UNIT 8 SCATTERING

Structure

8.1 INTRODUCTION

In Unit 7 you have learnt to apply the concepts of mechanics to many-particle systems. You are familiar with the phenomenon of collisions, which you have studied in Unit 3. It is also called scattering. In this unit we intend to study scattering in more detail. As you know, it involves two or more particles interacting with each other for a brief time. Collisions of particles or scattering of particles is an important feature of our physical universe. On a larger scale, we wonder if the earth's collision with an asteroid led to the extinction of dinosaurs. Galaxies also collide with each other giving rise to new formations. Much of our knowledge of atomic and nuclear structure and elementary particles comes from scattering experiments. These microscopic bodies are bombarded with microscopic particles and the number of particles scattered in various directions is measured. The angular distribution of scattered particles is expressed in terms of scattering cross-sections.

In this unit we shall begin our discussion with scattering cross-sections. The cross-sections are ***calculated*** in the ***centre-of-mass*** frame of reference but ***experimentally determined*** in the laboratory frame of reference. So you will study these two frames of reference and determine the relationship of the relevant physical quantities as observed from each of them. The impact parameter method makes the study of many a scattering phenomenon fairly easy. So you will learn this method and study two of its applications, namely, scattering of two hard spheres and Rutherford scattering. Rutherford scattering is one of the most dramatic scattering experiments. Performed in 1911 by Geiger and Marsden it led to the nuclear model of the atom, In Unit 9 you will learn to apply the concepts of mechanics to the rotational motion of rigid bodies.

Objectives

After studying this unit you should be able to

- distinguish between the c.m. and laboratory frames of reference
- compute differential and total scattering cross-sections in c.m. and laboratory frames of reference
- apply the impact parameter method to solve problems based on elastic scattering of two hard spheres and Rutherford scattering.

8.2 SCATTERING CROSS-SECTIONS

You already know what a collision or scattering of particles is. Recall the collision of two particles with which you are familiar (see Sec. 3.4). We can identify three distinct stages in the entire scattering process. We show these three stages of the collision process in Fig. 8.1. The first stage shown in Fig. 8.1a, corresponds to a time long before the interaction of the colliding particles. At this stage each particle is effectively free, i.e. its energy is positive. As the particles approach each other (Fig. 8.1b), interaction forces much larger than any other force acting on them come into play. Finally, long after the interaction (Fig. 8.1c), the emerging particles are again free and move along straight lines with new velocities in new directions. The emerging particles may or may not be the same as the original particles.

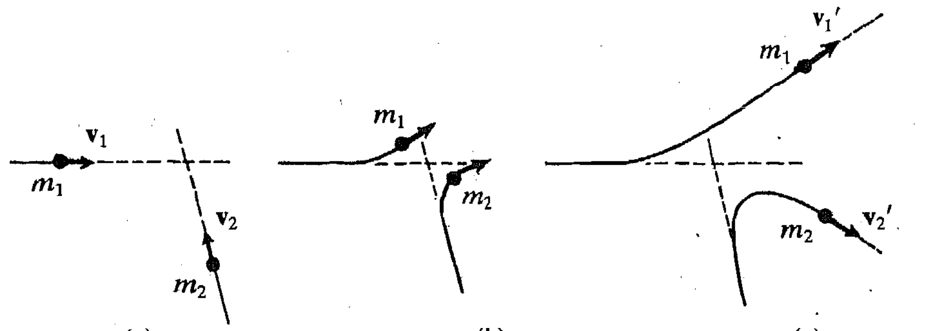


Fig. 8.1: Scattering of two particles

In a typical scattering experiment, a parallel beam of particles, also called projectiles, of given energy and momentum is incident upon a target (Fig. 8.2). The particles interact with the target for a short time, which deflects or scatters them in various directions. Evehtually, these particles are detected at large distances from the target. The scattered particles may or may not have the same energies and momenta.

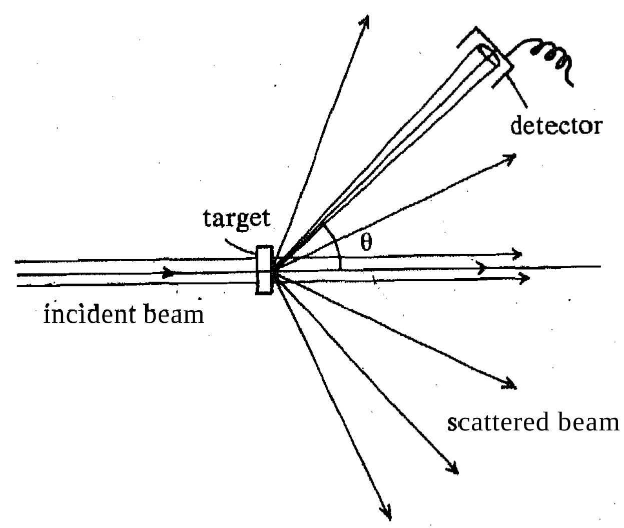
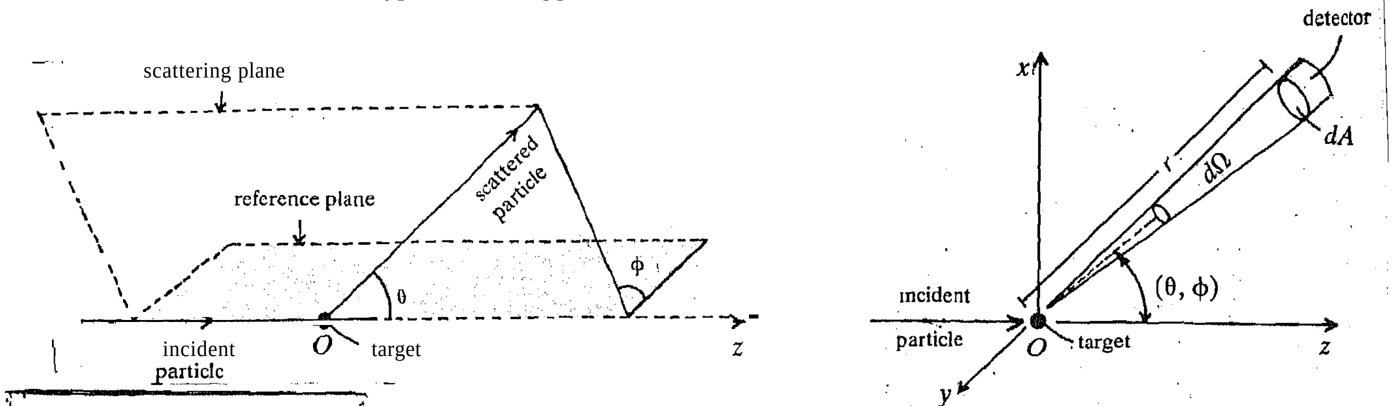


Fig. 8.2: A typical scattering process

An experimenter may be interested in knowing the velocities, linear momenta and energies of the particles before and after scattering. Then the changes brought about in these quantities can be determined. As you have studied in Unit 3, the principles of conservation of linear momentum and total energy allow us to determine these parameters. For example, James Chadwick discovered the neutron by making use of similar information about scattering of these unknown particles. When a beam of these particles was bombarded on the hydrogenous material paraffin, the protons had maximum recoil velocity of $3.3 \times 10^7 \text{m s}^{-1}$. When these were bombarded on the nitrogenous material para-cyanogen, the maximum recoil velocity of nitrogen nuclei was $4.7 \times 10^6 \text{ m s}^{-1}$. Using the methods you have studied in Unit 3, the mass of these particles was calculated and it was found to be a totally different and new particle, the neutron.

There is another aspect of interest in scattering. We may want to know how likely a particle's motion in a given direction is, after its interaction with the target. In other words, we may want to know the probability of scattering in a given direction. This is important because it gives us information about the nature of force between the projectiles and the target, and also their internal structures. For example, the size of the electron was determined by electron-electron scattering experiments. Similarly, electron-atom scattering experiments give us information about the internal structure of the target atom, i.e. their energy levels, configurations etc. The probability of scattering in a given direction is found by determining the scattering cross-sections. Let us now define the scattering cross-sections for a typical scattering process.



(a) (b)

Fig. 8.3: (a) Schematic diagram of a scattering event showing the angles (θ,ϕ); (b) $d\Omega$ is the solid angle about the angle (θ,ϕ). The particles scattered into the solid angle $d\Omega$ are received by the detector. dA is the cross-sectional area of the detector.

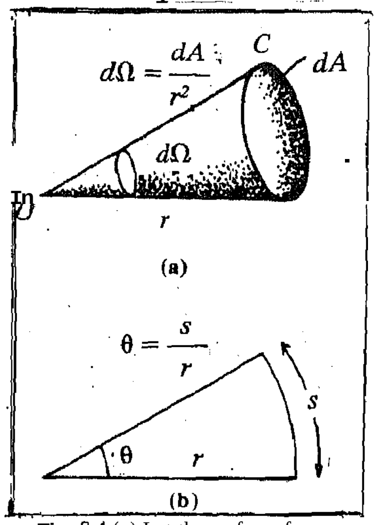


Fig. 8.4: (a) Let the surface of area dA be bounded by a closed curve C shown in the figure. The lines from the point O to the points of C generate a cone. Now visualise the area of a unit sphere about O (or the area of a sphere of radius r about O, divided by r^2). This area intercepted by the cone is called the solid angle subtended at O by the portion dA of the sphere's surface, enclosed by C. You can think of the solid angle as the space enclosed by a cone. The measure of a solid angle is defined as the ratio of the subtended area (dA) to the radius (r) squared, i.e. $d\Omega = \frac{dA}{r^2}$. Its unit is called steradian (sr). You can see that it plays the same role for a sphere as the angle (in radians) for a circle; (b) as you know an angle in the plane is the space between two intersecting lines. The measure of an angle, in radians, is defined as the ratio of the subtended arc length to the radius, i.e. $\theta = \frac{s}{r}$.

