UNIT 5 STANDARD FORM AND SOLUTIONS

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5.1 INTRODUCTION

In Block 1, you learnt how some real life problems e.g. **Product mix problem, Investment problem, Diet problem, Inspection problem** etc. could be formulated as linear programming problems. You have also studied the graphical method for solving these linear programming problems and found that the graphical method is useful only for the linear programming problems involving two decision variables. It does not help to solve linear programming problems involving three or more variables. **How** to **solve such problems?** Simplex method was invented for this purpose The Simplex Method is applicable to any problem that can be formdlated in terms of a linear objective function subject to a set of linear constraints. As you learn the method, you will see there is no limit on the number of decision variables or constraints in a problem.

Note that the term Simplex has nothing to do with the method as it is now used; it had its origin in a special problem that was studied in early development of the method. We shall have a thorough discussion of this method in Unit 6. However, to discuss Simplex Method, you need to know some basic concepts and results. You ought to know the two important forms of a general linear programming problem namely the standard fonn and canonical form. You should be familiar with some special types of variables namely the **slack** and **surplus variables.** Also, you should identify the nature of solutions such as **feasible solutions, basic solutions, optimal solutions,** etc. In this unit we shall introduce you to these basic requirements of linear programming leading to the Simplex Method.

Objectives

After you have completed this unit, you should be able to:

- convert inequality constraints into equations
- write a given linear programming problem in a standard form and in a canonical form
- identify a feasible solution, a basic solution, a basic feasible solution and an optimal solution (basic or non-basic).

5.2 STANDARD FORM OF A LPP

In Unit 3, we discussed how to formulate a mathematical model for linear' programming problems involving two variables only. In Unit 4, we extended the method to construct a mathematical model for problem involving three or more variables. In all the examples discussed there, we were concerned mainly with two types of problems :

i) **Maximization problems**

ii) **Minimization problems**

Also, you know that the constraints are either '≤' or '=' or '≥' type, and variables in general, are non-negative. But these variables sometimes are unrestricted in the sign of ≤, =, ≥. Thus to deal with all types of variables, restricted or unrestricted or both, we restate **General Linear Programming Model** (GLPM) as follows:

Maximize (or Minimize) the objective function

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \ (\le, =, \ge)\ b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\le, =, \ge)\ b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\le, =, \ge)\ b_m$$

With the restrictions

$$x_1 \ge 0, \quad x_2 \ge 0 \quad \dots\dots\dots\dots\dots \ x_n \ge 0.$$

where c_j, b_i, a_{ij} (i = 1, ... m; j = 1, 2, ... n) are constants and x_j, j = 1, 2,...... ... n are decision variables. Only one sign (≤, =, ≥) holds for each constraint.

In the above problem the vector formed by the coefficients of the variable x_j (j^{th} variable) in all the constraints is called the vector associated to the variable x_j and is denoted by A_j, where

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, j = 1, 2, \dots, n$$

Thus to each variable x_j in a linear programming model, we associate a unique vector $A_j \in E^m$, where, E^m denotes the Eulidean space generated by the m constraints $a_{1j}, a_{2j}, \dots, a_{mj}$. The vectors A_j, j = 1, 2, ... , n are called **Activity Vectors.** The Column vector formed by the quantities $b_1, b_2, \dots b_m$ is called the **Requirement Vector** and is denoted by B. The coefficients c_1, $c_2, \dots c_n$ in the objective function are known as the prices associated with the variables $x_1, x_2, \dots x_n$ respectively. Also vector $C = (c_1, c_2, \dots c_n)$ is known as the **Price Vector,**

After formulating **a** linear programming model, our next step is to solve the model. You have seen in Unit 3 that linear programming models are presentable in various forms (maximization or minimization problems and constraints ≤, = or ≥ or mixed). It is therefore necessary to modify these forms so that we are able to develop a standard solution procedure that we shall be discussing in the next unit. For the development of such a solution procedure, we have to compute the linear programming model into well-known forms namely **the standard form** and **the canonical form.** The canonical form will be especially useful in discussing the **duality theory** which will be presented in Unit 7. The standard form will be used to develop the general procedure for solving any linenr programming problem.

Characteristics of the Standard Form of a LPP

A standard form of a linear programming problem has the following characteristics :

1. All constraints are equations.
2. All variables are non-negative.
3. Objective function is of the maximization or of the minimization type.
4. The right hand side element of each constraint equation is non-negative.

In general, it is much more convenient to work with equations than with inequalities. For this reason we would like to convert the inequalities, if any. in the constraints of a given linear programming problem into equations. This conversion can be carried out very simply by introducing some additional variables, which are called slack or surplus variables.

5.2.1 Slack Variables

We discuss the meaning of a slack variable by means of the following example :

EXAMPLE 1 : The following product mix linear programming problem that we formulated in Unit 3 had both constraints '≤' type.

Maximize

$$Z = 4x_1 + 5x_2$$

subject to

$$2x_1 + 34 \le 12$$
$$3x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0.$$

Reduce its constraints to equations.

