UNIT 8 THE INTEGRAL

Structure

8.1 INTRODUCTION

So far we concentrated only on that part of calculus which is based on the operation of the derivative, namely, 'differential calculus'. The second major operation of the calculus is integral calculus. The name 'integral calculus' originated in the process of summation, and the word 'integrate' literally means 'find the sum of'. Historically, the subject arose in connection with the determination of areas of plane regions. But in the seventeenth century it was realised that integration can also be viewed as the inverse of differentiation. Integral calculus consists in developing methods for the determination of integrals of any given function.

The relationship between the derivative and the integral of a function is so important that mathematicians have labelled the theorem that describes this relationship as the **Fundamental Theorem of Integral Calculus.**

In this unit, we will introduce the notions of antiderivative, indefinite integral and the notion of definite integral as the limit of a sum. The Fundamental Theorem of Integral Calculus is also discussed in this unit.

Objectives

After reading this unit, you should be able to:

- compute the antiderivative of a given function,
- use the properties of indefinite integrals to compute integrals of simple functions,
- compute the definite integral of a function as the limit of a sum,
- compute the definite integral of a function using the Fundamental Theorem of Integral Calculus.

8.2 ANTIDERIVATIVES

So far, we have been occupied with the 'derivative problem', that is, the problem of finding the derivative of a given function. Some of the important applications of the calculus lead to the inverse problem, namely, given the derivative of a function, is it possible to find the function? This process is called **antidifferentiation** and the result of antidifferentiation is called an **antiderivative**. The importance of the antiderivative results partly from the fact, that scientific laws often specify the rates of change of quantities. The quantities themselves are then found by antidifferentiation.

To get started, suppose we are given that $f'(x) = 5$. Can we find $f(x)$? It is easy to see that one such function f is given by $f(x) = 5x$, since the derivative of $5x$ is 5. Before making any definite decision, consider the functions

$5x + 3$, $5x - 8$, $5x + \sqrt{2}$

Each of these functions has 5 as its derivative. Thus, not only can f(x) be 5x, but it can also be 5x + 3 or 5x − 8 or 5x + $\sqrt{2}$. Not enough information is given to help us determine which is the correct answer.

Let us look at each of these possible functions a bit more carefully. We notice that each of these functions differs from another only by a constant. Therefore, we can say that if $f'(x) = 5$ then f(x) must be of the form $f(x) = 5x + c$, where c is a constant. We call 5x + c the antiderivative of 5.

More generally, we have the following definition.

Definition 1 : Suppose f is a given function. Then a function F is called an **antiderivative** of f, if $F'(x) = f(x) \ \forall \ x$.

We now state an important theorem without giving its proof.

Theorem 1 : If F_1 and F_2 are two antiderivatives of the same function, then F_1 and F_2 differ by a constant, that is,

$F_1(x) = F_2(x) + c$.

Remark : From Theorem 1, it follows that we can find all the antiderivatives of a given function, once we know one antiderivative of it. For instance, in the above example, since one antiderivative of 5 is 5x, all antiderivatives of 5 have the form 5x + c, where c is a constant.

Let us now do a few examples.

Example 1 : Find all the antiderivatives of 2x.

Solution : We have to look for a function F such that $F'(x) = 2x$. Now, an antiderivative of 2x is x^2(check that $\frac{d}{dx}(x^2) = 2x$). Thus, by Theorem 1, all antiderivatives of 2x are given by $x^2 + c$, where c is a constant.

Example 2 : Find all the antiderivatives of $\sqrt{x}$.

Solution : We have to look for a function F such that $F'(x) = \sqrt{x}$. Since an antiderivative of $\sqrt{x} = \frac{x^{3/2}}{3/2}$ therefore, all the antiderivatives of x are given by $\frac{2}{3}x^{3/2} + c$, where c is a constant.

You may now try this exercise.

E1) Find all the antiderivatives of each of the following functions.

(a) $f(x) = x$, (b) $f(x) = 9x^8$ (c) $f(x) = -3x$

Let us now view integration as an inverse of differentiation.

8.3 INTEGRATION AS INVERSE OF DIFFERENTIATION

In the last section we have said that all the antiderivatives of 2x and $\sqrt{x}$ are given by $x^2 + c$ and $\frac{2}{3}x^{3/2} + c$, respectively, where c is a constant. In general, for any given function f, we use the symbol $\int f(x)\,dx$ to denote the antiderivative of f with respect to x. The symbol $\int$ is called **integral** sign and the process of **antidifferentiation** is referred to as an **integration. In other words, the process of integration and differentiation are inverses of one another.** Thus, we write $\int f(x)\,dx = F(x) + c$, where F denotes the antiderivative of f. In the above equation, $\int f(x)\,dx$ is called an **indefinite integral** (or simply an integral) of the function f, c is called the **constant of integration** and x is the **variable of interest**.

