x', y', z') changes its orientation with respect to (x, y, z) as the motion proceeds.

Fig. 9.7 shows a case of combined translational and rotational motion of a rigid body. It can be considered as a schematic extension of Fig. **7.12.** Study Fig. **9.7** and work out the following SAQ.

SAQ 4

Compare Figs. **9.3** and 9.7. Mention very briefly the distinctive features in respect of the observer's reference axes (x, y, z) and the body-fixed axes (x', y', z').

Now that you have worked out SAQ 4 you can realise that determining the location O' in Fig. 9.7 is the good old problem of the motion of c.m. which we have studied in detail. As stated earlier our chief concern in this unit is to suitably study the rotational aspect. For this we have to develop a formalism to analyse rotational motion of a rigid body. Now, in Unit **4** of Block 1, you have already studied the dynamics of angular motion of a particle. We shall only make an extension of that study here.

Recall from **Sec. 4.3.3** of Block 1 that a particle executing rotational motion possesses a moment of inertia (denoted by I). For rotational motion I plays the same role as the mass of the particle plays for translational motion. It is very important to understand the meaning of moment of inertia of a rigid body for its rotational motion. So let us now learn about the ***'moment of inertia'*** of a rigid body. We shall start by determining the angular momentum of a rotating rigid body about the axis of rotation.

9.3 MOMENT OF INERTIA

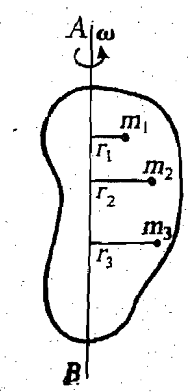


Fig. 9.8: A rigid body rotating about an axis AB.

We know that the earth rotates about the line joining the poles which passes through the centre of earth. How can we calculate its angular momentum about the axis of rotation? We know that when the earth rotates about its axis, every point on it executes a uniform circular motion about this axis. The radius of this circle decreases with the latitude of the point. The circle along which New Delhi moves has a smaller radius than that along which Trivandrum moves. So the linear velocity of each point is in general different. In order to determine the angular momentum of a body we shall first have to determine the angular momentum of each particle in it. And as angular momentum is a vector quantity we shall add vectorially the individual angular momenta to get the angular momentum of the body. Let us now consider a general situation.

Refer to Fig. **9.8.** A rigid body is rotating about an axis AB fixed in an inertial frame with a uniform angular speed ω. Three point masses m_1, m_2, m_3 at distances r_1, r_2, r_3, respectively,, from AB have been shown. m_1 moves along a circle of radius r_1 and let its velocity be $\mathbf{v}_1$. Using Eq. 4.23 of Unit 4, Block 1, we may say that the angular momentum $\mathbf{L}_1$, of m_1 is given by

$$\mathbf{L}_1 = m_1 \mathbf{r}_1 \times \mathbf{v}_1$$

Now the mass m_1 is rotating along a circle of radius r_1 whose plane is perpendicular to AB. In fact every point mass is moving along a circle whose plane is perpendicular to AB. Using **Eq. 4.13a** of Unit 4, Block 1, we get

$$\mathbf{v}_1 = \dot{r}_1 \hat{\mathbf{r}}_1 + r_1 \dot{\theta}_1 \hat{\boldsymbol{\theta}}_1,$$

where $\hat{\mathbf{r}}_1$ is the unit vector along r, and $\hat{\boldsymbol{\theta}}_1$ is perpendicular to $\hat{\mathbf{r}}_1$ in the sense of increasing angle θ_1 . You may recall that the directions of $\hat{\mathbf{r}}_1$ and $\hat{\boldsymbol{\theta}}_1$ change with time. Again $\dot{\theta}_1 = \omega$, which is same for all the point masses. So we get,

$$\mathbf{L}_1 = m_1 r_1 \hat{\mathbf{r}}_1 \times (\dot{r}_1 \hat{\mathbf{r}}_1 + r_1 \dot{\theta}_1 \hat{\boldsymbol{\theta}}_1)$$

Now, $\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_1 = \mathbf{0}$ and $\hat{\mathbf{r}}_1 \times \hat{\boldsymbol{\theta}}_1 = \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector along BA (See SAQ 3c of Unit 4 of Block 1).

$$\therefore \quad \mathbf{L}_1 = m_1 r_1^2 \omega \hat{\mathbf{n}} = m_1 r_1^2 \boldsymbol{\omega}.$$

Similarly $\mathbf{L}_2 = m_2 r_2^2 \boldsymbol{\omega}$, $\mathbf{L}_3 = m_3 r_3^2 \boldsymbol{\omega}$ and so on.

So the angular momentum of the body is given by

$$\begin{aligned} \mathbf{L} &= \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \ldots\ldots. \\ &= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots.)\, \boldsymbol{\omega} \\ &= I\boldsymbol{\omega}, \end{aligned} \tag{9.2}$$

where $$I = \sum_i m_i r_i^2 \tag{9.3}$$

is called the **moment of inertia** of the body about the given axis of rotation. Here the summation extends over all the point masses that constitute the body. The SI unit of moment of inertia is kg m^2.

If the mass of the body be M then we can express I as $I = Mk^2$, (9.4)

where k is a quantity having the dimension of length. This quantity is called the *radius of* gyration. If we compare **Eq.** 9.4 with **Eq.** 4.21b of Unit 4, we find that k is equivalent to the distance from the axis of rotation of the point where the entire mass of the body can be considered to be concentrated. In other words it is the distance between the axis of rotation and the c.m. of the body.

Did you notice the similarity between the expression (9.2) and the same for linear momentum (i.e. $M\mathbf{v}$). Since $\boldsymbol{\omega}$ is analogous to v (see Table 1.1), I must be analogous to M. In other words, I is the rotational analogue of mass about which you have read in Sec. 4.3.3. This analogy also becomes evident from the expression of K.E. of rotation K which you may work out in the following SAQ.

As stated in Sec. 9.2.3 the general motion of a rigid body can be considered as a translational motion of *its* c.m. and a rotational motion about its c.m. Hence, the considerations of this unit apply also to rotations about an axis that is *not* fixed in an inertial frame, provided the axis passes through the c.m. and the moving axis always has the same direction in space.

In situations involving asymmetric objects, L and $\boldsymbol{\omega}$ may be in different directions. In that case I cannot be expressed as a single number but in a more complicated mathematical form called tensor.

SAQ 5

Show that for the body in Fig. 9.8 the K.E. of rotation is given by

$$K = \frac{1}{2} I \omega^2 . \tag{9.5}$$

Compare the expression for K with that of K.E. of linear motion and find the rotational analogue of mass.

So far in this section we have considered a case where the axis of rotation lies within the body. The above analysis also holds if the axis lies outside the body; e.g. the bob of a conical pendulum (Fig. 9.9).

Now that you have understood **the** meaning of **the** term 'moment of inertia' we may proceed **to study the method for** its determination.

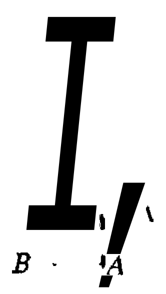


Fig. *9.9: B* **is** *the* **boh** of *a* **conical pendulum and OA, the axis** of **rotation.**

9.3.1 Determination of Moment of Inertia of a Rigid Body

We shall now put Eq. 9.3 to use. To start with let us try to determine the moment of inertia of a dumb-bell (Fig. 9.10a). We shall assume that the thin rod joining the masses m_1 and m_2 is of negligible mass. We shall also consider m_1 and m_2 as point masses. These assumptions may appear oversimplifying. But this model finds many applications in molecular spectroscopy as this can represent a diatomic molecule. Let us first work out the following example related to the determination of moment of inertia of the dumb-bell. Then we shall study an application of this model.