SOLUTION : Consider for example, the first constraint $2x, + 3x_2 \le 12$. 'To convert this inequality constraint into an equation, we introduce a new variable

$$x_3 = 12 - 2x_1 - 3x_2 \ge 0.$$

We call x_3 a slack variable, so that by introduction of a new variable $x_3 \ge 0$, the first constraint of our example reduces to

$$2x_2 + 3x_2 + x_3 = 12.$$

Similarly, corresponding to the second constraint we introduce a slack variable $x_4 \geq 0$; and the second constraint reduces to

$$3x_1 + x_2 + x_4 = 8$$

The given linear programming problem may now be written as

Maximize

$$Z = 4x_1 + 5x_2$$

Subject to

$$2x_1 + 3x_2 + x_3 = 12$$
$$3x_1 + x_2 + x_4 = 8$$

where $x_1, x_2, x_3, x_4 \geq 0$.

In general, if in any linear programming problem, we have a constraint of the type

$$a_1x_1 + a_2 x_2 + \ldots + a_n x_n \leq b, \; b \geq 0$$

then this inequality can be converted into an equation by adding some non-negative variable x_{n+1} to the left hand side. This **new variable is called a slack variable** and the constraint is transformed into an equation as

$$a_1x_1 + a_1x_2 + \ldots + a_nx_n + x_{n+1} = b, \text{ where } x_{n+1} \geq 0$$

Thus, **a non-negative variable added to left-handside of a less than or equal to** type **of constraint that converts it into an equation is called a slack variable.** The value of this variable can be interpreted as the amount of unused resource.

'You can try the following similar exercise :

EXERCISE 1 : Given the linear programming problem

Maximize

$$Z = 4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 3$$
$$x_1 + 4x_2 \leq 5$$
$$x_1, x_2, \geq 0$$

reduce its constraints into equations by introducing slack variables.

5.2.2 Surplus Variables

Again to define surplus variables, we consider the following example :

EXAMPLE 2 : Consider a linear programming problem with all '≥' type of constraints say

Minimize

$$Z = 2x_1 + 2x_2$$

subject to

$$3x_1 + 2x_2 \geq 5$$
$$x_1 + 5x_2 \geq 6$$
$$x_1,\ x_2 \geq 0$$

Reduce its constraints to equations by introducing surplus variables.

SOLUTION : For this problem, consider the first constraint

$$3x_1 + 2x_2 \geq 5 \text{ or } 3x_1 + 2x_2 - 5 \geq 0,$$

We now introduce a new variable $x_3 \geq 0$ defined by

$$x_3 = 3x_1 + 2x_2 - 5$$

so that the first constraint of this example reduces to

$$3x_1 + 2x_2 - x_3 = 5.$$

This new variable x_3 which has enabled us to convert an inequality ('2' type) into an equation is subtracted from left-handside of the 'first constraint and is called a **surplus 'variable.**

Similarly, we can convert the second constraint of this problem into an equation by introducing a surplus variable $x_4 \geq 0$ so that second constraint can now be written as

$$x_1 + 5x_2 - x_4 = 6.$$

The given problem, then, reduces to

Minimize

$$Z = 2x_1 + 2x_2$$

Subject to

$$3x_1 + 2x_2 - x_3 = 5$$
$$x_1 + 5x_2 - x_4 = 6$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

In general, let us understand the concept of **surplus variable** by means of a constraint

$$q_1 x_1 + q_2 x_2 + \ldots + q_n x_n \geq q,\ q \geq 0$$

Now something non-negative shpuld be subtracted from the left-handside to convert this inequality into an equation, i.e. we **can** have

$$q_1 x_1 + q_2 x_2 + \cdots + q_n x_n - x_{n+1} = q$$

where $x_{n+1} \geq 0$ is called a surplus variable.

Thus **a non-negative variable subtracted from the left-hand side of a greater than or equal to type of constraint to convert the constraint into an equation is called a surplus variable.** The value of this variable can be interpreted as the amount over and above the required minimum level.

Now try the following exercise yourself:

EXERCISE **2** : The following diet problem formulated in Unit 3 had both constraints $\geq$ type

Minimize

$$Z = 3x_1 + 2.5x_2$$

subject to

$$2x_1 + 4x_2 \geq 40$$
$$3x_1 + 2x_2 \geq 50$$
$$x_1, x_2 \geq 0$$

reduce its constraints into equations by introduction of surplus variables.

In majority of practical linear programming problems as formulated in the GLPM, we shall find that $b_i \geq 0$, $i = 1, 2, \ldots, m$. Suppose that in the original formulation of some constraint i, $b_i \leq 0$. Multiplying both sides of this constraint by -1, we obtain $-b_i \geq 0$. Thus it is a simple matter to convert all constraints into ones with $b_i \geq 0$. Hence let us assume that GLPM represents a set of constraints with each $b_i \geq 0$.

To convert these constraints into the corresponding linear simultaneous equations, you can rearrange these constraints in the following way.

Write (say) the first p constraints with $\leq$ sign. Write the next q constraints (say) with 2 sign and then write the remaining constraints which are originally equations.