We always record the variable of interest together with the letter d. For example, if the variable of interest is t rather than x, then we write $\int f(t)\,dt$ for the integral of f(t).

Look at the following example.

Example 3 : Integrate cos x with respect to x.

Solution : Since we know that $\frac{d}{dx}(\sin x) = \cos x$, an antiderivative of cos x is sin x.

Therefore, $\int \cos x\,dx = \sin x + c$, where c is any constant.

We now state two properties of indefinite integrals which allow us to find many more integrals.

General Properties of Indefinite Integrals

The following two properties of indefinite integrals are useful when evaluating the integral of a function which is composed of the sum or difference of two or more functions.

A) $\int K\,f(x)\,dx = K \int f(x)\,dx$, where K is a constant.

That is, the integral of a constant multiplied by a function is constant multiplied by the integral of that function.

B) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.

The integral of the sum or difference of two functions is equal to the sum or difference of their integrals.

The properties (A) and (B) can be verified by differentiation. But we will not worry about the actual verification here. They are the properties corresponding, respectively, to constant multiple and sum or difference rules for derivatives (see Unit 6).

Note that (B) also hold good for a finite number of functions, that is,

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots\ldots \pm f_n(x)]\,dx$$

$$= \int f_1(x)dx \pm \int f_2(x)dx \pm \int f_3(x)dx \pm \ldots\ldots \pm \int f_n(x)dx$$

Let us now use these properties in solving the following examples.

Example 4 : Evaluate $\int (3x + 4x^3)\,dx$

Solution : Now $\int (3x + 4x^3)dx = \int 3x\,dx + \int 4x^3\,dx$ (by rule B)

$= 3\int x\,dx + 4\int x^3\,dx$ (by rule A)

$= 3\frac{x^2}{2} + 4\frac{x^4}{4} + c$, (where c is a constant of integration)

$= \frac{3}{2}x^2 + x^4 + c.$

Remember, we can always check the result obtained by differentiating it. Since $\frac{d}{dx}\left(\frac{3}{2}x^2 + x^4 + c\right) = \frac{3}{2}2x + 4x^3 = 3x + 4x^3$.

Therefore, our answer of Example 4 is correct.

Let us look at another example.

Example 5 : Evaluate $\int (2e^x - 3\sqrt{x})\, dx$

Solution : We have,

$$\int (2e^x - 3\sqrt{x})\, dx = \int 2e^x\, dx - \int 3\sqrt{x}\, dx \text{ (rule B)}$$

$$= 2\int e^x\, dx - 3\int x^{1/2}\, dx \text{ (rule A)}$$

$$= 2e^x - \frac{3x^{3/2}}{3/2} + c$$

(Since $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(x^{3/2}) = \frac{3}{2}x^{1/2}$)

Therefore, $\int (2e^x - 3\sqrt{x})\, dx = 2e^x - 2x^{3/2} + c$, c being the constant of integration.

You may now try the following exercises.

E2) Find the following integrals:

a) $\int (x^2 - x - 1)\, dx$ b) $\int \sin x\, dx$

c) $\int (\frac{1}{\sqrt{x}} - 3\sqrt{x})\, dx$ d) $\int (e^x + 5)\, dx$

e) $\int (4 - 5e^{-5t} - \frac{e^{2t}}{3})\, dt$

E3) Integrate the following :

a) $3x^5$ b) $3x + 4x^2$ c) $e^x + 2\sin x - 3\cos x$

d) $5\cos x + 2x - 10$ e) $\frac{(x+x^2)^3}{x^3}$ f) x^n, $n \neq -1$.

We have so far regarded integration as inverse of differentiation and defined the indefinite integral of a given function. In the next section we shall be defining the definite integral of a given function. It will be shown that a definite integral can also be represented as the limit of the sum of a certain number of terms, when the number of terms tends to infinity.

8.4 DEFINITE INTEGRAL AS THE LIMIT OF THE SUM

Suppose somebody asks you; what is meant by the areas of a geometrical figure? You will at once answer that it is a measurement that gives the size of the region enclosed by the figure. For instance, the area of a rectangle is the product of its length and width, the area of a triangle is half the product of the lengths of the base and the altitude, and so on. However, how do we define the area of a region in a plane if the region is bounded by a curve? We begin with the definition of the area of such region, and we shall be using this definition to motivate the definition of the definite integral.

