**b)** From Eq. 6.17, we get

$$T = 2\pi\sqrt{\frac{a^3}{GM}}.$$

where M is the mass of the sun. So on putting the values of a, G and M, we get

$$T = 2\pi\sqrt{\frac{(2.7 \times 10^{12})^3\ m^3}{(6.673 \times 10^{-11}\ Nm^2\ kg^{-2}) \times (1.99 \times 10^{30}\ kg)}}$$

$= 2.42 \times 10^9 s = 76.7$ years.

# UNIT 7 MANY-PARTICLE SYSTEMS

## Structure



## 7.1 INTRODUCTION

So far you have studied the motion of single particles. In Unit 6 we did take up the example of a planet moving in the sun's gravitational field. However, we assumed that the sun was at rest. You may have wondered as to why only the planet moves due to their mutual gravitational attraction. Should not the sun also move ? Indeed, as wc shall find in this unit the sun also has a motion. Then why did we neglect it in Unit 6 ? We can answer this question if we analyse the motion of the two-body system of the sun and the planet.

In this unit we shall first study the motion of two bodies moving under the influence of their mutual interaction force. We shall, of course, be applying the basic concepts of mechanics to this system. In addition, you will learn the concepts of the motion of centre-of-mass and the relative coordinates and apply them to two-body systems. We shall then determine the other dynamical variables like the linear and the angular momenta and the K.E. of each system.

We shall next extend these concepts to study the motion of many-particle systems. The Solar System made up of planets and their satellites, asteroids and comets is one such system. Gas filled in a cylinder, is also a many-particle system if its molecules can be regarded as point masses in a given problem. Objects such as exploding stars, an acrobat, a javelin thrown in air, a cup of tea, a planet, a car, a ball arc all systems composed of many particles. In some systems, e.g. a solid metallic sphere the distances between the particles remain fixed. We shall study the motion of such systems in Unit 9. In other systems the constituent particles move with respect to one another. In this unit you will learn the basic concepts needed to understand these more complex and realistic systems. However, predicting the motion of even more complicated many-particle systems, such as air masses that determine earth's weather, is still very difficult. We need supercomputers to apply these concepts to such systems.

In the next unit we shall use the concepts of mechanics to study the phenomenon of scattering,

### Objectives

After studying this unit you should be able to

- define the centre-of-mass and relative coordinates, and reduced mass
- solve problems involving motion of two-body systems
- derive and explain the physical significance of the expressions of linear and angular momenta and K.E. of a many-particle system.

## 7.2 MOTION OF TWO-BODY SYSTEMS

The motion of a planet around the sun is an example of a two-body motion. In Unit **6** we had approximated this motion as a one-body motion around a stationary sun, for reasons you will study in this section. However, when the masses of the two bodies are comparable, such an approximation cannot be made. Such is the case for the earth-moon system or the system of two charges. For these systems we need to solve the equation of motion of both the bodies moving under each other's influence. In this section we will study a method of solving these equations.

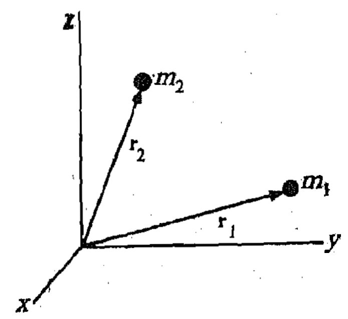



Fig. 7.1: A two-body system

Let us consider the motion of a system of two particles 1 and 2 of masses $m_1$ and $m_2$, respectively. Let their position vectors be $\mathbf{r}_1$ and $\mathbf{r}_2$ at time $t$ with respect to an origin $O$ in an inertial frame of reference (Fig. *7.1*). We will study the case when no external force acts on the system. The only forces responsible for their motion are the mutual action and reaction forces. For example, planets interact via gravitational attraction and molecules interact via inter-molecular forces. Two charged bodies carrying like charges repel each other. In all these cases no external force acts on the systems.

Let the force on 1 due to 2 be $\mathbf{F}_{21}$, then the force on 2 due to 1 is $\mathbf{F}_{12}$ $(= -\mathbf{F}_{21}$, from Newton's third law of motion). The equations of motion for the two particles are

$$m_1\, \ddot{\mathbf{r}}_1 = \mathbf{F}_{21}. \qquad (7.1a)$$

$$m_2\, \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}\,. \qquad (7.1b)$$

We need to solve these two differential equations in order to determine the path of the two particles. However, we can reduce these two equations to a single differential equation of motion. We will use another set of coordinates, namely the centre-of-mass and relative coordinates to arrive at that single equation of motion.

### 7.2.1 Equation of Motion in Centre-of-mass and Relative Coordinates

Refer to **Fig.** 7.2a. We define the position of the centre-of-mass (c.m.) of this system to be

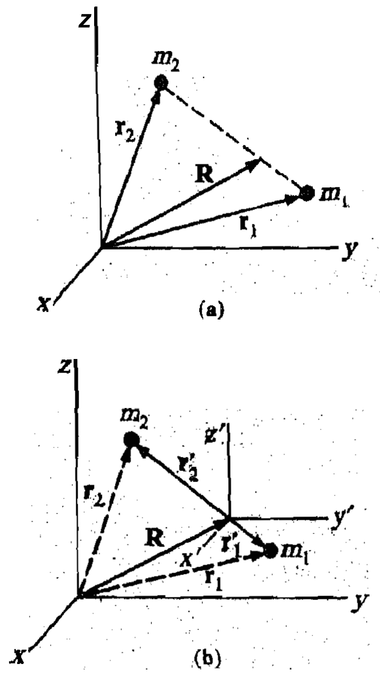



Fig. 7.2: (a) The centre-of-mass of a two-body system: (b) Position vectors of the two bodies with respect to the c.m.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\,. \qquad (7.2)$$

R is referred to as **the centre-of-mass coordinate.** The relative coordinate of m, with respect to $m_2$ is defined as

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad (7.3)$$

The position vectors (Fig. 7.2b) of the particles with respect to the c.m. are given by

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\,\mathbf{r}, \qquad (7.4a)$$

$$\mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\,\mathbf{r}, \qquad (7.4b)$$

where $M = m_1 + m_2$.

Eq. **7.2** defines the centre-of-mass coordinate which together with the relative coordinate of Eq. **7.3** constitutes a new coordinate system to study the two-body motion. Let us now express the equations of motion in terms of these coordinates,

Adding Eqs. **7.1a** and 7.1b, we get

$$m_1\, \ddot{\mathbf{r}}_1 + m_2\, \ddot{\mathbf{r}}_2 = \mathbf{0},$$

$$\text{or} \quad \frac{d^2}{dt^2}\,(m_1\,\mathbf{r}_1 + m_2\,\mathbf{r}_2) = \mathbf{0}.$$

From Eq. *7.2*, we get

$$\frac{d^2}{dt^2}\,\{(m_1 + m_2)\,\mathbf{R}\} = 0,$$

or $$M \frac{d^2\mathbf{R}}{dt^2} = \mathbf{0}. \tag{7.5a}$$

Again from Eqs. 7.1 a and 7.1b, we get

$$\ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{21}}{m_1}, \quad \ddot{\mathbf{r}}_2 = -\frac{\mathbf{F}_{21}}{m_2},$$

or $$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \mathbf{F}$$

From **Eq.** 7.3 we can see that $\mathbf{r} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$.

$$\therefore \quad \ddot{\mathbf{r}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \mathbf{F}_{21}. \tag{7.5b}$$

Let us now introduce a quantity μ such that

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2},$$

$$\text{i.e.} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{7.6}$$

μ is called the **reduced mass** of the system. So Eq. **7.5b** becomes

$$\mu \ddot{\mathbf{r}} = \mathbf{F}_{21}.$$

The equations of motion for particles 1 and 2 given by Eqs. **7.1a** and 7.1b are thus equivalent to

$$M\ddot{\mathbf{R}} = \mathbf{0}, \tag{7.7}$$

$$\text{and} \quad \mu \ddot{\mathbf{r}} = \mathbf{F}_{21}. \tag{7.8}$$

Let us now study the significance of these two equations.

### Centre-of-mass motion

Eq. 7.7 describes the motion of the centre-of-mass. This can be integrated to give

$$\text{MR} = \text{constant}. \tag{7.9}$$

Since M is a constant, we have R = a constant, **i.e.** the centre-of-mass moves with constant velocity. Let us now choose an inertial frame of reference which is moving with respect to the present frame with a velocity R. Using Eq. 1.37 of Unit 1 we find that the **c.m.** will be at rest in this new frame.