Example 1

Refer to Fig.9.10b. AB is perpendicular to the line joining the masses m_1 and m_2 and it passes through C, the c.m. Using the assumptions stated above show that the moment of inertia of the system described in Fig. 9.10b is μr^2, where μ is the reduced mass of the system and r is the distance between the masses.

For the given system the summation of Eq. 9.3 will have two terms. i.e.

$$I = m_1 r_1^2 + m_2 r_2^2.$$

Since C is the c.m. we have $m_1 r_1 = m_2 r_2$

$$\text{or} \quad \frac{r_1}{m_2} = \frac{r_2}{m_1} = \frac{r_1 + r_2}{m_2 + m_1} \quad \text{(by addendo)}$$

$$\therefore \quad r_1 = \frac{m_2 r}{m_1 + m_2}, \; r_2 = \frac{m_1 r}{m_1 + m_2} \quad (\because r = r_1 + r_2)$$

$$I = m_1 \left(\frac{m_2}{m_1 + m_2} r\right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} r\right)^2 = \frac{m_1 m_2}{m_1 + m_2} r^2.$$

Hence, using Eq. 7.6 we get,

$$I = \mu r^2.$$

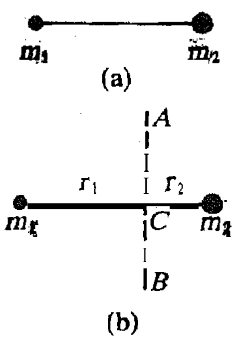


Fig. 9.10: (a) A dumb-bell having masses m_1 and m_2 at its ends; (b) the determination of moment of inertia of the dumb-bell in (a) about an axis passing through the c.m. of m_1 and m_2 and perpendicular to the line joining them.

You may now like to study an application of Example 2 by working out the following **SAQ.**

SAQ 6

The atoms in the oxygen molecule (O_2) may he considered to be point masses separated by a distance of 1.2Å. The molecular speed of an oxygen molecule at s.t.p. is 460m s^{-1}. It is known that the rotational K.E. of the molecule is 2/3 of its translational K.E. Calculate its angular velocity at s.t.p. assuming that molecular rotation takes place about an axis through the c.m. of, and perpendicular to the line joining the atoms.

We have just now applied Eq. 9.3 to determine the moment of inertia of a system made up of discrete particles. In each of the systems (dumb-bell and diatomic molecule) the total mass is distributed among particles which are not attached to one another, i.e. the particles that comprise the system can be enumerated. We shall now take up the case of systems where there is a continuous distribution of matter. Here the particles cannot be enumerated. For example, we have bodies like a uniform rod, a sphere, a cylinder and so on. For that we shall modify Eq. 9.3 in the following manner.

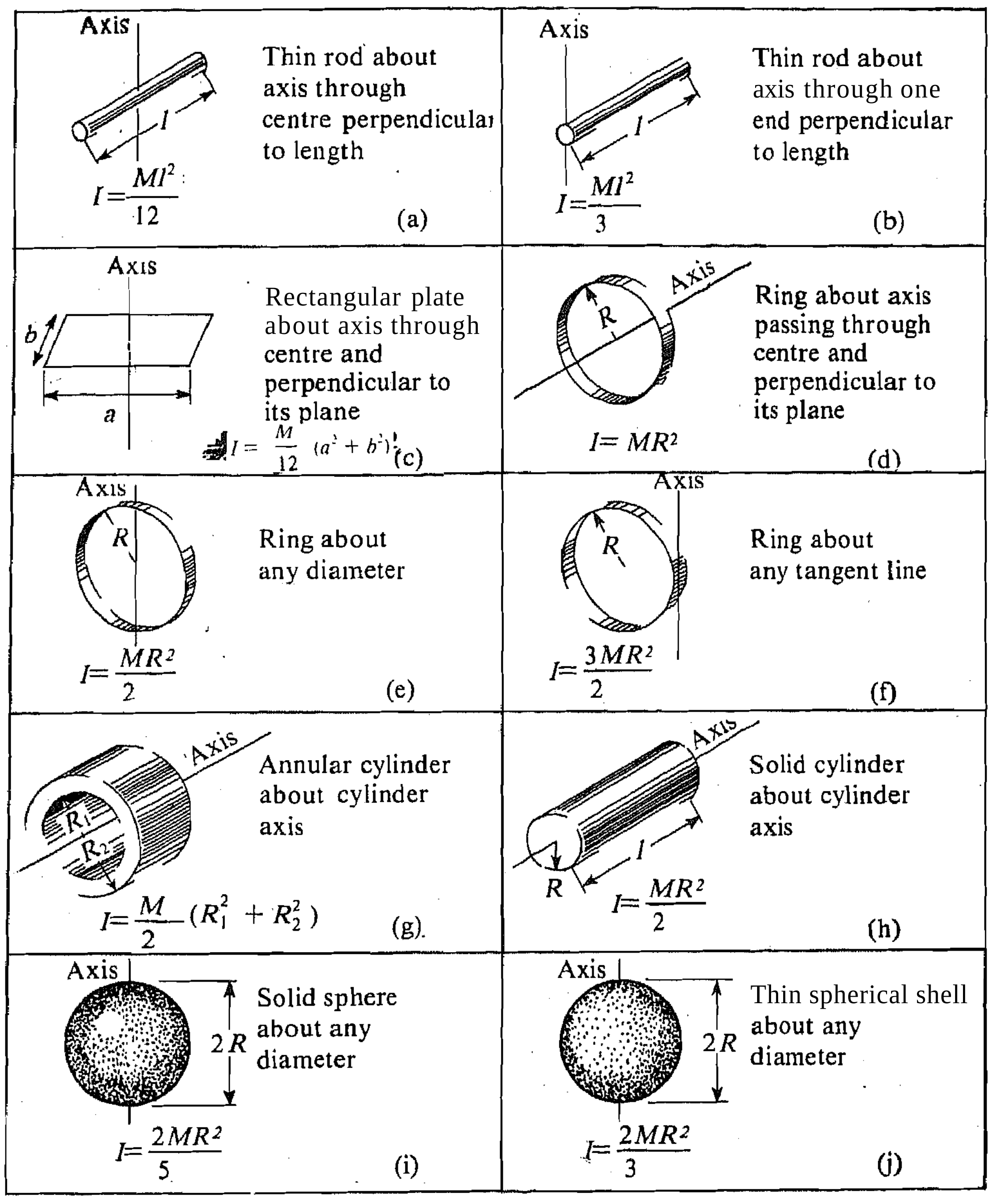
Let r be the perpendicular distance of an infinitesimal mass Δm of the body from the axis. Then from Eq. 9.3, we get

$$I = \lim_{\Delta m \to 0} \Sigma r^2 \, \Delta m = \int r^2 \, dm \qquad (9.6)$$

where Δm gets replaced by dm, the differential of mass and the sutnmation by integral. The integral is a definite one extending over the entire body. Using Eq. 9.6 the moments of inertia of symmetrical bodies about certain axes can be determined.

The moments of inertia about certain axes of a few common symmetrical bodies have been given in Table 9.1 (In all cases M represents the mass of the body in the diagram). We have derived these results in Appendix B.

Table 9.1



So you have understood the meaning of the term 'moment of inertia'. You have also come to know the value of moments of inertia of several bodies about certain axes. So we may proceed to study the dynamics of rotational motion.

9.4 ROTATIONAL DYNAMICS OF A RIGID BODY

You know that dynamics is the study of accelerated motion and its causes. For translational motion it is governed by Newton's second law, i.e.