Accordingly, then the system of constraints in the GLPM can be written as follows:

The first p constraints with $\leq$ sign are

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p1}x_1 + a_{p2}x_2 + \ldots a_{pn}x_n \leq b_p$$

The next q constraints with $\geq$ sign are

$$a_{p+11}x_1 + a_{p+12}x_2 + \cdots + a_{p+1n}x_4 \geq b_{p+1}$$
$$\vdots$$
$$a_{p+q1}x_1 + a_{p+q2}x_2 + \cdots + a_{p+qn}x_4 \geq b_{p+q}$$

The remaining constraints i.e. equations are

$$a_{p+q+11}x_1 + a_{p+q+12}x_2 + \dots + a_{p+q+1n}x_4 \ge b_{p+q+1}$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Or

$$\sum_{j=1}^{n} a_{ij}\, x_j \le b_i\ , i = 1, 2, \dots , p$$

$$\sum_{j=1}^{n} a_{ij}\, x_j \ge b_i \quad i = p\ 1, \dots, p + q$$

and

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad i = p + q\ , \ \dots , m.$$

Now first p-constraints have '≤' sign, therefore add $x_{n+i} \ge 0, i = 1, 2, \dots,\ p$, p-slack variables one to each of the left hand side of first p-constraints. You may rewrite these constraints as the equations

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i,\ i = 1,\dots, p$$

Similarly, introduce q-surplus variables one to each of the next q-constraints that are ≥ type, and you have the following equations

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i,\ i = p+1,\dots, p + q.$$

Thus, all constraints in GLPM are converted into system of simultaneous linear equations of the form

$$\sum_{j=1}^{n} a_{ij}\, x_j + x_{n+i} = b_i \quad i = 1, \dots , p$$

$$\sum_{j=1}^{n} a_{ij}\, x_j - x_{n+i} = b_i \quad i = p + 1, \dots , p + q$$

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad i = p + q + 1, \dots , m.$$

Let us illustrate it by the following example :

EXAMPLE 3 : **Let us convert the system of constraints**

$$3x_1 + 2x_2 \ge 6$$
$$7x_1 + x_2 \le 5$$
$$x_1 + x_2 = 1$$

into a set of three simultaneous equations

SOLUTION : Note that all entries on the right hand side of the given three constraints are positive. Introduce a surplus variable $x_3 \geq 0$ into the first constraint to obtain .

$$3x_1 + 2x_2 - x_3 = 6.$$

Also introduce a slack variable $x_4 \geq 0$ into the second constraint to obtain

$$7x_1 + x_2 + x_4 - 5.$$

The third constraint is already in an equation form. Therefore the resulting system of equations can be written as

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 6 \\ 7x_1 + x_2 + x_4 &= 5 \\ x_1 + x_2 &= 1 \end{aligned}$$

EXERCISE 3 : Rewrite the following inequalities in the form of equations by introducing Slack or surplus variables

$$\begin{aligned} 1x_1 - 2x_2 &\leq 6 \\ 2x_1 + 5x_2 &\geq 8 \\ x_1 - 3x_2 &\geq -6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5.2.3 Unrestricted Variables

You are familiar with the linear programming problems having non-negative decision variables. What happens if you have a linear programming model in which some of the decision variables are unrestricted in sign i.e. variables are positive, negative or zero?

For this, you have to put each unrestricted variable by two non-negative variables. This is possible because we can write

$$(-3) = (+2) - (+5)$$

In other words, a linear programming model with unrestricted decision variables can be recast into usual standard formulation by such replacements.

Study the following examples which will help us in clarifying this concept.

EXAMPLE 4 : Maximize

$$Z = 3x_1 + 2x_2 - 4x_3$$

subject *to*

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 5 \\ x_1 + 2x_2 + x_3 &\leq 3 \\ 3x_1 + 5x_2 - 2x_3 &\leq 2 \\ x_1 \geq 0,\ x_2 &\geq 0, \end{aligned}$$

While x_3 is unrestricted in sign.

SOLUTION : We write the unrestricted variable x_3 as

$$x_3 = x_3' - x_3'' \text{ where } x_3' \geq 0,\ x_3'' \geq 0$$

Then the given problem reduces to

Minimize
$$Z = 3x_1 + 2x_2 - 4x_3' + 4x_3''$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3' - 3x_3'' &\geq 5 \\ x_1 + 2x_2 - x_3' + x_3'' &\leq 3 \\ 3x_1 + 5x_2 - 2x_3' + 2x_3'' &\leq 2 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' &\geq 0 \end{aligned}$$

EXAMPLE 5: Maximize
$$Z = x_1 - 2x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\quad 10 \\ x_1 - x_2 &\geq -2 \\ -x_1 + 2x_2 &= 7 \end{aligned}$$

x_1, x_2 unrestricted.

SOLUTION: Write $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$ so that $x_1', x_1'', x_2', x_2'' \geq 0$

The problem reduces to

Maximize
$$Z = x' - x_1'' - 2x_2' + 2x_2''$$

subject to

$$\begin{aligned} 3x_1' - 3x_1'' + 2x_2' - 2x_2' &\leq 0 \\ x_1' - x_1'' - x_2' + x_2'' &\geq -2 \\ -x_1' + x_1'' + 2x_2' - 2x_2'' &= 7 \\ x_1', x_1'', x_2', x_2'' &\geq 0. \end{aligned}$$

Thus in general an unrestricted variable x_j can be written as

$$x_j = x_j' - x_j''$$

where $x_j' \geq 0,\ x_j'' \geq 0.$

If $\mathbf{A}_j$ be the vector associated with the variable x_j, then in the restated problem we should write

$$x_j \mathbf{A}_j = (x'_j - x''_j)\mathbf{A}_j = x'_j \mathbf{A}_j + x''_j(-\mathbf{A}_j)$$

Thus in the new problem $\mathbf{A}_j$ and $-\mathbf{A}_j$ are the vectors associated with the variables x'_j and x''_j respectively. In the objective function, we replace $c_j x_j$ by $c_j x'_j - c_j x''_j$.