So we have found an inertial frame in which **the c.m.** is at rest. Such a frame of reference is called the **centre-of-mass frame of reference.** Its origin lies at the **c.m.** In this frame we need not solve Eq. 7.7. So it is very convenient to describe the motion in the **c.m.** frame of reference. The position vectors of 1 and 2 with respect to the **c.m.,** are given by $\mathbf{r}_1'$ and $\mathbf{r}_2'$ as given by **Eqs. 7.4a** and **7.4b.** Now, if we want to arrive at the solution in any other frame of reference we may use Eqs. **7.4a** and **7.4b** to find $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{r}_1'$, $\mathbf{r}_2'$ and **R.** These may also be used for determining the velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$.

### Relative motion

In the **c.m.** frame we have to solve only Eq. 7.8. It is the equation of motion for a single fictitious particle of mass μ moving under the force $\mathbf{F}_{21}$. If we solve this differential equation, we **get** $\mathbf{r}(t)$, which describes the relative motion of particle 1 with respect to the particle 2. We can also determine the paths of the two particles 1 and 2 by solving for $\mathbf{r}_1$ and $\mathbf{r}_2$ using **Eqs. 7.4a** and **7.4b.**

So, by introducing the concept of **c.m.** we have reduced the task of solving two second order differential equations (**7.1a** and 7.1b) to solving a single equation 7.8. If we can solve this one-body problem then we can **also** solve the two-body problem. Thus, the motion of a **two-body system** is equivalent to a one-body system. All the concepts and laws concerning single particle motion which you studied in Block 1 can now be applied, once the mutual interaction force is known. If it is a central conservative force then the concepts that you have studied in Unit 6 will apply. Note that a mutual force need not always be central as you have worked out in the SAQ 1(c) of Unit 6.

You may now like to work out an SAQ.

**SAQ 1**

a) Verify the relations (7.4a) and (7.4b).

b) Write down Eqs. 7.1a and 7.1b when an external force alongwith the mutual forces of interaction, acts on the system. Recast these equations using the centre-of-mass and relative coordinates. Does it still reduce to an equivalent one-body problem ?

c) What happens if the external force in (b) is the force of gravity ?

Now that you have solved this SAQ, you must have realised the following fact. The reduction of two-body problem to an equivalent one-body problem is possible if no external force acts on the system. The farce of gravity, of course, is an exception.

Let us now consider a system in which the mass of one particle, say $m_1$, is very large compared with the other, so that $\frac{m_2}{m_1} \ll 1$ as in the case of the earth and the sun. Then

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_2}{1 + \frac{m_2}{m_1}} \simeq m_2 \text{ and} \tag{7.10 a}$$

$$\mathbf{R} = \frac{\mathbf{r}_1 + \frac{m_2}{m_1}\mathbf{r}_2}{1 + \frac{m_2}{m_1}} \simeq \mathbf{r}_1. \tag{7.10 b}$$

So, the reduced mass is equal to the smaller mass. And the centre-of-mass is located almost at the position of the greater mass, which can then be regarded as fixed. The motion of the two-body system is thus equivalent to the motion of the lighter body around the heavier one. Let us consider the example of a planet orbiting the sun. In Unit 6, we should have, in principle, determined the planet's orbit by solving Eq. 7.8. Instead we regarded the sun as fixed and solved the equation of motion of the planet with respect to the sun. Can that method be criticised? We know that the sun is much more massive than any other planet, the ratio $\frac{m_2}{m_1}$ being $2.5 \times 10^{-4}$ for the most massive planet Jupiter. So you can apply Eqs. 7.10a and 7.10b, and see that the approximate method which we adopted in Unit 6 is quite valid.

However, even when one particle is very heavy, its motion should be considered and we should use Eq. 7.8. Note that if $m_1$ and $m_2$ occur in the expression of the force $\mathbf{F}_{21}$, then they should be retained as such and no replacement with $\mu$ is to be made ! Let us now work out an example to have a comparative study between the use of Eq. 7.8 and the method adopted in Unit 6 towards the analysis of the planetary motion problem.

**Example 1**

Write down Eq, 7.8 for the case of a two-body system comprising a planet of mass $m$ and the sun of mass M. Hence explain how Eq. 6.17 (Kepler's third law) of Unit 6 will be modified.

Let the relative coordinate of the planet with respect to the sun be $\mathbf{r}$. Then Eq. 7.8 takes the form

$$\mu\ddot{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}},$$

where $\mu = \frac{Mm}{M+m}$.

So we get,

$$\mu\ddot{\mathbf{r}} = -\frac{GM_0\mu}{r^2}\hat{\mathbf{r}},$$ where $M_0 = M + m$ = the sum of the masses of the planet and the sun.

This is similar to Eq. 6.9b of Unit *6* with ***M*** replaced by $M_0$. So we can solve this equation in the same way as we did in Sec. *6.3.*

Then in place of Eq. 6.17 we shall obtain the following :

$$T^2 = \frac{4\pi^2 a_0^3}{GM_0},$$

where $a_0$ is the semi-major axis of the relative orbit (Fig. 7.3) and $M_0 = M + m$.

So the orbital period does not depend only on the semi-major axis. It also depends on the mass of the planet. Hence, Kepler's third law is only approximately true.

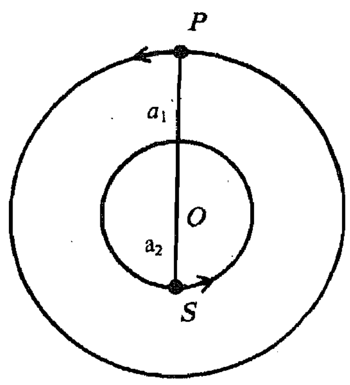


**Fig. 7.3: Orbits of a planet and the sun,** $OP = a_1$, $OS = a_2$, $SP = a_1 + a_2 = a_0$.

You may now like to work out an **SAQ** on these concepts.

**SAQ 2**

One of the most massive stars known at present is a binary or double star, i.e. it consists of two stars bound together by gravitation. It is known from spectroscopic studies that

(a) The period of revolution of the stars about their c.m. is 14.4 days ($1.2 \times 10^6$s).

(b) Each component has a velocity of about 220 km $s^{-1}$. Since both components have nearly equal, but opposite velbcities we may infer that they are at nearly the same distance from the centre-of-mass, and so their masses are nearly equal.

(c) The orbit is nearly circular.

From this data calculate the reduced mass and the separation of the two components.

We have thus seen that a two-body motion can be reduced to the centre-of-mass and relative motion. In such cases, various kinematical quantities, like linear and angular momenta and kinetic energy of the two bodies can also be expressed in terms of c.m. and relative coordinates. We can also say that these quantities are redistributed in the centre-of-mass and relative motion. Let us see how this is done.

### 7.2.2 Linear and Angular Momentum and Kinetic Energy

From Eq. 2.20 of Unit 2, the total linear momentum of the two-body system of Fig. 7.1 is given as

$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the linear momenta of 1 and 2,

$$\text{or} \quad \mathbf{p} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2. \qquad (7.11a)$$

Differentiating Eq. 7.2 with respect to time we get

$$(m_1 + m_2)\,\dot{\mathbf{R}} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2,$$

$$\text{or} \quad \mathbf{p} = M\dot{\mathbf{R}}. \qquad (7.11b)$$

According to Eq. 7.9, $M\dot{\mathbf{R}}$, i.e. p is a constant provided no external force acts on the system, Thus, we arrive at the principle of conservation of linear momentum for a two-body system which is as follows:

**The total linear momentum of a two-body system remains constant provided no external force acts on it.** If the mass of the two-body system remains constant, then it leads to the following statement:

***The velocity of the centre-of-Mass of a two-body system remains constant provided no external force acts on it.***

Let us now work out a simple example on this concent

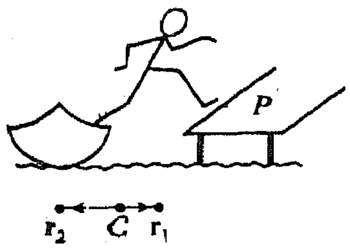


Fig 7.4

**Example 2**

A 70 kg man tries to step out of a 35 kg boat, initially at rest, onto a platform P beside a lake (Fig. 7.4). What happens if he tries to step 1m sideways from the boat without holding on to the platform ?

The boat has no keel. So we can assume that the reaction of the water on the boat, sideways to it is negligible for the brief time in which the action takes place. Thus, the net external force on the two-body system (man and boat) is zero and the velocity of c.m. of the system remains constant. Before the man jumped from the boat, the c.m. of the system was at rest. Therefore, it should remain at rest, i.e.

