UNIT 8 LIMIT OF A FUNCTION

STRUCTURE

8.1 INTRODUCTION

In Unit 5, we dealt with sequences and their limits. As you know, sequences are functions whose domain is the set of natural numbers. In this unit, we discuss the limiting process for the real functions with domains as subsets of the set R of real numbers and range also a subset of R. What is the **precise meaning for the intuitive idea of the values f(x) of a function f tending to or approaching a number A as x approaches the number a?** The search for an answer to this question shall enable you to understand the concept of the limit which you have used in calculus. We shall give a rigorous meaning to the intuitive idea of the limit of a function in Section 8.2. The relation between the limit of a function and the limit of a sequence is established in Section 8.3. The effect of algebraic operations of addition, subtraction, multiplication and division on the limits of functions is examined in Section 8.4. It will then be extended to study the effect of these algebraic operations on the continuity of a function in Unit 9.

Objectives

Thus after studying this unit, you should be able to

- define limit of a function at a point and find its value
- know sequential approach to limit of a function
- find the limit of sum, difference, product and quotient of functions.

8.2 NOTION OF LIMIT

The intuitive idea of limit was used both by Newton and Leibnitz in their independent invention of Differential Calculus around 1675. Later this notion of limit was also developed by D'Alembert. **"When the successive values attributed Io a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others."**

Consider a simple example in which a function f is defined as

$$f(x) = 2x + 3, \qquad \forall x \in R, x \neq 1.$$

Give x the values which are near to 1 in the following way:

When x = 1.5, 1.4, 1.3, 1.2, 1.1, 1.01, 1.001

f(x) = 6, 5.8, 5.6, 5.4, 5.2, 5.02, 5.002

When x = .5, .6, .7, .8, .9, .99, .999

f(x) = 4, 4.2, 4.4, 4.6, 4.8, 4.98, 4.998

The limit of a function f at a point a is meaningful only if a is a limit point of its domain. That is, the condition f(x) → α as n → a would make sense only when there does not exists a nbd. U of a for which the set U ∩ Dom(f)\{a} is empty i.e., a ∈(Dom (f))'.

You can form a table for these values as follows:

x	.5	.6	.7	.8	.9	.99	.999	1.001	1.01	1.1	1.2	1.3	1.4	1.5
f(x)	4	4.2	4.4	4.6	4.8	4.98	4.998	5.002	5.02	5.2	5.4	5.6	5.8	6

You see that as the values of x approach 1, the values of f(x) approach 5.This is expressed by saying that limit of f(x) is 5 as x approaches 1. You may note that when we consider the limit of f(x) as x approaches 1, we do not consider the value of f(x) at x = 1. .

Thus, in general, we can say as follows:
Let f be a real function defined in a neighbourhood of a point x = a except possibly at a. Suppose that as x approaches a, the values taken by f approach more and more closely a value A. In other wirds, suppose that the numerical difference between A and the values taken by f can be made as small as we please by taking values of x sufficiently close to a. Then we say that f tends to the limit A as x tends to a. We write

$$f(x) \to A \text{ as } x \to a \quad \text{or} \quad \lim_{x\to a} f(x) = A.$$

This intuitive idea of the limit of a function can be expressed mathematically as formulated by the German mathematician Karl Veierstrass in the 18th Century. Thus, we have the following definition:

DEFINITION 1 : Limit of a Function
Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. The function f is said to tend to or'approach a number A as x tends to or approaches a number 'a' if for any ε > 0, there exists a number δ > 0 such that

$$|f(x) - A| < \varepsilon \text{ for } 0 < |x - a) < 6.$$

We write it as $\lim_{x\to a}$ f(x) = A. You may note that

$$|f(x) - A| < \varepsilon \text{ for } 0 < |x - a| < 6.$$

can be equivalently written as (see Unit 3)

$$f(x) \in]A - \varepsilon, A + \varepsilon[\text{ for } x \in]a - 6, a + \delta[\text{ and } x \neq a.$$

Geometrically, the above definition says that, for strip S_A of any given width around the point A, if it is possible to find a strip S_a of some width around the point a such that the values that f(x) takes, for x in the strip S_a (x ≠ a), lies in the shaded box formed by the intersection of strips S_A and S_a, then $\lim_{x\to a} f(x) = A$.

This is shown geometrically in Figure 1 below. The inequality 0 < |x − a| < 6 determines the interval] a − 6, a + δ[minus the point 'a' along the x-axis and the inequality |f(x) − A| < ε determines the interval] A − ε, A + ε[along the y-axis.

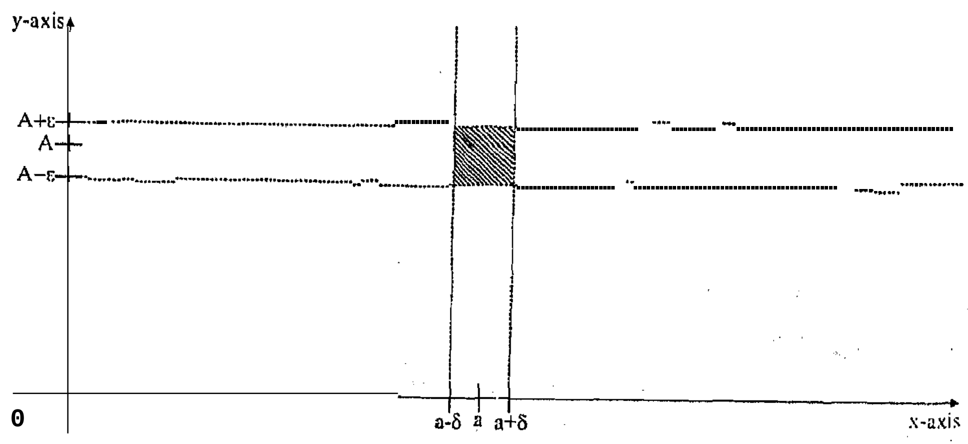


Fig. 1

EXAMPLE 1 : Let a function f: **R** → R be defined as

$$f(x) = x^2, \ \forall \ x \in \mathbf{R}.$$

Find its limit when $x \to 2$.

SOLUTION : By intuition, it follows that

$$\lim_{x\to2} f(x) = \lim_{x\to2} x^2 = 4.$$

Let us verify this with the help of ε- 6 definition. In other words, we have to show that for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 41 < \varepsilon.$$

Suppose that an $\varepsilon > 0$ is fixed. Then consider the quantity $|f(x) - 4|$, which we can write as

$$|f(x) - 4| = |x^2 - 4| = |(x - 2)(x + 2)|.$$

Note that the term $|x - 2|$ is exactly the same that appears in the 6–inequality in the definition. Therefore this term should be less than 6. In other words,

$$|x - 2| < \delta$$
$$\Rightarrow 2 - \delta < x < 2 + \delta$$
$$\Rightarrow x \in]\, 2 - \delta, 2 + \delta[.$$

We restrict 6 to a value 2 so that x lies in the interval $]\, 2 - 6, 2 + \delta[\subset]\, 0, 4[$. Accordingly, then $|x + 2| < 6$. Thus, if $6 \le 2$, then

$$|x - 2| < 2 \Rightarrow 0 < |x + 2| < 6,$$

and further that

$$|x - 2| < \delta \le 2 \Rightarrow |x + 2|\, |x - 2| < 6\, |x - 2| < 6\delta.$$

If 6 is small then so is 66. In fact it can be made less than ε by choosing δ suitably.