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

The rotational analogue of Newton's second law of motion, as you know (see Eq. 4.24 of Unit 4, Block 1) is given by

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

where $\boldsymbol{\tau}$ is the torque acting on the particle and L its angular momentum and I the moment of inertia about the axis of rotation. You may recall that you have studied the dynamics of angular motion of a particle in Sec. 4.3 of Block 1. We shall now apply the concepts you have studied in Unit 4 of Block 1 mostly to a rigid body. You have studied the necessary principles and laws there. We shall now list them in the Table 9.2. Here we have shown the equivalent aspects of translational and rotational motion. A few spaces have been left **blank** which you may fill in.

Table 9.2

S.No.	Translational Motion	Rotational Motion
i)	Position, r	Angular position, θ
ii)	velocity, $\mathbf{v} = \frac{}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
iii)	Acceleration, $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$	Angular acceleration. $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
iv)	Mass, m	Moment of inertia, I
v)	Linear momentum, p = mv	Angular momentum. $L = I\omega$
vi)	Force, **F**	Torque, τ
vii)	Newton's second law $\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$	Analogue of second law $\tau = \frac{d\mathbf{L}}{dt}$ = ...
viii)	Work done $= \int \mathbf{F}.d\mathbf{r}$	Work done $= \int \tau.d\theta$
ix)	K.E. $= \frac{1}{2}mv^2$	K.E. = ...
x)	Principle of conservation of linear momentum: When the net external force acting on a body is zero the linear momentum of its c.m. remains constant.	....
xi)	Impulse $= \int_{t_1}^{t_2} \mathbf{F}(t)dt = \mathbf{p}(t_2) - \mathbf{p}(t_1)$	Angular Impulse = ...

SAQ 7

Fill in the blank spaces of Table 9.2.

Now that you have studied Table 9.2 and worked out SAQ 7, we can discuss some applications of the principles of rotational dynamics. We shall start with the rotational analogue of Newton's second law.

9.4.1 Rotational Analogue of Newton's Second Law

We have used the equation $F = \frac{d\mathbf{p}}{dt}$ to describe the dynamics of linear motion of a body. For a system having constant mass this equation becomes $\mathbf{F} = m\mathbf{a}$. To study the rotational dynamics of a body we first need to know its moment of inertia I about the axis of rotation. Then we shall use the rotational analogue of the above equation, i.e.

$$\tau = \frac{d\mathbf{L}}{dt}$$

Now, we know from Eq. 9.2 that, $L = I\omega$. $\therefore \tau = \frac{d}{dt}(I\omega)$.

For a system having constant I. we get

$$\tau = I\frac{d\omega}{dt} = I\alpha \qquad (9.7)$$

τ *in* Eq. 9.7 is the net torque acting on the body. So we must take care to determine all torques that act on the body and take their vector sum *to* obtain the net ***torque.***

We have studied about the linear motion of a many-particle system in Sec. 7.3. There we found that only external forces matter. The internal forces cancel in pairs according to Newton's third law. Now, let us see what happens in the case of internal torques. Refer to Fig. 9.11. It shows two particles 1 and 2 of a rigid body. The internal force on 1 due to 2 is $\mathbf{F}_{21}$ and that on 2 due to 1 is $\mathbf{F}_{12}$. Let us find out the total internal torque about a point O due to these forces. You may recall from Eq. 1.16 of Block 1 that this total internal torque is given by,

$$\boldsymbol{\tau}_{int} = \mathbf{r}_1 \times \mathbf{F}_{21} + \mathbf{r}_2 \times \mathbf{F}_{12}$$

Now, $\mathbf{r}_1 \times \mathbf{F}_{21} = r_1 F_{21} \sin(\pi - \theta_1)\, \hat{\mathbf{n}} = F_{21}\, r_1 \sin\theta_1\, \hat{\mathbf{n}}$,

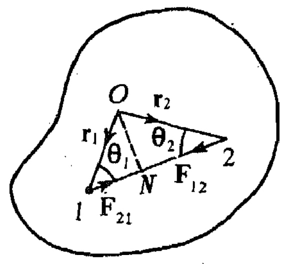


Fig. 9.11: Internal forces on two particles of a rigid body.

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane of this page and pointing towards you. And,

$$\mathbf{r}_2 \times \mathbf{F}_{12} = r_2 F_{12} \sin(\pi - \theta_2)\,(-\hat{\mathbf{n}}) = -F_{12}\, r_2 \sin\theta_2\, \hat{\mathbf{n}}$$

$$\therefore \quad \boldsymbol{\tau}_{int} = (F_{21}\, r_1 \sin\theta_1 - F_{12}\, r_2 \sin\theta_2)\, \hat{\mathbf{n}}. \qquad (9.8)$$

We know from Newton's third law that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal and opposite. So $F_{12} = F_{21}$. Again, we can see from Fig. 9.11 that,

$$r_1 \sin\theta_1 = r_2 \sin\theta_2 = ON,$$

where ON is the length of the perpendicular drawn from O on the line joining the points 1 and 2. Hence, from Eq. 9.8, we get

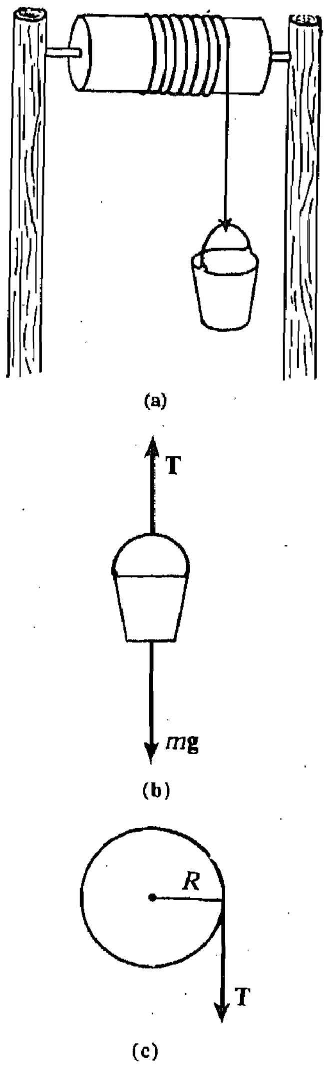
$$\boldsymbol{\tau}_{int} = 0$$

So we see that internal torques cancel in pairs. Thus, the torque in Eq. **9.7** is the net external torque.

Let us now work out an example to illustrate Eq. **9.7.** You will find that the situation is analogous to the case of accelerated linear motion as the applied torque and the angular velocity of the rotating body are in the same direction.

Example 2

A solid cylinder of mass M is mounted on a horizontal axle over a well (Fig. 9.12a). A rope is wrapped around the cylinder and a bucket of mass m is suspended from the rope. Find in terms of m, M and g an expression for the acceleration of the bucket as it falls down. Neglect the mass of the rope and any friction between the axle and the cylinder. Assume that **the** rope does not slip over the cylinder as it unwinds.



Pig. 9.12: Diagram for Example 2

If the bucket were not connected to the cylinder it would have accelerated downward at the rate g. But now there is an upward tension T on the bucket due to the rope. It reduces the net downward force on the bucket. It also exerts a torque on the cylinder, The magnitude of the downward force on the bucket (Fig. 9.12b) is given by

$$F = mg - T.$$

But $F = ma$, where a is the linear acceleration of the bucket.

$$\therefore \; ma = mg - T. \qquad (9.9)$$

If we take the end view of the cylinder (Fig 9.12c), we see that the rope exerts a torque of magnitude $\tau (= RT)$ on the cylinder. This gives rise to an angular acceleration α given by Eq. 9.7 as

$$\alpha = \frac{\tau}{I} = \frac{RT}{I}, \qquad (9.10)$$

where I is the moment of inertia of the cylinder about the axis.