Areas and Integrals

In order to find the area of a region bounded by a curve, we shall be considering the sums of many terms and so it is convenient to make use of a notation $\sum$ to denote the sum. $\sum$ is a Greek Letter. For instance, the Notation $\sum_{i=1}^{5} i^2$ denotes the sum of squares of the first five integers, that is,

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

Formally, we can have the following definition.

Definition 2 : $\sum_{i=m}^{n} F(i) = F(m) + F(m+1) + \ldots\ldots + F(n-1) + F(n)$

where, m and n are integers, $m \leq n$.

In Definition 2 above the number m is called the **lower limit** of the sum, and n is called **upper limit** of the sum. The symbol i is called the **index of summation**. It is a "dummy" symbol, because any other letter can be used for this purpose. For instance $\sum_{k=3}^{5} k^2$ and $\sum_{i=3}^{5} i^2$ are both equivalent to $(3^2 + 4^2 + 5^2)$.

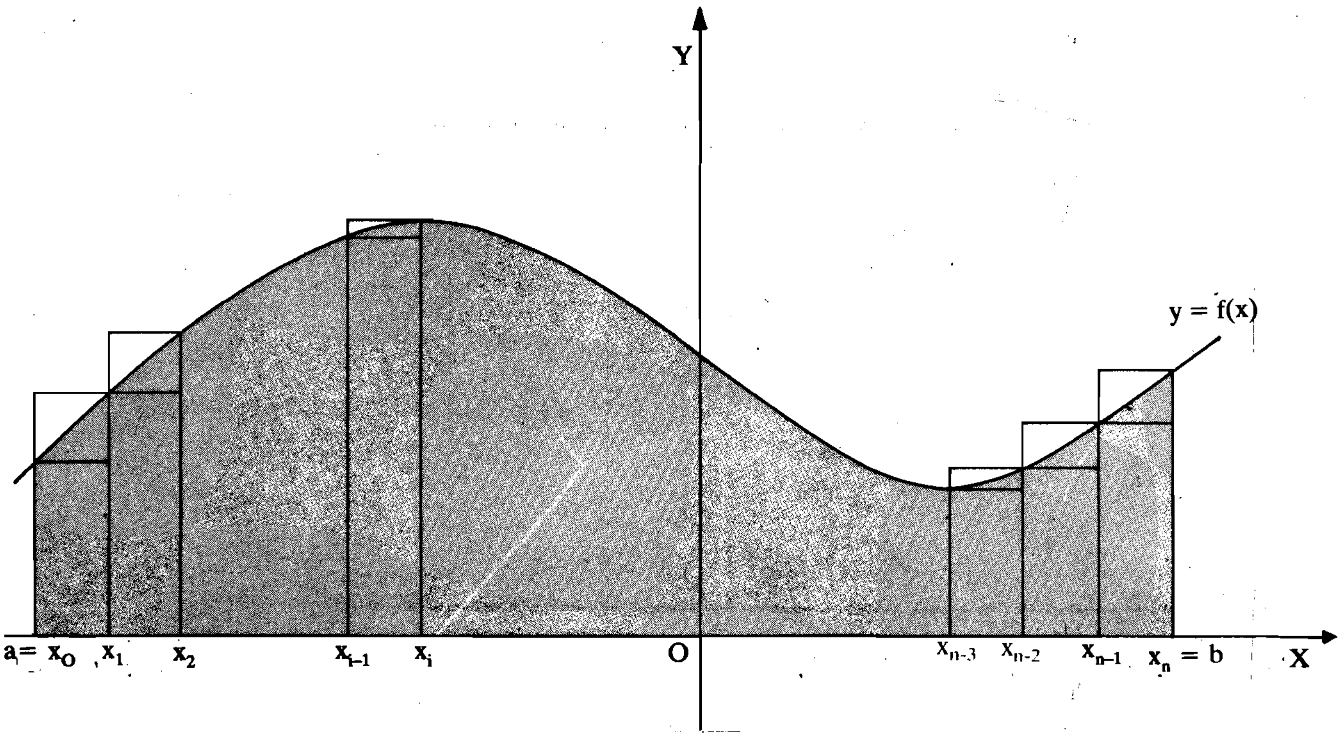

Fig. 1 : Approximation of an area

We now describe the method of finding the area by finding sums in terms of what we call the lower sum and the upper sum.

Consider a shaded region R in the plane as shown in Fig. 1.

The region R is bounded by the x-axis, the lines x = a, x = b, and the curve having the equation y = f(x), where f is continuous on the closed interval [a, b]. For simplicity, we have assumed $f(x) \geq 0$ for all x in [a, b]. Let A denote the area of a region R.