Let us, therefore, select 6 such that $\delta = \min.(2, \varepsilon/6)$. Then

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < 6\, |x - 2| < 6.\ \delta \le 6.\ \varepsilon/6 = \varepsilon.$$

This completes the 'solution.

Note that the first step is to manipulate the term $|f(x) - A|$ by using algebra. The second setp is to use a suitable strategy to manipulate $|f(x) - A|$ into the form

$$|x - a|\ \text{(trash)}$$

where the 'trash' is some expression which has the property that: it is bounded provided that 6 is sufficiently small. Why we use the term 'trash' for the expression as a multiple of $|x - a|$? The reason is that once we know that it is bounded, we can replace it by a number and forget about it.

In Example 1, the number 6 arose by virtue of this 'trash'. If you take $6 \le 3$ (instead of $6 \le 2$), you can still show that 6 will be replaced by 7. In that case you can set δ as

$$\delta = \min.(3, \varepsilon/7)$$

and the proof will be complete. Thus, there is nothing special about 6. The only thing is that such a number (whether 6 or 7) has to be evaluated by the restriction placed on 6.

Finally, note that in **general, 6** will **depend upon** ε.
Now you should be able to solve the following exercises:

EXERCISE 1

For a function f: R → R defined by $f(x) = x^2$, find its limit when x tends to 1 by the ε − δ approach.

EXERCISE 2

Show that $\lim_{x\to2} \dfrac{x^2 - x + 18}{3x - 1} = 4$, **using the ε − δ definition.**

In Unit 5, we proved that a convergent sequence cannot have more than one limit. In the same way, a function cannot have more than one limit at a single point of its domain. We prove it in the following theorem:

THEOREM 1 : If $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} f(x) = B$, then $A = B$.

PROOF : In short, we have to show that if $\lim_{x \to a} f(x)$ has two values say A and B, then $A = B$. Since $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} f(x) = B$, given a number $\varepsilon > 0$, there exists numbers $\delta_1, \delta_2 > 0$ such that

$$|f(x) - A| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta_1$$

and

$$|f(x) - B| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta_2.$$

If we take δ equal to minimum of δ_1 and δ_2, then we have

$$|f(x) - A| < \varepsilon/2 \text{ and } |f(x) - B| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta.$$

Choose an x_0 such that $0 < |x_0 - a| < \delta$. Then

$$|A - B| = |A - f(x_0) + f(x_0) - B| \leq |A - f(x_0)| + |f(x_0) - B|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

ε is arbitrary while A and B are fixed. Hence $|A - B|$ is less than every positive number ε which implies that $|A - B| = 0$ and hence $A = B$. (For otherwise, if $A \neq B$ then $A - B = C \neq 0$ (say). We can choose $\varepsilon < |C|$ which will be a contradiction to the fact that $|A - B| < \varepsilon$ for every $\varepsilon > 0$.)

In the example considered before defining limit of a function, we allowed x to assume values both greater and smaller than 1. If we consider values of x greater than 1 that is on the right of 1, we see that values of f(x) approaches 5. We say that f(x) tends to 5 as x tends to 1 from the right. Similarly you see that values of f(x) approach 5 as x tends to 1 from the left i.e. through values smaller than 1. This leads us to define right hand and left hand limits as under :

DEFINITION 2 : Right hand limits and Left hand limits

Let a function f be defined in a **neighbourhood** of a point **'a'** except **possibly** at 'a'. It is said to tend to a number A as x tends to a number 'a' from the right or through values greater than 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for } a < x < a + \delta.$$

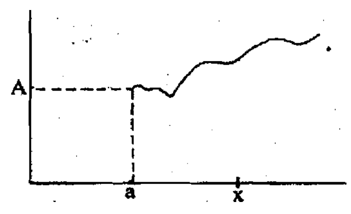


Fig. 2(a)

We write it as

$$\lim_{x \to a+} f(x) = A \text{ or } \lim_{x \to a+0} f(x) = A \text{ or } f(a+) = A.$$

See figure **2(a)**.

The function f is said to tend to a number A as x tends to 'a' from the left or through values smaller than 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for } a - \delta < x < a$$

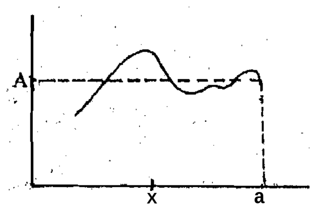


Fig. 2(b)

We write it as

$$\lim_{x \to a-} f(x) = A \text{ or } \lim_{x \to a-0} f(x) = A \text{ or } f(a-) = A.$$

See figure **2(b)**.

Since the definition of limit of a function employs only values of x different from 'a' it is totally immaterial what the value of the function is at $x = a$ or whether f is defined at $x = a$ at all. Also it is obvious that $\lim_{x \to a} f(x) = A$ if and only if $f(a+) = A$, $f(a-) = A$.

This we prove in the next theorem. First we consider the following example to illustrate it.

EXAMPLE 2 : Find the limit of the function f defined by

$$f(x) = \frac{x^2 + 3x}{2x} \text{ for } x \neq 0$$

when $x \to 0$

SOLUTION : The given function is not defined at $x = 0$ since $f(0) = \frac{0}{0}$ which is meaningless.

If $x \neq 0$, then $f(x) = \frac{x+3}{2}$. Therefore

Right Hand Limit $= \lim_{x \to 0+0} f(x)$

$= \lim_{h \to 0} \frac{(0+h)+3}{2} \quad (h > 0)$

$= 3/2.$

Left Hand **Limit** $= \lim_{x \to 0-0} f(x)$

$= \lim_{h \to 0} f(x) = \frac{(0-h)+3}{2} \quad (h > 0)$

$= 3/2.$

Since both the right hand and left hand limits exist and are equal,

$\lim_{x \to 0} f(x) = 3/2.$

You can similarly solve the following exercise.

EXERCISE 3

Find the limit of the function f defined as

$f(x) = \frac{2x^2 + x}{3x}$, **$x \neq 0$ when x tends to 0.**

We, now, discuss the theorem concerning the existence'of limit and that of right and the left hand limits.

THEOREM 2 : Let f be a **real function. Then**

$\lim_{x \to a} f(x) = A$ if **and only if** $\lim_{x \to a+} f(x)$ **and** $\lim_{x \to a-} f(x)$

both exist and are equal to A.

PROOF : If $\lim_{x \to a+} f(x) = A$, then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|f(x) - A| < \varepsilon$ whenever $0 < |x - a| < \delta$

i.e. $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a + \delta,\ x \neq a$

This implies that $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a$

and $|f(x) - A| < \varepsilon$ whenever $a < x < a + \delta$,

Hence both the **left** hand and right hand **limits** exist and are equal to A.
Conversely, if $f(a+)$ and $f(a-)$ exist and are equal to A say, then corresponding to $\varepsilon > 0$, there exist positive numbers δ_1 and δ_2 such that

$|f(x) - A| < \varepsilon$ whenever $a < x < a + \delta_1$

and

$|f(x) - A| < \varepsilon$ whenever $a - \delta_2 < x < a$.