Now try the following exercise :

EXERCISE 4 : Write the following linear programming problem in a form in which variables are all non-negative.

Maximize
$$Z = 4x_1 + 2x_2 - 3x_3$$

subject to
$$2x_1 + 3x_2 + 4x_3 \le 7$$
$$5x_1 + x_2 + 2x_3 \ge 9$$
$$x_1, x_2 \ge 0, x_3 \quad \text{unrestricted.}$$

A general linear programming problem in its **standard form** can be stated as follows :

Maximize (or Minimize)
$$Z = c_1x_1 + c_2x_2 + \dots c_nx_n$$

subject to
$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$
$$x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0$$

or in the matrix notation, we have:

Maximize (or Minimize) $\mathbf{Z} = CX$

Subject to $\mathbf{AX} = \mathbf{B}$
$\mathbf{X} \ge \mathbf{0}$.

Where $\mathbf{C} = (c_1, c_2, \dots, c_n)$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and $$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

A standard form of a LPP is

- a maximization or a minimization problem
- all constraints are in the form of equations
- all variables are non-negative.

EXAMPLE **6 : Put the following linear programming problem into its standard form**

Minimize

$$Z = -3x_1 + 4x_2 - x_3$$

Subject to

$$-x_1 - 3x_2 + 2x_3 \geq -2$$
$$2x_1 + x_2 - 4x_3 \leq 10$$
$$3x_1 - x_2 + 2x_3 \geq 5$$
$$2x_1 + x_2 + 2x_3 = 3$$
x_1 is unrestricted; $x_2, x_3 \geq 0$.

SOLUTION : Multiply the first constraint by (–1) to ensure that all entries on the right hand side of the constraints are positive,

Then problem takes the form

Minimize

$$Z = -3x_1 + 4x_2 - x_3$$

Subject to

$$+x_1 + 3x_2 - 2x_3 \leq 2$$
$$2x_1 + x_2 - 4x_3 \leq 10$$
$$3x_1 - x_2 + 2x_3 \geq 5$$
$$2x_1 + x_2 + 2x_3 = 3$$
x_1 unrestricted; $x_2, x_3 \geq 0$

Let now $x_1 = x_1' - x_1''$, so that $x_1' \geq 0$, $x_1'' \geq 0$ and problem can be rewritten as

Minimize

$$Z = -3x_1' + 3x1' + 4x_2 - x_3$$

Subject to

$$x_1' - x_1'' + 3x_2 - 2x_3 \quad \leq 2$$
$$2x_1' - 2x_1'' + x_2 - 4x_3 \quad \leq 10$$
$$3x_1' - 3x_1'' - x_2 + 2x_3 \quad 2\ 5$$
$$2x_1' - 2x_1'' + x_2 + 2x_3 \quad = 3$$
$$x_1',\ x_1'',\ x_2,\ x_3 \quad 2\ 0$$

Now, introduce slack variables $x_4 \geq 0$, $x_5 \geq 0$ respectively to the first two constraints and surplus variable $x, \geq 0$ to the third constraint, you may rewrite the above problem in the standard form as

Minimize

$$Z = -3x_1' + 3x_1'' + 4x_2 - x_3$$

Subject to

$$x_1' - x_1'' + 3x_2 - 2x_3 + x_4 = 2$$
$$2x_1' - 2x_1'' + x_2 - 4x_3 + x_5 = 10$$
$$3x_1' - 3x1' - x_2 + 2x_3 - x, = 5$$
$$2x_1' - 2x_1'' + x_2 + 2x_3 = 3$$
$$x_1',\ x_1'',\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0$$

Now try the following exercise :

EXERCISE 5 : Reduce the following linear programming problems respectively into their standard forms

Minimize

[redacted] $= 3x_1 - 2x_2 + 4x_3$

Subject to

$$x, - 2x_2 + 3x_3 \quad \geq 2$$
$$2x_1 + 3x_2 + x, \quad \geq -2$$
$$4x_1 + 2x_2 \quad \geq 2$$
$$x_1, x_2, x_3 \quad \geq 0$$

5.3.4 The Canonical Form of a LPP

The general linear programming problem (GLPP) can always be put in the following form :

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i , \qquad i = 1, 2, \dots , m.$$

This is also called the **canonical form.** The characteristics of this form are :

1. all decision variables are nonnegative,
2. all constraints are of the $(\le)$ type,
3. the objective function is of the maximization type.

Any linear programming problem can be put in the canonical form by the use of following five elementary transformations :

I. The minimization of the objective function Z is equivalent to the maximization of the negative expression of this function, $-Z$.

For example, the linear objective functions,

$$\text{Minimize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n,$$

is equivalent to

$$\textbf{Maximize } Z' = -Z = -c_1x_1 - c_2x_2 - \dots - c_nx_n$$

Consequently, in any linear programming problem the objective function can be put in the maximization form.