The **first stage** of the method is to divide the closed interval [a, b] into n sub-intervals. We assume that each of these sub-intervals are of equal length, say of length h. Therefore, $h = \frac{(b-a)}{n}$. Denote the end points of these sub-intervals by $x_0, x_1, x_2, \ldots\ldots, x_{n-1}, x_n$, where, $x_0 = a$, $x_1 = a+h$, ... $x_i = a+ih$, $x_{n-1} = a + (n-1)h$, $x_n = a + nh = b$.

Secondly, in each of the sub-intervals we approximate the area under the curve by two types of rectangles, each having the sub-interval as its base. The first type of rectangle has the minimum value of the function on the sub-interval as height, and the second, the maximum value of the function on the sub-interval as its height.

Thirdly, we sum the areas of the rectangles of the first type, which lie within the region and which produces an under-estimate (lower sum) for the area A, and sum the areas of the second type of rectangle which gives an over-estimate (upper sum) for the area A.

Finally, the required area A is found by seeing what happens to these upper and lower sums as the lengths of the sub-intervals, that is, the width of the rectangles, tend to 0. (Since the n sub-intervals have equal width, this is equivalent to letting $n \to \infty$.)

Consider the interval $[x_{i-1}, x_i]$. Let M_i be the maximum value of the function f on this sub-interval and m_i be the minimum value of f on this sub-interval. Then the area A_i bounded by $x = x_{i-1}$, $x = x_i$, $y = 0$ and $y = f(x)$ is approximated as

$$m_i h \le A_i \le M_i h$$

Proceeding in the same way in other sub-intervals and then adding the results we conclude that

$$\sum_{i=1}^{n} m_i h \le \sum_{i=1}^{n} A_i \le \sum_{i=1}^{n} M_i h,$$

or $s_n \le A \le S_n$,

where $s_n = \sum_{i=1}^{n} m_i h$ is the lower sum that is, an under-estimate for the area and

$S_n = \sum_{i=1}^{n} M_i h$, is the upper sum, the over-estimate for the area.

The area we are interested in is squeezed between the lower sum and upper sum. When the upper and lower sums have the same limit as $n \to \infty$ (as $n \to \infty$ we have $h \to 0$) we get the required area.

Symbolically we write

$$A = \lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n$$

Remember that there may be cases in which upper and lower sums may not have the same limit as $n \to \infty$.

In the above discussion we assumed $f(x) \ge 0$ in each sub-interval. But this discussion is also valid for other situations.

We can now give the following definition.

Definition 3 : If f is a continuous function defined on the closed interval [a, b] then the **definite integral** of f from a to b, denoted by $\int_a^b f(x)\,dx$, is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} m_i h = \lim_{n\to\infty} \sum_{i=1}^{n} M_i h.$$

If these limits exist, where m_i and M_i are respectively the minimum and maximum values of the function on each sub-interval $[x_{i-1}, x_i]$ of [a, b].

Note that in the above definition the existence of $\int_a^b f(x)\,dx$, depends on the existence of the limit. In case the upper and lower sums do not tend to a unique limit as $n \to \infty$, we say that $\int_a^b f(x)\,dx$, does not exist.

In the notation for the definite integral $\int_a^b f(x)\,dx$, f(x) is called **integrand**, a is called the **lower limit**, and b is called the **upper limit.**

Remark : You may notice that by defining definite integral as the limit of a sum we have shown the equality between $\int_a^b f(x)\,dx$, and the area A considered earlier.

In other words $\int_a^b f(x)\,dx$, represents geometrically the area founded by the curve $y = f(x)$, the x-axis and the two ordinates $x = a$ and $x = b$.

Let us now do few examples using Definition 3. As it is obvious that finding m_i and M_i for a given function on each sub-interval of the given interval is not an easy task. We thus, assume in the following examples the function f to be an increasing function in the given interval. In that case for each sub-interval $[x_{i-1}, x_i]$ of the given interval the minimum value of the function will be attained at x_{i-1}, and the maximum value will be attained at x_i (ref. Unit 7). In such cases we will have

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} h\, f(x_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^{n} h\, f(x_i),\ h = \frac{b-a}{n}$ provided these limits exist.

Example 6 : Evaluate $\int_1^3 x^2\,dx$ as the limit of a sum.

Solution : Consider an equal partition of the closed interval [1, 3] into n sub-intervals. Then $h = \frac{3-1}{n} = \frac{2}{n}$. If we choose x_i, $1 \le i \le n$ as the right end point of each sub-interval, we have $x_1 = 1 + \frac{2}{n}$, $x_2 = 1 + 2\left(\frac{2}{n}\right), \ldots\ldots, x_i + i\left(\frac{2}{n}\right), x_n = 1 + n\left(\frac{2}{n}\right)$.