Let us suppose that a uniform parallel beam of n particles, all of the same mass and energy, is incident upon a target containing N number of identical particles or scattering centres. Such scattering centres might, for example, be the positive nuclei of atoms in a thin metal foil which could be bombarded by α particles. Let us assume that the particles in the beam do not interact with each other and the scattering centres in the target are sufficiently far apart. With these assumptions we can regard the incident particles and target particles to be sufficiently far apart. Then we can think of this scattering event as if at a given time only one projectile was being scattered by one target particle, without being affected by the presence of other particles. So, effectively at any instant we deal with a two-body collision process. For convenience we choose the origin of the coordinate system at the position of the target and one of the axes, say z-axis, in the direction of the incident beam.

The direction of scattering is given by the angles (θ,ϕ) as shown in Fig. 8.3a. The angle 8, called the ***angle of scattering,*** is the angle between the scattered and the incident directions. These two directions define the plane of scattering. The angle ϕ specifies the orientation of this plane with respect to some reference plane containing the z-axis. The shaded plane in Fig. 8.3a is a reference plane. The probability of the scattering of a particle in a given direction (θ,ϕ) is measured in terms of the differential cross-section. So let us understand what it is.

8.2.1 Differential Cross-Section

Let F be the number of projectiles incident per unit area per unit time on the target. F represents the incident flux. Let ***An*** be the number of particles scattered into a small solid angle $d\Omega$ about the angle (θ,ϕ) in time Δt(Fig. 8.3b). Study Fig. 8.4a and read its caption carefully to understand what a solid angle is. Then the number of scattered particles by a single target particle in time ***At,*** must be proportional to the incident flux F, the duration Δt and also the solid angle in which they are scattered, i.e.

$$An \propto F\,(d\Omega)(\Delta t). \qquad (8.1a)$$

The, constant of proportionality is defined as the differential **scattering cross-section** and is denoted by the symbol $\frac{d\sigma}{d\Omega}$. We also write it as dcs, in short. So

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right) F \left(d\Omega\right) \left(\Delta t\right), \qquad \text{or} \qquad \frac{d\sigma}{d\Omega} = \frac{\Delta n}{F \Delta t \; d\Omega} \tag{8.1b}$$

Thus, we can also express the differential cross-section as the following ratio :

$$\frac{d\sigma}{d\Omega} = \frac{\text{The number of particles scattered per unit time in a solid angle } d\Omega \text{ in the direction } (\theta,\phi)}{\text{Incident flux, i.e. the number of particles incident on the target per unit area per unit time.}}$$

You can see that defined as a ratio like this, the differential cross-section (dcs) gives a probability. In fact, it is a measure of the probability that an incident particle will be scattered in solid angle $d\Omega$ in the direction (θ,ϕ). You can also see that $\frac{d\sigma}{d\Omega}$ has the dimension of area. This explains the use of the term 'cross-section'. Therefore, it can also be thought of as the 'effective' area offered by the scatterer to the incident particle. More precisely, $\frac{d\sigma}{d\Omega}$ is equal to the cross-sectional area of the incident beam that contains the number of particles scattered into the solid angle $d\Omega$ by a single target particle. The unit of dcs is $m^2 \, sr^{-1}$. The dcs depends only on the parameters of the incident particle, nature of the target and the nature of the interaction between the two.

Cross-section literally means the surface formed by cutting through something, especially at right angles. Areas, as you know, are associated with surfaces.

So far we have discussed the scattering of particles from a single scattering centre in the target. For the N scattering centres the number of particles scattered will be just N times the number scattered by a single scattering centre. Thus for N scattering centres, the number of particles scattered is

$$\Delta n' = \frac{d\sigma}{d\Omega} NF \, d\Omega \, \Delta t. \tag{8.1c}$$

Of course, Eq. 8.1c is valid only when the target scattering centres are far enough apart so that the same particle is not scattered by two of them. Having defined the differential cross-section we will introduce you to the total scattering cross-section.

8.2.2 Total Cross-Section

Let us place the detector at all possible values of (θ,ϕ) and count the total number of scattered particles entering all the corresponding solid angles. Then we will get the total **scattering cross-section** (tcs, in short). It is denoted by σ. It can also be calculated from the differential scattering cross-sections by integrating over all possible values of $d\Omega$. Thus,

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega. \tag{8.2a}$$

So the tcs represents the number of particles scattered in all directions per unit flux of incident particles. It has the dimension of area. So its unit is m^2. Now, we also define the solid angle subtended by an area to be $d\Omega = \sin\theta \, d\theta \, d\phi$, where the limits of θ and ϕ are 0 to π and 0 to 2π, respectively. You will learn about these relations in the course on Mathematical Methods in Physics-I.. If you wish to understand their proofs now, you may read the last book given in the references. Using these relations we get,

$$\sigma = \int_0^{\pi} \int_0^{2\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta \, d\theta \, d\phi, \tag{8.2b}$$

We can show that for the cases in which the force is central and its magnitude depends only on r, $\frac{d\sigma}{d\Omega}$ is independent of ϕ. We will not prove this result here. In such cases, we can integrate over ϕ so that

$$\sigma = 2\pi \int_0^{\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta \, d\theta \tag{8.2c}$$

In the discussion that follows, we shall limit ourselves to the cases in which $\frac{d\sigma}{d\Omega}$ does not depend on ϕ. i.e. it is the same for all values of ϕ. We will now work out an example based on these concepts. Then you may like to work out an SAQ to concretise the concepts you have just studied.

Example 1

A beam of α-particles with a flux of $3 \times 10^8\ m^{-2}\ s^{-1}$ strikes a thin foil of aluminium, which contains 10^{21} atoms. A detector of cross-sectional area 400 mm^2 is placed 0.6 m from the target in a direction at right angles to the direction of the incident beam. If the rate of detection of a-particles is $8.1 \times 10^3\ s^{-1}$, compute the dcs.

Here we shall use Eq. 8.1c to compute the dcs. It is given that the flux $F = 3\times10^8 m^{-2}s^{-1}$, $\theta = 90°$, the rate of detection of a-particles is $\frac{\Delta n}{\Delta t} = 8.1 \times 10^3\ s^{-1}$ and the number of target atoms, $N = 10^{21}$. From Eq. 8.1 c

$$\frac{d\sigma}{d\Omega} = \left(\frac{\Delta n}{\Delta t}\right)\left(\frac{1}{NF}\right)\left(\frac{1}{d\Omega}\right)$$

In this case, $d\Omega$ is the solid angle subtended by the detector at the target for $\theta = 90°$. You know from Fig. 8.4a that

$$d\Omega = \frac{dA}{L^2},$$

where dA is the area of the detector and L , its distance from the target.

Thus $d\Omega = \frac{(400 \times 10^{-6})m^2}{(0.6m)^2} = 1.1 \times 10^{-3}$ sr.

Therefore $\frac{d\sigma}{d\Omega} = \frac{8.1 \times 10^3 s^{-1}}{10^{21} \times (3\times10^8 m^{-2}s^{-1}) \times 1.1\times10^{-3} sr} = 2.4\times10^{-23}\ m^2 sr^{-1}$

SAQ 1

A beam of neutrons is passed through paraffin. Its incident flux is $5\times10^{10}\ m^{-2}\ s^{-1}$. The dcs is measured to be $1.5 \times 10^{-26}\ m^2 sr^{-1}$ at an angle 60°. Compute the number of particles scattered per unit time by (i) a single paraffin molecule and (ii) 10^{22} paraffin molecules, into a solid angle 10^{-3} sr.

So far we have defined the dcs and tcs. We would next like to find out how these can be determined for various scattering processes. To do this we need some additional information about the cross-sections. Let us see what it is !

Experimentalists measure these cross-sections in laboratory experiments. Theoretical physicists make model's of the forces of interaction and calculate these cross-sections. If the calculated values agree well with the experimental values then those models are held to be valid.

When a scattering experiment is performed in the laboratory, the target is taken to be at rest. But for calculating the cross-sections it is easier to use the frame of reference in which the c.m. is at rest because then the two-body problem can be reduced to a one-body problem (recall Sec. 7.2.1). Then we have to deal with only the relative motion of the target and the projectile. So the first question is how to compare the measured cross-sections with the calculated ones ? For this we need to define these frames of reference and determine the relationship of the cross-sections as observed or calculated in them.

8.2.3 Laboratory and Centre-of-mass Frames of Reference

In the laboratory frame of reference (Fig. 8.5a), the target particle of mass m_2 is taken to be at rest before the collision. It is taken to be situated at O, the origin of the coordinate system. Let the projectile of mass m_1 approach the target with velocity $\mathbf{u}_1$. After collision, let the two particles have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$ with respect to O at any instant t. From Eq 7.2 the position and velocity vectors of the c.m. in the laboratory frame of reference after collision, are given by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \tag{8.3a}$$

$$\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} = \frac{m_1 \mathbf{u}_1}{m_1 + m_2} \tag{8.3b}$$

since from conservation of linear momentum, $m_1 \mathbf{u}_1 = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$.

