UNIT 5 LINEAR TRANSFORMATIONS - I

Structure

5.1 INTRODUCTION

You have already learnt about a vector space and several concepts related to it. In this unit we initiate the study of certain mappings between two vector spaces, called linear transformations. The importance of these mappings can be realised from the fact that, in the calculus of several variables, every continuously differentiable function can be replaced, to a first approximation, by a linear one. This fact is a reflection of a general principle that every problem on the change of some quantity under the action of several factors can be regarded, to a first approximation, as a linear problem. It often turns out that this gives an adequate result. Also, in physics it is important to know how vectors behave under a change of the coordinate system. This requires a study of linear transformations.

In this unit we study linear transformations and their properties, as well as two spaces associated with a linear transformation, and their dimensions. Then, we prove the existence of linear transformations with some specific properties. We discuss the notion of an isomorphism between two vector spaces, which allows us to say that all finite-dimensional vector spaces of the same dimension are the 'same', in a certain sense.

Finally, we state and prove the Fundamental Theorem of Homomorphism and some of its corollaries, and apply them to various situations.

Since this unit uses concepts developed in Units 1, 3 and 4, we suggest that you revise these units before going further.

Objectives

After reading this unit, you should be able to

- verify the linearity of certain mappings between vector spaces;
- construct linear transformations with certain specified properties;
- calculate the rank and nullity of a linear operator;
- prove and apply the Rank Nullity Theorem;
- define an isomorphism between two vector spaces;
- show that two vector spaces are isomorphic if and only if they have the same dimension;
- prove and use the Fundamental Theorem of Homomorphism.

5.2 LINEAR TRANSFORMATIONS

In Unit 2 you came across the vector spaces $\mathbf{R}^2$ and $\mathbf{R}^3$. Now consider the mapping $f: \mathbf{R}^2 \to \mathbf{R}^3: f(x,y) = (x,y,0)$ (see Fig.1).