Let us now calculate the upper sum and lower sum.

The sum of first n natural numbers is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, and the sum of the square of the first n natural numbers $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{2.3}$

$$\text{Upper sum} = \lim_{n\to\infty} \sum_{i=1}^{n} h\ f(x_i)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{n+2i}{n}\right)^2 \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} (n^2 + 4ni + 4i^2)$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \left[n^2 \sum_{i=1}^{n} 1 + 4n \sum_{i=1}^{n} i + 4 \sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \left[n^2.n + 4n\,\frac{n(n+1)}{2} + 4\,\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \left[n^3 + 2n^3 + 2n^2 + \frac{2n(2n^2+3n+1)}{3}\right]$$

$$= \lim_{n\to\infty} \left[6 + \frac{4}{n} + \frac{8n^2+12n+4}{3n^2}\right]$$

$$= \lim_{n\to\infty} \left[6 + \frac{4}{n} + \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right]$$

$$= 6 + 0 + \frac{8}{3} + 0 + 0$$

$$\left(\text{since } \lim_{n\to\infty} \frac{1}{n} = 0\right) = \frac{26}{3}$$

Similarly, you can show that lower sum $= \lim_{n\to\infty} \sum_{i=1}^{n} h\ f(x_{i-1}) = \frac{26}{3}$

$$\int_1^3 f(x)\,dx = \frac{26}{3}$$

E4) Show that in Example 6, lower sum $= \frac{26}{3}$.

In definition 3, the closed interval [a, b] is given, and so we assume that a < b. To consider the definite integral of a function f from a to b when a > b or when a = b we have the following definitions.

Definition 4 : If a > b, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \text{ if } \int_b^a f(x)\,dx \text{ exists.}$$

In Example 6, we showed that $\int_1^3 x^2 dx = \frac{26}{3}$. Therefore, from Definition 4,

$$\int_3^1 x^2\,dx = -\int_1^3 x^2\,dx = -\frac{26}{3}$$

On the basis of our earlier discussion, we now give another definition

Definition 5 : If f(a) exists, then

$$\int_a^a f(x)\,dx = 0.$$

From this definition, $\int_1^1 x^2\,dx = 0$.

You may now try the following exercise.

E5) Evaluate the following definite integral, as the limit of a sum.

a) $\int_2^7 3x\,dx$ b) $\int_2^0 x^2\,dx$ c) $\int_1^4 (x^2+4x+5)\,dx$

d) $\int_0^4 (x^2+x-6)\,dx$ e) $\int_1^1 x^2\,dx$

Evaluating a definite integral from the definition by actually finding the limit of a sum, as was done in Section 8.4, is usually quite tedious and frequently almost impossible.

In the following section we state the result which completely avoids the use of upper or lower sums. This result is known as the "Fundamental Theorem of Integral Calculus

8.5 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

This theorem is "Fundamental" since it expresses the integral in terms of an antiderivative and establishes the key link between differentiation and integration. But before we state the theorem we first need to give some properties of the definite integral.

Simple Properties of Definite Integrals

P 1 : If f is the constant function with f(x) = k on [a, b], then

$$\int_a^b k\,dx = k\,(b-a)$$

Example 7 : Evaluate $\int_{-3}^{5} 4\, dx$

Solution : Applying P1

$$\int_{-3}^{5} 4\, dx = 4(5 - (-3)) = 4(8) = 32.$$

Note that, if $k > 0$ in P1, then this calculation gives the area of a rectangle of height k and base b−a.

P 2 : If $\int_a^b f(x)\, dx$ exists and k is any real constant, then

$$\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$$

Here if $f(x) = 1$, then we get P1 as a particular case of P2.

Example 8 : Evaluate $\int_1^3 4x^2\, dx$.

Solution : Applying P2

$$\int_1^3 4x^2\, dx = 4 \int_1^3 x^2\, dx = 4\, \frac{26}{3} \text{ (from Example 6)} = \frac{104}{3}$$

P 3 : If $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ exists, then

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Note the similarity of the properties P2 and P3 to limit Theorem 1 of Unit 6 on the limit of a function multiplied by a constant and the limit of the sum or difference of two function.

Let us now do the following example.

Example 9 : Use the result of Example 6, and the fact that $\int_1^3 x\, dx = 4$ to evaluate $\int_1^3 (3x^2-5x+2)\, dx$.