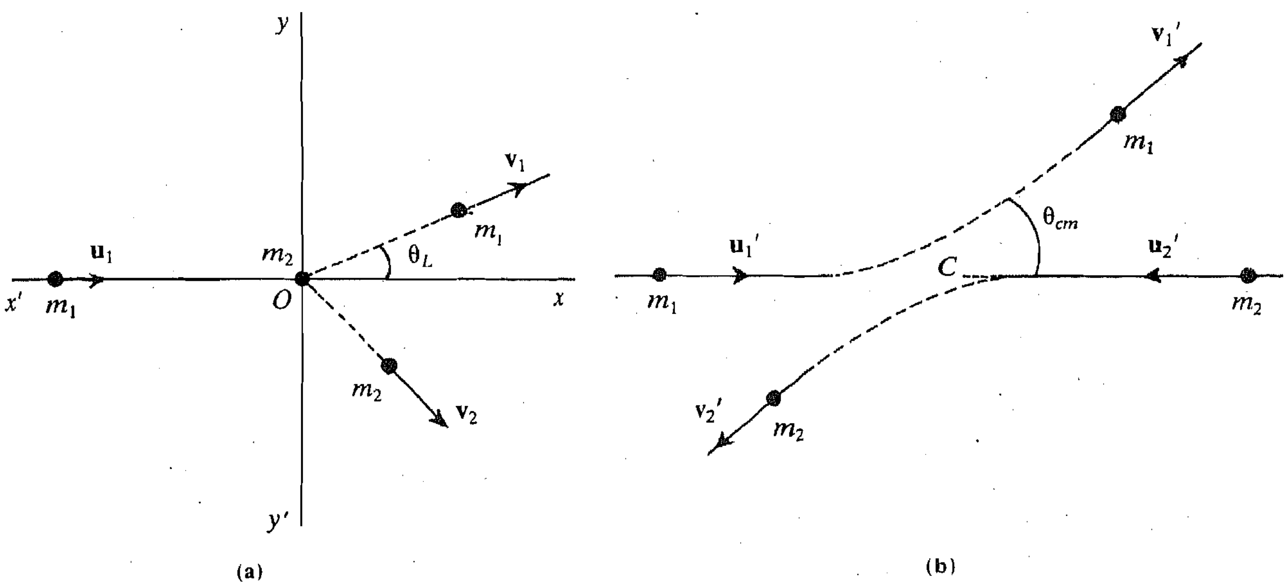


Fig. 8.5: (a) The laboratory frame of reference in which the target particle of mass m_2 is initially at rest: (b) centre-of-mass frame of reference in which the c.m. (C) is initially and always at rest.

It is convenient to study collisions using the c.m. frame of reference. As you know, in this frame, the c.m. is initially and always taken to be at rest (Fig. 8.5b). The origin of the coordinate system is located at the c.m. Since the c.m. is at rest always, its linear momentum and so the linear momentum of the entire system is zero before and after collision. Therefore, the c.m. frame of reference is also known as the **zero momentum frame of reference.** For the two particle system, let the velocities of the particles in the c.m. frame of reference be $\mathbf{u}'_1$ and $\mathbf{u}'_2$ before collision. Let their velocities after collision be $\mathbf{v}'_1$ and $\mathbf{v}'_2$. Then putting the velocity of c.m. equal to zero, we gel

$$m_1 \mathbf{u}_1' + m_2 \mathbf{u}_2' = 0 = m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2', \tag{8.4a}$$

or

$$-\frac{\mathbf{u}_2'}{\mathbf{u}_1'} = \frac{m_1}{m_2} = -\frac{\mathbf{v}_2'}{\mathbf{v}_1'} \tag{8.4b}$$

Thus, the colliding particles have equal and opposite momenta before and after collision in the c.m. frame of reference. You can see from Eqs. 8.4a and b that for elastic collisions, the magnitudes of the particles' velocities will remain the same after scattering. In fact, you can work out this result yourself in the following SAQ.

SAQ 2

Show that for elastic collisions $u_1' = v_1'$, $u_2' = v_2'$ in the c.m. frame of reference. (Hint : Recall the definition of an elastic collision from Sec. 3.4 of Block 1 and use the condition of the conservation of kinetic energy alongwith Eqs. 8.4a and b).

We have specified the laboratory and c.m. frames of reference. We would now like to determine the relationship between the angles and the differential scattering cross-sections in the two frames of reference. For this, let us look at the relation between the position and velocity vectors of the particles after scattering in these two frames of reference.

Recall that we have chosen the incident direction along the z-axis. The coordinates of m_1 and m_2 as measured from the origin O of the lab system after collision are $\mathbf{r}_1$ and $\mathbf{r}_2$. The c.m. **(C)** has the coordinate R with respect to O. Let $\mathbf{r}_1'$ and $\mathbf{r}_2'$ be the coordinates of m_1 and m_2 with respect to C after collision. As you know from its definition, the c.m. lies on the line joining m_1 and m_2. Thus $\mathbf{r}_1'$ and $\mathbf{r}_2'$ lie along the same line. So, we can relate the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{R}$, $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in a vector diagram as shown in Fig. 8.6a.

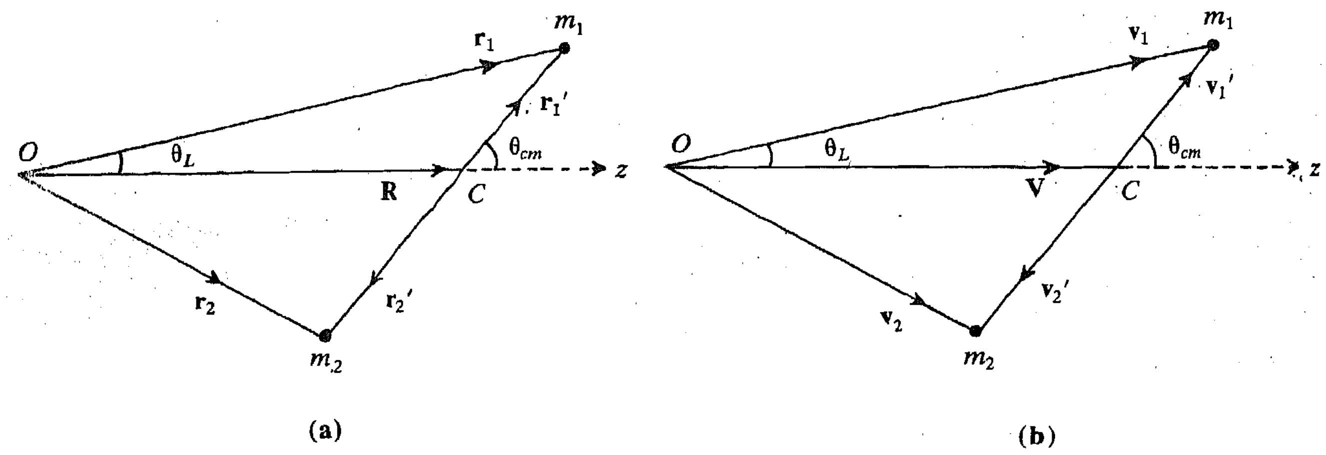


Fig. 8.6: Relation between (a) the position vectors and (b) velocities of the colliding particles in lab and c.m. frames of reference, after collision.

From Fig. 8.6a we have

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_1' \quad , \quad \mathbf{r}_2 = \mathbf{R} + \mathbf{r}_2' \tag{8.5a}$$

The relative coordinate of particle 1 with respect to particle 2 is $\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2$. From Eq. 8.5a you can see that

$$\mathbf{r}_{21} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1' - \mathbf{r}_2' \equiv \mathbf{r}, \text{ say.} \tag{8.5b}$$

Thus, the separation of the two particles is the same in both frames of reference. Using Eqs. 8.3a, 8.5a and b, we can write $\mathbf{r}_1'$ and $\mathbf{r}_2'$ in terms of $\mathbf{r}$:

$$\mathbf{r}_1' = \frac{m_2}{m_1 + m_2}\mathbf{r} \quad , \quad \mathbf{r}_2' = \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{8.6}$$

Differentiating Eq, 8.5a we can relate the velocity vectors in both frames of reference after scattering :

$$\mathbf{v}_1 = \mathbf{V} + \mathbf{v}_1', \tag{8.7a}$$

$$\mathbf{v}_2 = \mathbf{V} + \mathbf{v}_2'. \tag{8.7b}$$

Similar relations can be derived for the position and velocity vectors of particles 1 and 2 before scattering, so that

$$\mathbf{u}_1 = \mathbf{V} + \mathbf{u}_1' \tag{8.7c}$$

$$\mathbf{u}_2 = \mathbf{V} + \mathbf{u}_2'. \tag{8.7d}$$

Since the particle 2 is initially at rest in the lab system, $\mathbf{u}_2 = 0$ and we have

$$\mathbf{u}_2' = -\mathbf{V}. \tag{8.7e}$$

Using Eqs. 8.3 to 8.7 we can determine the relations between the angles of scattering and scattering cross-sections in the laboratory and c.m. frames of reference.