f is a well defined function. Also notice that

i) $f((a,b) + (c,d)) = f((a + c, b + d)) = (a + c, b + d,0) = (a,b,0) + (c,d,0)$

$= f((a,b)) + f((c,d))$, for $(a,b), (c,d) \in \mathbf{R}^2$, and

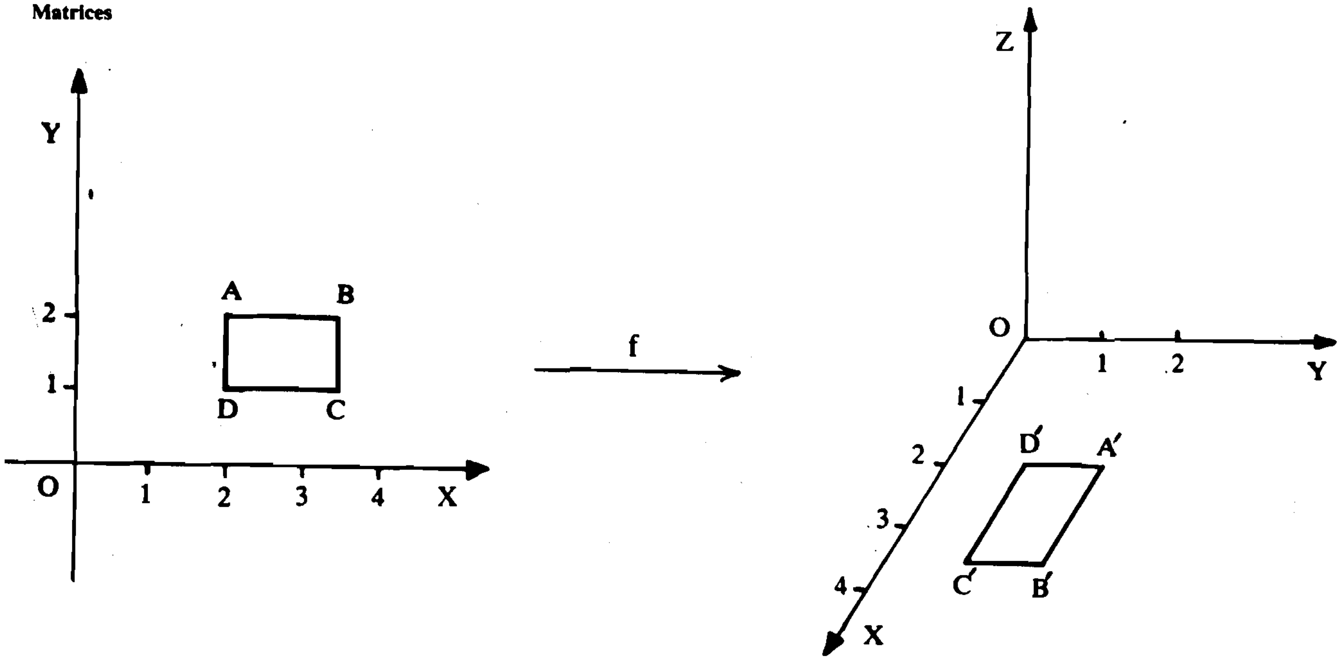

Fig. 1: f transforms ABCD to A'B'C'D'.

ii) for any $\alpha \in \mathbf{R}$ and $(a,b) \in \mathbf{R}^2$, $f(\alpha(a,b)) = f((\alpha a, \alpha b)) = (\alpha a, \alpha b, 0) = \alpha(a,b,0)$

$$= \alpha f((a,b)).$$

So we have a function f between two vector spaces such that (i) and (ii) above hold true.

(i) says that the sum of two plane vectors is mapped under f to the sum of their images under f. (ii) says that a line in the plane $\mathbf{R}^2$ is mapped under f to a line in $\mathbf{R}^3$.

The properties (i) and (ii) together say that f is linear, a term that we now define.

Definition: Let U and V be vector spaces over a field **F**. A **linear transformation** (or **linear operator**) from U to V is a function $T: U \rightarrow V$, such that

LT1) $T(u_1 + u_2) = T(u_1) + T(u_2)$, for $u_1, u_2 \in U$, and

LT2) $T(\alpha u) = \alpha T(u)$ for $\alpha \in \mathbf{F}$ and $u \in U$.

The conditions LT1 and LT2 can be combined to give the following equivalent condition.

LT3) $T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2)$, for $\alpha_1, \alpha_2 \in \mathbf{F}$ and $u_1, u_2 \in U$.

What we are saying is that [LT1 and LT2] $\Leftrightarrow$LT3. This can be easily shown as follows:

We will show that LT3 $\Rightarrow$ LT1 and LT3 $\Rightarrow$ LT2. Now, LT3 is true $\forall \alpha_1, \alpha_2 \in \mathbf{F}$. Therefore, it is certainly true for $\alpha_1 = 1 = \alpha_2$, that is, LT1 holds.

Now, to show that LT2 is true, consider $T(\alpha u)$ for any $\alpha \in \mathbf{F}$ and $u \in U$. We have $T(\alpha u) = T(\alpha u + 0.u) = \alpha T(u) + 0.T(u) = \alpha T(u)$, thus proving that LT2 holds.

You can try and prove the converse now. That is what the following exercise is all about!

E E1) Show that the conditions LT1 and LT2 together imply LT3.

Before going further, let us note two properties of any linear transformation $T:U \rightarrow V$, which follow from LT1 (or LT2, or LT3).

LT4) $T(\mathbf{0}) = \mathbf{0}$. Let's see why this is true. Since $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$ (by LT1), we subtract $T(\mathbf{0})$ from both sides to get $T(\mathbf{0}) = \mathbf{0}$.

LT5) $T(-\mathbf{u}) = -T(\mathbf{u})\ \forall u \in U$. Why is this so? Well, since $\mathbf{0} = T(\mathbf{0}) = T(u - u)$
$= T(u) + T(-u)$, we get $T(-u) = -T(u)$.

E 2) Can you show how LT4 and LT5 will follow from LT2?

Now let us look at some common linear transformations.

Example 1: Consider the vector space U over a field **F**, and the function $T : U \to U$ defined by $T(u) = u$ for all $u \in U$.

Show that T is a linear transformation. (This transformation is called the **identity transformation,** and is denoted by $\mathbf{I}_U$, or just **I**, if the underlying vector space is understood.)

Solution: For any $\alpha, \beta \in \mathbf{F}$ and $u_1, u_2 \in U$, we have

$T(\alpha u_1 + \beta u_2) = \alpha u_1 + \beta u_2 = \alpha T(u_1) + \beta T(u_2)$.