Since the rope unwinds without slipping, a is related to α. Using Eq. 4.11a of Block 1 we get from Eq. 9.10 that,

$$a = \alpha R = \frac{R^2 T}{I}. \qquad (9.11)$$

Hence, from Eqs. 9.9 and 9.11 we get

$$ma = mg - \frac{I\,a}{R^2}$$

$$\therefore \left[m + \frac{I}{R^2}\right] a = mg$$

or
$$a = \frac{mg}{m + \frac{I}{R^2}} \qquad (9.12a)$$

We know from result (h) of Table 9.1 that for the cylinder $I = \frac{1}{2}MR^2$. So, we can rewrite Eq. 9.12a as

$$a = \frac{mg}{m + \frac{M}{2}} \qquad (9.12b)$$

Eq. 9.12b indicates that if $M << m$, then $a = g$. In other words if the mass of the cylinder is very small compared to that of the bucket then the rotation of the cylinder does not matter. The acceleration of the bucket is simply equal to g.

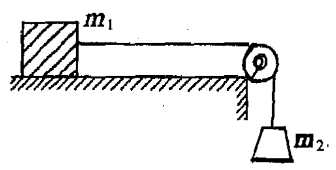


Fig. 9.13: The **falling mass m_2 can provide horizontal acceleration to** m_1.

However, in general we can say that the gravitational force on the bucket does not only provide its linear acceleration, it also gives rise to the angular acceleration of the cylinder. As a result, the linear acceleration of the bucket-decreases. **A** falling mass can provide horizontal acceleration to another mass (Fig. 9.13). From Example 2, we have just now seen that a falling mass can also generate angular acceleration in another body.

So we have learnt to apply the rotational analogue of Newton's second law of motion. This study throws some light on the concept of equilibrium of a body. You may recall that we have studied about equilibrium of forces in Sec 2.2.2. of Block 1. A body had been said to be in equilibrium if the vector sum of all the forces acting on it is zero. This is equivalent to saying that the linear acceleration of the c.m. of the body is zero. But we know that the general motion of a rigid body is a combination of the translational motion of the c.m. and a rotational motion about an axis passing through the c.m. So we can say that our study in **Sec. 2.2.2** of Block 1 was restricted to the case of translational equilibrium only. The general condition of equilibrium of a body must include therotational aspect too. We shall study briefly about this now.

Equilibrium of a Rigid Body

A rigid body is said to be in mechanical equilibrium if with respect to an inertial frame (i) the linear acceleration $\mathbf{a}_{cm}$ of its c.m. is zero and (ii) its angular acceleration α about any axis fixed in this frame is zero.

The above conditions do not imply that the body must be at rest with respect to the frame. It should only be unaccelerated. Its c.m., for example, may be moving with a constant velocity $\mathbf{v}_{cm}$ and the body may be rotating about a fixed axis with a constant angular velocity.

The translational motion of the body, as you know is governed by the equation

$$\mathbf{F}_e = M\,\mathbf{a}_{cm},$$

where $\mathbf{F}_e$ is the net external force acting on the body of mass M. So condition (i) may be expressed as follows : **The vector sum of** all **the external forces acting on the** body is **zero.** In other words, if a rigid body is in translational equilibrium under the action of several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and so on, we may write the above condition as

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots.. = \mathbf{0}. \qquad (9.13a)$$

The other condition is given by $\alpha = 0$ for any axis. We know that the angular acceleration of a rigid body is related to the net external torque τ as

$$\tau = I\alpha,$$

where I is the moment of inertia of the body about the axis of rotation. So condition (ii) may be expressed as follows : ***The vector*** sum **of** all ***the* external *torques acting on*** the ***body* is *zero.*** In other words if a rigid body is in rotational equilibrium under the action of several torques τ_1, τ_2, τ_3 and so on we may express this condition as

$$\tau_1 + \tau_2 + \tau_3 + ... = 0. \quad (9.13b)$$

Hence, a rigid body is said to be in mechanical equilibrium if both the conditions 9.13 a and b hold.

Let us take the example of a man standing an a ladder (Fig. 9.14). Suppose that the entire system is in equilibrium when the man is at the point M of the ladder AB. We shall first find out what are the forces acting on the system. The weight of the man acting vertically downwards through M is w. The weight **W** of the ladder acts vertically downward through its mid point G . $\mathbf{N}_1$ and $\mathbf{N}_2$ are the normal reactions at the points of contact A and B of the ladder with the vertical and horizontal surfaces, respectively. Since the point A has a tendency to slip towards O, the force of friction $\mathbf{F}_1$ at A acts along OA. Again B has a tendency to slip along OB. So the force of frictian $\mathbf{F}_2$ at B is along BO. So condition (9.13a) demands that

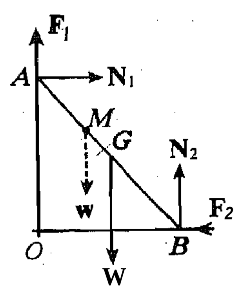


Fig. 9.14: A ladder in equilibrium

$$\mathbf{w} + \mathbf{W} + \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F}_1 + \mathbf{F}_2 = 0.$$

Now let us define the Cartesian x and y-axes along OB and OA, respectively. Then the above condition may be written as

$$-w\hat{\mathbf{j}} - W\hat{\mathbf{j}} + N_1\hat{\mathbf{i}} + N_2\hat{\mathbf{j}} + F_1\hat{\mathbf{j}} - F_2\hat{\mathbf{i}} = 0$$

$$\text{or} \quad (N_1 - F_2)\hat{\mathbf{i}} + (N_2 + F_1 - w - W)\hat{\mathbf{j}} = 0.$$

Hence, we get

$$N_1 - F_2 = 0, \; N_2 + F_1 - w - W = 0$$

$$\text{i.e.} \quad N_1 = F_2 \text{ and } N_2 + F_1 = w + W. \quad (9.14a)$$

Now, we shall take care of the condition (9.13b). For this we have to determine the total torque acting on the system about any point. The choice of this point is quite important. A proper choice helps us in getting the final condition in a simple form. Let us see how. If we select the point A, the torques of $\mathbf{F}_1$ and $\mathbf{N}_1$ vanish. Similarly for the point B, the torques of $\mathbf{F}_2$ and $\mathbf{N}_2$ vanish. So if we select any one of these two points, we may get rid of the expressions of torques of a pair of forces while writing the condition (9.13b). This considerably simplifies the final condition. However, the meaning of the condition is independent of the choice of the point about which the torques are being determined.

So, let us now write (9.13b) with reference to the point B. We have,

$$\mathbf{AB} \times \mathbf{F}_1 + \mathbf{AB} \times \mathbf{N}_1 + \mathbf{MB} \times \mathbf{w} + \mathbf{GB} \times \mathbf{W} = 0$$

Now, let $AB = 2l$, $BM = a$ and $LOBA = \theta$ (Fig. 9.14). So we get.

$$2lF_1 \sin(90° + 0)\hat{\mathbf{k}} + 2lN_1 \sin 0\,\hat{\mathbf{k}} - aw\sin(90° - 0)\hat{\mathbf{k}} - lW\sin(90° - 0)\hat{\mathbf{k}} = 0,$$

where $\hat{\mathbf{k}}$ is the unit vector perpendicular to the xy - plane and pointing towards you.

$$\text{or } 2lF_1 \cos\theta + 2lN_1 \sin\theta - aw\cos\theta - lW\cos\theta = 0,$$

$$\text{or } \cot\theta = \frac{2lN_1}{aw + lW - 2lF_1}. \quad (9.14b)$$

So for the equilibrium of the system (ladder and man) both the equations (9.14a and 9.14b) should hold good.