8.2.4 Relations Between Angles and Scattering {Cross-Sectionsin the Lab and C.M. Frarnes of Reference

Let θ_L and θ_{cm} be the angles of scattering in the laboratory and c.m. frames of reference.. respectively (see Fig. 8.6). Resolving Eq. 8.7a into its components along the initial z-direction and perpendicular to it (see Fig. 8.6b), we get

$$v_1 \cos\theta_L = v_1' \cos\theta_{cm} + V, \tag{8.8a}$$

$$v_1 \sin\theta_L = v_1' \sin\theta_{cm} \quad (8.8b)$$

Dividing Eq. 8.8b by Eq. 8.8a gives

$$\tan\theta_L = \frac{v_1' \sin\theta_{cm}}{v_1' \cos\theta_{cm} + V} = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \frac{V}{v_1'}}$$

$$\text{or} \quad \tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \gamma} \quad \text{with} \quad \gamma = \frac{V}{v_1'} . \quad (8.9)$$

You can see that γ is the ratio of the speed of the c.m. in the laboratory system to the speed of the observed particle in the c.m. system. The value of γ can be determined for both elastic and inelastic scattering: We shall limit ourselves to the case of elastic scattering. Let us find $\frac{V}{v_1'}$ for elastic scattering.

γ for elastic scattering

You have already shown in SAQ 2 that $v_1' = u_1'$. We can obtain u_1' in terms of V from Eqs. 8.7c and 8.3b as follows:

$$\mathbf{u}_1' = \mathbf{u}_1 - \mathbf{V} = \frac{(m_1 + m_2)}{m_1}\mathbf{V} - \mathbf{V} = \frac{m_2}{m_1}\mathbf{V},$$

$$\text{or} \quad u_1' = \frac{m_2}{m_1} V . \quad (8.10a)$$

Thus, for elastic scattering

$$\gamma = \frac{V}{v_1'} = \frac{V}{u_1'} = \frac{m_1}{m_2} . \quad (8.10b)$$

You may now like to apply these relations to solve a problem.

SAQ 3

An experiment is to be designed to measure the differential scattering cross-section for elastic pion-proton scattering. In the c.m. frame, the scattering angle is 70° and kinetic energy of the pion is 490 keV. (The eV is the atomic unit of energy.) Find the corresponding angle in the lab at which the scattered pions should be detected and the required lab kinetic energy in eV of the pion beam. The ratio of pion to proton mass is 1/7.

$1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$

Let us now determine the relation between the differential scattering cross-sections in the lab and c.m. frames of reference. The incident flux F and the number of particles (Δn) scattered per unit time in the solid angle $d\Omega$, will be the same in the laboratory and the c.m. systems. So Eq. 8.1a gives us the condition that

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)_{lab} F\,(d\Omega)_{lab}\,(\Delta t) = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F\,(d\Omega)_{cm}\,(\Delta t),$$

$$\text{or} \quad \left(\frac{d\sigma}{d\Omega}\right)_{lab} (d\Omega)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} (d\Omega)_{cm} . \quad (8.11a)$$

We know from Eqs. 8.2a and 8.2b that $d\Omega = \sin\theta\, d\theta\, d\phi$. Since we are dealing with situations in which the cross-sections are independent of ϕ, we can write $d\phi_{lab} = d\phi_{cm}$, so that

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \frac{\sin\theta_{cm}\, d\theta_{cm}}{\sin\theta_L\, d\theta_L},$$

$$\text{or} \quad \left(\frac{d\sigma}{d\Omega}\right)_{lab} = \left(\frac{d\sigma}{d\Omega}\right)_{cm} \frac{d(\cos\theta_{cm})}{d(\cos\theta_L)}, \quad (\because d(\cos\theta) = -\sin\theta\, d\theta) . \quad (8.11b)$$

We can use Eq. 8.9 to simplify Eq. 8.11b further as follows:

$$\text{Since } \tan\theta_L = \frac{\sin\theta_{cm}}{\cos\theta_{cm} + \gamma}$$

you can verify that

$$\cos\theta_L = \frac{\cos\theta_{cm} + \gamma}{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{1/2}}$$

and

$$\frac{d(\cos\theta_L)}{d(\cos\theta_{cm})} = \frac{(1 + \gamma\cos\theta_{cm})}{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{3/2}}$$

Thus, we get the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{(1 + \gamma^2 + 2\gamma\cos\theta_{cm})^{3/2}}{(1 + \gamma\cos\theta_{cm})} \left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.11c)$$

It is $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ which is obtained from theory. Eq. 8.11c tells us how to transform it to the laboratory system to compare with experimental data.

For elastic scattering, $\gamma = \frac{m_1}{m_2}$ and we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta_{cm}\right)^{3/2}}{\left(1 + \frac{m_1}{m_2}\cos\theta_{cm}\right)} \left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.12)$$

If the masses of the target and projectile are equal, i.e. $m_2 = m_1$, then Eq. 8.12 reduces to

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = 4\cos\frac{\theta_{cm}}{2} \cdot \left(\frac{d\sigma}{d\Omega}\right)_{cm} \qquad (8.13)$$

The total scattering cross-sections will be the same in both the frames of reference.

You will get some practice on these equations if you work out the following SAQ.

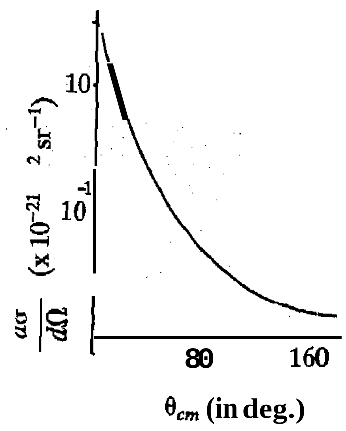


Fig. 8.7

SAQ 4

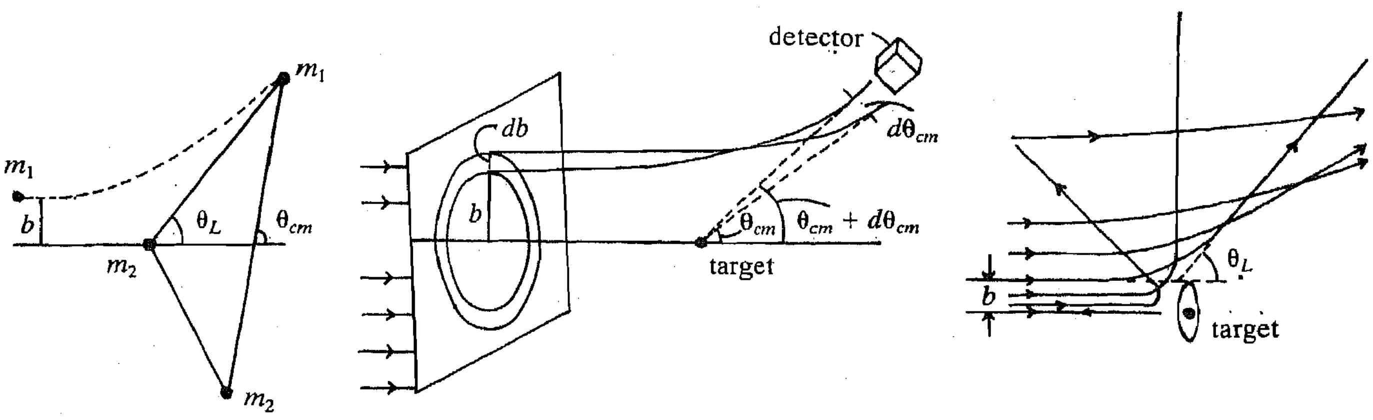
a) The differential scattering cross-sections in a proton-proton elastic scattering experiment are measured to be 2.3×10^{-27} $m^2 sr^{-1}$ and 2.6×10^{-28} $m^2 sr^{-1}$ at the scattering angles 30° and 60°. Find the corresponding quantities in the c.m. frame of reference.

b) Fig. 8.7 shows the variation of the differential cross-section (dcs) with the angle of scattering for the elastic scattering of electrons by lithium atoms in the c.m. frame of reference. What is the corresponding curve in the lab system?

So far we have defined scattering cross-sections and established the relations between the scattering angles and cross-sections in the laboratory and c.m. frames of reference. Let us now determine the cross-sections for a few scattering processes. One of the methods commonly used for this purpose is the method involving impact parameters, which we shall now study,

8.3 IMPACT PARAMETERS

Let us suppose that the projectile does not make a head-on collision with the target. Instead, it travels along a path, which if continued in a straight line, would pass the target at a distance b (Fig. 8.8a). This, indeed, is the case most of the times. The distance b is known as the **impact parameter.** You can see that h is the perpendicular distance between the projectile's initial path and the target.



(a) (b)

Fig. 8.8: (a) The impact parameter b; (b) the particles having impact parameters between h and $h + db$ are scattered into angles between θ_{cm} and $\theta_{cm} + d\,\theta_{cm}$; (c) the scattering angle decreases with increasing impact parameter.