Let δ be the minimum of δ_1 and δ_2. Then

$|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a + \delta,\ x \neq a$

i.e. $|f(x) - A| < \varepsilon$ if $0 < |x - a| < \delta$

which proves that

$\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = A$.

EXAMPLE 3 : Consider the function f defined by

$$f(x) = \frac{x^2 - 1}{x - 1},\ x \in R,\ x \neq 1$$

Find its limit as $x \to 1$.

SOLUTION : Note that f(x) is not defined at $x = 1$. **(Why ?)**.

For any $x \neq 1$, $f(x) = \frac{x^2 - 1}{x - 1} = x + 1$.

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (x + 1) = 2$

$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x + 1) = 2$

Since $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$, by Theorem 2, $\lim_{x \to 1} f(x) = 2$,

$\lim_{x \to 1} f(x) = 2$ can be seen by ε-δ definition as follows :

Corresponding to any number $\varepsilon > 0$, we can choose $\delta = \varepsilon$ itself. Then, it is clear that

$$0 < |x - 1| < \delta = \varepsilon \Rightarrow$$

$$|f(x) - 2| = \left| \frac{x^2 - 1}{x - 1} - 2 \right| = |x + 1 - 2| = |x - 1| < \varepsilon.$$

From Theorem 2, it follows that $f(1+)$ and $f(1-)$ also exist and are both equal to 2.

EXAMPLE 4 : Let $f: R \to R$ be defined as

$$f(x) = \begin{cases} |x|, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

Find its limit when $x \to 0$.

SOLUTION : You are familiar with the graph of f as given in Unit 4. It is easy to see that $\lim_{x \to 0} f(x) = 0 = f(0+) = f(0-)$. The fact that $f(0) = 3$ has neither any bearing on the existence of the limit of f(x) as x tends to 0 nor on the value of the $\lim_{x \to 0} f(x)$.

Now try the following exercise:

EXERCISE 4

Find, if possible, the limit of the following functions.

(i) $f(x) = \frac{|x - 2|}{x - 2},\ x \neq 2$

when x tends to 2.

(ii) $f(x) = \frac{-1}{e^{1/x} + 1},\ x \neq 0$

when x tends to 0.

EXAMPLE 5 : Define f on the whole of the real Line as follows:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Find its limit when x tends to 0.

SOLUTION : Since $f(x) = 1$ for all $x > 0$,

$$f(0+) = \lim_{x \to 0+} f(x) = +1.$$

Similarly $f(0-) = -1$. Since $\lim_{x \to 1+} f(x) \neq \lim_{x \to 1-} f(x)$, $\lim_{x \to 0} f(x)$ does not exist.

Now, we give another proof using E – S definition.

For, if $\lim_{x \to 0} f(x) = A$, then for a given E > 0, there must exist some 6 > 0, such that

$|f(x) - A| <$ E. Let us choose $x_1 > 0$, $x_2 < 0$ such that $|x_1| <$ S and $|x_2| <$ 6. Then

$$2 = |f(x_1) - f(x_2)|$$

$$\leq |f(x_1) - A| + |A - f(x_2)|$$

$$< 2\varepsilon, \qquad \text{(because } |x_1 - 0| < \delta \text{ and } |x_2 - 0| < \delta)$$

for every ε which is clearly impossible if E < 1. Non-existence of $\lim_{x \to 0} f(x)$ also follows from Theorem 2, since $f(0+) \neq f(0-)$.

The above example shows clearly that the existence of both $f(a+)$ and $f(a-)$ alone is not sufficient for the existence of $\lim_{x \to 0} f(x)$. In fact, for lirn $\lim_{x \to a} f(x)$ to exist, they both should be equal.

Now consider, the function f defined by $f(x) = \frac{1}{x}$ for $x \neq 0$.

The graph of f looks as shown in the Figure 3. You know that it is a rectangular hyperbola. Here none of the lirn $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0+} f(x)$ exists. Hence $\lim_{x \to 0} f(x)$ does not exist

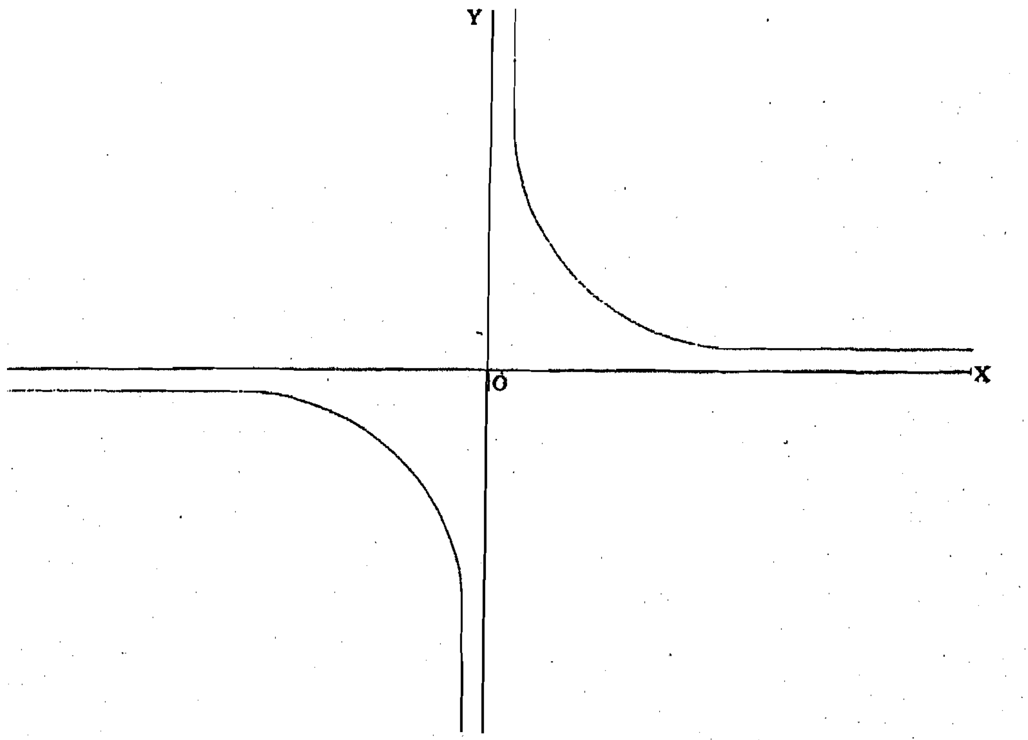


Fig. 3

This can be easily seen from the fact that $1/x$ becomes very large numerically as x approaches 0 either from the left or from the right. If x is positive and takes up larger and larger values, then values of $1/x$ i.e. $f(x)$ is positive and becomes smaller and smaller. This is expressed by saying that $f(x)$ approaches 0 as x tends to ∞. Similarly if x is negative and numerically takes up larger and larger values, the values of $f(x)$ is negative and numerically becomes smaller and smaller and we say that $f(x)$ approaches 0 as x tends to –∞. These two observations are related to the notion of the limit of a function at infinity.

Let us now discuss the behaviour of a function f when x tends to ∞.