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \text{a constant},$$

where $m_1$, $m_2$ and $\mathbf{r}_1$, $\mathbf{r}_2$ are the masses and position vectors of the man and the boat, respectively.

If we select the origin of the coordinate system at the position of the c.m., as in Fig. 7.2b, then R = 0, i.e.

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0.$$

Substituting the values of $m_1, m_2$ and $\mathbf{r}_1$ we get

(70 kg) (1m) $\hat{\mathbf{r}}_1 = -$ (35 kg) $\mathbf{r}_2$, where $\hat{\mathbf{r}}_1$ is the unit vector in the direction of $\mathbf{r}_1$.

$$\text{or} \quad \mathbf{r}_2 = -2\text{m}\ \hat{\mathbf{r}}_1.$$

The boat will thus move 2 m in a direction opposite to the man. So the man has to hold on to something or bring the boat nearer, otherwise he will be in danger of falling in the lake.

You may now like to work out an SAQ.

**SAQ 3**

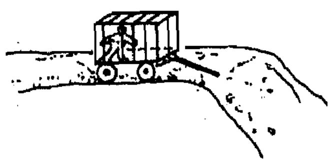


Fig 7.5

Suppose in a nightmare you find yourself locked in a light cage on rollers on the edge of a cliff (Fig. 7.5) ! Assuming that no external forces act on the system consisting of you and the cage, what could you do to move the cage away from the edge ? What must you avoid doing ? If you weigh *60* kg and the cage weighs 90 kg and is 2m long, how far can you move the cage ?

So far we have discussed the linear momentum of a two-body system. Let us now find an expression for the angular momentum of a two-body system.

The total angular momentum of the two-body system is the vector sum of the angular momentum of each body.

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$$

$$= \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2$$

$$\text{or} \quad \mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2.$$

Substituting $\mathbf{r}_1$ and $\mathbf{r}_2$ from Eqs. **7.4** a and b, we get

$$\mathbf{L} = m_1\left(\mathbf{R} + \frac{m_2}{M}\mathbf{r}\right) \times \mathbf{v}_1 + m_2\left(\mathbf{R} - \frac{m_1}{M}\mathbf{r}\right) \times \mathbf{v}_2$$

$$= \mathbf{R} \times (m_1\mathbf{v}_1 + m_2\mathbf{v}_2) + \frac{m_1m_2}{M}(\mathbf{r} \times \mathbf{v}_1 - \mathbf{r} \times \mathbf{v}_2).$$

Using Eqs. 7.3, 7.6 and *7.11* we get

$$\mathbf{L} = M(\mathbf{R} \times \dot{\mathbf{R}}) + \mu\mathbf{r} \times \mathbf{v},$$

$$\therefore \quad \mathbf{L} = \mathbf{R} \times M\mathbf{V} + \mu\mathbf{r} \times \mathbf{v}, \qquad (7.12)$$

where $\mathbf{V} = \dot{\mathbf{R}}$ and $\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$.

You may now try an SAQ. The first part is concerned with Eq. 7.12 and the second part is associated with the K.E. of the two-body system.

**SAQ 4**

a) Use Eq. 7.12 to prove that the angular momentum of a two-body system is conserved provided that no external force acts on it and they move only under their mutual interaction force which is central.

b) Express the K.E. of the two-body system as $T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$ and use the relevant equations to show that

$$T = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} \mu v^2. \tag{7.13}$$

You can see from Eqs. 7.11b, 7.12 and 7.13 that to obtain the values of dynamical variables in any frame from those in c.m. frame, we only need to add the contribution of a particle of mass M located at the c.m. R.

So far we have analysed the motion of a two-body system. We saw that the introduction of the centre-of-mass and relative coordinates made it easier to study this system. We could treat the motion of individual bodies as equivalent to the motion of one body relative to the centre-of-mass. Can we extend such an analysis to a many-particle system ? Let us find out.

## 7.3 DYNAMICS OF MANY-PARTICLE SYSTEMS

To begin with, let us consider a system of three particles $A$, B, $C$ of masses $m_1$, $m_2$ and $m_3$ and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively, at a time $t$ with respect to an origin $O$ in a given frame of reference (Fig. 7.6). We will analyse the motion of this system and extend each result to an $N$-particle system. We define the position vector of the c.m. of the three-body system as

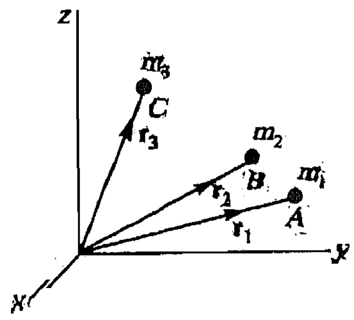


Fig. 7.6: Three-particle system

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_1 + m_2 + m_3} = \frac{\sum\limits_{i=1}^{3} m_i \mathbf{r}_i}{M}, \tag{7.14a}$$

where $\Sigma$, as you know, represents the sum over the three terms in the numerator of Eq. 7.14a and

$$M = m_1 + m_2 + m_3 = \sum_{i=1}^{3} m_i .$$

We also define the relative coordinate of the $j^{th}$ particle with respect to the $i^{th}$ particle as

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i = -\mathbf{r}_{ji} . \tag{7.14b}$$

Let us now write down the equation of motion for particle 1 of the system. In general this particle may be subjected to an external force $\mathbf{F}_{e1}$ and the mutual forces of interaction due to the other two particles in the system (Fig. 7.7).

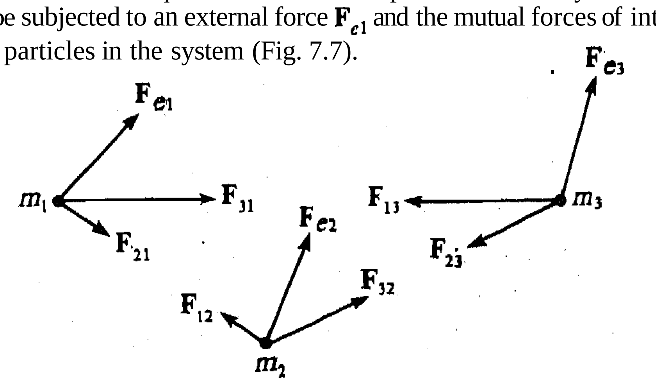


Fig 7.7: Forces acting on a three-particle system

So the net force experienced by particle 1 is

$$\mathbf{F}_1 = \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31}. \tag{7.15}$$

Since the particle does not exert a force on itself, the term $\mathbf{F}_{11}$ does not appear in Eq. 7.15. The equation of motion for particle 1 then becomes

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31}. \tag{7.16}$$

You can now work out the following SAQ to obtain the equations of motion for particles **2** and **3.**

**SAQ 5**

a) Write down the relative coordinates of each particle with respect to the other for the three-particle system of Fig. 7.7.

b) Let the external forces acting on the particles **2** and 3 be $\mathbf{F}_{e2}$ and $\mathbf{F}_{e3}$, respectively. Write down the equations of motion for these two particles.

Now that you have written the equations of motion for the other two particles, let us add the equations of motion of all the three particles. This gives

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 = \; & \mathbf{F}_{e1} + \mathbf{F}_{21} + \mathbf{F}_{31} \\ & + \mathbf{F}_{e2} + \mathbf{F}_{12} + \mathbf{F}_{32} \\ & + \mathbf{F}_{e3} + \mathbf{F}_{13} + \mathbf{F}_{23} \end{aligned}$$

Now the mutual forces of interaction between each pair of particles are equal and opposite, so that

$$\mathbf{F}_{12} = -\mathbf{F}_{21},\; \mathbf{F}_{13} = -\mathbf{F}_{31},\; \mathbf{F}_{23} = -\mathbf{F}_{32} \text{ and we get}$$

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3},$$

$$\text{or } \sum_{i=1}^{3} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{3} \mathbf{F}_{ei} = \mathbf{F}_e, \tag{7.17a}$$

where $\mathbf{F}_e$ is the net external force acting on the system. Now if we differentiate Eq. 7.14a twice with respect to time, we obtain $M\ddot{R} = \sum_{i=1}^{3} m_i \ddot{\mathbf{r}}_i$, provided $m_1, m_2, m_3$ are constant. Eq. 7.17a then becomes

$$M\ddot{R} = \mathbf{F}_e. \tag{7.17b}$$

This is the equation of motion of a single particle of mass M situated at R under an external force $F_e$. So, the introduction of the centre-of-mass allows us to apply Newton's second law to the entire system rather than to each individual particle. As far as its overall motion is concerned, the system acts as if its entire mass were concentrated at the centre-of-mass. However, using Eqs. 7.17a and b we cannot obtain a general analytical solution for the individual motion of the three bodies.