Let us now express the differential scattering cross-sections in terms of the impact parameter. We will study the scattering process in the c.m. frame of reference with θ_{cm} as the angle of scattering (Fig. 8.8b). What is the number of particles incident on the target during time Δt having impact parameters between b and $b + dh$? Let us consider a circular ring having radii between h and $h + db$. The area of the ring is $2\pi bdb$ for infinitesimal values of db. If the incident flux is F then,

The number of incident particles having an impact parameter between h and $(h + db)$

$$= F(\Delta t)\,(2\pi\, b\, db). \qquad (8.14)$$

Let us suppose that these particles are scattered into angles between θ_{cm} and $\theta_{cm} + d\theta_{cm}$. The particles with larger h will be scattered through smaller angles as shown in Fig. 8.8c. This happens because larger h means lesser interaction, i.e. less scattering. For very large b, scattering will be minimal and the particles will go almost undeflected in a straight line. Now in the c.m. frame of reference the number of particles scattered in the solid angle $d\Omega$ in time Ar is given from Eq. 8.1a as

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F\,(d\Omega)_{cm}\,\Delta t\,.$$

This is the same as the number of incident particles in time At having impact parameters between b and $h + db$, given by Eq. 8.14, i.e.

$$F\,(\Delta t)\,2\pi bdb = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F(d\Omega)_{cm}\,\Delta t,$$

$$\text{or} \qquad 2\pi bdb = -\left(\frac{d\sigma}{d\Omega}\right)_{cm} 2\pi \sin\theta_{cm}\, d\theta_{cm}. \qquad (8.15a)$$

Here we have assumed that $\left(\frac{d\sigma}{d\Omega}\right)$ is independent of ϕ. Taking into account all values of ϕ in $d\Omega$, we have $d\Omega = 2\pi \sin 8\; d\theta$. The negative sign expresses the fact that as b increases, θ_{cm} decreases, i.e. db and $d\theta_{cm}$ have opposite signs. From Eq. 8.15a we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{h}{\sin\theta_{cm}} \left|\frac{db}{d\theta_{cm}}\right|. \qquad (8.15b)$$

We have not written the negative sign in Eq. 8.15b because $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ has the dimension of area and its magnitude has to be positive. So, if we know h as a function of scattering angle θ_{cm}, we can calculate the differential scattering cross-section using Eq. 8.15b

How do we determine b as a function of θ_{cm}? We will not study any general method for finding b (θ_{cm}). Instead, we will study two specific cases, namely, the hard sphere scattering and Rutherford scattering as applications of Eq. 8.15b.

8.3.1 Elastic Scattering of Two Nard Spheres

Let us consider the elastic scattering of a sphere of mass m_1 and radius R by a target sphere of mass m_2 and radius R_s (Fig. 8.9a). Let the distance between the centres of the two spheres at any instant be r.

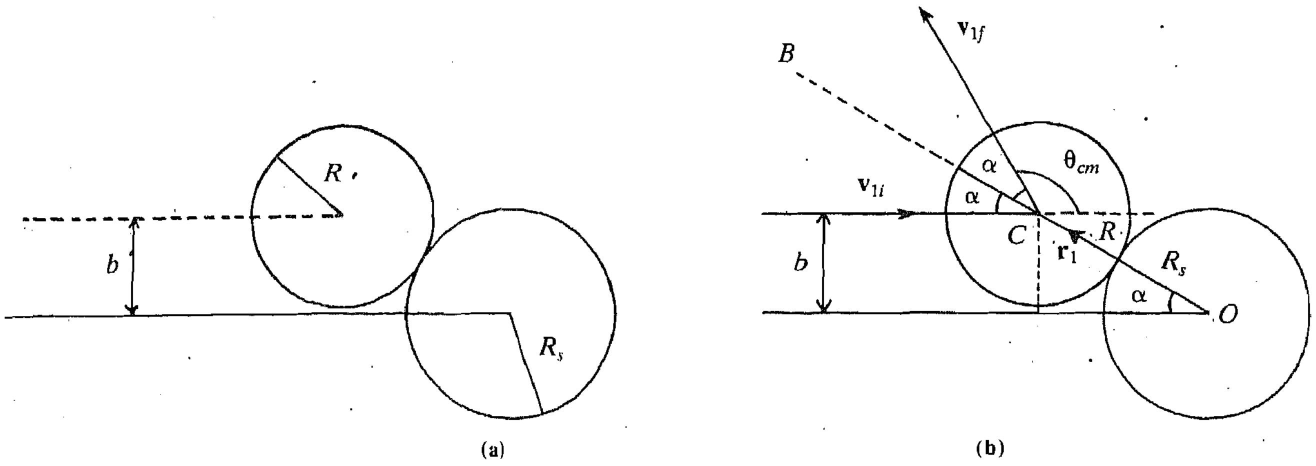


Fig. 8.9: (a) Scattering of two hard spheres: (b) the incident hard sphere rebounds at the same angle as the incident angle after scattering from the target sphere.

The incident hard sphere will get scattered after rebounding from the target hard sphere. What do we mean by the term 'hard sphere'? This means that the spheres cannot penetrate a distance smaller than R + R_s. So we can say that the force or potential is infinite for $r <$ (R + R_s). For a distance $r > (R + R_s)$, the spheres are free to move both before and after the collision, i.e. there is no force between them. Mathematically we can express such a situation in terms of a potential V (r) such that

$$V(r) = \infty \quad \text{for} \quad r < (R + R_s), \tag{8.16}$$
$$= 0 \quad \text{for} \quad r > (R + R_s) .$$

You know that $F = -\frac{dV}{dr}$. So you can see that the force on the spheres corresponding to such a potential is infinite for $r < (R + R_s)$ and zero for $r >$ (R + R_s). This means that the torque is zero for $r >$ (R + R_s). Since the torque = $\frac{d\mathbf{L}}{dt}$, the total angular momentum will remain constant before and after the collision.

Let us now find out the relation between b and θ_{cm}. Refer to Fig. 8.9b. For an elastic collision, K.E. is conserved. You have already worked out in SAQ 2 that for elastic scattering the target and projectile velocities remain the same before and after collision. Let α be the angle between the direction of the initial velocity $\mathbf{v}_{1i}$ of m_1 and the line joining the centres of the two spheres at the time of impact as shown in Fig. 8.9b. Let $\mathbf{r}_1$ be the position vector of the centre of sphere of radius R with respect to the centre of sphere of radius R_s. The magnitude of the angular momentum of m_1 with respect to the centre of m_2 just before the impact is

$$L_1 = m_1 \left| (\mathbf{v}_{1i} \times \mathbf{r}_1) \right| = m_1 v_{1i} r_1 \sin(\pi - \alpha) = m_1 v_{1i} r_1 \sin\alpha .$$

Just after the impact it is

$$L_1 = m_1 \left| (\mathbf{v}_{1f} \times \mathbf{r}_1) \right| = m_1 v_{1f} r_1 \sin \angle ACB .$$

Since from SAQ 2, $v_{1i} = v_{1f}$ for elastic scattering we have that $m_1 v_{1i} \sin a = m_1 v_{1i}$ sin LACB, i.e. $\angle ACB = a$.

Thus, the sphere m_1 will bounce off the sphere m_2 at an angle to the normal, equal to the incident angle **a.** So from Fig. 8.9b you can see that

$$\theta_{cm} = \pi - 2\alpha . \tag{8.17}$$

Now, we can relate the impact parameter b to a using Fig. 8.9b as follows:

$$b = r_1 \sin\alpha = (R + R_s) \sin \alpha$$

$$= (R + R_s) \sin \frac{\pi - \theta_{cm}}{2}, \text{ using Eq. 8.17,}$$

or $$b = (R + R_s) \cos \frac{\theta_{cm}}{2}, \quad (8.18a)$$

$$\therefore \; \frac{db}{d\theta_{cm}} = -\frac{R + R_s}{2} \sin \frac{\theta_{cm}}{2}. \quad (8.18b)$$

Then from Eq. 8.15b we get the differential scattering cross-section as

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}} \frac{R + R_s}{2} \sin \frac{\theta_{cm}}{2}$$

$$= \left(\frac{b}{2\cos\frac{\theta_{cm}}{2}}\right) \left(\frac{R + R_s}{2}\right).$$

Using $\dfrac{b}{\cos\frac{\theta_{cm}}{2}} = R + R_s$ from Eq. 8.18a, we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_s)^2}{4} \quad (8.19)$$

The total scattering cross-section is

$$\sigma = 2\pi \int_0^\pi \left(\frac{d\sigma}{d\Omega}\right)_{cm} \sin\theta_{cm}\, d\theta_{cm}$$

$$= 2\pi \int_0^\pi \frac{(R + R_s)^2}{4} \sin\theta_{cm}\, d\theta_{cm}$$

or $$\sigma = 2\pi \frac{(R + R_s)^2}{4} \times 2 = \pi (R + R_s)^2. \quad (8.20)$$

If the projectile is a point particle instead of a sphere, then the total scattering cross-section is πR_s^2 which is the cross-sectional area of the target sphere. You may like to work out an SAQ applying the ideas of this section.

SAQ **5**

A beam of point particles strikes a wall. Each atom in the wall behaves like a sphere of radius 3×10^{-15} m. The mass of each particle is much less than that of an atom. What is the dcs, tcs and the impact parameter of the particles entering a detector placed at an angle of 60° to the direction of the beam?

Let us now study another application of Eq. 8.15b, namely the Rutherford scattering.

8.3.2 Rutherford Scattering

The Rutherford scattering experiment was an important milestone in understanding the structure of the atom. Until the early twentieth century Thomson's plum pudding model of the atom was believed to be valid. J.J. Thomson had proposed, in 1898, that atoms were uniform spheres of positively charged matter in which electrons were embedded (Fig. 8.10a). It was almost 13 years later that a definite experimental test of this model was made. Now, the most direct way to find out what is inside a plum pudding is to plunge a finger into it! A similar technique was used in the classic experiment performed in 1911, by Geiger and Marsden who were working with Lord Rutherford. They bombarded thin foils of various materials with α-particles (helium nuclei) and recorded the angular distribution of the scattered α-particles (see Fig. 8.11).