II. An inequality in one direction $(\le$ or $\ge)$ may be changed to an inequality in the opposite direction $(\ge$, or $\le)$ by multiplying both sides of the inequality by –1. For example, the linear constraint

$$a_1x_1 + a_2x_2 \ge b ,$$

is equivalent to

$$-a_1x_1 - a_2x_2 \le -b ,$$

Also,

$$p_1x_1 + p_2x_2 \le q ,$$

is equivalent to

$$-p_1x_1 - p_2x_2 \ge -q.$$

III. An equation may be replaced by two weak inequalities in opposite directions, For example

$$a_1x_1 + a_2x_2 = b$$

is equivalent to the two simultaneous constraints,

$$a_1x_1 + a_2x_2 \le b \text{ and } a_1x_1 + a_2x_2 \ge b,$$

or

$$a_1x_1 + a_2x_2 \le b \text{ and } -a_1x_1 - a_2x_2 \le -b.$$

IV. A variable which is unrestricted in sign (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unrestricted in sign, it can be replaced by $(x' - x'')$ where x' and x'' are both non-negative; that is, $x' \ge 0$, and $x'' \ge 0$.

EXAMPLE 7: Consider the linear programming problem :

Minimize

$$Z = 8x_1 - 8x_2 + 9x_3,$$

subject *to*

$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 &\le 60, \\ x_1 + 8x_2 - 7x_3 &\ge 70, \\ 5x_1 + 3x_2 &= 30, \\ x_1 \ge 0,\ x_2 &\ge 0, \end{aligned}$$

where x_3 is unrestricted in sign.

This problem can be put in the canonical form as follows. By the fourth transformation you can have

$$x_3 = x_3' - x_3'',$$

where $x_3' \ge 0$ and $x_3'' \ge 0$. Thus, the canonical form is given by :

Maximize

$$Z' = (-Z) = -8x_1 + 8x_2 - 9(x_3' - x_3''),$$

Subject to

$$\begin{aligned} 2x_1 + 4x_2 + 3(x_3' - x_3'') &\le 60, \\ -x_1 - 8x_2 - 7(x_3' - x_3'') &\le -70, \\ 5x_1 + 3x_2 &\le 30, \\ -5x_1 - 3x_2 &\le -30, \end{aligned}$$

where

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3' \ge 0, \quad x_3'' \ge 0.$$

Note that in this case that the only difference between the original and the canonical forms occurs in the objective function where Z in the original problem becomes equal to (–Z') in the canonical form. The values, however, of the variables are the same in both cases, since the constraints are essentially the same.

5.4 TYPES OF SOLUTIONS OF A LPP

In this, section, we introduce some standard definitions and describe a few characteristics of a solution to the general linear programming problem.

Any vector $X = (x_1, x_2, \ldots, x_n)$ that satisfies the constraints of a linear programming problem is called a **solution** to **a** linear programming problem, If in addition to the constraints the vector $X = (x_1, x_2, \ldots, x_n)$ also satisfies non-negativity restrictions, then it is called a **feasible solution** of a linear programming problem.

For instance in exercise **1** of this unit,

$x_1 = 3, x_2 = -3$ is a solution, where as $x_1 = 1, x_2 = 1$ is **a** feasible solution.

Also, you can check, $x_1 = 1, x_2 = 1$ is **a** feasible solution in example **2.**

Consider a system of two simultaneous equations in two unknowns namely

$$3x_1 + 5x_2 = 13$$
$$2x_1 + x_2 = 4$$

which can be written in matrix notation as

$$\begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13 \\ 4 \end{pmatrix},$$

i.e. this system is of the form **AX** = B

$$\text{where } A = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} \; X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 13 \\ 4 \end{pmatrix}.$$

Note that the rank of the coefficient matrix is

$\rho(A) = 2.$ (Refer to Unit 1)

We can solve this system uniquely and solution is given by

$$X = A^{-1}B$$

i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{pmatrix} \begin{pmatrix} 13 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus

$$x_1 = 1, x_2 = 2$$

which gives the solution of the system of the equations. Note that you could obtain a unique solution to this system since A^{-1} is unique and the number of unknowns is equal'to number of equations.

Now, the question arises : **How to solve a system of two equations in three or more variables?**

For example let us consider the following system of **two** equations in **four** unknowns

$$2x_1 + x_2 + x_3 = 3$$
$$x_1 + 4x_2 + x_4 = 5.$$

This system is again of the form AX = B where now

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

It is clear, $\rho(A) = 2$, i.e. number of linearly independent columns of A is two out of the four columns in the coefficient matrix A. Let us select any two linearly independent columns say the last two columns $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Form a 2 x 2 non-singular matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

from A. Now put x_1, x_2 (the variables that are not associated with columns of this submatrix S of A) equal to zero. Then the solution of the resulting system is unique and is. given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

or

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

i.e. $x_3 = 3, x_4 = 5$

Thus, $x_1 = 0, x_2 = 0, x_3 = 3, \ x_4 = 5$ is a solution of the original system with two variables equal to zero. This solution is called a **basic solution** variables x_3 and x_4 that are non-zero are called **basic variables.**

In general, if you are given a system of m-simultaneous equations in n unknown as AX = B where A is an mxn matrix with $\rho(A) = m$ (<n). Select a $m \times m$ non-singular submatrix S from A.Set all those components of X, that

are not associated with columns of S equal to zero. Then the solution of the resulting system is called a Basic **Solution.** The 111-variables which can be different from zero are called Basic **Variables.**