However, we can use the concept of the centre-of-mass to explain many important features of the motion of a three-body system. We will illustrate this with the three-body system of the earth, moon and the sun.

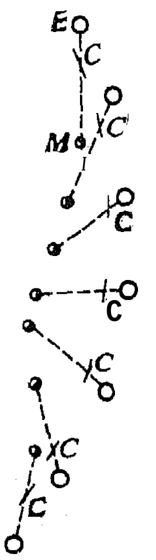


**Fig. 7.8:** The earth and the moon execute elliptical motion about their c.m.

**Example 3: A three-body system — earth, moon and sun**

Recall your study of the Lwo-body problem. If we consider the earth-moon (E.M.) system only, then both the bodies would execute elliptical motion about their centre-of-mass (Fig. 7.8). Let us see what happens when we include the sun in the system. The c.m. of the earth-moon-sun system lies at

$$\mathbf{R} = \frac{M_e\mathbf{R}_e + M_m\mathbf{R}_m + M_s\mathbf{R}_s}{M_e + M_m + M_s} \tag{7.18}$$

where $M_e, M_m, M_s$ are the masses and $\mathbf{R}_e, \mathbf{R}_m, \mathbf{R}_s$ the position vectors of the earth, moon and the sun, respectively. Dividing the numerator and denominator by $M_s$, we can show that $\mathbf{R} \approx \mathbf{R}_s$, since $M_s$ is much larger than $M_e$ and $M_m$. Thus, to a good approximation, the c.m. of this three-body system lies at the centre of the sun (Fig. 7.9a). Now, the external forces due to the gravitational attraction of other celestial bodies are negligible. So, the c.m. moves with a constant velocity. We have seen in Sec. 7.2.1 that the c.m. will be at rest in an inertial frame of reference moving with the same velocity as that of the c.m. Thus, in such an inertial frame, the sun is effectively at rest and we can use a coordinate system with its origin at the centre of the sun, so that R = **0 (Fig. 7.9b).** Then we need to consider the motion of the earth and the moon about the sun.

Let $\mathbf{r}_e$ and $\mathbf{r}_m$ *be* the positions of the earth and moon with respect to the sun. Their c.m. lies at

$$\mathbf{R}_{em} = \frac{M_e\mathbf{r}_e + M_m\mathbf{r}_m}{Me+ M_m}$$

The external force on the earth-moon system is the gravitational attraction of the sun given as

$$\mathbf{F} = -GM_s\left(\frac{M_e}{r_e^2}\,\hat{\mathbf{r}}_e + \frac{M_m}{r_m^2}\,\hat{\mathbf{r}}_m\right)$$

The equation of motion of the c.m. is

$$(M_e + M_m)\,\ddot{\mathbf{R}}_{em} = \mathbf{F}.$$

Now you can verify from the table of physical constants that the earth and moon are very close to each other when compared with their distances from the sun. So we can assume to a good approximation that $\mathbf{r}_e \approx \mathbf{r}_m \approx \mathbf{R}_{em}$.

With this approximation the equation of motion of c.m. becomes

$$(M_e + M_m)\,\ddot{\mathbf{R}}_{em} = \frac{-GM_s}{R_{em}^2}\,(M_e\hat{\mathbf{r}}_e + M_m\hat{\mathbf{r}}_m) = \frac{-GM_s}{R_{cm}^2}\,(M_e + M_m)\hat{\mathbf{R}}_{em}.$$

So the c.m. of the earth-moon system moves around the sun like a planet of mass $(M_e + M_m)$. Its orbit can be determined to be an ellipse by the method used in Unit 6 (See Fig. *7.10).*

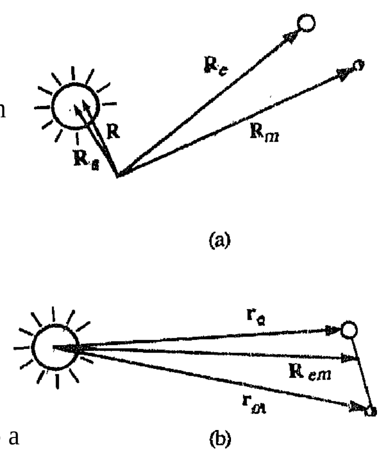



**Fig. 7.9: The earth-moon-sun system.**

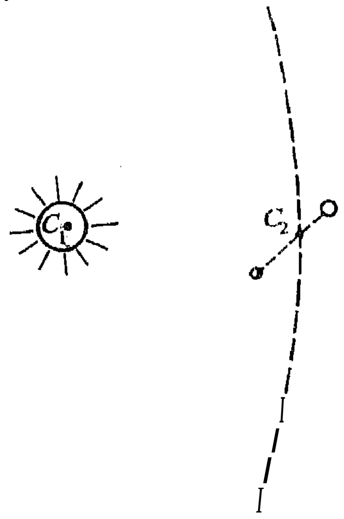



**Fig. 7.10: The c.m. of the earth-moon-sun system ($C_1$) lies at the sun. The c.m. of the earth-moon system ($C_2$) moves around the sun,**

### *N-particle system*

Let us now study the motion of a system of N -particles of masses $m_1, m_2, m_3$ ......., $m_N$ located at positions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, ........., $\mathbf{r}_N$ at an instant *t* with respect to an origin *O*. The position of the c.m. of this N-particle system is given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \dots + m_N\mathbf{r}_N}{m_1 + m_2 + \dots + m_N} = \frac{\sum_{i=1}^{N} m_i\mathbf{r}_i}{M}, \quad (7.19)$$

where $M = \sum_{i=1}^{N} m_i.$

Let $\mathbf{F}_{e1}, \mathbf{F}_{e2}, \dots, \mathbf{F}_{eN}$ be the external forces acting on the particles 1,2, ..., *N*, respectively. These particles are also subject to the mutual forces of interaction. We can now write the equations of motion for all members of the N - particle system. Each particle is subjected to an external force, and forces of interaction due to the other (*N*–1) particles. Thus, we have

$$\begin{aligned}
m_1\ddot{\mathbf{r}}_1 &= \mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_{41} + \dots + \mathbf{F}_{N1} + \mathbf{F}_{e1},\\
m_2\ddot{\mathbf{r}}_2 &= \mathbf{F}_{12} + \mathbf{F}_{32} + \mathbf{F}_{42} + \dots + \mathbf{F}_{N2} + \mathbf{F}_{e2},\\
m_3\ddot{\mathbf{r}}_3 &= \mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{43} + \dots + \mathbf{F}_{N3} + \mathbf{F}_{e3},\\
m_4\ddot{\mathbf{r}}_4 &= \mathbf{F}_{14} + \mathbf{F}_{24} + \mathbf{F}_{34} + \dots + \mathbf{F}_{N4} + \mathbf{F}_{e4}, \qquad (7.20)\\
&\vdots\\
m_N\ddot{\mathbf{r}}_N &= \mathbf{F}_{1N} + \mathbf{F}_{2N} + \mathbf{F}_{3N} + \mathbf{F}_{4N} + \dots + \mathbf{F}_{eN}
\end{aligned}$$

Now, from, Newton's third law we have

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \quad \text{for } i = 1, 2, 3, \dots, N \text{ and } j = 1, 2, 3, \dots, N.$$

Thus, $\mathbf{F}_{12} = -\mathbf{F}_{21}$, $\mathbf{F}_{13} = -\mathbf{F}_{31}$, ...., $\mathbf{F}_{1N} = -\mathbf{F}_{N1}$ and so on.

Now, if we add all these equations, the terms due to mutual interaction of the particles cancel out and we get

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + \ldots + m_N\ddot{\mathbf{r}}_N = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \ldots + \mathbf{F}_{eN}. \quad (7.21a)$$

Using the summation notation we can write Eq. 7.21a in a compact form as

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_{ei} = \mathbf{F}_e \quad (7.21b)$$

where $\mathbf{F}_e$ is the net external force on the N-particle system.

These equations may appear difficult to you in the first instance. Do nor feel scared. You don't have to memorise them. Try to understand the reasoning behind them. The following SAQ may also help you in this regard.

SAQ *6*

Draw the mutual forces of action and reaction acting on each member of the four-particle system shown in Fig. 7.11. Write down the equation of motion for this system.