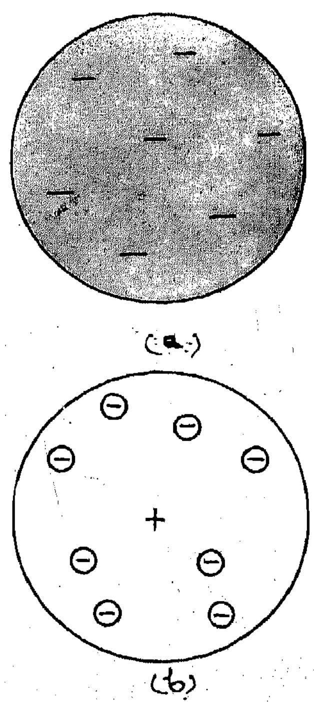


Fig. 8.10: (a) Thomson's plum pudding model of the atom; (b) Rutherford's nuclear model.

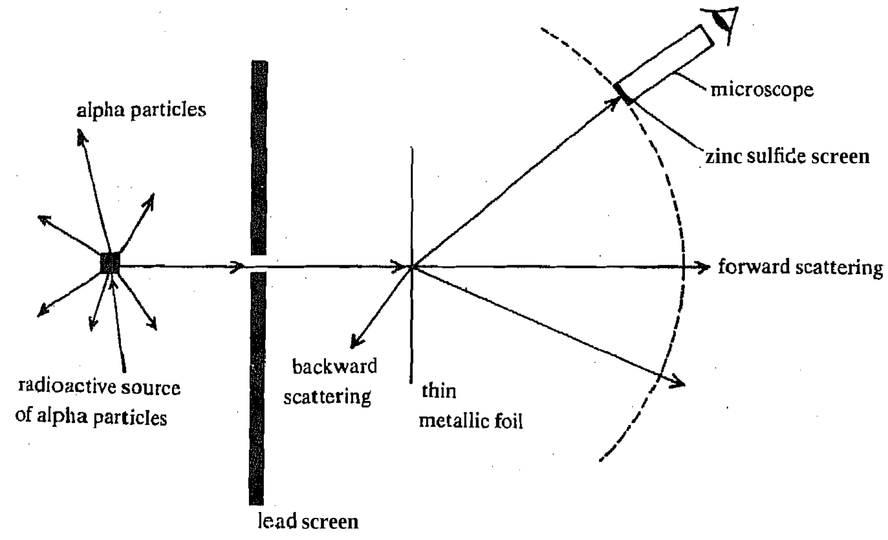


Fig. 8.11 Rutherford scattering experiment. A source of a-particles is placed behind a lead screen with a small hole, so that a narrow beam is directed at a thin metallic foil. A movable zinc sulphide screen is placed at the other side of the foil. When an a-particle strikes the screen it gives off a flash of light.

The thickness of the foils used by Geiger and Marsden was of the order of 10^{-7}m. Compare this with the human hair which is about 10^{-4}m in diameter.

On seeing these results. Rutherford remarked, "It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

It was found that most of the a-particles pass through the foil (i.e. scattering angle $\theta < 90°$). However, about 1 in 6.17×10^6 alpha particles was scattered backward, i.e. deflected through an angle greater than 90°. This result was unexpected according to Thomson's model. It was anticipated that the alpha panicles would go right through the foil with only slight deflections. This follows from the Thomson model. If this model were correct, only weak electric forces would be exerted on alpha particles passing through a thin metal foil. In such a case their initial momenta should be enough to make them go through with only slight deflections. It would indeed need strong forces to cause such considerable deflections of α-particles as were observed.

In order to explain these results Rutherford proposed a nuclear model of the atom. Using this model he calculated the dcs. In doing so, he reasoned that the backward scattering could not be caused by electrons in the atom, The alpha particles are so much more massive than electrons that they would hardly be scattered by them. He assumed that the positive charge in the atom was concentrated in a very small volume, which he termed the nucleus, rather than being spread out over the volume of the atom. So the scattering of alpha particles was due to the atomic nucleus. As you know the force of interaction between the α - particles and the nucleus is simply the repulsive inverse square electrostatic force. On the basis of this model, Rutherford calculated the differential cross-sections.

There was a striking agreement between the calculated and observed cross-sections. This established the nuclear model of the atom, i.e. the positive charge of the atom is concentrated in the nucleus, which is surrounded by electrons [Fig. 8.10b).

Let us consider the scattering of a particle carrying charge q by the atomic nuclei having charge q'. For this scattering process Rutherford derived the relation between the impact parameter b and the angle of scattering θ_{cm} to be

$$b = \frac{r_0}{2} \cot \frac{\theta_{cm}}{2}, \tag{8.21}$$

where $r_0 = \frac{qq'}{4\pi \varepsilon_0 E_{cm}}$. E_{cm} is the total mechanical energy of the projectile and the target in the c.m. system.

ε_0 is known as the permittivity of free space. Its value is $8.8 \times 10^{-12}\, C^2 N^{-1} m^{-2}$.

The differential scattering cross-section in the c.m. system for Rutherford scattering is then given from Eq. 8.15b as

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}}\left(\frac{db}{d\theta_{cm}}\right)$$

$$= \frac{r_0 \cot\frac{\theta_{cm}}{2}}{2\sin\theta_{cm}}\frac{r_0}{2} \cdot \frac{1}{2}\,\text{cosec}^2\frac{\theta_{cm}}{2}$$

$$= \frac{r_0^2 \cot\frac{\theta_{cm}}{2}}{16 \sin\frac{\theta_{cm}}{2}\cos\frac{\theta_{cm}}{2}}\,\text{cosec}^2\frac{\theta_{cm}}{2},$$

or $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\,\text{cosec}^4\frac{\theta_{cm}}{2}, \text{ where } r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}} \qquad (8.22)$$

This is the Rutherford scattering cross-section. For scattering of an a-particle by a nucleus of atomic number Z, qq' = (2e) (Ze) = $2Ze^2$, where e is the electronic charge. You can see that the Rutherford scattering cross-section is strongly dependent on both the energy of the incoming particle and the scattering angle. Also we expect the number of particles scattered to increase as Z^2 with increasing atomic number. Let us now apply the ideas discussed in this section to a concrete situation.

Example 2

In one of their experiments on scattering of a-particles, Geiger and Marsden bombarded 7.7 MeV a-particles on a gold target, for which Z = 79. Its atomic weight is 197 amu. Find the impact parameters and differential scattering cross-sections of the a-particles which are scattered elastically through angles equal to (i) 10°, (ii) 90" and (iii) 150°.

1 MeV = 10^6 eV

It is given here that the **K.E.** of the incident α-particles in the laboratory system is 7.7 MeV, i.e. 1.2 x 10^{-12}**J.** The angles of scattering in the lab system are (i) $\theta_L = 10°$, (ii) $\theta_L = 90°$ and (iii) $\theta_L = 150°$. In order to apply Eqs. 8.21 and 8.22 we must determine the scattering angle θ_{cm} and total mechanical energy E_{cm} in the c.m. frame of reference. We have also to find out r_0.

The total mechanical energy E_L in the lab system is simply the initial **K.E.** of the a-particles, since the target is initially at rest and the two particles are free. This is given to be 1.2 x 10^{-12}J. We have to determine E, in terms of E_L. As you know the total mechanical energy in the c.m. frame before scattering is

$$E_{cm} = \frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2$$

Now from Eqs. 8.10a, 8.7e and 8.3b, $u_1' = \frac{m_2}{m_1}V$, $u_2' = -V$ and

$$V = \frac{m_1 u_1}{m_1 + m_2}, \text{ so that}$$

$$E = \frac{1}{2}m_1 \frac{m_2^2}{m_1^2} \cdot \frac{m_1^2}{(m_1+m_2)^2}u_1^2 + \frac{1}{2}m_2 \frac{m_1^2 u_1^2}{(m_1+m_2)^2}$$

or $$E_{cm} = \frac{1}{2} m_1 u_1^2 \left[\frac{m_2^2}{(m_1+m_2)^2} + \frac{m_2 m_1}{(m_1+m_2)^2}\right] = E_L \frac{m_2}{m_1+m_2} = \frac{E_L}{\left(1+\frac{m_1}{m_2}\right)}$$

For α-particle scattering by gold atoms, we have m_1 4 amu and m_2 ... amu

1amu = 1.67 × 10^{-27}kg

$$\therefore \quad E_{cm} = \frac{1.2 \times 10^{-12}\text{J}}{\left(1 + \frac{4}{197}\right)} = 1.2 \times 10^{-12}\text{ J}$$

From Eq. 8.22, $r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}} = \frac{2Ze^2}{4\pi\varepsilon_0 E_{cm}}$

$$= \frac{2 \times 79 \times (1.6 \times 10^{-19}C)^2}{(4\pi) \times (8.8 \times 10^{-12}\,C^2N^{-1}m^{-2}) \times (1.2 \times 10^{-12}\,J)}$$

or $\quad r_o = 3.0 \times 10^{-14}$ m.

Let us now use Eqs, 8.9, 8.21 and 8.22 to calculate θ_{cm}, b and $\left(\frac{d\sigma}{d\Omega}\right)$ for (i) $\theta_L = 10°$, (ii) $\theta_L = 90°$, (iii) $\theta_L = 150°$, respectively.

Since $\frac{m_1}{m_2} \doteq .02 \ll 1$, we can neglect it, so that $\theta_{cm} \simeq \theta_L$. In fact, you can verify this yourself by calculating the exact value of θ_{cm} using Eq. 8.9.