So far we have studied how to apply the rotational analogue of Newton's second law of motion, In Unit 3 of Block 1 yau have read about 'Work and Energy', as applied to linear motion. We shall now study about these quantities with reference to rotational motion of a rigid body.

9.4.2 Work and Energy in Rotational Motion

In general work done by a force F during linear motion is given by

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

where $d\mathbf{r}$ is an infinitesimal displacement. τ is the rotational analogue of F. The angular displacement θ is the analogue of **r.** So work done in rotational motion by a torque can be obtained by replacing F with τ and r with θ in the above expression of W. It is given by

$$W_{rot} = \int \tau \cdot d\theta \qquad (9.15a)$$

For a constant torque acting in the direction of the angular displacement, we get

$$W_{rot} = \tau \Delta\theta \qquad (9.15b)$$

where $\Delta\theta$ is the overall angular displacement.

Let us now apply Eq. 9.15b to a simple example.

Example 3

An automobile engine develops 72kW of power when rotating at a rate of 1800 r.p.m. What torque does it deliver?

Power is the rate of doing work. Now if the work W_{rot} $(= \tau\Delta\theta)$ is done in a time *At,* then the power will be given by

$$P = \frac{\tau \Delta\theta}{\Delta t}$$

where $\frac{\Delta\theta}{At} = \omega$ = the angular speed,

$$\text{or} \quad \tau = \frac{P}{\omega}$$

For this example, $P = 72 \times 10^3 \text{W} = 72 \times 10^3 \text{kg m}^2 \text{s}^{-3}$

and $\omega = 2\pi \times \frac{1800}{60} \text{rad s}^{-1} = 60\pi \text{ rad s}^{-1}$

$$\therefore \quad \tau = \frac{72 \times 10^3 \text{ kgm}^2 \text{s}^{-3}}{60\pi \text{ rad s}^{-1}} = 382 \text{ Nm.}$$

You may recall that here Nm is not equivalent to joule.

Let us now discuss the K.E. of rotation. We have derived the expression for the K.E. of a rotating body in Sec. 9.3. It is given by

$$K_{rot} = \frac{1}{2} I\omega^2, \qquad (9.16)$$

where I is the moment of inertia of the body about the axis of rotation and ω is its angular speed. We shall now apply **Eq.** 9.16 to discuss briefly about the motion of rolling objects.

Rolling Objects

A rolling object exhibits both rotational and translational motion. As the object moves forward, it rotates about a point that is itself moving along a straight line. How do we express the total K.E. of such a rolling object? The expression must contain both the translational and rotational K.E. So the total K.E. is given by

$$K = K_{trans} + K_{rot}$$

$$K_{trans} = \frac{1}{2} M v_{cm}^2 \quad , \quad K_{rot} = \frac{1}{2} I_{cm}\omega^2,$$

where M is the mass of the object. v_{cm} is the speed of the c.m., I_{cm} is the moment of inertia of the object about an axis passing through the c.m. and ω the angular speed.

Thus: $K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm}\omega^2$.

Now, if the object has a radius R and it is **rolling without** slipping, then $\omega = v_{cm}/R$. Hence for an object which is rolling without slipping,

$$K = \frac{1}{2}\left(M + \frac{I_{cm}}{R^2}\right) v_{cm}^2. \qquad (9.17)$$

Let us apply Eq. 9.17 to work out the following example.

Example 4

A solid cylinder and a solid sphere, each of the same mass M and radius R, start from rest and roll without slipping down an inclined plane (Fig. 9.15). Which one reaches the bottom of the incline first?

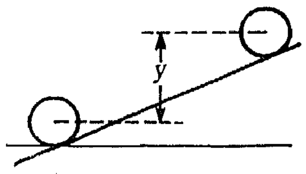


Fig. 9.15: Diagram for Example 4

Let the finishing line be at a vertical distance y below the starting line. The object whose c.m. finishes with greater speed reaches first. Using Eq. 9.17 and applying the principle of conservation of energy, we get

$$Mgy = \frac{1}{2}\left[M + \frac{I_{cm}}{R^2}\right] v_{cm}^2$$

$$\text{or} \quad v_{cm}^2 = \frac{2gy}{1 + \dfrac{I_{cm}}{MR^2}}. \qquad (9.18)$$

For the solid cylinder, $I_{cm} = \frac{1}{2} MR^2$ or $v_{cm}^2 = \frac{4}{3} gy$

and for the solid sphere $I_{cm} = \frac{2}{5} MR^2$ or $v_{cm}^2 = \frac{10}{7} gy$.

Since (10/7) > (4/3), we find that the sphere reaches first. You may like to work out an SAQ based on the above concept.

SAQ 8

A spherical ball rolls without slipping down a slope of vertical height 35 cm, and reaches the bottom moving at $2\,m\,s^{-1}$. Is the ball hollow or solid?

So far you have studied some applications of the principles of rotational dynamics. You may recall from Sec. 4.4.2 of Block 1 that the principle of conservation of angular momentum is used widely in physics. We have already studied some applications of this principle in Unit 4 of Block 1. The law of equal areas which you have read in Unit 6 is also an application of this principle. We shall now review the principle of conservation of angular momentum and study some other of its applications.

9.4.3 Conservation of Angular Momentum and its Applications

Now, you are quite familiar with the relation

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

You may recall that we have proved this result for a single particle right at the beginning of Sec. 4.4 of Block 1. For a many-particle system $\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i$ and $\mathbf{L} = \sum_i \mathbf{L}_i$, where $\boldsymbol{\tau}_i$ and $\mathbf{L}_i$ are the torque experienced and the angular momentum, respectively, of the ith particle. Now, we know that,

$$\boldsymbol{\tau}_i = \frac{d\mathbf{L}_i}{dt}$$

Again, $\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(\sum_i \mathbf{L}_i\right) = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \boldsymbol{\tau}_i =$ the sum of torques acting on the particles.

But we have seen in Sec. 9.4.1 that internal torques cancel in pairs. So the sum of the torques is equal to the net external torque.

$$\therefore \quad \frac{d\mathbf{L}}{dt} = \tau_e, \qquad (9.19)$$

where τ_e is the net external torque.

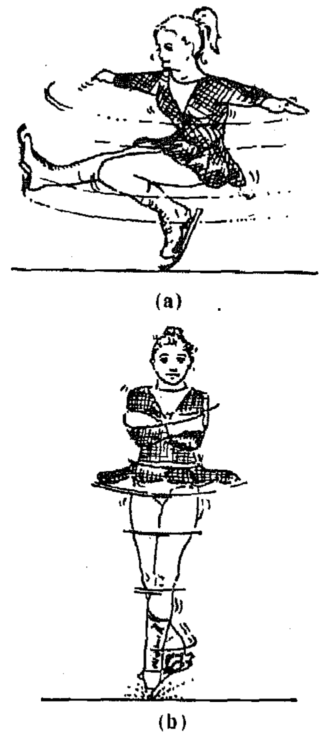


Fig. 9.16: Motion of a figure skater. (a) The I of the skater is large and ω is small (b) I is small and ω is large

When there is no external torque on a system, Eq. 9.19 tells us that $\frac{d\mathbf{L}}{dt} = 0$, or the angular momentum is constant. This is the principle of conservation of angular momentum. It implies that the angular momentum of an isolated system cannot change. We shall now study some applications of this principle.