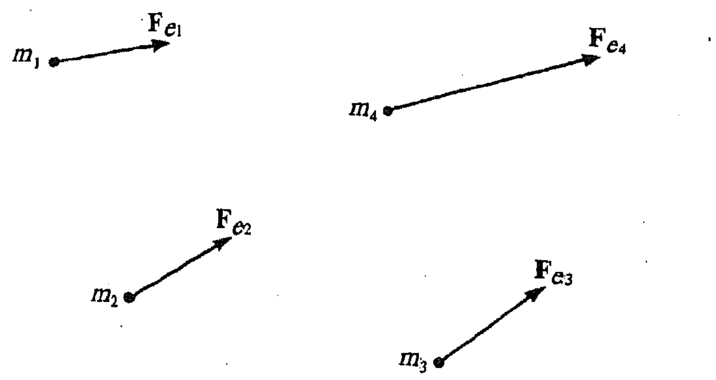



**Fig. 7.11: A four-particle system**

We can again differentiate Eq. 7.19 twice to obtain

$$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i \text{ so that Eq. 7.21b becomes}$$

$$M\ddot{\mathbf{R}} = \mathbf{F}_e \quad (7.22)$$

which is the equation of motion of the c.m. of the system, Again, as long as we are interested only in the motion of a body as a whole, we may replace it by a particle of mass M located at the centre-of-mass. In Fig. 7.12, you can see an example of this result in action for the external force of gravity. In this case we can apply the Eq. 7.35 from the solution of SAQ 1c. The solution of this equation tells us that the centre-of-mass of a complex object follows a simple parabolic path under gravity (see Sec. 2.2.2).

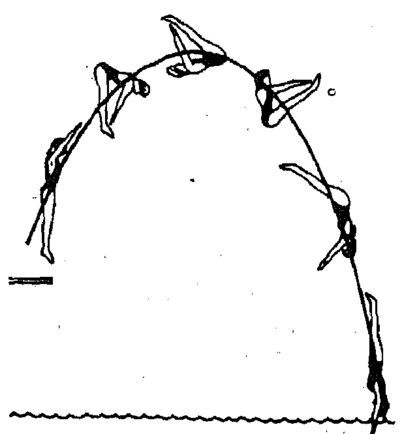


**Fig. 7.12: Centre-of-mass of the diver follows a parabolic path, even though the diver rotates while moving through the air.**

Let us now determine the expressions of the linear and angular momenta and the kinetic energy of an N-particle system in terms of the c.m. coordinate.

### 7.3.1 Linear Momentum, Angular Momentum and Kinetic Energy of an N-Particle System

We can extend Eq. 7.11 for a two-body system to express the total linear momentum of an N-particle system as

$$\mathbf{P} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 + m_3\dot{\mathbf{r}}_3 + \ldots + m_N\dot{\mathbf{r}}_N = \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i \quad (7.23a)$$

Again differentiating Eq. 7.19 with respect to time we get

$$M\dot{\mathbf{R}} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i, \text{ so that } \mathbf{P} = M\dot{\mathbf{R}}. \quad (7.23b)$$

Now, if the net external force acting on the system is zero then from Eq. 7.22 we get

$$\mathbf{P} = M\dot{\mathbf{R}} = \text{constant}. \quad (7.24)$$

This is the principle of conservation of linear momentum which can also be stated as follows:

The velocity of the *c.m.* of an N-particle system remains constant provided no external forces act on it.

You can now apply these ideas to work out an **SAQ.**

**SAQ 7**

Consider a system of three particles, each of mass m, which remain always in the same plane. The particles interact among themselves in a manner consistent with Newton's third **law.** The three particles $A, B, C$ have positions at various times as given in Table 7.1, **i.e.** it shows the $(x,y)$ components (in metres) of their position vectors at three instants.

**Table 7.1**

| Time(s) | A | B | C |
|---|---|---|---|
| 0 | (1,1) | (2,2) | (3,3) |
| 1 | (1,0) | (0,1) | (3,3) |
| 2 | (0,1) | (1,2) | (2,0) |

**Determine** whether any external forces are acting on the system.

The total angular momentum of the N-particle system about any origin $O$ is the vector sum of the angular momenta of individual particles about that origin, **i.e.**

$$\mathbf{L} = m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{r}_2 \times \mathbf{v}_2 + \ldots + m_N\mathbf{r}_N \times \mathbf{v}_N = \sum_{i=1}^{N} m_i\mathbf{r}_i \times \mathbf{v}_i. \quad (7.25)$$

The value of L depends on the choice of the origin $O$, just as it did for a single particle. We can express L in terms of R by subtracting and adding the quantity $\sum_i m_i \mathbf{R} \times \mathbf{v}_i$ from **Eq.** 7.25. Thus,

$$\mathbf{L} = \sum_{i=1}^{N} m_i (\mathbf{r}_i - \mathbf{R}) \times \mathbf{v}_i + \sum_{i=1}^{N} m_i \mathbf{R} \times \mathbf{v}_i.$$

$(\mathbf{r}_i - \mathbf{R}) = \mathbf{r}_i'$, say, is the position **vector** of the $i^{th}$ **particle** about the **c.m.** So $m_i \mathbf{r}_i' \times \mathbf{v}_i$ is the angular **momentum** of the $i^{th}$ particle about the **c.m.** Thus, the first term is the **sum** of the angular momenta of the particles about the **c.m.** It can be denoted by $\mathbf{L}_{cm}$.

Since **R** is constant, from Eq. **7.23a** the second term can be expressed as

$$\mathbf{R} \times \sum_{i=1}^{N} m_i\mathbf{v}_i = \mathbf{R} \times \mathbf{P}.$$

Therefore, the total angular momentum of the N-particle system can be expressed as

$$\mathbf{L} = \mathbf{L}_{cm} + \mathbf{R} \times \mathbf{P}. \quad (7.26)$$

If no net external force acts on the **system,** then as we have seen in Example **2,** the **c.m.** can be taken to be at rest. So we can choose the origin of the coordinate system at the **c.m., i.e.** R = **0.** In this'case the expression of L further simplifies to

$$\mathbf{L} = \mathbf{L}_{cm} = \sum_{i=1}^{N} m_i\mathbf{r}_i \times \mathbf{v}_i \quad (7.27)$$

Note that in the Eq. **7.27**, $\mathbf{r}_i$ ***is the position vector of the $i^{th}$ particle with respect to the centre-of-mass.*** We can make use of the expression to estimate the angular momentum of the Solar System.

**Example** 4: Angular momentum **of the Solar System**

The sun is very massive when compared with the planets. So according to Example **3,** the c.m. of the Solar System is very nearly at the position of the sun. Thus, according to Eq. ***7.27*** the total angular momentum of the Solar System is the sum of the angular momenta of the planets and that of the sun about the centre of the sun. Let us make an estimate of the angular momentum of one of the planets, say Jupiter. Since Jupiter's orbit is very nearly circular, the magnitude of its angular momentum about the centre of the sun is

$$L_j = M_j\, \omega_j r_j^2$$

where $M_j$, $\omega_j$, $r_j$ are Jupiter's mass, angular speed and mean distance from the sun, respectively. But $\omega_j = 2\pi/T_j$, where $T_j$ is the time period of revolution of Jupiter around the sun. Substituting the numerical values $M_j = 1.90 \times 10^{27}$ kg, $T_j = 11.9$ years, $r_j = 7.78 \times 10^{11}$m, we get

$$L_j = (1.9 \times 10^{27}\,\text{kg}) \times \left(\frac{2\pi}{(11.9)\times(365.25)\times(86400)\text{s}}\right) \times (7.78 \times 10^{11}\text{m})^2$$

$$= 1.92 \times 10^{43}\,\text{kg m}^2\,\text{s}^{-1}.$$

Likewise, the angular momenta of other planets can be estimated by assuming circular orbits.

The angular momentum of the sun about its axis is approximately $6 \times 10^{41}\,\text{kg m}^2\,\text{s}^{-1}$. Now, all planets move in the same sense around the sun and the sun moves in that sense about its axis. So the directions of the angular momenta of the planets and the sun are the same.

Therefore, the magnitude of the total angular momentum of the solar system about the centre of the sun is obtained by simply adding the magnitude of the planets' and the sun's angular momenta. It is $3.2 \times 10^{43}\,\text{kg m}^2\,\text{s}^{-1}$, which is **a** constant. **It** can be seen that a huge torque (which may act for a small duration of time) will be required to disrupt this system. You can also see that the sun's angular momentum about an axis through its centre is less than 2% of the total angular momentum of the Solar System. **A** typically hotter star may carry about 100 times as much angular momentum as that of the sun. Thus the process of formation of a planetary system is apparently a mechanism for carrying off angular momentum from a cooling star.