Hence, LT3 holds, and T is a linear transformation.

Example 2: Let $T : U \to V$ be defined by $T(u) = \mathbf{0}$ for all $u \in U$.

Check that T is a linear transformation. (It is called the null or **zero transformation**, and is denoted by **0**.)

Solution: For any $\alpha, \beta \in \mathbf{F}$ and $u_1, u_2 \in U$, we have

$T(\alpha u_1 + \beta u_2) = \mathbf{0} = \alpha.\mathbf{0} + \beta.\mathbf{0} = \alpha T(u_1) + \beta T(u_2)$.

Therefore, T is a linear transformation.

Example 3: Consider the function $pr_1 : \mathbf{R}^n \to \mathbf{R}$, defined by $pr_1[(x_1, \ldots\ldots, x_n)] = x_1$. Show that this is a linear transformation. (This is called the **projection** on the first coordinate. Similarly, we can define $pr_i : \mathbf{R}^n \to \mathbf{R}$ by $pr_i[(x_1, \ldots, x_{i-1}, x_i, \ldots\ldots, x_n)] = x_i$ to be the **projection on the ith coordinate** for $i = 2, \ldots, n$. For instance, $pr_2 : \mathbf{R}^3 \to \mathbf{R} : pr_2(x,y,z) = y$.)

Solution : We will use LT3 to show that pr_1 is a linear operator. For $\alpha, \beta \in \mathbf{R}$ and $(x_1,\ldots\ldots,x_n)$, $(y_1,\ldots\ldots,y_n)$ in $\mathbf{R}^n$, we have

$pr_1[(\alpha(x_1,\ldots\ldots,x_n) + \beta(y_1,\ldots\ldots, y_n)]$

$= pr_1(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots\ldots, \alpha x_n + \beta y_n) = \alpha x_1 + \beta y_1$

$= \alpha pr_1[(x_1,\ldots\ldots,x_n)] + \beta pr_1[(y_1,\ldots\ldots, y_n)]$.

Thus pr_1 (and similarly pr_i) is a linear transformation.

Before going to the next example, we make a remark about projections.

Remark : Consider the function $p : \mathbf{R}^3 \to \mathbf{R}^2 : p(x,y,z) = (x,y)$. This is a projection from $\mathbf{R}^3$ on to the xy-plane. Similarly, the functions f and g, from $\mathbf{R}^3 \to \mathbf{R}^2$, defined by $f(x,y,z) = (x,z)$ and $g(x,y,z) = (y,z)$ are projections from $\mathbf{R}^3$ onto the xz-plane and the yz-plane, respectively.

In general, any function $\theta : \mathbf{R}^n \to \mathbf{R}^m$ $(n > m)$, which is defined by dropping any $(n - m)$ coordinates, is a projection map.

Now let us see another example of a linear transformation that is very geometric in nature.

Example 4: Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(x,y) = (x,-y)\ \forall x,y \in \mathbf{R}$.
Show that T is a linear transformation.

(This is the **reflection** in the x-axis that we show in Fig. 2.)

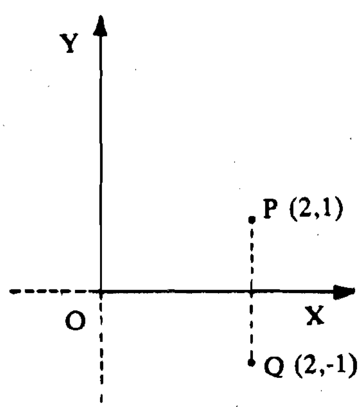

Fig. 2: Q is the reflection of P in the x-axis.

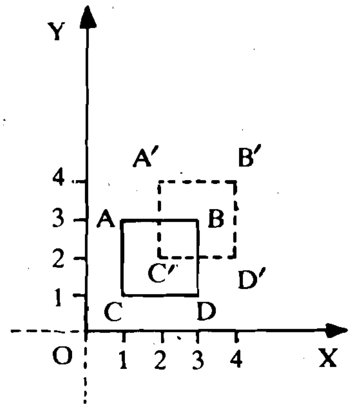

Fig. 3: A′B′C′D′ is the translation of ABCD by (1, 1).