Solution : In Example 6, we have shown that $\int_1^3 x^2\, dx = \frac{26}{3}$.
Using P2 and P3 we get,

$$\int_1^3 (3x^2-5x+2)\, dx = \int_1^3 3x^2 dx - \int_1^3 5x\, dx + \int_1^3 2\, dx$$

$$= 3\int_1^3 x^2 dx - 5\int_1^3 x\, dx + 2\int_1^3 dx$$

$$= 3\left(\frac{26}{3}\right) - 5(4) + 2(3-1)$$

$$= 26 - 20 + 4 = 10.$$

P 4 : If $\int_a^b f(x)\, dx$, $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ exists, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ where } a < c < b$$

Applying P4 to $\int_{-3}^{5} 4\,dx$, we can write it as

$$\int_{-3}^{5} 4\,dx = \int_{-3}^{4} 4\,dx + \int_{4}^{5} 4\,dx, \text{ because } -3 < 4 < 5.$$

Now, $\int_{-3}^{5} 4\,dx = 4\,[4-(-3)] + 4\,[5-4]$

$= 4.7 + 4 = 28 + 4 = 32.$
which is the required result.

P 5 : If $\int f(x)\,dx$ exists on a closed interval containing the three real numbers, a, b and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

regardless of the order of a, b and c.

That is, the integral $\int_{-3}^{5}$ 4 dx taken above can also be evaluated as a sum of integrals.

$$\int_{-3}^{6} 4\,dx + \int_{6}^{5} 4\,dx = 4\,[6-(-3)] + 4[5-6] = 4.9 - 4 = 32.$$

P 6 : If $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ exists, and if $f(x) \geq g(x)$ for all x in [a, b], then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

To illustrate P6, consider $f(x) = x^2$ and $g(x) = x$. Now, we know that $x^2 \geq x \; \forall \; x$ in [1,3].

Also, $\int_1^3 x^2\,dx = \frac{26}{3}$ (ref. Example 6) and $\int_1^3 x\,dx = 4$ (ref. Example 9) which shows that $\int_1^3 x^2\,dx \geq \int_1^3 x\,dx$.

P 7 : Suppose that the function f is continuous on the closed interval [a, b]. If m and M are respectively the absolute minimum and absolute maximum values of f on [a, b] so that

$m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Example 10 : Apply P7 to find a closed interval containing the value of

$$\int_{1/2}^{4} (x^3-6x^2+9x+1)\,dx$$

Solution : If $f(x) = x^3-6x^2+9x+1$ then it can be easily checked that f has a minimum value 1 at $x = 3$ and maximum value 5 at $x = 1$. The absolute minimum value of f on $[\frac{1}{2}, 4]$ is 1, and the absolute maximum value is 5 (Ref. Unit 7). Taking m = 1 and M = 5 in P 7, we have

$$1\left(4-\frac{1}{2}\right) \le \int_{1/2}^{4} (x^3-6x^2+9x+1)\, dx \le 5\left(4-\frac{1}{2}\right)$$

$$\frac{7}{2} \le \int_{1/2}^{4} (x^3-6x^2+9x+1)\, dx \le \frac{35}{2}$$

Therefore, the closed interval $[\frac{7}{2}, \frac{35}{2}]$ contains the value of the definite integral.

You can now do these exercises easily.

E6) Evaluate the following definite integrals.

a) $\int_{2}^{5} 4\, dx$ b) $\int_{-2}^{2} \sqrt{5}\, dx$ c) $\int_{5}^{-1} 6\, dx$ d) $\int_{3}^{3} dx$

E7) Evaluate the given definite integrals by using the following results :

$$\int_{-1}^{2} x^2\, dx = 3 ; \quad \int_{-1}^{2} x\, dx = \frac{3}{2} ; \quad \int_{0}^{\pi} \sin x\, dx = 2$$

$$\int_{0}^{\pi} \cos x\, dx = 0 ; \quad \int_{0}^{\pi} \sin^2 x\, dx = \frac{\pi}{2}$$

a) $\int_{-1}^{2} (2x^2-4x+5)$ b) $\int_{2}^{-1} (2x+1)^2\, dx$ c) $\int_{-1}^{2} (x-1)(2x+3)\, dx$

d) $\int_{0}^{\pi} (2\sin x + 3\cos x + 1)\, dx$ e) $\int_{0}^{\pi} (\cos x + 4)^2\, dx$

We now give the statement of the fundamental theorem of integral calculus.

We shall not be giving you the proof of this theorem as it is beyond the scope of this course.

Theorem 2 : If f is a continuous function defined on [a, b] and F is an antiderivative of f, that is, if $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_{a}^{b} f(x)\, dx = F(b) - F(a) = F(x) \Big|_{a}^{b}$$

Note that the fundamental theorem does not specify which antiderivative to use. However, we know that if F_1 and F_2 are two antiderivatives of f on [a, b] then they differ by a constant (see Section 8.2) that is, $F_1(x) = F_2(x) + c$, so

$$F_1(b) - F_1(a) = [F_2(b) + c] - [F_2(a) + c] = F_2(b) - F_2(a).$$

The c's cancel. Thus, all choices of F give the same result.