---

The total kinetic energy of the system of N particles is

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2. \tag{7.28}$$

We can express the total kinetic energy in terms of the c.m. coordinates. In our discussion on angular momentum we have defined the position vector of the $i^{th}$ particle with respect to the c.m. as

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}, \quad i = 1, 2, \ldots\ldots, N. \tag{7.29}$$

From the definition of the c.m. we yet the condition

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = \sum_{i=1}^{N} m_i \mathbf{R},$$

$$\text{or} \quad m_1(\mathbf{r}_1 - \mathbf{R}) + m_2(\mathbf{r}_2 - \mathbf{R}) + \ldots + m_N(\mathbf{r}_N - \mathbf{R}) = 0$$

$$\text{or} \quad \sum_i m_i(\mathbf{r}_i - \mathbf{R}) = \sum_i m_i \mathbf{r}'_i = 0. \tag{7.30}$$

Differentiating Eqs. 7.29 and ***7.30*** with respect to tinne, we also get

$$\mathbf{v}'_i = \mathbf{v}_i - \dot{\mathbf{R}}, \tag{7.31a}$$

$$\text{and} \quad \sum_i m_i \mathbf{v}'_i = 0. \tag{7.31b}$$

Substituting $(\mathbf{v}_i' + \dot{\mathbf{R}})$ for $\mathbf{v}_i$ (from **Eq. 7.31a)** in **Eq. 7.28** we get

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i (\mathbf{v}_i \cdot \mathbf{v}_i) = \frac{1}{2}\sum_{i=1}^{N} m_i [v_i'^2 + \dot{R}^2 + 2\mathbf{v}_i' \cdot \dot{\mathbf{R}}]$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2 + \frac{1}{2}\sum_{i=1}^{N} m_i \dot{R}^2 + \left(\sum_{i=1}^{N} m_i \mathbf{v}_i'\right) \cdot \dot{\mathbf{R}}$$

($\because$ R is a constant independent of $i$.)

The last term in this expression is zero in view of Eq.7.31b. Again as **R** does not depend on i the second term is simply $\frac{1}{2} M\dot{R}^2$.

Hence we can express total kinetic energy as

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\sum_{i=1}^{N} m_i v_i'^2 \qquad (7.32)$$

The first term in **Eq. 7.32** depends on the total mass $M$ and on the motion of the c.m.

The second term depends on the internal coordinates and velocities of the system. Eq. **7.32** implies that a certain amount of **K.E.** is locked up, as it was in the motion of **c.m.** In the absence of external forces, R remains constant and thus the first term does not change. This means that during the collision of two objects, only a certain fraction of their total **K.E.** is available for cohversion to other purposes. Let us now consider a simple example to explain the above fact.

---

Example 5

Show that if a moving object of mass $m_1$ (=2 units) strikes a stationary object of mass $m_2$ (= 1 unit), then **66.7%** of the initial **K.E.** is **locked up** in the motion of **c.m.** and only the remaining is available for the purpose of producing **deformations** and so **on, when** the objects collide.

We have from **Eq. 7.32** that

$$T = \underline{\frac{1}{2}(m_1 + m_2)\dot{R}^2} + \frac{1}{2}(m_1 v_1'^2 + m_2 v_2'^2) \qquad (7.33)$$

where the underlined part is the contribution of the first term.

Here $m_1 = 2, m_2 = 1, \mathbf{v}_1 = \mathbf{v}$ (say), $\mathbf{v}_2 = 0$.

$\therefore\ m_1 + m_2 = 3$ and $\dot{R} = \frac{2}{3} v$.

Again from Eq. 7.31a,

$$\mathbf{v}_1' = \mathbf{v}_1 - \frac{2\mathbf{v}_1}{3} = \frac{\mathbf{v}}{3} \text{ and } \mathbf{v}_2' = \mathbf{v}_2 - \frac{2\mathbf{v}_1}{3} = -\frac{2\mathbf{v}}{3}$$

$\therefore$ From Eq. 7.33,

$$T = \frac{3}{2} \times \frac{4v^2}{9} + \frac{1}{2}\left(\frac{2v^2}{9} + \frac{4v^2}{9}\right)$$

$$\text{or } T = \frac{2v^2}{3} + \frac{v^2}{3} = v^2.$$

Hence the **first** term is equal to **2/3** of the total. Or in other words (2/3) × 100, **i.e.** 66.7 per cent of the initial **K.E.** is locked up in the motion of **c.m.** and the remaining is available for conversion to other purposes.

---

While studying this unit you must have noticed the remarkable similarities between the results for a single particle and a many-particle system. The **analogy** is direct for the expression of linear momentum and equations of motion under an applied force. In the expressions for angular momentum and K.E. of a many-particle system we find an additional term for each. We are presenting these similarities in a compact form in Table 7.2.

Table 7.2

| Results | Single particle | Many-particle system |
|---|---|---|
| Linear momentum | $\mathbf{p} = m\,\dot{\mathbf{r}}$ | $\mathbf{P} = M\,\dot{\mathbf{R}}$ |
| Equation of motion | $\dot{\mathbf{p}} = m\,\ddot{\mathbf{r}} = \mathbf{F}_e$ | $\dot{\mathbf{P}} = M\,\ddot{\mathbf{R}} = \mathbf{F}_e$ |
| Equation of motion when external force is absent | $\dot{\mathbf{p}} = 0$ | $\dot{\mathbf{P}} = 0$ |
| Angular momentum | $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ | $\mathbf{L} = \mathbf{R} \times \mathbf{P} + \mathbf{L}_{cm}$ |
| Kinetic Energy | $K.E. = \frac{1}{2} m\dot{r}^2$ | $K.E. = \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\sum_i m_i v_i'^2$ |

Let us now summarise what you have learnt in this unit.

## 7.4 SUMMARY

- For two bodies of masses $m_1$, $m_2$ and position vectos $\mathbf{r}_1$, $\mathbf{r}_2$, respectively, the coordinates of c.m. and the relative coordinate of $m_1$ with respect to $m_2$ are given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

- The differential equation of motion for each particle in a two-body system under their mutual interaction force can be expressed as

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21}, \quad m_2\,\ddot{\mathbf{r}}_2 = \mathbf{F}_{12} = -\mathbf{F}_{21}$$

These can be reduced effectively to a single differential equation of motion given by

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21} \quad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$\mu$ is known as the reduced mass of the system.

- The expressions of linear and angular momenta and K.E. of a two-bodv system are given by

$$\mathbf{p} = M\mathbf{V}$$

$$\mathbf{L} = \mathbf{R} \times M\mathbf{V} + \mu\mathbf{r} \times \mathbf{v},$$

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2}\mu v^2,$$

where $M = m_1 + m_2$,

$$\mathbf{V} = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2} = \text{the velocity of c.m.}$$

$\mathbf{v} = \dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2$ = the relative velocity of $m_1$ with respect to $m_2$.

- The c.m. coordinate of an $N$-particle system having masses $m_1$, $m_2$, $m_3$, ..... and position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, .... is given by

$$\mathbf{R} = \sum_{i=1}^{N} m_i\mathbf{r}_i \Big/ \sum_{i=1}^{N} m_i$$

- The differential equation of motion of an N-particle system under the influence of a total external force $\mathbf{F}_e$ and mutual interaction forces is given as

$$M \ddot{\mathbf{R}} = \mathbf{F}_e \text{ where } M = \sum_{i=1}^{N} m_i$$

This indicates that the motion of the system is equivalent to the motion of its c.m. with mass $M$ under the influence of the external force only.

- The linear and angular momenta and K.E. of an N-particle system are given ,by

$$\mathbf{P} = M\dot{\mathbf{R}} \ ,$$

$$\mathbf{L} = \mathbf{L}_{cm} + \mathbf{R} \times \mathbf{P} \ ,$$

where $\mathbf{L}_{cm}$ = Angular momentum of the System about the c.m. and

$$T = \frac{1}{2} M\dot{R}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i v_i'^2 \text{ where } \mathbf{v}_i' = \mathbf{v}_i - \dot{\mathbf{R}}$$

## 7.5 TERMINAL QUESTIONS

1. Two particles P and Q of masses 0.1 kg and 0.3 kg, respectively, are initially at rest 1m apart. They attract each other with a constant force 1 N. No external force acts on the system. Describe the motion of the c.m. At what distance from P's original position do the particles collide?

2. a) Two astronauts (Fig. 7.13) each having a mass of 80 kg are connected by a light rope 8 m long. They are isolated in space, orbiting their c.m. (C ) at a speed of 5 m s$^{-1}$. Treat the astronauts as particles and a) calculate the angular momentum and K.E. of the system.'