Let a function f be defined for all values of x greater than a fixed number c. That is to say that f is defined for all sufficiently large values of x. Suppose that as x increases indefinitly, f(x) takes a succession of values which approach more and more closely a value A. Further suppose that the numerical difference between A and the values f(x) taken by the function can be made as small as we please by taking values of x sufficiently large. Then we say f tends to'the limit A as x tends to infinity. More precisely, we have the following definition:

DEFINITION 3 : A function f tends to a **limit** A, as x tends to **infinity** if having chosen a positive **number** ε, there **exists** a positive number k such **that**

$$|f(x - A)| < \varepsilon \ \forall \ x \geq k.$$

The number E can be made as small as we like. Indeed, however small ε we may take, we can always find a number k for which the above inequality holds. We rewrite this definition in the following way:

A function $f(x) \to A$ as $x \to \infty$ if for every $\varepsilon > 0$, there exists $k > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for all } x \geq k.$$

We write it as

$$\lim_{x \to \infty} f(x) = A.$$

This notion of the limit of a function needs a slight modification when x tends to $-\infty$. This is as follows:

We say that $\lim_{x \to -\infty} f(x) = A$, if for a given $\varepsilon > 0$, there exists a number $k < 0$ such that

$$|f(x) - A| < \varepsilon \text{ whenever } x \leq k.$$

We write it as $\lim_{x \to -\infty} f(x) = A.$

Instead of f(x) approaching a real number A as x tends to $+\infty$ or $-\infty$, we may also have f(x) approaching $+\infty$ or $-\infty$ as x tends to a real number 'a'. For example, if $f(x) = 1/x^2$, $x \neq 0$ and x takes values near 0, the values of f(x) becomes larger and larger. Then we say that f(x) is tending to $+\infty$ as x tends to 0. We can also have f(x) tending to $+\infty$ or $-\infty$ as x tends to $+\infty$ or $-\infty$. For example $f(x) = x$ tends to $+\infty$ or $-\infty$ as x tends to $+\infty$ or $-\infty$. Again: the function $f(x) = -x$ tends to $+\infty$ or $-\infty$ as x tends to $-\infty$ or $+\infty$. We formulate the following definition to cover all such cases of infinite limits.

DEFINITION 4 : Infinite Limits of a Function

Suppose a is a real number. We **say** that **a** function f tends to $+\infty$ when x tends to a, if for a given positive real number M there exists a positive number δ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = +\infty.$$

In this case we say that the function becomes unbounded and tends to $+\infty$ as x tends to a.

In the same way, f is said to $-\infty$ as x tends to a if for every real number $-M$, there is a positive number δ such that

$$f(x) < -M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = -\infty.$$

In this case also f(x) is unbounded and tends to $-\infty$ as x tends to a. You can give similar definitions for $f(a+) = +\infty$, $f(a-) = +\infty$, $f(a+) = -\infty$, $f(a-) = -\infty$.

Now we define $\lim_{x \to \infty} f(x) = \infty$.

f is said to tend to ∞ as x tends to ∞ if given a number $M > 0$, there exists a number $k > 0$ such that

$f(x) > M$ for $x \geq k$.

We may similarly define

$\lim_{x \to -\infty} f(x) = +\infty, \quad \lim_{x \to +\infty} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = -\infty.$

In all such cases we say that the function f becomes unbounded as x tends to $+\infty$ or $-\infty$ as the case may be.

It is easy to see from the definition of limit of a function that the limit of a constant function at any point in its domain is the constant itself. Similarly if $\lim_{x \to a} f(x) = A$, then $\lim_{x \to a} cf(x) = cA$ for any constant c where c is a real number.

EXAMPLE 6 : Justify that

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty.$$

SOLUTION : You have to verify that corresponding to a given positive number M, there exists a positive number 6, such that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < 6.$$

Indeed for $x \neq 2$,

$$\frac{1}{(x-2)^2} > M \Rightarrow (x-2)^2 < \frac{1}{M}$$

$$\Rightarrow |x-2| < \frac{1}{\sqrt{M}}.$$

Take $\delta = \frac{1}{\sqrt{M}}$. Then you can see that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < 6.$$

Hence

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty.$$

Now try the following exercise.

EXERCISE 5

(i) Consider $f(x) = |x|$, $x \in R$. Show that $\lim_{x \to +\infty} f(x) = +\infty$,
and $\lim_{x \to -\infty} f(x) = +w$
and $f(0+) = f(0-) = 0 = f(0)$

(ii) Let $f(x) = -|x|$, $x \in R$. Prove that $\lim_{x \to +\infty} f(x) = -\infty$
and $\lim_{x \to -\infty} f(x) = +\infty$
and $f(0) = f(0+) = f(0-) = 0$.

We have already stated that if a function f is defined by $f(x) = 1/x$, $x \neq 0$, then the limits $f(0+)$ and $f(0-)$ and $\lim_{x \to 0} f(x)$ do not exist, It simply means that these limits do not exist as real numbers. In other words, there is no (finite) real number A such that $f(0+) = A$ $f(0-) = A$, or $\lim_{x \to 0} f(x) = A$.

You can easily solve the following exercise:

EXERCISE 6

(i) Let $f(x) = \frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x\to 0+} f(x) = +\infty$, $\lim_{x\to 0-} f(x) = \infty$ and $\lim_{x\to 0} f(x) = +\infty$.

(ii) Let $f(x) = -\frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x\to 0+} f(x) = -\infty$, $\lim_{x\to 0-} f(x) = -\infty$ and $\lim_{x\to 0} f(x) = -\infty$.

(iii) Let $f(x) = \frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x\to 0+} f(x) = +\infty$, $\lim_{x\to 0-} f(x) = -\infty$

(iv) Let $f(x) = -\frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x\to 0+} f(x) = -\infty$, $\lim_{x\to 0-} f(x) = \infty$.

8.3 SEQUENTIAL LIMITS

In Unit 5, you studied the notion of the limit of a sequence. You also know that a sequence is also a function but a special type of function. What is special about a sequence? Do you remember it? Recall it from Unit 5. Naturally, you would like to know the relationship of a sequence and an arbitrary real function in terms of their limit concepts. Both require us to find a fixed number **A** as a first step. Both assume a small positive number ε as a test for closeness. For functions we need a positive number δ corresponding to the given positive number E and for sequences we need a positive integer m which depends on ε. So, then what is the difference between the two notions? The only difference is in their domains in the sense that the domain of a sequence is the set of natural numbers whereas the domain of an arbitrary function is any subset of the set of real numbers. In the case of a sequence, there are natural numbers only which exceed any choice of m. But for a function with a domain as an arbitrary set of real numbers, this is not necessary the case. Thus in a way, the notion of the limit of a function at infinity is a generalization of that of limit of a sequence.

Let us now, therefore, examine the connection between the limit of a function and the limit of a sequence called the sequential limit. We state and prove the following theorem for this purpose:

THEOREM 3 : Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. Then f(x) tends to a limit A as x tends to 'a' if and only if for every sequence (x_n), $x_n \neq a$ for any natural number n, converging to 'a', $f(x_n)$ converges to A.

PROOF : Let $\lim_{x\to a} f(x) = A$. Then for a number E > 0, there exists a 6 > 0 such that for $0 < |x - a| < 6$ we have

$$|f(x) - A| < \varepsilon$$

Let (x_n) be a sequence ($x_n \neq a$ for any $n \in N$) such that (x_n) converges to a i.e. $x_n \to a$.