To make the method more simple, partition A into two submatrices S and T where S is m×m non-singular and T consists of those columns of A which have not been included in S. Let X_S and X_T be the vectors of the variables associated with columns of S and T respectively. Then, you may put the system

$$AX = B$$

as

$$(S, T)\begin{pmatrix} X_S \\ X_T \end{pmatrix} = B$$

$$SX_S + TX_T = B$$
$$SX_S = B - TX_T$$

Since S is non-singular, therefore S^{-1} exists. Premultiply this last relation by S^{-1}, so that you can have

$$S^{-1} SX_S = S^{-1}B - S^{-1}T X_T$$
$$X_S = S^{-1}B - S^{-1}T X_T$$

Now set

$$X_T = O,$$

where O is a zero-matrix. Then, it follows that

$$X_S = S^{-1} B.$$

This is called a **Basic solution** and m-variables included in X_S are called **Basic variables.** Non-basic variables in X_T are zero variables. In general, (X_S, O) is a Basic Solution of the system of equations AX = B, where O'is a Zero matrix. **"How many basic solutions are possible in a system of m-equations in n-unknowns?"** The answer to this-question is the number of combinations of n things taken m at a time i.e.

$$n_{C_m} = \frac{n!}{m!\,(n-m)!}$$

is the maximum number of basic solutions that we may get from the system AX = B.

Let us discuss an example to illustrate the method :

EXAMPLE 8 : Find all basic solutions to the following system of equations

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

SOLUTION : The given system is of the form AX = B where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Here $\rho(A) = 2$. Therefore a basic solution will have two components different from zero. Number of basic solutions will be

$$^3C_2 = \frac{3\,!}{2\,!\,1\,!} = 3 ,$$

since m = 2, n = **3.**

To find all basic solutions let us first set $x_3 = 0$, the system reduces to the form

$$x_1 + 2x_2 = 4$$
$$2x_1 + x_2 = 5$$

i.e. we have

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Here $X_T = (x_3) = (0)$, $X_S = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In other words

$$x_1 = 2, x_2 = 1, x_3 = 0$$

is a basic solution, where x_1 and x_2 are basic variables.

Now, again set $x_2 = 0$, therefore the original system reduces to

$$x_1 + x_3 = 4$$
$$2x_1 + 5x_3 = 5$$

We may now solve this uniquely as

$$\begin{pmatrix}1 & 1\\2 & 5\end{pmatrix}\begin{pmatrix}x_1\\x_3\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$$

or $$\begin{pmatrix}x_1\\x_3\end{pmatrix} = \begin{pmatrix}1 & 1\\2 & 5\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}\frac{5}{3} & -\frac{1}{3}\\-\frac{2}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$$

Therefore $x_1 = 5, x_2 = 0, x_3 = -1$. This is another basic solution to the given system.

Similarly, we can set $x_1 = 0$ and the system reduces to

$$2x_2 + x_3 = 4$$
$$x_2 + 5x_3 = 5$$

or $$\begin{pmatrix}2 & 1\\1 & 5\end{pmatrix}\begin{pmatrix}x_2\\x_3\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$$

$$\begin{pmatrix}x_2\\x_3\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 5\end{pmatrix}^{-1}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}\frac{5}{9} & -\frac{1}{9}\\-\frac{1}{9} & \frac{2}{9}\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}\frac{5}{3}\\\frac{2}{3}\end{pmatrix}$$

and $x_1 = 0, x_2 = \frac{5}{3}, x_3 = \frac{2}{3}$ is a basic solution to the given system. Thus **we** are able to determine all the three basic solutions to the given system which are

$$x_1 = 2, \quad x_2 = 1, \quad x_3 = 0$$
$$x_1 = 5, \quad x_2 = 0, \quad x_3 = -1$$
$$x_1 = 0, \quad x_2 = \frac{5}{3}, \quad x_3 = \frac{2}{3}.$$

Here you must note something special regarding these three solutions. The basic variables in the first and third solutions here satisfy the non-negativity restrictions, whereas in the second solution, one of the basic variables is negative. Hence first and third solutions above are **basic feasible solutions whereas-the secqnd one is basic non-feasible.** Thus we **can** now define a **basic feasible solution to be a basic solution that satisfies the non-negativity restrictions i.e. all basic variables are non-negative.**

Hence a basic solution $(X_B, 0)$ to the system $AX = B$ is basic feasible if $X_B \geq 0$.

EXERCISE 6 : Find the basic solutions of

$$2x_1 + x_2 - x_3 = 2$$
$$3x_1 + 2x_2 + x_3 = 3$$

EXERCISE 7 : Find the basic feasible solution of

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Any feasible solution which optimizes the objective function is called an optimal solution.

A basic feasible solution $(X_s, 0)$ to a linear programming problem

Maximize (or Minimize)

$$Z = CX$$

subject to

$$AX = B$$
$$X \geq 0$$

is said to be **optimal** if it gives us the maximum (or minimum) value of the objective function.

Before we conclude this unit, we discuss a few fundamental results of linear programming which will serve as a tool for the development of the **Simplex Algorithm** to be discussed in the next unit. We give these results in the form of the following theorems:

'Theorem 1 : The set of all feasible solutions to a linear programming problem is a convex set.

Proof : You know that the constraints of a linear programming problem can be converted into equations by means of introduction of slack or surplus variables. Let us therefore consider the constraint system of any given linear programming problem of the form

$$AX = B, \quad X \geq 0$$

where A is an mxn matrix, X is $n \times 1$ matrix and B is an mxl matrix.