Solution: For $\alpha, \beta \in \mathbf{R}$ and $(x_1,y_1), (x_2,y_2) \in \mathbf{R}^2$, we have

$$T[\alpha(x_1,y_1) + \beta(x_2,y_2)] = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (\alpha x_1 + \beta x_2, -\alpha y_1 - \beta y_2)$$
$$= \alpha(x_1, -y_1) + \beta(x_2, -y_2)$$
$$= \alpha T(x_1,y_1) + \beta T(x_2,y_2).$$

Therefore, T is a linear transformation.

So far we've given examples of linear transformations. Now we give an example of a very important function which is **not linear**. This example's importance lies in its geometric applications.

Example 5: Let u_0 be a fixed non-zero vector in U. Define $T : U \to U$ by $T(u) = u + u_0 \forall u \in U$. Show that T is not a linear transformation. (T is called the **translation** by u_0. See Fig. 3 for a geometrical view.)

Solution: T is not a linear transformation since LT4 does not hold. This is because $T(\mathbf{0}) = u_0 \neq \mathbf{0}$.

Now, try the following exercises.

E E3) Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the reflection in the y-axis. Find an expression for T as in Example 4. Is T a linear operator?

E E4) For a fixed vector (a_1, a_2, a_3) in $\mathbf{R}^3$, define the mapping $T: \mathbf{R}^3 \to \mathbf{R}$ by $T(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3$. Show that T is a linear transformation. Note that $T(x_1,x_2,x_3)$ is the dot product of (x_1,x_2,x_3) and (a_1, a_2, a_3) (ref. Sec. 2.4).

E E5) Show that the map $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 - x_2, x_2 + 2x_3)$ is a linear operator.

You came across the real vector space $\mathbf{P}_n$, of all polynomials of degree less than or equal to n, in Unit 4. The next exercise concerns it.

E E6) Let $f \in \mathbf{P}_n$ be given by

$f(x) = \alpha_0 + \alpha_1 x + \ldots\ldots + \alpha_n x^n, \alpha_i \in \mathbf{R} \ \forall i.$

We define $(Df)(x) = \alpha_1 + 2\alpha_2 x + \ldots + n\,\alpha_n x^{n-1}$

Show that $D:\mathbf{P}_n \to \mathbf{P}_n$ is a linear transformation. (Observe that Df is nothing but the derivative of f. D is called the **differentiation operator.**)

In Unit 3 we introduced you to the concept of a quotient space. We now define a very useful linear transformation, using this concept.

Example 6: Let W be a subspace of a vector space U over a field **F**. W gives rise to the quotient space U/W. Consider the map $T:U \to U/W$ defined by $T(u) = u + W$.
T is called the **quotient map** or the **natural map.**
Show that T is a linear transformation.

Solution: For $\alpha, \beta \in \mathbf{F}$ and $u_1, u_2 \in U$ we have

$$T(\alpha u_1 + \beta u_2) = (\alpha u_1 + \beta u_2) + W = (\alpha u_1 + W) + (\beta u_2 + W)$$
$$= \alpha(u_1 + W) + \beta(u_2 + W)$$
$$= \alpha T(u_1) + \beta T(u_2)$$

Thus, T is a linear transformation.

Now solve the following exercise, which is about plane vectors.

E E7) Let $u_1 = (1, -1)$, $u_2 = (2, -1)$, $u_3 = (4, -3)$, $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (1, 1)$ be 6 vectors in $\mathbf{R}^2$. Can you define a linear transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ such that $T(u_i) = v_i$, $i = 1, 2, 3$?

(Hint: Note that $2u_1 + u_2 = u_3$ and $v_1 + v_2 = v_3$.)

You have already seen that a linear transformation $T:U \to V$ must satisfy $T(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2)$, for $\alpha_1, \alpha_2 \in \mathbf{F}$ and $u_1, u_2 \in U$. More generally, we can show that,

LT6 : $\mathbf{T(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \ldots + \alpha_n T(u_n)}$,

where $\alpha_i \in \mathbf{F}$ and $u_i \in U$.