Then corresponding to $\delta > 0$, there exists a natural number m such that for all $n \geq m$

$$|x_n - a| < \delta.$$

Consequently, we have

$$|f(x_n) - A| < \varepsilon, \forall n \geq m.$$

This implies **that** $f(x_n)$ converges to **A.**

Conversely, let $f(x_n)$ converge to **A** for every sequence x_n which converges to a, $x_n \neq a$ for any n.

Suppose $\lim_{x \to a} f(x) \neq A$.

Then there exists at least one ε, say $\varepsilon = \varepsilon_0$ such that for any $\delta > 0$ we have an x_δ such that

$0 < |x_\delta - a| < \delta$

and

$|f(x_\delta) - A| \geq \varepsilon_0$.

Let $\delta = \frac{1}{n}$, n = 1, 2, 3......

We get a a sequence (x_n) such that $x_n = x_\delta$ where $\delta = 1/n$ and

$0 < |x_n - a| < \frac{1}{n}$ for n = 1, 2,.....

and

$|f(x_n) - A| \geq \varepsilon_0$.

$0 < |x_n - a| \Rightarrow x_n \neq a$ for any n.

Since $\frac{1}{n} \to 0$ and $|x_n - a| < \frac{1}{n}$, it follows that $x_n \to a$.

But $|f(x_n) - A| \geq \varepsilon_0 \Rightarrow f(x_n) \not\to A$ i.e. $f(x_n)$ does not tend to A.

Therefore $x_n \neq a$ $\forall$ a and x_n tends to a as n tends to ∞ whereas $f(x_n)$ does not converge to A, contradicting our hypothesis. This completes the proof of the theorem.

You may note that the above theorem is true even when either a or A is infinite or both a and A are infinite (i.e. $+\infty$ or $-\infty$).

By applying this theorem, we can decide about the existence or non-existence of limit of a function at a point. Consider the following examples:

EXAMPLE 7 : Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Show thrt at no point a in the real line R $\lim_{x \to a} f(x)$ exists.

SOLUTION : Consider any point 'a' of the real line. Let (p_n) be a sequence of rational numbers converging to the point 'a'. Since p_n is a rational number, $f(p_n) = 0$ for all n and consequently $\lim f(p_n) = 0$. Now consider a sequence (q_n) of irrational numbers converging to 'a'. Since q_n is an irrational number, $f(q_n) = 1$ for all n and consequently $\lim f(q_n) = 1$. So for two sequences (p_n) and (q_n) converging to 'a'; sequences $(f(p_n))$ and $(f(q_n))$ do not converge to the same limit. Therefore $\lim_{x \to a} f(x)$ cannot exist for if it exists and is equal to A, then both $(f(p_n))$ and $(f(q_n))$ would have converged to the same limit A.

EXAMPLE 8 : Show thnt for the function f: R $\to$ R defined by
$f(x) = x$ Q $x \in R$, $\lim_{x \to a} f(x)$ exists for every $a \in R$.

SOLUTION : Consider any point $a \in R$. Let (x_n) be a sequence of points of R converging to 'a'. Then $f(x_n) = x_n$ and consequently $\lim f(x_n) = \lim (x_n) = a$. So for every sequence $<x_n>$ converging to 'a' $(f(x_n))$ converges to 'a'. So by Theorem 3, $\lim_{x \to a} f(x) = a$. Consequently $\lim_{x \to a} f(x)$ exists for every $a \in R$.

Now try, the following exercises.

EXERCISE 7

Show that for the function f: R $\to$ R defined by

$f(x) = x^2$,

$\lim_{x \to a} f(x)$ exists for every $a \in R$.

EXERCISE 8

Show that $\lim_{x \to 1} 2^x = 2$ by proving that for any sequence (x_n), $x_n \neq 1$, converging to 1, 2^{x_n} converges to 2.

8.4 ALGEBRA OF LIMITS

We discussed the algebra of limits of sequences in Unit 5. In this section we apply the same algebraic operations to limits of functions. This will enable us to solve the problem of finding limits of functions. In other words we discuss limits of sum, difference, product and quotient of functions. Before we do this, let us first recall the meanings of the sum, difference, product, quotient of two functions which you have studied in Unit 4.

DEFINITION 5 : Algebraic Operations on Functions

Let f and g be two functions with domain $D \subset R$. Then the sum, difference, product, quotient of f and g denoted by $f + g$, $f - g$, fg, f/g are functions with domain D defined by

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x).\,g(x)$$

$$(f/g)(x) = f(x)/g(x)$$

provided in the last case $g(x) \neq 0$ for all x in D.

Now we prove the theorem.

THEOREM 4

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where A and B are real numbers,

(i) $\lim_{x \to a} (f + g)(x) = A + B = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$,

(ii) $\lim_{x \to a} (f - g)(x) = A - B = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} (f \cdot g)(x) = A \,.\, B = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.

(iv) If further $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} f/g\,(x)$ exists and

$$\lim_{x \to a} \frac{f}{g}(x) = A/B = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

PROOF : Since $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, corresponding to a number $\varepsilon > 0$. There exist numbers

$\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2 \qquad (1)$$

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon/2 \qquad (2)$$

Let δ = minimum (δ_1, δ_2). Then from (1) and (2) we have that

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (A + B)| \leq |f(x) - A| + |g(x) - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Which shows that $\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + g(x) = A + B$

This proves part (i).

The proof of (ii) is exactly similar. Try it yourself.

(iii) $|f(x)\, g(x) - AB| = |(f(x) - A)\, g(x) + A\,(g(x) - B)|$

$$\leq |f(x) - A|\, |g(x)| + |A| .\, |(g(x) - B)| . \qquad (3)$$

Since $\lim_{x \to a} g(x) = B$ corresponding to 1, there exists a number $\alpha_0 > 0$

such that

$0 < |x - a| < \alpha_0 \Rightarrow |g(x) - B| < 1.$

which implies that $|g(x)| \leq |g(x) - B| + |B| \leq 1 + |B| = K$ (say) (4)

Since $\lim_{x\to a} f(x) = A$, corresponding to $\varepsilon < 0$, there exists a number $\delta_1 > 0$ such that

number $\delta_1 < 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2K \qquad (5)$$

Since $\lim_{x\to a} g(x) = B$, corresponding to a number $\varepsilon > 0$, there exists a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - B| < \frac{\varepsilon}{2(|A| + 1)} \qquad (6)$$

Let $\delta = \min(a, , \delta_1, \delta_2)$.Then using (4), (5) and (6) in (3), we have for $0 < |x - a| < 6$,

$$\begin{aligned} |f(x)\, g(x) - AB| &\leq |f(x) - A|\, |g(x)| + |A|\, |g(x) - B| \\ &\leq |f(x) - A|.\, K + |A|\, |(g(x) - B)| \\ &< \frac{\varepsilon}{2K}\, .\, K + \frac{\varepsilon}{2(|A| + 1)}\, |A| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $\lim_{x\to a} g(x) = AB$ i.e. $\lim_{x\to a} (fg)(x) = AB = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$, which proves

part (iii) of the theorem.

(iv) First we show that g does not vanish in a neighbourhood of a.

$\lim_{x\to a} g(x) = B$ and $B \neq 0$. Therefore $|B| > 0$. Then corresponding to

$\frac{|B|}{2}$ we have a number $\mu > 0$ such that for $0 < |x - a| < \mu$,

$|g(x) - B| < \frac{|B|}{2}$.