Did you notice that while deriving Eq. 9.19, we did not require that the system in question be a rigid body? So conservation of angular momentum also applies to systems that undergo changes in configuration, and hence in moment of inertia. A common example is that of a figure skater, who starts spinning relatively slowly with her arms extended (Fig. 9.16a) and then pulls her arms in to spin much more rapidly (Fig. 9.16b). Let us find out why this happens. As her arms move in, her mass gets concentrated more towards the axis of rotation. In other words in the expression $\sum mr^2$ of I, r's become small. So I decreases. But the angular momentum $I\omega$ is conserved. Hence ω increases. The principle also applies to the case of the diver in Fig. 7.12. A schematic representation of Fig. 7.12 is shown in Fig. 9.17. At the positions A, E and F the value of I is high and so ω is low, whereas at the positions B, C and D, I is low And ω is high. So the diver utilises the principle of conservation of angular momentum to do somersaults in mid-air and enter the pool with head and hands down.

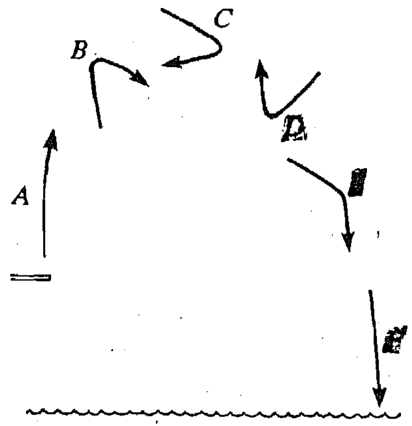


Fig. 9.17: Different stages of the motion of a diver

You may now like to work out an SAQ on the above concept.

SAQ 9

The earth is suddenly condensed so that its radius becomes half of its usual value without its mass being changed. How will the period of daily rotation change ?

We have studied the application of the principle of conservation of angular momentum. We know that the angular momentum vector changes when an external torque is applied to the system. The change in the angular momentum vector when the applied torque is perpendicular to the direction of the angular momentum presents an interesting situation. The resulting motionfs called 'precession' about which we shall study now.

9.4.4 precession

At some time you must have played with a top. You must have seen that the axis of rotation of a spinning top slowly rotates about the vertical. This means that the directiod of L of the top (which lies along its axis of rotation) changes. This must be due to a torque acting on the top. You can also observe this effect if you carry out the following activity.

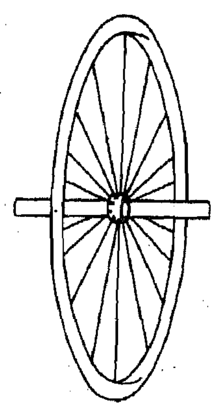


Fig. 9.18: **A** bicycle wheel

Activity

Turn a bicycle upside down and make it stand on its seat and handle. Rotate its front wheel. When the wheel is rotating reasonably fast, lift it upwards by applying force at the tip of the axle (Fig. 9.18). What happens if you do this?

In doing this activity you must have seen that when you applied the force, the wheel turned, i.e. its axis of rotation changed. Why does this happen? To understand this, study Fig. 9.19.

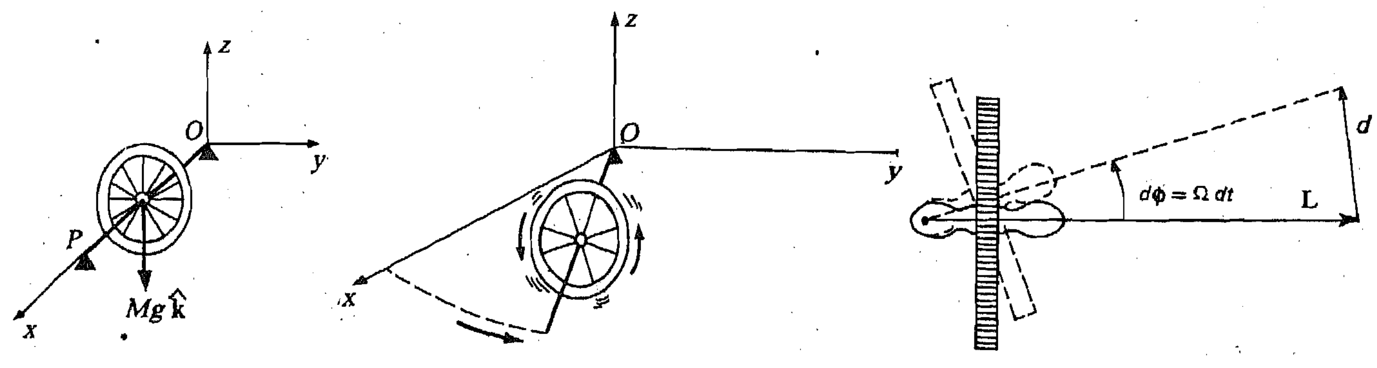


Fig. 9.19 : a) Axle of the wheel is supported at both ends; b) A rapidly spinning wheel does not fall on removing the support at P, but exhibits precession; c) top view of the precessing wheel.

Fig. 9.19 shows a free wheel with an axle. Initially the axle is supported at both end points *O* and P (Fig. 9.19a). If the support at P is removed, the torque due to force of gravity *m*g causes the wheel to fall. Now suppose you rotate this wheel anticlockwise and remove the support at P. What happens in this case? This time the wheel does not fall. Instead the axle remains 'almost horizontal and-begins to revolve about the z-axis (Fig. 9.19b). Why does this happen?

This happens because the torque due to gravity acts on the wheel and changes its angular momentum ($\because \boldsymbol{\tau} = d\mathbf{L}/dt$). Since L is along the axis of rotation, the axis of rotation also turns. We can calculate the angular velocity $\boldsymbol{\Omega}$ at which the axis of rotation moves using tlie relation $\boldsymbol{\tau} = d\mathbf{L}/dt$. Let the axis of rotation turn by an angle $d\phi$ during time interval dt, then

$$\Omega = \frac{d\phi}{dt}$$

Let the angular speed of the wheel (ω) be constant. Then since $L = I\omega$, the magnitude of L is constant and only its direction changes. From Fig. 9.19c we have

$$d\phi = \frac{dL}{L}, \quad \Omega = \frac{d\phi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{\tau}{L}. \qquad (9.20)$$

The direction in which the axis of rotation turns will be along $d\mathbf{L}$, i.e. along the torque's direction. Now if r be the distance of the point of support to the centre of the wheel then'

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = (r\,\hat{\mathbf{i}}) \times (-Mg\,\hat{\mathbf{k}}) = rMg\,(\hat{\mathbf{k}} \times \hat{\mathbf{i}}) = rMg\,\hat{\mathbf{j}}\,.$$

Substituting $L = I\omega$ and $\tau = rMg$ in Eq. 9.20 we get

$$\Omega = \frac{rMg}{I\omega}. \qquad (9.21)$$

Eq. 9.21 indicates that Ω increases as ω decreases. As rotational energy is lost due to friction, ω will decrease and the wheel's axis of rotation will change faster.

Such a motion in which the axis of rotation changes is called *precession*. $\boldsymbol{\Omega}$ is termed as the **angular velocity of precession,** i.e. the velocity at which the axis of rotation precesses.

SAQ 10

Perform the activity suggested in this section once again In the light of what we have discussed in this section attempt the following question giving reasons for your answers.

a) In which direction will the wheel turn when you apply an upward force at P, if as seen from P, it were rotating (i) clockwise and (ii) anticlockwise?

b) If you applied upward forces at both P and Q, would the wheel's axis of rotation change?

Let us now summarise what we have studied in this unit.