Let us show this by induction, that is, we assume the above relation for n = m, and prove it for m + 1. Now,

$$T(\alpha_1 u_1 + \ldots + \alpha_m u_m + \alpha_{m+1} u_{m+1})$$
$$= T(u + \alpha_{m+1} u_{m+1}), \text{ where } u = \alpha_1 u_1 + \ldots + \alpha_m u_m$$
$$= T(u) + \alpha_{m+1} T(u_{m+1}), \text{ since the result holds for } n = 2$$
$$= T(\alpha_1 u_1 + \ldots + \alpha_m u_m) + \alpha_{m+1} T(u_{m+1})$$
$$= \alpha_1 T(u_1) + \ldots + \alpha_m T(u_m) + \alpha_{m+1} T(u_{m+1}), \text{ since we have assumed the result for } n = m.$$

Thus, the result is true for n = m + 1. Hence, by induction, it holds true for all n.

Let us now come to a very important property of any linear transformation $T:U \to V$. In Unit 4 we mentioned that every vector space has a basis. Thus, U has a basis. We will now show that T is completely determined by its values on a basis of U. More precisely, we have

Theorem 1: Let S and T be two linear transformations from U to V, where $\dim_F U = n$. Let $\{e_1, \ldots, e_n\}$ be a basis of U. Suppose $S(e_i) = T(e_i)$ for $i = 1, \ldots, n$. Then $S(u) = T(u)$ for all $u \in U$.

Proof: Let $u \in U$. Since $\{e_1, ..., e_n\}$ is a basis of U, u can be uniquely written as $u = \alpha_1 e_1 + ... + \alpha_n e_n$, where the α_i are scalars.

Then, $S(u) = S(\alpha_1 e_1 + ... + \alpha_n e_n)$

$= \alpha_1 S(e_1) + ... + \alpha_n S(e_n)$, by LT6

$= \alpha_1 T(e_1) + ... + \alpha_n T(e_n)$

$= T(\alpha_1 e_1 + ... + \alpha_n e_n)$, by LT6

$= T(u)$.

What we have just proved is that once we know the values of T on a basis of U, then we can find T(u) for any $u \in U$.

Note: Theorem 1 is true even when U is not finite-dimensional. The proof, in this case, is on the same lines as above.

Let us see how the idea of Theorem 1 helps us to prove the following useful result.

Theorem 2: Let V be a real vector space and $T:\mathbf{R} \to V$ be a linear transformation. Then there exists $v \in V$ such that $T(\alpha) = \alpha v \ \forall \alpha \in \mathbf{R}$.

Proof: A basis for **R** is $\{1\}$. Let $T(1) = v \in V$. Then, for any $\alpha \in \mathbf{R}$, $T(\alpha) = \alpha T(1) = \alpha v$.

Once you have read Sec. 5.3 you will realise that this theorem says that T(**R**) is a vector space of dimension one, whose basis is $\{T(1)\}$.

Now try the following exercise, for which you will need Theorem 1.

E E8) We define a linear operator $T:\mathbf{R}^2 \to \mathbf{R}^2$: $T(1,0) = (0,1)$ and $T(0,5) = (1,0)$. What is T(3, 5)? What is T(5,3)?

Now we shall prove a very useful theorem about linear transformations, which is linked to Theorem 1.

Theorem 3: Let $\{e_1, ..., e_n\}$ be a basis of U and let $v_1, ..., v_n$ be any n vectors in V. Then there exists one and only one linear transformation $T:U \to V$ such that $T(e_i) = v_i$, $i = 1, ..., n$.

Proof: Let $u \in U$. Then u can be uniquely written as $u = \alpha_1 e_1 + ... + \alpha_n e_n$ (see Unit 4, Theorem 9).

Define $T(u) = \alpha_1 v_1 + ... + \alpha_n v_n$. Then T defines a mapping from U to V such that $T(e_i) = v_i$ $\forall i = 1, ..., n$. Let us now show that T is linear. Let a, b be scalars and $u, u' \in U$. Then $\exists$ scalars $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n$ such that $u = \alpha_1 e_1 + ... + \alpha_n e_n$ and $u' = \beta_1 e_1 + ... + \beta_n e_n$.

Then $au + bu' = (a\alpha_1 + b\beta_1) e_1 + ... + (a\alpha_n + b\beta_n) e_n$.

Hence, $T(au + bu') = (a\alpha_1 + b\beta_1) v_1 + ... + (a\alpha_n + b\beta_n) v_n = a(\alpha_1 v_1 + ... + \alpha_n v_n) + b(\beta_1 v_1 + ... + \beta_n v_n) = aT(u) + bT(u')$.