Now by triangle inequality, we have

$$|\,|g(x)| - |B|\,| \leq |g(x) - B| < \frac{|B|}{2}.$$

i.e., $|B| - \frac{|B|}{2} < |g(x)| < |B| + \frac{|B|}{2}$.

In other words, $0 < |x - a| < \mu \Rightarrow |g(x)| > \frac{|B|}{2}$. (7)

Again since $\lim_{x\to a} g(x) = B$, for a given number $\varepsilon > 0$, we have a number $\mu' > 0$ such

that $0 < |x - a| < \mu'$ implies that

$$|g(x) - B| < \frac{|B|^2\varepsilon}{2}.$$

Let $6 = \min(\mu, p')$. Then if $0 < |x - a| < \delta$, from (7) and (8) we have

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|B - g(x)|}{|g(x)||B|} < \frac{2|B - g(x)|}{|B|^2} < \frac{2|B|^2\varepsilon}{2|B|^2} = \varepsilon$$

This proves that $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{B}$.

Now by part (iii) of this theorem, we get that

$$\begin{aligned} \lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} f(x).\frac{1}{g(x)} = \lim_{x\to a} f(x). \lim_{x\to a} \frac{1}{g(x)} \\ &= A.\, \frac{1}{B} = A/B. \end{aligned}$$

$$\text{i.e., } \lim_{x\to a} \left(\frac{f}{g}\right)(x) = A/B = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

This completes the proof of the theorem. You may note the theorem is true even when $a = \pm\infty$. You may also see that while proving (iv), we have proved that if

$\lim_{x \to a} g(x) = B \neq 0$, then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{B}$.

Before we solve some examples, we prove two more theorems.

THEOREM 5 : Let f and g be defined in the domain D and let $f(x) \leq g(x)$ for all x in D. Then if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist,

$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

PROOF : Let $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} g(x) = B$. If possible, let $A > B$.

for $\varepsilon = \frac{A - B}{2}$, there exist $\delta_1, \delta_2 > 0$ such that

$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon$

and $0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon$.

If $\delta = \min. (\delta_1, \delta_2)$, then for $0 < |x - a| < \delta$, $g(x) \in]B - \varepsilon, B + \varepsilon[$ and $f(x) \in]A - \varepsilon, A + \varepsilon[$. But $B + \varepsilon = A - \varepsilon = \frac{A + B}{2}$. Therefore $g(x) < f(x)$ for $0 < |x - a| < \delta$ which contradicts the given hypothesis. Thus $A \leq B$.

THEOREM 6 : Let S and T be non-empty subsets of the real set R, and let $f: S \to T$ be a function of S onto T. Let $g: U \to R$ be a function whose domain $U \subset R$ contains T. Let us assume that $\lim_{x \to a} f(x)$ exists and is equal to b and $\lim_{y \to b} g(y)$ exists and is equal to c. Then $\lim_{x \to a} g(f(x))$ exists and is equal to c.

PROOF : Since $\lim_{y \to b} g(y) = c$, given a number $\varepsilon > 0$, there exists a number $\alpha_0 > 0$ such that

$0 < |y - b| < \alpha_0 \Rightarrow |g(y) - c| < \varepsilon$.

Since $\lim_{x \to a} f(x) = b$, corresponding to $\alpha_0 > 0$, there exists $\delta > 0$ such that

$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \alpha_0$.

Hence taking $y = f(x)$ and combining the two we get that for

$0 < |x - a| < \delta$, $|g(f(x)) - c| = |g(y) - c| < \varepsilon$

(since $|f(x) - b| < \alpha_0$).

This completes the proof of the theorem. Finally we give one more result without proof.

RESULT : If $\lim_{x \to a} f(x) = A$, $A > 0$ and $\lim_{x \to a} g(x) = B$ where A and B are finite real numbers then

$\lim_{x \to a} f(x)^{g(x)} = A^B$.

Now we discuss some examples. You will see how the above results help us in reducing the problem of finding limit of complicated functions to that of finding limits of simple functions.

EXAMPLE 9

Find $\lim_{x \to \infty} \frac{(2x + 7)(3x - 11)(4x + 5)}{4x^3 + x - 1}$

SOLUTION

$\lim_{x \to \infty} \frac{(2x + 7)(3x - 11)(4x + 5)}{4x^3 + x - 1}$

$= \lim_{x \to \infty} \frac{x^3\left[\left(2 + \frac{7}{x}\right)\left(3 - \frac{11}{x}\right)\left(4 + \frac{5}{x}\right)\right]}{x^3\left(4 + \frac{1}{x^2} - \frac{1}{x^3}\right)}$

We divide the numerator and denominator by x^3 since x^3 is neither zero nor ∞.

$$= \lim_{x \to \infty} \frac{(2x+7)(3x-11)(4x+5)}{4x^3+x-1}$$

$$= \lim_{x \to \infty} \frac{\left(2+\frac{7}{x}\right)\left(3-\frac{11}{x}\right)\left(4+\frac{5}{x}\right)}{4+\frac{1}{x^2}-\frac{1}{x^3}} = \frac{2\times3\times4}{4} = 6;$$

EXAMPLE 10 : Find $\lim_{x \to 3} \frac{x^2-9}{x^2-4x+3}$

SOLUTION : $\lim \frac{x^2-9}{x^2-4x+3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)(x-1)}$

Hence $\lim_{x \to 3} \frac{x^2-9}{x^2-4x+3} = \lim_{x \to 3} \frac{x+3}{x-1}$

$$= \frac{\lim_{x \to 3}(x+3)}{\lim_{x \to 3}(x-1)} = \frac{6}{2} = 3$$

The function $f(x) = \frac{x^2-9}{x^2-4x+3}$ is not defined at $x = 3$. But we are considering only the values of the function at those points x in a neighbourhood of 3 for which $x \neq 3$ and hence we can cancel $x - 3$ factor

EXAMPLE 11 : Evaluate $\lim_{x \to 0} \frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1}$

SOLUTION : To make the problem easier, we make a substitution which enables us to get rid of fractional powers 1/2 and 1/3. L.C.M. of 2 and 3 is 6. So, we put $1 + x = y^6$

Then we have

$$\lim_{x \to 0} \frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1} = \lim_{y \to 1} \frac{y^3-1}{y^2-1} = \lim_{y \to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)}$$

$$= \lim_{y \to 1} \frac{y^2+y+1}{y+1} = \frac{3}{2}.$$

Try the following exercises:

EXERCISE 9

Find

(i) $\lim_{x \to \infty} \frac{(2x+3)^3(3x-2)^2}{x^5+5}$

(ii) $\lim_{x \to \infty} \frac{(x^3+1)^{1/3}}{x+1}$

EXERCISE 10

If $g(x) = \begin{cases} 2x & \text{for } 0 \le x < 1 \\ 4 & \text{for } x = 1 \\ 5-3x & \text{for } 1 < x \le 2 \end{cases}$

find $\lim_{x \to 1} g(x)$

EXERCISE 11

Find

(i) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

(ii) Find $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

(iii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(iv) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$

(v) $\lim_{x \to \infty} \left(\frac{x - 1}{x + 1}\right)^x$

(vi) $\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^{1+x}$

(vii) $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1}\right)^{x^2}$

8.5 SUMMARY

In this unit, you have been introduced to the concept of a limit of a function. In Section 8.2, we started with the intuitive idea of a limit of a function. Then we derived the rigorous definition of the limit of a function, popularly called $\varepsilon - \delta$ definition of a limit. Further, we gave the notion of right and left hand limits of a function. It has been proved that $\lim_{x \to a} f(x) = A$ if and only if both right hand and left hand limits are equal to A i.e. $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = A$. In the same section we discussed the limit of a function as x tends to $+\infty$ or $-\infty$. Also we discussed the infinite limit of a function. In Section 8.3, we studied the idea of sequential limit of a function by connecting the idea of limit of an arbitrary function with the limit of a sequence. It has been shown how this relationship helps in finding the limits of functions. In Section 8.4, we defined the algebraic operations of sum, difference, product, quotient of two functions. We proved that the limit of the sum, difference, product and quotient of two functions at a point is equal to the sum, difference, product and quotient of the limits of the functions at the point provided in the case of quotient, the limit of the function in the denominator is non-zero. Finally in the same section, the usefulness of the algebra of limits in finding the limits of complicated functions has been illustrated.