9.5 SUMMARY

- A rigid body is one in which the relative separation between any two of its constituent particles always remains constant.
- A rigid body is said to execute pure translational motion if each particle in it undergoes the same displacement as every other particle in any given interval of time.

  A rigid body is said to execute rotational motion if each particle in it moves in a circle, the centres of which lie on a straight line called the axis of rotation.
- The general motion of a rigid body is a combined effect of the translation of its c.m. and a rotation about an axis passing through the c.m.
- The rotational analogue of mass is moment of inertia. It measures the resistance of a body to changes in rotational motion. It depends on the mass of a body and on the distribution of mass about the axis of rotation. It is given by

  $$I = \sum_i m_i r_i^2$$

  for a body consisting of discrete masses, and by

  $$I = \int r^2 dm$$

  for a continuous distribution of matter.
- Torque is the rotational analogue,of force. Torque, moment of inertia and angular acceleration are related by the rotational analogue of Newton's second law

  $$\tau = I\alpha$$

- A rigid body is said to be in mechanical equilibrium if

  $$\Sigma F = 0, \Sigma\tau = 0$$

- The work done during a rotational motion by a torque is given by

  $$W_{rot} = \int \tau \cdot d\theta$$

- The expression for K.E.of rotation is similar to that of K.E. for linear motion with mass replaced by I and linear speed by angular speed. It is given by

  $$K_{rot} = \frac{1}{2} I\omega^2$$

- The total K.E. of a rolling object may be written as the sum of the translational K.E. of its c.m. and its rotational K.E. about an axis through its c.m.
- The expression for angular momentum of a rigid rotating object is given by

  $$\mathbf{L} = I\boldsymbol{\omega}$$

- The rotational analogue of Newton's second law may be written in terms of angular momentum as

  $$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

- In the absence of external torques, the angular momentum of a system is conserved.
- When a torque is applied perpendicular to the angular momentum vector, then the axis of rotation exhibits a precessional motion.

9.6 TERMINAL QUESTIONS

1. a) Explain with reasons whether the mass of a body can be considered as concentrated at its c.m. for the purpose of computing its moment of inertia ?

   b) Two circular discs of the same mass and thickness are made from metals having different densities. Which disc will have the larger moment of inertia about its central axis ?

c) Comment on the following statement : "The melting of polar icecaps is a possible cause of the variation in the time period of rotation of earth."

2. Refer to Fig. 9.20. It shows a satellite of mass 960 kg. Assume that it is in the form of a solid cylinder of 1.6m diameter and that the total mass is uniformly distributed throughout its volume. Now, suppose that the satellite is spinning at 10 r.p.m. about its axis and it has to be stopped so that a space shuttle crew can make necessary repairs. Two small gas jets are mounted diametrically opposite on the satellite as shown in Fig. 9.20. The jets aim tangentially to the surface of the satellite and each of them produces a thrust of 20N. How long must the jets be fired in order to stop the rotation of the satellite?

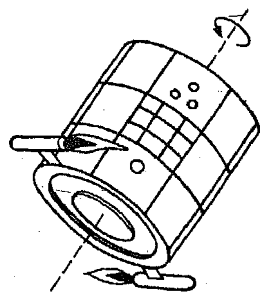


Fig. 9.20 : A spinning satellite

3. The rotational energy of the earth is decreasing steadily because of tidal friction. Estimate the change in the rotational energy of the earth in a day. It is given that the rotational period of the earth decreases by about 10 microseconds in a year. Assume the earth to be a solid sphere.

9.7 ANSWERS

SAQs

1. (a), (c), (e).

2. a) Each has a magnitude of 4.2 cm.

   b) i) A stone falling freely under gravity.

   ii) The motion of a block on a table when it is given a push.

3. The required differential equation (see Eq. 7.22) would be $M\mathbf{R} = \mathbf{F}_e$, where R is the position vector of the c.m. of the body and R is its acceleration.

4. In Fig. 9.3 the x', y', z' - axes are always parallel to the x, y, z - axes, whereas in Fig. 9.7 the former continually changes its orientation with respect to the latter. In case of Fig 9.3 the location of the body can be obtained only by locating O', the c.m. of the body while in Fig. 9.7 one has to know in addition the orientation of x', y', z' - axes with respect of the x, y, z - axes.

5. From Sec. 4.3.4 of Block 1 we may say that the K.E. of rotation K_1 of the point mass m_1 is given by

$$K_1 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly the K.E.s of m_2 and m_3 are $K_2 = \frac{1}{2} m_2 r_2^2 \omega^2$,

$K_3 = \frac{1}{2} m_3 r_3^2 \omega^2$. So the K.E. of rotation of the body is given by

$$K = K_1 + K_2 + K_3 + \ldots\ldots$$

$$= \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots)\omega^2 = \frac{1}{2} I\omega^2.$$

The expression for the K.E. of linear motion is $\frac{1}{2}Mv^2$ and since ω is analogous to v, I must be the rotational analogue of M.

6. Let the mass in kg of each atom be m. Then from Eq. 7.5 we get $\mu = m/2$. Here $r = 1.2 \times 10^{-10}$ m. If the required angular speed be ω, then from Eq. 9.2 and Example 1, the rotational K.E. is given by

$$E_R = \frac{1}{2}\left[\frac{m}{2}\,\text{kg}\right](1.2 \times 10^{-10}\,\text{m})^2 \omega^2.$$

The translational K.E. is giv'en by

$$E_T = 2 \times \frac{1}{2} mv^2 = mv^2, \text{ where } v = 460\ \text{ms}^{-1}.$$

It is given that $E_R = \frac{2}{3} E_T$

$$m\,(0.36 \times 10^{-20}\ \text{kgm}^2)\,\omega^2 = \frac{2}{3} m\,(460)^2\ \text{kgm}^2\,\text{s}^{-2},$$

or $\omega = 6.3 \times 10^{12}$ rad s^{-1}

7. vii) $\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = I\alpha$

ix) K.E. $= \frac{1}{2} I\omega^2$

x) Principle of conservation of angular momentum : When the net torque acting on a body is zero, its angular momentum remains conserved.

xi) Angular impulse $= \int_{t_1}^{t_2} \boldsymbol{\tau}(t)\,dt = \mathbf{L}(t_2) - \mathbf{L}(t_1)$.

8. For (a) a hollow ball. $I_{cm} = \frac{2}{3} MR^2$,

and (b) a solid ball, $I_{cm} = \frac{2}{5} MR^2$.

Now, from Eq. 9.18, we get for (a), $(v_{cm}^2)_a = \frac{6}{5} gy$

and for (b) $(v_{cm}^2)_b = \frac{10}{7} gy$.

For our problem y = 0.35 m and we put $g = 9.8\ \text{m s}^{-2}$

So $(v_{cm}^2)_a = 4.1\ \text{m}^2\,\text{s}^{-2}$, $(v_{cm}^2)_b = 4.9\ \text{m}^2\,\text{s}^{-2}$. The observed value of $v_{cm}^2 = 4\text{m}^2\text{s}^{-2}$ which .agrees more closely with (a). Hence the ball is liollow.

9. From the principle of conservation of angular momentum, we get,

$$I_1\omega_1 = I_2\omega_2$$

Here $I_1 = \frac{2}{5} MR_1^2$, $I_2 = \frac{2}{5} MR_2^2$ and $R_2 = \frac{R_1}{2}$

$$\therefore \quad \frac{2}{5} MR_1^2\,\omega_1 = \frac{2}{5} M\,\frac{R_1^2}{4}\,\omega_2$$

or $\omega_2 = 4\omega_1$

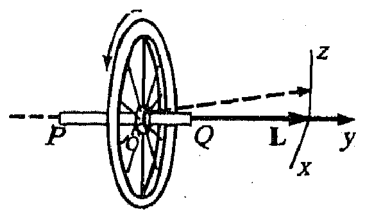
But $\omega_1 = \frac{2\pi}{T_1}$ and $\omega_2 = \frac{2\pi}{T_2}$ where T_1 and T_2 are the usual and changed time periods of daily rotation of earth.