(i) For $\theta_{cm} = 10°$, $b = \frac{r_0}{2} \cot 5° = \frac{(3.0 \times 10^{-14}\text{ m}) \times 11.4}{2} = 1.7 \times 10^{-13}$ m.

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(3.0 \times 10^{-14}\text{ m})^2}{16} \operatorname{cosec}^4 (5°) = \frac{(3.0 \times 10^{-14}\text{ m})^2}{16} \times (11.5)^4$$

$$= 9.8 \times 10^{-25}\text{ m}^2\text{ sr}^{-1}$$

(ii) For $\theta_{cm} = 90°$, $b = 1.5 \times 10^{-14}$ m, $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = 2.2 \times 10^{-28}\text{ m}^2\text{ sr}^{-1}$

(iii) For $\theta_{cm} = 150°$, $b = 4 \times 10^{-15}$ m, $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = 6.5 \times 10^{-29}\text{ m}^2\text{ sr}^{-1}$

Let us again understand the physical significance of the Rutherford scattering cross-section in the light of what we have studied so far. The distance of closest approach between the alpha particle and the nucleus is given by $r_{min} = \frac{r_0 + \sqrt{r_0^2 + 4h^2}}{2}$, where

$$r_o = \frac{q\,q'}{4\pi\,\varepsilon_o\,E_{cm}}.$$

So to investigate the structure of the atom at small distances, E_{cm} should be large because only for those values of E_{cm}, r_{min} would be su ciently small. Thus we should bombard the target with high energy particles and examine large angle scattering for which b is small.

You can see from Example 2 that the cross-section is large for small values of scattering angles. But physically we are interested in large angle scattering. This is because of the fact that only very strong forces acting at very short distances can give rise to scattering at such large angles. On what basis can we say this? Let us find out.

If the positive nuclear charge were spread out over a larger volume as proposed by Thomson, the force would be inverse-square law force only down to a distance equal to the radius of the charge distribution. Beyond this point it would decrease as we go to even smaller distances. (Recall the Example 2 in Sec. 5.4 of Unit 5, Block 1. A force law with a similar r-dependence would hold for a charge placed inside a spherical charge distribution. The constants would, of course, change). As a result, charged particles which penetrate inside the charge distribution would experience a weaker force than the inverse square force. Thus, particles with smaller b and smaller r_{min} would be scattered through smaller angles. But this does not turn out to be true, experimentally.

This was why Rutherford assumed the nuclear charge to be concentrated in a very small volume. Only in such a case the strong inverse square force would act at very small distances of the order of r_{min}, giving rise to large deflections, The agreement of theory and experiment vindicated Rutherford's nuclear model. Thus, Rutherford is credited with the 'discovery' of atomic nucleus. In fact, if we neglect electrons completely in Thomson's model, the electric field intensity at the atom's surface is calculated to be about 10^{13}V m^{-1}. On the other hand using Rutherford's model, the electric field intensity at the surface of the nucleus exceeds 10^{21}V m^{-1}. This is greater by a factor of 10^8, enough to reverse the direction of alpha particles.

An interesting aspect of this scattering experiment is that it determines an upper limit to the' dimensions of atomic nuclei. This is none else than the parameter r_o, since for $b = 0$,

$r_{min} = r_0$. For the typical α-particle scattering discussed in Example 2, $r_0 = 3.0 \times 10^{-14}$m. The radius of gold nucleus is, therefore, less than 3.0×10^{-14}m. In recent years, however, α-particles of higher energies have been used to determine nuclear dimensions. It has been found that the Rutherford scattering formula does fail to agree with experiment. From these experiments the radius of gold nucleus comes out to be 1/6 of the values of r_0 found in Example 2.

Another interesting feature of the dcs of Eq. 8.22 is that the corresponding total cross-section is infinite. This is because of the infinite range of the Coulomb force. Even if a particle is very far away from the nucleus, it experiences some force and is scattered through a non-zero (though small) angle. So the total number of particles scattered is indeed infinite.

From these applications you must have realised that scattering is an important tool for invesrigating the microscopic structure of matter. Let us now summarise what we have studied in this unit.

8.4 SUMMARY

- When a beam of particles strikes a target, the angular distribution of scattered particles for different values of (θ, ϕ) may be found from the differential scattering cross-section $\frac{d\sigma}{d\Omega}$. The total scattering cross-section is obtained by integrating the dcs over all values of 8 and ϕ.

- The dcs are measured in the laboratory frame of reference but calculated in the c.m. frame of reference. For elastic scattering the relations between the scattering angle and the dcs in the lab and c.m. frames are

$$\tan \theta_L = \frac{\sin \theta_{cm}}{\cos\theta_{cm} + \frac{m_1}{m_2}}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = \frac{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta_{cm}\right)^{3/2}}{\left(1 + \frac{m_1}{m_2}\cos\theta_{cm}\right)} \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

- If we know the impact parameter b as a function of θ_{cm} we can calculate the dcs for any given scattering process using the relation

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}} \left| \frac{db}{d\theta_{cm}} \right|$$

- For the elastic scattering of two hard spheres

$$b = (R + R_s)\cos\frac{\theta_{cm}}{2}, \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_s)^2}{4} \text{ and } \sigma = \pi (R + R_s)^2$$

- For the scattering of a point charge q from another point charge q',

$$b = \frac{r_0}{2}\cot\frac{\theta_{cm}}{2}, \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\operatorname{cosec}^4\frac{\theta_{cm}}{2}, \text{ where } r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}}.$$

  This is known as the Rutherford scattering cross-section. The tcs is infinite due to the infinite range of Coulomb forces.

8.5 TERMINAL QUESTIONS

1. Show that for Rutherford scattering the total cross-section for particles scattered through any angle θ' greater than a lower limit θ_0 is

$$\sigma(\theta' > \theta_0) = \frac{\pi r_0^2}{4}\cot^2\frac{\theta'}{2}.$$

2. At low energies neutrons and protons behave roughly like hard spheres of radius about 1.3×10^{-12} cm. A parallel beam of neutrons with a flux of 3×10^6 neutrons $cm^{-2}\, s^{-1}$

strikes a target containing 4 x 10^{22} protons. A circular detector of radius 2 cm is placed *70* cm away from the target. Calculate the rate of detection, i.e. $\Delta n/\Delta t$ of neutrons for a scattering angle $\theta_L = 30°$.

3. Find the dcs of *7* MeV a-particles scattered from a lead target (Z = *82,* atomic weight = 207 amu) for $\theta_L = 30°$, given that *1* amu = 1.67×10^{-27} kg.

8.6 ANSWERS

SAQs

1. (i) Here we will use Eq. 8.1b. We have to calculate $\Delta n/\Delta t$ given $F = 5 \times 10^{10} m^{-2}s^{-1}$.

$$\frac{d\sigma}{d\Omega} = 1.5 \times 10^{-26}\, m^2\, sr^{-1} \text{ and } d\,\Omega = 10^{-3} sr.$$

From Eq. *8.*1b

$$\frac{\Delta n}{\Delta t} = \left[\frac{d\sigma}{d\Omega}\right](F)(d\,\Omega)$$

$$= (1.5 \times 10^{-26}\, m^2 sr^{-1})\,(5 \times 10^{10}\, m^{-2}s^{-1})\,(10^{-3} sr)$$

$$= 7.5 \times 10^{-19}\, s^{-1}$$

(ii) For N = 10^{22}, we have

$$\frac{\Delta n}{\Delta t} = \left[\frac{d\sigma}{d\Omega}\right](NF)(d\Omega) = (7.5 \times 10^{-19}\, s^{-1}) \times 10^{22} = 7.5 \times 10^{3}\, s^{-1}.$$

2. For elastic collisions, the total kinetic energy of the system remains constant. Its value for the entire system is the same before and after collision. Thus, we have that

$$\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_1'^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2.$$

Substituting for u_2' and v_2' in terms of u_1' and v_1' from Eq. 8.4b we have

$$\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2\left[\frac{m_1^2}{m_2^2}u_1'^2\right] = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2\left[\frac{m_1^2}{m_2^2}v_1'^2\right]$$

$$\text{or}\quad \frac{1}{2}u_1'^2\left[m_1 + \frac{m_1^2}{m_2}\right] = \frac{1}{2}v_1'^2\left[m_1 + \frac{m_1^2}{m_2}\right],$$

$$\text{or}\quad u_1 = v_1\,.$$

Similarly you can show that $u_2' = v_2'$.

3. Here $\gamma = 1/7$, $\theta_{cm} = 70°$, E_{cm} for pion = *490* keV. From *Eq.* 8.9

$$\tan\theta_L = \frac{\sin 70°}{\cos 70° + 1/7} = 1.94 \text{ or } \theta_L = 62.7°.$$

The pion K.E.s in laboratory and c.m. frames of reference are $E_L = \frac{1}{2}m_1 u_1^2$ and

$$E_{cm} = \frac{1}{2}m_1 u_1'^2, \text{ respectively.}$$

From Eqs. 8.10a and 8.3b we have

$$E_{cm} = \frac{1}{2}m_1\frac{m_2^2}{m_1^2}\left[\frac{m}{m_1 + m_2}\right]^2 u_1^2 = \frac{1}{2}m_1 u_1^2\frac{m_2^2}{(m_1+m_2)^2} = \frac{E_L}{\left[1 + \frac{m_1}{m_2}\right]^2} \qquad (8.24)$$

Thus $E_L = (1 + \frac{1}{7})^2 \times 490 \text{ keV} = 640 \text{ keV}$.