8.6 ANSWERS/HINTS/SOLUTIONS

E1) We claim that $\lim_{x \to 1} f(x) = 1$. To verify this, let $\varepsilon > 0$ be a fixed real number. Then

$$|f(x) - 1| = |x^2 - 1| = |x - 1|\,|x + 1|.$$

Suppose $0 < \delta \le 1$. Then $|x - 1| < 1 \Rightarrow 0 < |x + 1| < 3$ and also

$$|x - 1| < 1 \Rightarrow |x + 1|.\,|x - 1| < 3.\,|x - 1|.$$

Choose $\delta = \text{Min}\{1, \varepsilon/3\}$. Then

$$\begin{aligned} 0 < |x - 1| < \delta \Rightarrow |f(x) - 1| &= |x^2 - 1| \\ &< 3.\,|x - 1| \\ &< 3.\,\delta \\ &\le 3.\,\varepsilon/3 = \varepsilon \end{aligned}$$

which proves the claim.

E2) $\frac{x^2 - x + 18}{3x - 1} - 4 = \frac{x^2 - 13x + 22}{3x - 1} = (x - 2)\frac{x - 11}{3x - 1}$

If x is near 2 then $(x - 2)$ is near zero. If x is near 2 and away from 1/3, then $\left|\frac{x - 11}{3x - 1}\right|$ is not very large. If $\delta \le 1$ and $0 < |x - 2| < \delta$ then

$2 - \delta < x < 2 + \delta$, $x \neq 2$ i.e. $1 < x < 3$, $x \neq 2$. Then $-10 < x - 11 < -8$ and $2 < 3x - 1 < 8$ so that $|x - 11| < 10$ and $|3x - 1| > 2$

Thus $\left|\frac{x-11}{3x-1}\right| < 5$. Now if $\varepsilon > 0$ is given and if simultaneously

$|x - 2| < \varepsilon/5$ and $\left|\frac{x-11}{3x-1}\right| < 5$ then $\left|\frac{x^2 - x + 18}{3x-1} - 4\right| < \varepsilon$.

Hence we can choose $\delta = \min(\varepsilon/5, 1)$

Then for $0 < |x - 2| < \delta$ we have

$$\left|\frac{x^2 - x + 18}{3x - 1} - 4\right| < \varepsilon.$$

In fact, in this problem, f(2) is defined and takes the value 4.

E3) When $x \neq 0$, $f(x) = \frac{2x+1}{3}$. Therefore

Right hand limit $= \lim_{x \to 0+} f(x)$

$= \lim_{h \to 0} \frac{2(0+h)+1}{3}$ $(h > 0)$

$= \frac{1}{3}$.

Left hand limit $= \lim_{x \to 0} f(x)$

$= \lim_{h \to 0} \frac{2(0-h)+1}{3}$ $(h > 0)$

$= \frac{1}{3}$.

Since both the right hand and left hand limits exist and are equal, therefore $\lim_{x \to 0} f(x) = \frac{1}{3}$.

E4) (i) Right hand limit $= f(2+) = \lim_{h \to 0} \frac{|2+h-2|}{2+h-2}$ $(h > 0)$

$= 1$.

Similarly left hand limit $= -1$

$\therefore \lim_{x \to 2} f(x)$ does not exist.

(ii) $f(0+) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$ $(h > 0)$

$= \lim_{h \to 0} \frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} = 1$

$f(0-) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = -1$

Since $f(0+) \neq f(0-)$, $\lim_{x \to 0} f(x)$ does not exist.

E5) (i) When x is positive or zero $f(x) = x$, and when x is negative, $f(x) = -x$.

$f(0+) = \lim_{x \to 0+} x = 0$ and $f(0-) = \lim_{x \to 0-} -x = 0$. Also $f(0) = 0$

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} |x| = \lim_{x \to \infty} x = \infty$. In fact for any $M > 0$, $f(x) > M$ if $x \geq k$ with $k = M + 1$.

Similarly $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} |x| = \lim_{x \to -\infty} -x = \infty$.

(ii) It is similar to (i).

E6) (i) Let $h > 0$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} \frac{1}{|0 + h|} = \lim_{h \to 0} \frac{1}{h} = \infty.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} \frac{1}{|0 - h|} = \lim_{h \to 0} \frac{1}{h} = \infty.$$

Hence it follows that $\lim_{x \to 0} f(x) = \infty$. It can also be proved as follows:

If $M > 0$ is any number, $f(x) = \frac{1}{|x|} > M$ if $0 < |x| < \frac{1}{M}$ that is $f(x) > M$ if $0 < |x| < \delta$ where $\delta = \frac{1}{M}$.

Hence $\lim_{x \to 0} f(x) = \infty$.

(ii) is similar to (i).

(iii) $\lim_{x \to 0+} f(x) = \lim_{h \to 0} \frac{1}{h} = \infty$ and $\lim_{x \to 0-} f(x) = \lim_{h \to 0} \frac{1}{-h} = -\infty$

(iv) is similar to (iii).

E7) If (x_n) be a sequence converging to 'a', $f(x_n) = x_n^2 \to a^2$ and so by Theorem 3,

$\lim_{x \to a} f(x) = a^2$.

E8) Let (x_n) be any sequence belonging to the domain of definition of f converging to 1 and such that $x_n \neq 1$ for any n. Given $\varepsilon > 0$ we want to find an M such that for all $n \geq M$

$$2 - \varepsilon < 2^{x_n} < 2 + \varepsilon$$

Choose $\varepsilon_1 = \log_2 (1 + \varepsilon/2)$. It is clear that $\varepsilon_1 > 0$.

Since $\lim_{n \to \infty} x_n = 1$, therefore corresponding to $\varepsilon_1 > 0$, there exists a positive integer M such that

$$1 - \varepsilon_1 < x_n < 1 + \varepsilon_1, \text{ for } n \geq M.$$

Thus for $n \geq M$, we have

$2^{x_n} < 2^{1+\varepsilon_1} = 2.2^{\varepsilon_1} = 2.2^{\log_2(1+\varepsilon/2)} = 2(1 + \varepsilon/2) = 2 + \varepsilon.$

and $2^{x_n} > 2^{1-\varepsilon_1} = 2.2^{-\varepsilon_1} = \frac{2}{2^{\log_2(1+\varepsilon/2)}} = \frac{2}{1 + \varepsilon/2} = \frac{4}{2 + \varepsilon}$

$$> \frac{4 - \varepsilon^2}{2 + \varepsilon} = 2 - \varepsilon.$$

$\therefore 2 - \varepsilon < 2^{x_n} < 2 + \varepsilon$, for $n \geq M$

i.e., $|2^{x_n} - 2| < \varepsilon$, for $n \geq M$.