We now illustrate this theorem through some examples :

Example 11 : Evaluate $\int_{1}^{3} x^2\, dx$.

Solution : Here $f(x) = x^2$. An antiderivative of x^2 is $\frac{1}{3}x^3$. So, we choose $F(x) = \frac{x^3}{3}$, and then using the theorem, we get

$$\int_{1}^{3} x^2\, dx = \frac{x^3}{3}\Big|_{1}^{3} = \frac{1}{3}[(3)^3 - (-1)^3]$$

$$= \frac{1}{3}[27-1] = \frac{26}{3}.$$

Example 12 : Evaluate $\int_{1/2}^{4} (x^3-6x^2+9x+1)\,dx$

Solution :

$$\int_{1/2}^{4} (x^3-6x^2+9x+1)\,dx = \int_{1/2}^{4} x^3\,dx - 6\int_{1/2}^{4} x^3\,dx + 9\int_{1/2}^{4} dx + \int_{1/2}^{4} dx$$

$$= \frac{x^4}{4} - 6\frac{x^3}{3} + 9\frac{x^2}{2} + x \Big]_{1/2}^{4}$$

(Using the antiderivative of each of the integrand.)

$$= \left[\left[\frac{256}{4} - \frac{6}{3}.64 + \frac{9}{2}.16 + 4\right] - \left[\frac{1}{4}\cdot\frac{1}{16} - \frac{6}{3}\cdot\frac{1}{8} + \frac{9}{2}\cdot\frac{1}{4} + \frac{1}{2}\right] \right]$$

$$= [64 - 128 + 72 + 4] - \left[\frac{1}{64} - \frac{1}{4} + \frac{9}{8} + \frac{1}{2}\right]$$

$$= \frac{679}{64}$$

Let us look at few more examples.

Example 13 : Evaluate $\int_{1}^{2} \frac{1}{x^4}\,dx$.

Solution : An antiderivative of $\frac{1}{x^4}$ is $\frac{1}{3x^3}$, since $\frac{d}{dx}(-\frac{1}{3}x^{-3}) = -\frac{1}{3}(-3)x^{-4} = x^{-4}$

Hence,

$$\int_{1}^{2} \frac{1}{x^4}\,dx = -\frac{1}{3x^3}\Big|_{1}^{2} = \left[-\frac{1}{3.2^2}\right] - \left[\frac{1}{3.1^3}\right]$$

$$= -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$$

Example 14 : Evaluate $\int_{0}^{\pi/2} (e^x\,dx - \cos x)\,dx$

Solution : $\int_{0}^{\pi/2} (e^x - \cos x)\,dx = \int_{0}^{\pi/2} e^x\,dx - \int_{0}^{\pi/2} \cos x\,dx$

$$= e^x\,dx \Big|_{0}^{\pi/2} - \sin x \Big|_{0}^{\pi/2}$$

$$= e^{\pi/2} - 1 - 1 = e^{\pi/2} - 2.$$

How about doing some exercises now?

E8) Compute the following integrals :

a) $\int_{4}^{6} 3x\,dx$ b) $\int_{1}^{8} (1 + \sqrt{x})\,dx$ c) $\int_{1}^{2} \frac{x^2+1}{x^2}\,dx$ d) $\int_{0}^{2} (x^2-1)\,dx$

E9) Evaluate the following integrals :

a) $\int_{a}^{b} x^{4/3}\,dx$ b) $\int_{-1}^{2} (x^4+8x)\,dx$ c) $\int_{1}^{2} 4\pi\, x^{2/3}\,dx$

d) $\int_{0}^{10} \left(\frac{x^4}{100} - x^2\right)dx$ e) $\int_{-1}^{2} (1 + x^2)^2\,dx$

E10) Evaluate the following integrals:

a) $\int_0^{\pi/2} (2-\cos x)\, dx$ b) $\int_{-\pi/2}^{\pi} (3\cos x - \frac{1}{3}\sin x)\, dx$

c) $\int_0^{\pi/2} (2x-3\sin x)\, dx$ d) $\int_0^{x} (3-\cos t)\, dt$

We end this unit by giving summary of what we have done in it.

8.6 SUMMARY

In this unit we have covered the following points.

1) Definition of an antiderivative of a function.
2) Meaning of an integral of a function.
3) Indefinite integral and its general properties.
4) Introduction of the definite integral as the limit of a sum.
5) Statement of simple properties of definite integrals.
6) Fundamental theorem of integral calculus.