Let us form a set $K = (X : AX = B, X \geq 0)$. The set K is called the set of all feasible solutions of the linear programming problem AX = B.

Recall the notion of convexity of sets discussed in Unit **2.** To establish conyexity of set K, let $X_1, X_2 \in K$. Then we have

$$AX_1 = B, X_1 \geq 0, AX_2 = B, X_2 \geq 0.$$

Consider $\lambda x_1 + (1 - 1) X_2$ for $0 \leq \lambda \leq 1$

and

$$A[\lambda X_1 + (1-\lambda)X_2] = \lambda AX_1 + (1-\lambda)AX_2$$
$$= \lambda B + (1-\lambda)B = B$$

Since X_1, X_2, λ, $1-\lambda$ are all non-negative, therefore

$$\therefore \lambda_1 X_1 + (1-\lambda)X_2 \geq 0$$

Thus $\lambda X_1 + (1-\lambda) X_2 \in K$ for $0 \le \lambda \le 1$

which implies the set K is a convex set.

We state the next result again in the form of the following theorem without proof :

Theorem 2 : If there is a feasible solution to the system of constraints

$$AX = B$$
$$X \ge 0,$$

then there also exists a basic feasible solution to this system.

Now we give an example to illustrate of the reduction of a feasible solution to a basic feasible solution.

EXAMPLE 9 : Consider the system of equations

$$2x_1 + 3x_2 - x_3 = 4$$
$$-5x_1 + 6x_2 + x_3 = 2$$

A feasible solution is $x_1 = 1$, $x_2 = 1$, $x_3 = 1$. Reduce this feasible solution to a basic feasible solution.

SOLUTION : The above system of equations may be put in matrix notations as

$$\begin{pmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

where AX = B

$$A = \begin{pmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Let the columns of A be denoted by

$$A_1 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, A_2 = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Here $\rho(A) = 2$, therefore a basic solution to the given system of equations exists with no more than two variables different from zero. Also, the column vectors A_1, A_2, A_3 are linearly dependent. Verify it yourself.

Therefore, there exist scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0$$

or

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix}\lambda_1 + \begin{pmatrix} 3 \\ 6 \end{pmatrix}\lambda_2 + \begin{pmatrix} -1 \\ +1 \end{pmatrix}\lambda_3 = 0$$

i.e.

$$2\lambda_1 + 3\lambda_2 - \lambda_3 = 0$$

$$-5\lambda_1 + 6\lambda_2 + \lambda_3 = 0$$

This is a system pf two equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$. Let us choose one of the λ_s arbitrarily say $\lambda_1 = 1$. Then we can uniquely calculate the values of λ_2 and λ_3 from

$$3\lambda_2 - \lambda_3 = -2$$
$$6h; + \lambda_3 = 5$$

we get $\lambda_2 = \frac{1}{3}$ and $\lambda_3 = 3$.

To reduce the number of positive variables, the variable to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \min_i \left\{ \frac{x_i}{\lambda_i} / \lambda_i > 0 \right\} = \min \left\{ \frac{1}{1}, \frac{1}{1/3}, \frac{1}{3} \right\} = \frac{1}{3}.$$

Thus, we can remove vector A_3 for which $\frac{x_3}{\lambda_3} = \frac{1}{3}$ and obtain new solution with not more than two non-negative variables. The values of new variables are

$$\hat{x}_1 = x_1 - \frac{\lambda_1}{\lambda_3} x_3 = 1 - 1 \cdot \frac{1}{3} = \frac{2}{3}$$

$$\hat{x}_2 = x_2 - \frac{\lambda_2}{\lambda_3} x_3 = 1 - \frac{1}{3} \cdot \frac{1}{3} = \frac{8}{9}.$$

Obviously, columns A, and **A,** of **A** corresponding to these non-zero variables are linearly independent. Hence a basic feasible solution to given system of equations is given by

$$x_1 = \frac{2}{3}, \; x_2 = \frac{8}{9}, \; x_3 = 0$$

If A_1 or A_2 are eliminated instead of A_3, then x_1 or x_2 are driven to zero. In these cases you will find that new solutions are not feasible.

Try the following exercises.

EXERCISES **8 :** Consider the system of equations

$$2x_1 + x_2 + 4x_3 = 11$$
$$3x_1 + x_2 + 5x_3 = 14.$$

A feasible solution is

$$x_1 = 2,\ x_2 = 3,\ x_3 = 1.$$

Reduce this feasible solution to a basic feasible solution.

EXERCISE **9 :** For the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 2x_3 - 2x_4 = 3,$$

Here, (1, 1, 1, 0) is a feasible solution. Find a basic feasible solution.

Theorem 3 : A basic feasible solution to a linear programming problem corresponds to an extreme point of the Convex set K of feasible solutions and conversely, every extreme point of K corresponds to a basic feasible solution to a linear programming problem.

We omit the proof of this theorem. However to illustrate this theorem, we consider the product mix. problem discussed in Unit **3** namely :

Maximize

$$Z = 4x_1 + 5x_2$$

Subject to

$$2x_1 + 3x_2 \le 12$$
$$3x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0.$$

Adding slack variables, the constraints become

$$2x_1 + 3x_2 + x_3 = 12$$
$$3x_1 + x_2 + x_4 = 8.$$

All basic feasible solutions can be found say x^1, x^2, x^3, x^4

$$x^1 = (0, 0, 12, 8), \qquad x^2 = \left(\frac{8}{3}, 0, \frac{20}{3}, 0\right),$$

$$x^3 = (0, 4, 0, 4), \qquad x^4 = \left(\frac{12}{7}, \frac{20}{7}, 0, 0\right)$$

Other two basic solutions (0, 8, –12, 0), (6, 0, 0, –10) are not feasible.