This proves that 2^{x_n} tends to 2. From theorem 3, it follows that $\lim_{x \to 1} 2^x = 2$.

E9) (i) $\lim_{x \to \infty} \frac{(2x + 3)^3 (3x - 2)^2}{x^5 + 5}$

$$= \lim_{x \to \infty} \frac{(2 + 3/x)^3 (3 - 2/x)^2}{1 + 5/x^5} = \frac{2^3.3^2}{1} = 72.$$

(ii) $\lim_{x \to \infty} \frac{\sqrt[3]{x^3 + 1}}{x + 1} = \lim_{x \to \infty} \frac{\sqrt[3]{1 + 1/x^3}}{1 + 1/x} = 1.$

E10) $g(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1 \\ 4 & \text{if } x = 1 \\ 5 - 3x & \text{for } 1 < x \leq 2. \end{cases}$

$\lim_{x \to 1+} g(x) = \lim_{x \to 1+} (5 - 3x) = 5 - \lim_{x \to 1+} 3x = 5 - 3 = 2$

and $\lim_{x \to 1-} g(x) = \lim_{x \to 1-} 2x = 2$

Hence $\lim_{x \to 1} g(x) = 2.$

E11) (i) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}$

$= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

(ii) $\lim_{x\to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

$= \lim_{x\to 2} \frac{(3x+5)(x-2)}{(x+7)(x-2)} = \lim_{x\to 2} \frac{3x+5}{x+7}$

$= \frac{11}{9}.$

(iii) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 x/2}{4.x^2/4} = \lim_{x\to 0} \frac{1}{2} \frac{\sin^2 x/2}{(x/2)^2}$

$= \frac{1}{2}$, since $\lim_{x\to 0} \frac{\sin^2 x/2}{(x/2)^2} = \lim_{x\to 0} \left(\frac{\sin x/2}{x/2}\right)^2 = 1$

(iv) $\lim_{x\to a} \frac{\sin x - \sin a}{x-a} = \lim_{x\to a} \frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$

$= \lim_{x\to a} \cos\frac{x+a}{2} \lim_{x\to a} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}$

$= \cos a.$

(v) $\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x$

$\lim_{x\to\infty} \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{1-1/x}{1+1/x} = 1.$

$\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x = \lim_{x\to\infty} \left[1 + \frac{x-1}{x+1} - 1\right]^x$

$= \lim_{x\to\infty} \left\{\left[1 + \left(\frac{-2}{x+1}\right)\right]^{-\frac{x+1}{2}}\right\}^{-\frac{2x}{1+x}}$

$= \lim_{x\to\infty} e^{-\frac{2x}{1+x}} = e^{-2}$

since $\lim_{x\to\infty} \left[1 + \left(\frac{-2}{x+1}\right)\right]^{-\frac{x+1}{2}} = e.$

or

$\lim_{x\to\infty} \left(\frac{x-1}{x+1}\right)^x = \frac{\lim_{x\to\infty} (1-1/x)^x}{\lim_{x\to\infty} (1+1/x)^x}$

$= \frac{\lim_{x\to\infty} \left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1}}{\lim_{x\to\infty} (1+1/x)^x} = \frac{e^{-1}}{e} = e^{-2}$

(vi) $\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^{1+x}$

$\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right) = 2$ and $\lim_{x\to 0} (1+x) = 1$

Hence $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)^{1+x} = 2^1 = 2.$

(vii) $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right)^{x^2}$

$\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right) = \frac{1}{2}$ and $\lim_{x \to \infty} x^2 = +\infty$

Hence $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right)^{x^2} = 0.$

UNIT 9 CONTINUITY

STRUCTURE

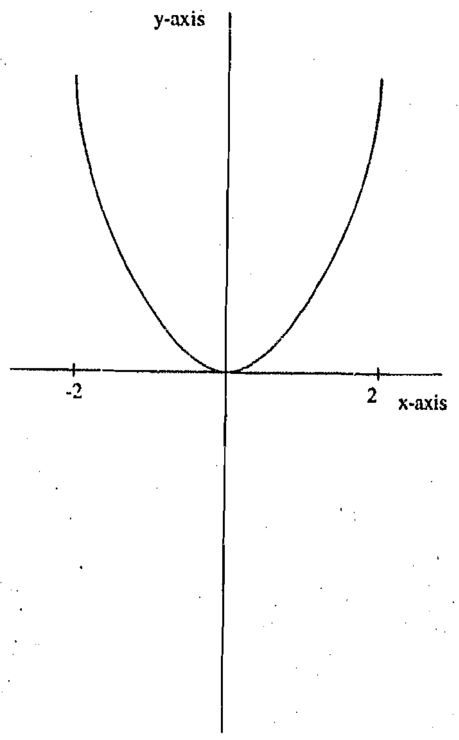
9.1 INTRODUCTION

Suppose that you have functions which are defined on an interval, either open; or closed. If you draw the graph of these functions, you will observe that some of these can be sketched down in one smooth **'continuous'** sweep of your pen, while others have many breaks or jumps. For example, draw the graph of the following two functions :

(a) $f(x) = x^2, x \in [-2, 2]$;

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \in [-2, 2], x \neq 0 \\ 0, & x = 0. \end{cases}$

These are as shown in figures 1(a) and 1(b).



Pig. 1(a)

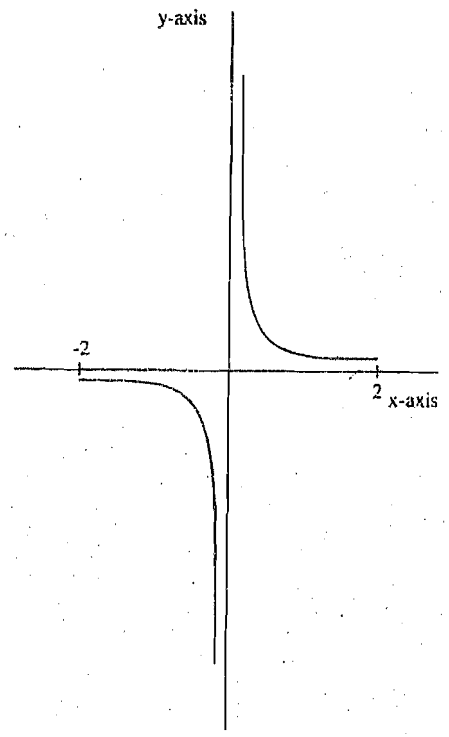


Fig. 1(b)

You can see that while the graph of the first fimction can be drawn in the **'continuous'** motion without lifting the pen from the paper while the graph of the other function cannot be drawn in this manner. This is an interesting property of the first function which is not possessed by the second function. It is, therefore, natural to wonder if it can be given some mathematical meaning. In fact, mathematicians of the past several centuries did confront this question, namely:

"Is there a way to specify those curves which can be drawn with a single stroke of one's pen?"