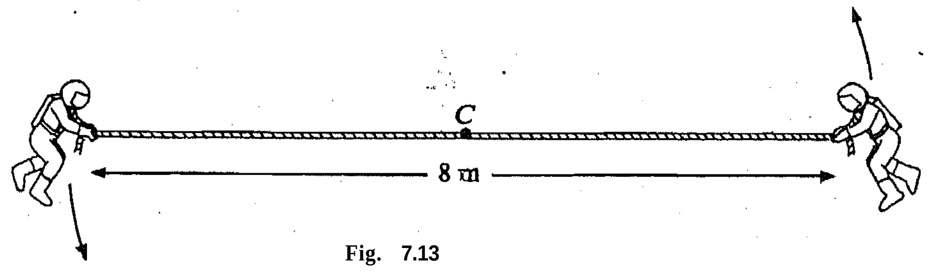



Fig. 7.13

By pulling the rope, the astronauts move closer to each other and their separation becomes 4 m.

b) What is the present angular momentum of the system ?

c) What are their new speeds ?

d) Does the K.E. of the system remain the same as that in case (a) ?

3. Two identical balloons are joined by a thin membrane (Fig. 7.14). Initially one is filled with gas while the other is in a collapsed state. The mass of the material of the balloons is negligible in comparison to the mass of the gas. At a certain instant the membrane ruptures, allowing the gas to fill the balloons equally. Assuming that there is no friction and that only horizontal motion can occur, determine which way (left or right) must the balloons move.

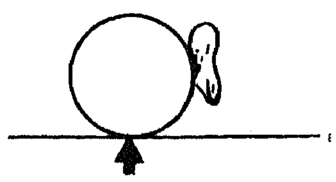


Fig. 7.14

## 7.6 ANSWERS

**SAQs**

1. a) Using Eq. **7.2,** we get

$$\mathbf{r}_1' = \mathbf{r}_1 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_2(\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2} = \frac{m_2\mathbf{r}}{M}$$

and $\mathbf{r}_2' = \mathbf{r}_2 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m_2(\mathbf{r}_2 - \mathbf{r}_1)}{m_1 + m_2} = -\frac{m_1\mathbf{r}}{M}$

b) If there is an additional external force then Eqs. 7.1a and 7.1b take the following forms :

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_{21} + \mathbf{F}_{e1}, \quad (7.34a)$$

$$m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_{12} + \mathbf{F}_{e2}. \quad (7.34b)$$

Adding Eqs. 7.34a and 7.34b and using $\mathbf{F}_{21} = -\mathbf{F}_{12}$ we get $m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_e$, where $\mathbf{F}_e = \mathbf{F}_{e1} + \mathbf{F}_{e2}$ = the net external force.

Hence using Eq. 7.2, we have

$$M\ddot{\mathbf{R}} = \mathbf{F}_e \quad (7.35)$$

where $M = m_1 + m_2$.

Again, from Eqs. 7.34a and 7.34b, we get

$$\ddot{\mathbf{r}}_1 = \frac{\mathbf{F}_{21}}{m_1} + \frac{\mathbf{F}_{e1}}{m_1}, \quad \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{12}}{m_2} + \frac{\mathbf{F}_{e2}}{m_2}$$

or $$\frac{d^2}{dt^2}(\mathbf{r}_1 - \mathbf{r}_2) = \left[\frac{1}{m_1} + \frac{1}{m_2}\right]\mathbf{F}_{21} + \left[\frac{\mathbf{F}_{e1}}{m_1} - \frac{\mathbf{F}_{e2}}{m_2}\right] \quad (\because \mathbf{F}_{21} = -\mathbf{F}_{12})$$

or $$\ddot{\mathbf{r}} = \frac{\mathbf{F}_{21}}{\mu} + \left[\frac{\mathbf{F}_{e1}}{m_1} - \frac{\mathbf{F}_{e2}}{m_2}\right] \quad (7.36)$$

So, we get two equations (7.35 and 7.36). Thus the case cannot be reduced to an equivalent one-body problem.

c) If the external force is that of gravity then $\mathbf{F}_{e1}/m_1 = \mathbf{F}_{e2}/m_2 = \mathbf{g}$. Hence Eq. 7.36 can be simplified to $\mu\ddot{\mathbf{r}} = \mathbf{F}_{21}$ which is the same as Eq. 7.8. The right-hand side of Eq. 7.35 still remains non-zero. But in this case it reduces to $\ddot{\mathbf{R}} = \mathbf{g}$ whose solution is quite well-known (see Eq. 2.9 of Block 1). It is given by $\mathbf{R} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 + \mathbf{B}$, where the symbols have their usual meanings. So effectively, this reduces to a one-body problem where we have to solve Eq. 7.36 only.

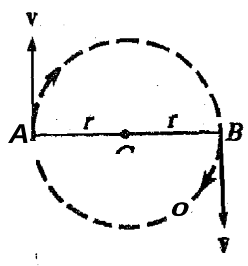



Fig. 7.15

2. Refer to Fig. 7.15. The two stars $A$ and B are moving in uniform circular motion about their common centre-of-mass $C$. Let their masses be $m_1$ and $m_2$. According to condition (b) $m_1 = m_2 = m$ (say), $AC = BC = r$ and the separation between the stars $= 2r$.

Now, if T be the time period of rotation of a star then

$$\frac{2\pi r}{T} = v$$

or $$2r = \frac{vT}{\pi} = \frac{(220 \times 1000\,\text{ms}^{-1}) \times (1.2 \times 10^6\,\text{s})}{\pi} = 8.4 \times 10^{10}\,\text{m}$$

Our next task is to calculate the reduced mass $\mu$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m^2}{2m} = \frac{m}{2}, \text{ or } m = 2\mu.$$

Again using Eq.7.8 we may write that

$$\mu a = \frac{G\,m_1 m_2}{(2r)^2}, \quad (7.37)$$

where a = the magnitude of the relative acceleration of $A$ and $B$. At any instant their accelerations are directed towards C. They are equal in magnitude ($=v^2/r$) and opposite in direction. So using Eq. 1.36 of Unit 1, we understand $a = 2v^2/r$. So on putting $m_1 = m_2 = m = 2\mu$ and $v = 2\pi r/T$, and using Eq. 7.37, we get

$$\frac{G\,4\mu^2}{4r^2} = \frac{2\mu}{r}\frac{4\pi^2 r^2}{T^2}$$

$$\text{or } \mu = \frac{8\pi^2 r^3}{GT^2} = \frac{\pi^2 (2r)^3}{GT^2}$$

$$\therefore \mu = \frac{\pi^2 \times (8.4 \times 10^{10}\,\text{m})^3}{(6.673 \times 10^{-11}\,\text{Nm}^2\,\text{kg}^{-2}) \times (1.2 \times 10^6\,\text{s})^2} = 6.1 \times 10^{31}\,\text{kg}.$$

**3.** As we have assumed that no external force acts on the system, the c.m. of yourself and the cage will remain at rest. Thus in order to take the cage away from the edge of a cliff you must move towards the edge. You must avoid moving towards the other side as in this case the cage will move towards the edge of the cliff.

For the final part of the problem we shall apply the equation $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$, ($\because$ the velocity of the c.m. is zero), where $m_1$ and $m_2$ are the masses of the cage and yourself, respectively. Now, your speed $v_2 = x_2/t$, where $x_2$ is the maximum distance that you can move in the cage, say in a time t, i.e., $x_2 = 2\text{m}$. Then $v_1 = x_1/t$, where $x_1$ is the maximum distance through which the cage can be moved in the opposite direction (as $\mathbf{v}_1$ must be opposite to $\mathbf{v}_2$) during the same time $t$. Thus

$$x_1 = \frac{m_2 x_2}{m_1} = \frac{(60\text{kg})\,(2\text{m})}{(90\text{kg})} = 1.3\text{ m}.$$

**4.** a) From Eq.7.12 on applying the result $\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$, we get

$$\frac{d\mathbf{L}}{dt} = \dot{\mathbf{R}} \times M\mathbf{V} + \mathbf{R} \times M\dot{\mathbf{V}} + \mu\dot{\mathbf{r}} \times \mathbf{v} + \mu\mathbf{r} \times \dot{\mathbf{v}}.$$

Now since $\dot{\mathbf{R}} = \mathbf{V}$ and $\dot{\mathbf{r}} = \mathbf{v}$, the first and the third term vanish.