You can see from the graphical solution of this problem that the extreme points of the convex set K of feasible solutions are

$$(0,\ 0)\ ,\quad \left(\frac{8}{3},\ 0\right),\quad (0,\ 4),\quad \left(\frac{12}{7},\ \frac{20}{7}\right)$$

Thus, clearly you can see the correspondence between basic feasible solutions and extreme points of the Convex set of feasible solution

x^1 corresponds to (0, 0)

x^2 corresponds to $\left(\frac{8}{3},\ 0\right)$

x^3 corresponds to (0, 4)

x^4 corresponds to $\left(\frac{12}{7},\ \frac{20}{7}\right)$

Also conversely, every extreme point of K corresponds to some basic feasible solutions.

We state yet another theorem (without proof).

Theorem 4 : The objective function of a linear programming problem attains its maximum (or minimum) at one of the extreme points of K.

EXAMPLE **10 :** Illustrate theorem 4 by using the Product Mix Problem of Unit 1.

SOLUTION : If the linear programming problem admits of an optimal solution, then optimal solution would coincide with at least one basic feasible solution of the problem.

The illustration of this theorem again follows from the same product-mix problem. You can obtain its optimal solution by graphical method in Unit 3 as

$$x_1 = \frac{12}{7},\quad x_2 = \frac{20}{7}$$

and optimal value of the objective function as $Z_{max} = \frac{148}{7} = 21.14$. You may notice that this value is attained at an extreme paint

$$x_4 = \left(\frac{12}{7},\ \frac{20}{7},\ 0,\ 0\right)$$

low try the following exercise.

EXERCISE.10 : Illustrate theorems 3 and 4 for the following linear programming problem (Diet problem formulated in Unit **3)**

Minimize

$$Z = 3x_1 + 2.5x_2$$

Subject to

$$2x_1 + 4x_2 \geq 40$$
$$3x_1 + 2x_2 \geq 50$$
$$x_1, x_2 \geq 0$$

5.5 SUMMARY

In this unit you have been introduced to

1. Standard and canonical form of a linear programming problem.
2. Introduction of slack and/or surplus variables to convert the inequality constraints of a linear programming problem into equality constraints.
3. Unrestricted variables.
4. Meanings of (i) Feasible solutions (ii) Basic solutions (iii) Basic feasible solutions (iv) Optimal solutions
5. Some important theorems in linear programming which are useful for the development of Simplex Algorithm to be discussed in the'next unit.

5.6 ANSWERS / HINTS / SOLUTIONS

E1 $2x_1 + x_2 + x_3 = 3, \quad x_3 \geq 0$
$x_1 + 4x_2 + x_4 = 5, \quad x_4 \geq 0$

E2 $2x_1 + 4x_2 - x_3 = 40, \quad x_3 \geq 0$
$3x_1 + 2x_2 - x_4 = 50, \quad x_4 \geq 0$

E3 $x_1 - 2x_2 + x_3 = 6, \quad x_3 \geq 0$
$2x_1 + 5x_2 - x_4 = 8, \quad x_4 \geq 0$
$x_1 - 3x_2 - x_5 = -6, \quad x_5 \geq 0$

E4 **Here x_3 is unrestricted. Therefore, write**

$x_3 = x_3' - x_3''$

where $x_3', x_3'' \geq 0$. Now complete the solution.

E5 **Minimize**

$$Z = 3x_1 - 2x_2 + 4x_3.$$

Subject to

$$x_1 - 2x_2 + 3x_3 - x_4 = 2$$
$$-2x_1 - 3x_2 - x_3 + x_5 = 2$$
$$4x_1 + 2x_2 - x_6 = 2$$
$$x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6 \geq 0.$$

E6) $(1, 0, 0)$, $\left(0, \frac{5}{3}, -\frac{1}{3}\right)$, $(1, 0, 0)$

$(1, 0, 0)$ **is a basic** feasible **solution)**

E7) **The** basic solutions **are** $\left(0, \frac{1}{2}, 0, 0\right)$, $\left(-2, \frac{7}{2}, 0, 0\right)$, $\left(\frac{8}{3}, 0, 0, -\frac{7}{3}\right)$

The only basic feasible solution is $(0, \frac{1}{2}, 0, 0,)$

E8) $x_1 = \frac{1}{2}, \ x_2 = 0, \ x_3 = \frac{5}{2}$

E9) $x_1 = 0, \ x_2 = \frac{1}{4}, \ x_3 = \frac{13}{8}, \ x_4 = 0$

E10) **Basic feasible solutions are**

$X^1 = (0, 25, 30, 0), \ X^2 = (15, \frac{5}{2}, 0, 0),$

$X^3 = (20, 0, 0, 10)$

[(0, 10, 0, −30), (0, 0, −40, −50), (50/3, 0, −20/3, 0) are non - basic]

Extreme points are (0, 25), $\left(15, \frac{5}{2}\right)$, (20, 0) optimal is **at** $\left(15, \frac{5}{2}\right)$ with Min Z = 51.25.