8.7 SOLUTIONS/ANSWERS

E1) a) $\frac{x^2}{2} + c$; b) $x^9 + c$; c) $\frac{-3x^2}{2} + c$

E2) a) $\frac{x^3}{3} - \frac{x^2}{2} - x + c$; b) $-\cos x + c$; c) $4x^{1/2} - 2x^{3/2} + c$

d) $e^x + 5x + c$; e) $4t + e^{-5t} + \frac{e^{2t}}{6} + c$

E3) a) $\frac{x^6}{6} + c$; b) $\frac{3x^2}{2} + \frac{4}{3}x^3 + c$ c) $e^x - 2\cos x - 3\sin x + c$

d) $5\sin x + x^2 - 10x + c$; e) $\frac{x^4}{4} + x^3 + \frac{3}{2}x^2 + x + c$

f) $\frac{x^{n+1}}{n+1} - c$

E4) Lower sum $= \lim_{n\to\infty} \sum_{i=1}^{n} h\, f(x_{i-1})$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[1 + \frac{2(i-1)}{n}\right]^2$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} \left[n+2(i-1)\right]^2$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} \left[n^2 + 4(i-1)^2 + 4n(i-1)\right]$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left[(n^2-4n+4)\sum_{i=1}^{n} 1 + 4(n-2)\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left[n(n^2-4n+4) + 4(n-2)\frac{n(n+1)}{2} + \frac{4n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \frac{2}{n^3}\left[\frac{13}{3}n^3 - 4n^2 + \frac{2}{3}n\right]$$

$$= \frac{26}{3}.$$

E5) a) $\frac{135}{2}$.

b) $\frac{-8}{3}$. Take $h = \frac{2}{n}$ and use the formula for the sum of the square of the first n natural numbers.

c) 66 ; d) $\frac{16}{3}$; e) 0

E6) a) 12 ; b) $4\sqrt{5}$; c) -36 ; d) 0

E7) a) 15 ; b) -21 ; c) $\frac{-3}{2}$; d) $4+\pi$; e) $\frac{33\pi}{2}$

E8) a) 30 ; b) $\frac{19}{3} + \frac{32\sqrt{2}}{3}$; c) $\frac{1}{2}$; d) $\frac{2}{3}$

E9) a) $\frac{3}{7}(b^{7/3} - a^{7/3})$; b) $\frac{93}{5}$; c) $(\frac{12\pi}{5})\ (3\sqrt{32-1})$

d) $\frac{400}{3}$. e) $\frac{78}{5}$.

E10) a) $\pi-1$; b) $\frac{8}{3}$; c) $\frac{\pi^2}{4} - 3$; d) $3x - \sin x$

UNIT 9 INTEGRATION OF ELEMENTARY FUNCTIONS

Structure

9.1 INTRODUCTION

In Unit 6, we developed techniques of differentiation which enable us to differentiate almost any function with comparative ease. Although, integration is the reverse process of differentiation as we have seen in Unit 8, yet integration is much harder to carry out. Recall that in the case of differentiation, if a function is an expression involving elementary functions (such as x^r, sin x, e^x, ...) then, so is its derivative. Although many integration problems also have this characteristic, certain ones do not. However, there are some elementary functions (e.g. e^{x^2}) for which an integral cannot be expressed in terms of elementary functions. Even where this is possible, the techniques for finding these integrals are often complicated. For this reason, we must be prepared with a broad range of techniques in order to cope with the problem of calculating integrals.

In this unit we will develop two general techniques, namely, integration by substitution and integration by parts for calculating both indefinite and definite integrals. We will also discuss their application for the integration of various classes of elementary and trigonometric functions.

Objectives

After reading this unit you should be able to :

- compute integrals of functions using standard integrals,
- use the method of substitution for integration,
- use the method of integration by parts for integration,
- compute integrals of various elementary and trigonometric functions.

9.2 STANDARD INTEGRALS

In many cases a function is at once recognised as the derivative of some other function and thus can be integrated easily. Such integrals are known as **Standard Integrals.** We now list such standard integrals for ready reference in Table 1. We shall be making use of these integrals every now and then while evaluating many more integrals. Before we give these integrals let us mention that throughout the discussion in this unit we shall be denoting the constant of integration by c.

It is important to note that when $n \neq -1$, the integral or x^n is obtained on increasing the index n by 1 and dividing by the increased index n+1. Thus, for example,

$$\int x^{1/2}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}x^{3/2},$$

and

$$\int \frac{dx}{x^2} = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} + c = -\frac{1}{x} + c.$$