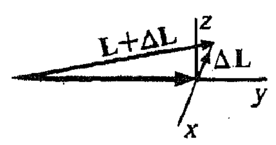
$$\therefore \quad T_2 = \frac{T_1}{4} = \frac{24}{4}\ \text{h} = 6\text{h}.$$

So the time period of daily rotation will become 6h.

10. a) (i) Refer to Fig. 9.21a. The direction of **L** is along the positive direction of y-axis. **A** vertically upward (i.e. along the positive direction of z-axis) force **F** is applied at P. The resulting torque (r x F) about O is along the negative direction of x-axis. So the change **AL** in the angular momentum vector is along that direction (Fig. 9.21b). Accordingly the new direction will be along **L + AL.** So the wheel will swerve so that the axle moves in the xy-plane in the sense $+x$ to $+y$ axis.

(ii) Following similar argument as in (i), we can draw the angular momentum vector **L**, its change **AL,** and the resulting vector L + **AL** as shown in Fig. 9.21c. So the wheel will again swerve in the xy-plane in the sense $+x$ to $-y$ axis.





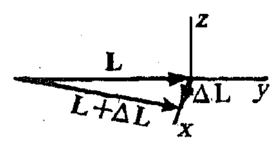


Fig. 9.21: (a) If a rotating bicycle wheel is lifted vertically, it swerves to the side; (b) *the* change in angular momentum vector for (i); (c) the change in angular momentum vector for (ii).

b) If upward forces are applied at both points P and Q, then the torques due to them about O will be equal and opposite. So the resulting torque is zero. Hence there would be no change in L. So the axis of rotation of the wheel will not turn.

Terminal Questions

1. a) $I = \Sigma m_i r_i^2$ and r_i is not same for all i. So the mass of a body cannot be considered **as** concentrated at its c.m. for the purpose of computing its moment of inertia.

b) A disc of thickness t, radius R and mass M is essentially a right circular cylinder of the same radius and of length t.

$$\therefore \qquad I = \frac{1}{2} MR^2 .$$

But $M = \pi R^2 t \rho$, where ρ = the density of the metal of which the disc is made.

$$I = \frac{\pi\rho t R^4}{2} = \frac{\pi\rho t}{2}\left[\frac{M}{\pi\rho t}\right]^2 = \frac{M^2}{2\pi\rho t}.$$

So we see that for same mass and thickness, I is inversely proportional to ρ. Hence the disc made of the metal having lower density will have larger moment of inertia.

c) When the polar icecap melts the water flows towards the equator. This leads to a redistribution of matter over the globe as a result of which I for the earth changes. But as the angular momentum of the earth remains constant its angular speed changes. But, $\omega = 2\pi/T$, where T is the time period of rotation. So T also changes.

2. The satellite's angular speed has to change by $\Delta\omega$ = 10r.p.m. If the angular acceleration a is constant then the time taken for the change is given by

$$\Delta t = \frac{\Delta\omega}{\alpha} = \frac{\Delta\omega I}{\tau} \qquad (\because I\alpha = \tau)$$

Since, the satellite is cylindrical, $I = \frac{1}{2}MR^2$, where M is the mass of the satellite and R its radius. The torque is exerted by two jets, each at a distance R from the rotational axis and directed perpendicular to the radius (Fig: 9.22). If F is the thrust of each jet we get, $\tau = 2RF$.

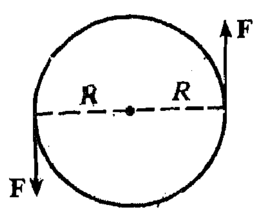


Fig. 9.22

$$\therefore \qquad \Delta t = \frac{\Delta\omega\left(\frac{1}{2}MR^2\right)}{2RF} = \frac{\Delta\omega MR}{4F}$$

$$\Delta t = \left[\frac{10}{60}\times 2\pi \text{ rads}^{-1}\right]\times\frac{(960\text{ kg})\times(0.8\text{ m})}{4\times 20N} = 10\text{s}.$$

3. The moment of inertia of the earth about its axis of rotation is given by

$$I = \frac{2}{5}MR^2, \text{ where } M = 5.97\times10^{24}\text{kg}, R = 6.37\times10^6\text{m}.$$

$$\therefore \qquad I = 9.7\times10^{37}\text{ kgm}^2.$$

The daily rotational period of earth is T = 24h = 86400s. Now the rotational **K.E.** is given by

$$E = \frac{1}{2}I\omega^2 = \frac{2\pi^2 I}{T^2} \qquad (\because \omega = \frac{2\pi}{T})$$

Now the relative changes in E and T are small in comparison to E and T themselves. So we can treat the changes as differentials dE and dT. We have,

$$dE = 2\pi^2 I\,(-2T^{-3}dT) = -\frac{4\pi^2 I dT}{T^3}$$

The change in T in one year ($\simeq$ 365 days) is 10×10^{-6} s, i.e. 10^{-5} s

$\therefore$ The change in a day is $dT = \frac{10^{-5}\,s}{365} = 2.7 \times 10^{-8}$ s.

Hence, the change in rotational K.E. will be

$$dE = -\frac{4\pi^2 \times (9.7 \times 10^{37}\,kg\,m^2) \times (2.7 \times 10^{-8}\,s)}{(86400\,s)^3}$$

$$= -1.6 \times 10^{17}\,kg\,m^2\,s^{-2}$$

So the rotational energy decreases by 1.6×10^{17} J per day.

UNIT 10 MOTION IN NON-INERTIAL FRAMES OF REFERENCE

Structure

10.1 INTRODUCTION

In the previous unit you have read about rigid body dynamics. The present unit will be the final one of our Elementary Mechanics course. We had introduced the concept of frame of reference in the very first unit of Block 1. In Unit 2 of Block 1 we introduced the idea of inertial and non-inertial observers. So far we have explained motion from the point of view of inertial observers. But as a matter of fact we live on a frame of reference (the earth) which is non-inertial. Moreover, we shall see that certain problems can be answered quite elegantly if we take the point of view of a non-inertial observer. So in this unit we shall study the description of motion relative to a non-inertial frame of reference. First we shall study what is meant by a non-inertial frame of reference.

You must have had the following experiences while travelling in a bus. You fall backward when the bus suddenly accelerates and forward when it decelerates. When the bus takes a turn you have sensation of an outward force. We shall explain these features by introducing the concept of inertial forces. Thereby we shall see how Newton's second law of motion gets 'modified in a non-inertial frame. This will be used to develop the concept of weightlessness.

Frames attached with rotating bodies like a merry-go-round, the earth and so on form the most interesting examples of non-inertial frames of reference. We shall derive the equation of motion of a body in such a frame of reference. Thereby we shall come across two inertial forces, namely, the centrifugal force and the Coriolis force. The former can be used to explain the action of a centrifuge. We will study a variety of applications of these forces in connection with the earth as a non-inertial frame of reference. Centrifugal force finds application in studying the variation of g with the latitude of a place.

Several natural phenomena like erosion of the banks of rivers, cyclones etc. can be explained using the concept of Coriolis force. Finally we shall study about Foucault's Pendulum experiment with a view to establishing the fact that the earth rotates about an 'axis passing through the poles.