4. (a) Here we have to apply Eqs. 8.9 and 8.13. For elastic scattering, $\gamma = m_1/m_2$ and in this case $m_1 = m_2$. So $\gamma = 1$.

Let us first determine the angles of scattering in c.m. frame of reference for (i) $\theta_L = 30°$ and (ii) $\theta_L = 60°$.

For $\gamma = 1$, $\tan\theta_L = \dfrac{\sin\theta_{cm}}{\cos\theta_{cm} + 1} = \dfrac{2\sin\frac{\theta_{cm}}{2}\cos\frac{\theta_{cm}}{2}}{2\cos^2\frac{\theta_{cm}}{2}} = \tan\dfrac{\theta_{cm}}{2}$

or $\theta_{cm} = 2\theta_L$.

So for $\theta_L = 30''$, $\theta_{cm} = 60''$ and for $\theta_L = 60°$, $\theta_{cm} = 120''$

We can now use Eq. 8.13 to obtain dcs in c.m. frame

(i) $\left(\dfrac{d\sigma}{d\Omega}\right)_{lab} = 2.3 \times 10^{-27}\ \text{m}^2\ \text{sr}^{-1}$ for $\theta_L = 30°$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4\cos 30°} \times (2.3 \times 10^{-27}\text{m}^2\ \text{sr}^{-1}) = 6.60 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}$$

(ii) $\left(\dfrac{d\sigma}{d\Omega}\right)_{lab} = 2.6 \times 10^{-28}\text{m}^2\ \text{sr}^{-1}$ for $\theta_L = 60°$.

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{4\cos 60°} \times (2.6 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}) = 1.3 \times 10^{-28}\ \text{m}^2\ \text{sr}^{-1}$$

b. In this case $m_1 \ll m_2$, since m_1 = mass of the electron = 9.31×10^{-31} kg and m_2 = mass of the Li atom $= 1.5 \times 10^{-26}$ kg.

So $\gamma = \dfrac{m_1}{m_2} = 8.09 \times 10^{-5}$, i.e. $\dfrac{m_1}{m_2} \ll 1$.

Thus from Eq. 8.9 $\tan\theta_L = \tan\theta_{cm}$, or $\theta_L = \theta_{cm}$.

Again from **Eq.** 8.12, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} \approx \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

since γ a 1 and $\gamma\cos\theta_{cm} \ll 1$ for all θ_{cm}.

Therefore, the dcs vs. θ curve in the lab system will be the same as in Fig. 8.7.

5. Since $m_1 \ll m_2$, we have $\theta_L \approx \theta_{cm}$.

Again since the incident particle is a point mass, we put R = 0 in Eqs. 8.19, 8.20 and 8.18a and get

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{R_s^2}{4} = \frac{(3 \times 10^{-15}\ \text{m})^2}{4}\ \text{sr}^{-1} = 2.2 \times 10^{-30}\,\text{m}^2\ \text{sr}^{-1}$$

$\sigma_{cm} = \pi R_s^2 = \pi \times (3 \times 10^{-15}\text{m})^2 = 2.8 \times 10^{-29}\ \text{m}^2$.

For $\theta_{cm} = 60''$

$b = R_s \cos\dfrac{\theta_{cm}}{2} = (3 \times 10^{-15}\text{m}) \times 0.87 = 2.6 \times 10^{-15}\text{m}$.

Terminal Questions

1. The dcs for Rutherford scattering is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\ \text{cosec}^4\frac{\theta_{cm}}{2}, \text{ where } r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}}.$$

We can use Eq. 8.2c to determine a, since the dcs does not depend on ϕ. Now instead of zero, the lower limit for integration over θ is any angle θ' greater than θ_0 in this question. Thus we have

$$\sigma = 2\pi \int_{\theta}^{\pi} \left(\frac{d\sigma}{d\Omega}\right)_{cm} \sin\theta_{cm}\, d\theta_{cm}$$

$$= 2\pi \int_{\theta'} \frac{r_o^2}{16} \operatorname{cosec}^4 \left[\frac{\theta_{cm}}{2}\right] \sin\theta_{cm}\, d\theta_{cm}$$

Now $\operatorname{cosec}^4\left(\frac{\theta_{cm}}{2}\right) = \frac{1}{\left(\sin^2\frac{\theta_{cm}}{2}\right)^2} = \frac{1}{\left(\frac{1-\cos\theta_{cm}}{2}\right)^2}$, [∵ cos 2θ = 1–2 sin²θ].

$$\therefore \quad \sigma = \frac{\pi r_o^2}{8} \int_{\theta'}^{\pi} \frac{4\sin\theta_{cm}\, d\theta_{cm}}{(1-\cos\theta_{cm})^2}$$

Putting $\cos\theta_{cm} = t$ we get

$$\sigma = \frac{\pi r_o^2}{2} \int_{-1}^{\cos\theta'} \frac{dt}{(1-t)^2} = \frac{\pi r_o^2}{2}\left[+\frac{1}{1-t}\right]_{-1}^{\cos\theta'}$$

$$= \frac{\pi r_o^2}{2}\left[\frac{1}{1-\cos\theta'} - \frac{1}{2}\right]$$

$$= \frac{\pi r_o^2}{4}\left[\frac{1+\cos\theta'}{1-\cos\theta'}\right] = \frac{\pi r_o^2}{4} \frac{2\cos^2\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}$$

or $\sigma(\theta') = \frac{\pi r_o^2}{4}\cot^2\frac{\theta'}{2}$.

2. From **Eq.** 8.1c, the rate of detection of the scattered neutrons is

$$\frac{\Delta n}{\Delta t} = \left(\frac{d\sigma}{d\Omega}\right) N F\, d\Omega = \left(\frac{d\sigma}{d\Omega}\right) N F \frac{dA}{L^2}$$

where dA is the cross-sectional area of the detector placed at a distance L from the target. We have been given the following data :

Incident flux F = 3 x 10^6 cm^{-2} s^{-1} = 3 x 10^{10}m^{-2}s^{-1}

Number of target scattering centres N = 4 x 10^{22}.

Cross-sectional area of the detector, $dA = \pi r^2 = \pi\,(0.02\text{m})^2$

$= 1.2 \times 10^{-3}$ m^2

Distance between the detector and the target, L = 70 cm = 0.7m Let us calculate the dcs using **Eq.** 8.19 for the elastic scattering of two hard spheres. In the c.m. frame it is

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{(R + R_t)^2}{4} = \frac{1}{4}(1.3\times10^{-14}\text{m} + 1.3\times10^{-14}\text{ m})^2\text{sr}^{-1}$$

$$= 1.7\times10^{-28}\text{ m}^2\text{ sr}^{-1}$$

We have to find out $\left(\frac{d\sigma}{d\Omega}\right)_{lab}$ for which we will use **Eq.** 8.13, wherein we also need θ_{cm}.

From **Eq.** 8.9, for $m_1 = m_2$ we have $\theta_L = \frac{\theta_{cm}}{2}$.

For $\theta_L = 30°$, $\theta_{cm} = 60°$, so that

$$\left(\frac{d\sigma}{d\Omega}\right)_{lab} = 4\cos 30° \left(\frac{d\sigma}{d\Omega}\right)_{cm}$$

$$= 2\sqrt{3}\times 1.7\times10^{-28}\text{m}^2\text{ sr}^{-1}$$

$$= 5.9\times10^{-28}\text{m}^2\text{sr}^{-1}$$

Therefore, the rate of detection of neutrons is

$$\frac{\Delta n}{\Delta t} = (5.9 \times 10^{-28}\text{m}^2\text{sr}^{-1}) \times (4 \times 10^{22}) \times (3 \times 10^{10}\text{m}^{-2}\text{s}^{-1}) \times \left[\frac{(1.2 \times 10^{-3}\text{m}^2)}{(0.7\text{ m})^2}\text{ sr}\right]$$

$$= 1.7 \times 10^3\text{ s}^{-1}$$

3. Here we have to essentially follow the method used in Example 2. It is given that

$$E_L = 7\ MeV = 7 \times 1.6 \times 10^{-13}\text{J}$$

$$= 1.1 \times 10^{-12}\text{J}$$

Putting $m_1 = 4$ a.m.u. and $m_2 = 207$ a.m.u. in Eq. 8.23, we get

$$E_{cm} = \frac{E_L}{1 + \frac{m_1}{m_2}} = \frac{1.1 \times 10^{-12}\text{J}}{\left(1 + \frac{4}{207}\right)} = 1.1 \times 10^{-12}\text{ J}.$$

From Eq. 8.22, $r_0 = \frac{2Ze^2}{4\pi\varepsilon_0 E_{cm}} = \frac{2 \times 82 \times (1.6 \times 10^{-19}\text{C})^2}{(4\pi) \times (8.8 \times 10^{-12}\text{ C}^2\text{N}^{-1}\text{m}^{-2}) \times (1.1 \times 10^{-12}\text{J})}$

$$= 3.4 \times 10^{-14}\text{m}$$

Since $\frac{m_1}{m_2} = 0.019 \ll 1$, $\theta_L \approx \theta_{cm}$.

For $\theta_L = 30°$, $\theta_{cm} \approx 30°$ and

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\text{cosec}^4\frac{\theta_{cm}}{2}$$

$$= \frac{(3.4 \times 10^{-14}\text{m})^2}{16}\text{ sr}^{-1}\text{ cosec}^4(15°)$$

$$= 1.6 \times 10^{-26}\text{m}^2\text{ sr}^{-1}$$