Again as no internal force acts on the system, the velocity of c.m. is constant and $\dot{\mathbf{V}} = 0$. So the second term also does not survive and we are left with

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mu\dot{\mathbf{v}},$$

Now $\mu\dot{\mathbf{v}} = \mu\ddot{\mathbf{r}} = \mathbf{F}_{21}$. From Eq. 7.7 it is given that $\mathbf{F}_{21}$ is central. So $\mathbf{F}_{21}$ is either parallel or anti-parallel to $\mathbf{r}$. Hence the cross product of r with $\mathbf{F}_{21}$ vanishes.

$$\therefore \frac{d\mathbf{L}}{dt} = 0 \text{ which means that L is conserved.}$$

b) $$T = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2$$

We know from Eqs. 7.4a and 7.4b that

$$\mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\mathbf{r} \quad\text{and}\quad \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\mathbf{r}.$$

$$\text{So } \dot{\mathbf{r}}_1 = \dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}} \quad\text{and}\quad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}.$$

$$\therefore \dot{r}_1^2 = \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = \dot{R}^2 + \frac{(m_2)^2}{M^2}\dot{r}^2 + \frac{2m_2}{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}$$

$$\text{and } \dot{r}_2^2 = \dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = \dot{R}^2 + \frac{(m_1)^2}{M^2}\dot{r}^2 - \frac{2m_1}{M}\dot{\mathbf{R}}\cdot\dot{\mathbf{r}}.$$

$$\therefore T = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{M^2}\dot{r}^2(m_1 + m_2)$$

$$\text{or } T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\frac{m_1 m_2}{M}v^2 \qquad (\because m_1 + m_2 = M,\ \dot{r} = v)$$

Now as $\mu = \frac{m_1 m_2}{M}$, we get $T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu v^2$.

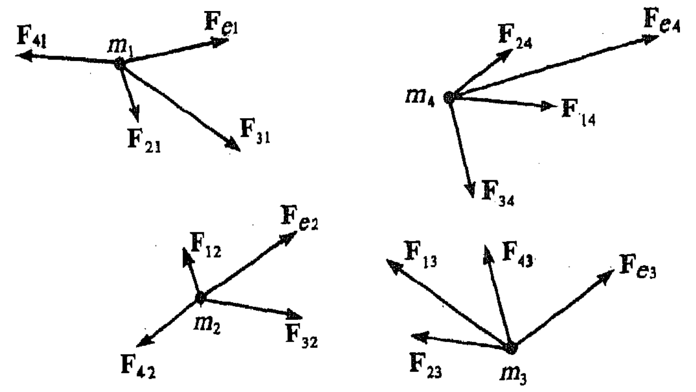
**5.** a) From Eq. 7.14, we get $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2$ and $\mathbf{r}_{31} = \mathbf{r}_1 - \mathbf{r}_3$.

b) The equation of motion for particles 2 and 3 will be as follows :

$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{e2} + \mathbf{F}_{12} + \mathbf{F}_{32}$$

$$\text{and} \qquad m_3\ddot{\mathbf{r}}_3 = \mathbf{F}_{e3} + \mathbf{F}_{13} + \mathbf{F}_{23}.$$

6.




**Pig.** 7.16: The forces acting on each member **of** the four-particle system

The equation of motion for this system is

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 + m_3\ddot{\mathbf{r}}_3 + m_4\ddot{\mathbf{r}}_4 = \mathbf{F}_{e1} + \mathbf{F}_{e2} + \mathbf{F}_{e3} + \mathbf{F}_{e4}.$$

7. We shall first construct a Table (7.3) following Table 7.1 to indicate the position vectors $\mathbf{r}_A$, r, and $\mathbf{r}_C$ of $A$, B, and C and the position vector $\mathbf{R}_t$ of the c.m. at $t$ = 0, .I and 2s, respectively. Remember that $m_A$ = m, $= m_C = m$.

**Table 7.3**

| $t$ | $\mathbf{r}_A$ | $\mathbf{r}_B$ | $\mathbf{r}_C$ | $\left[\mathbf{R}_t = \frac{m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C}{m_A + m_B + m_C}\right]$ |
|---|---|---|---|---|
| 0 | $\hat{i} + \hat{j}$ | $2\hat{i} + 2\hat{j}$ | $3\hat{i} + 3\hat{j}$ | $\mathbf{R}_0 = \frac{m(6\hat{i} + 6\hat{j})}{3m} = 2\hat{i} + 2\hat{j}$ |
| 1 | $\hat{i}$ | $\hat{j}$ | $3\hat{i} + 3\hat{j}$ | $\mathbf{R}_1 = \frac{m(4\hat{i} + 4\hat{j})}{3m} = \frac{4}{3}\hat{i} + \frac{4}{3}\hat{j}$ |
| 2 | $\hat{j}$ | $\hat{i} + 2\hat{j}$ | $2\hat{i}$ | $\mathbf{R}_2 = \frac{m(3\hat{i} + 3\hat{j})}{3m} = \hat{i} + \hat{j}$ |

Now the average velocity of the c.m. during the interval

$$t = 0 \text{ to } 1\text{s} = \frac{\mathbf{R}_1 - \mathbf{R}_0}{1 - 0} = \frac{-2}{3}(\hat{i} + \hat{j})$$

and that during the interval $t$ = 1 to 2s $\frac{\mathbf{R}_2 - \mathbf{R}_1}{2-1} = \frac{-1}{3}(\hat{i} + \hat{j})$ .

This indicates that the velocity of the c.m. has changed. Hence, some external force has acted on the system.

## Terminal Questions

1. Since there is no external force, the velocity of the c.m. remains constant. In other words, the c.m. remains at rest as its initial velocity is zero.

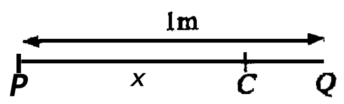



Fig. 7.17

Since the c.m. remains at rest, its position must coincide with that of the particles at the instant of their collision. So they collide at the position of their stationary c.m. $C$ (Fig.7.17). Our task is now to obtain PC.

Let PC $= x$ m.

Then CQ = $(1-x)$ m. As C is the c.m. we have

$m_P(PC) = m_Q(CQ)$

or $(0.1\text{ kg}) \times x\text{ m} = (0.3\text{ kg}) \times (1 - x)\text{ m}$

or $4x = 3$,

or $x = \frac{3}{4}$, i.e. $PC = 0.75$ m.

2a) Refer to Fig. 7.13. The angular momentum vectors of the astronauts are parallel (perpendicular to the plane of the paper and pointing towards us) and equal in magnitude. The magnitude of the angular momentum of the system is given by

$$L = L_1 + L_2 = mvr + mvr = 2mvr,$$

where $m = 80$ kg, $v = 5\,ms^{-1}$ and $r = (8/2)\,m = 4\,m$.

$$\therefore\ L = 2\,(80\text{ kg})\,(5ms^{-1})\,(4m) = 3200\text{ kg m}^2 s^{-1}$$

The K.E. of the system $= 2(\frac{1}{2}mv^2) = (80\text{ kg})\,(5\,ms^{-1})^2 = 2000\,J$.

b) The astronauts move close to each other due to equal and opposite internal forces that act along the line joining them. This means that the mutual force is central. And there is no external force. Hence the angular momentum of the system remains conserved, i.e. $3200\text{ kg m}^2 s^{-1}$ in the same direction as that stated in (a).

c) Let V and R be the new speed and radius, respectively.

Then we have,

$2mVR = 3200\text{ kg m}^2 s^{-1}$, where m = 80 kg and R = 4m/2 = 2m.

$$\therefore\ V = \frac{3200\text{ kg m}^2 s^{-1}}{2(80\text{ kg})\,(2m)} = 10\ ms^{-1}.$$

d) The new total K.E. $= 2(\frac{1}{2}mV^2) = (80\text{kg})\,(10m\,s^{-1})^2 = 8000\,J$.

So the new K.E. is greater than that in (a).

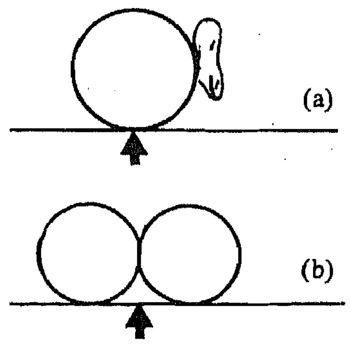


Fig. 7.18

**3.** Refer to Fig.7.18. Here the net external force is zero. So the velocity of c.m. remains invariant. Before the rupture (Fig.7.18a) the c.m. is at rest roughly at the centre of the gas filled balloon. So after the rupture (Fig. 7.18b) the c.m. must remain at the same position. Now, after rupture, the position of c.m. is at the meeting point of the balloons. The arrows in Fig. 7.18 indicate the positions of the c.m. before and after rupture. In order to maintain this fixed position of c.m. the balloons must move to the left.