Therefore, T is a linear transformation with the property that $T(e_i) = v_i \ \forall i$. Theorem 1 now implies that T is the only linear transformation with the above properties.

Let's see how Theorem 3 can be used.

Example 7: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ form the standard basis of $\mathbf{R}^3$. Let (1, 2), (2,3) and (3,4) be three vectors in $\mathbf{R}^2$. Obtain the linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(e_1) = (1,2)$, $T(e_2) = (2,3)$ and $T(e_3) = (3,4)$.

Solution: By Theorem 3 we know that $\exists\, T : \mathbf{R}^3 \to \mathbf{R}^2$ such that $T(e_1) = (1,2)$, $T(e_2) = (2,3)$ and $T(e_3) = (3,4)$. We want to know what T(x) is, for any $x = (x_1, x_2, x_3) \in \mathbf{R}^3$. Now, $x = x_1 e_1 + x_2 e_2 + x_3 e_3$.

Hence, $T(x) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

$= x_1(1,2) + x_2(2,3) + x_3(3,4)$

$= (x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + 4x_3)$

Therefore, $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 3x_2 + 4x_3)$ is the definition of the linear transformation T.

E9) Consider the complex field **C**. It is a vector space over **R**.

a) What is its dimension over **R**? Give a basis of **C** over **R**.

b) Let $\alpha, \beta \in \mathbf{R}$. Give the linear transformation which maps the basis elements of **C**, obtained in (a), onto α and β, respectively.

Let us now look at some vector spaces that are related to a linear operator.

5.3 SPACES ASSOCIATED WITH A LINEAR TRANSFORMATION

In Unit 1 you found that given any function, there is a set associated with it, namely, its range. We will now consider two sets which are associated with any linear transformation, T. These are the range and the kernel of T.

5.3.1 The Range Space and the Kernel

Let U and V be vector spaces over a field **F**. Let $T:U \to V$ be a linear transformation. We will define the range of T as well as the kernel of T. At first, you will see them as sets. We will prove that these sets are also vector spaces over **F**.

Definition: The **range of T**, denoted by **R(T)**, is the set $\{T(x) \mid x \in U\}$.
The **kernel** (or null space) **of T**, denoted by **Ker T**, is the set $\{x \in U \mid T(x) = \mathbf{0}\}$.
Note that $R(T) \subseteq V$ and $\text{Ker } T \subseteq U$.

To clarify these concepts consider the following examples.

Example 8: Let $I: V \to V$ be the identity transformation (see Example 1). Find R(I) and Ker I.

Solution: $R(I) = \{I(v) \mid v \in V\} = \{v \mid v \in V\} = V$. Also, $\text{Ker } I = \{v \in V \mid I(v) = \mathbf{0}\} = \{v \in V \mid v = \mathbf{0}\} = \{\mathbf{0}\}$.

Example 9: Let $T:\mathbf{R}^3 \to \mathbf{R}$ be defined by $T(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3$. Find R(T) and Ker T.

Solution: $R(T) = \{x \in \mathbf{R} \mid \exists\, x_1, x_2, x_3 \in \mathbf{R} \text{ with } 3x_1 + x_2 + 2x_3 = x\}$.
For example, $\mathbf{0} \in R(T)$, since $\mathbf{0} = 3.0 + 0 + 2.0 = T(0,0,0)$

Also, $1 \in R(T)$, since $1 = 3.1/3 + 0 + 2.0 = T(1/3, 0,0)$, or

$1 = 3.0 + 1 + 2.0 = T(0,1,0)$, or $1 = T(0,0,1/2)$, or $1 = T(1/6, 1/2,0)$.

Now can you see that R(T) is the whole real line **R**? This is because, for any $\alpha \in \mathbf{R}$,

$\alpha = \alpha.1 = \alpha T(1/3,0,0) = T\left(\frac{\alpha}{3}, 0, 0\right) \in R(T)$.

$\text{Ker } T = \{(x_1,x_2,x_3) \in \mathbf{R}^3 \mid 3x_1 + x_2 + 2x_3 = 0\}$.

For example, $(0,0,0) \in \text{Ker } T$. But $(1,0,0) \notin \text{Ker } T$. $\therefore \text{Ker } T \neq \mathbf{R}^3$. In fact, Ker T is the plane $3x_1 + x_2 + 2x_3 = 0$ in $\mathbf{R}^3$.

Example 10: Let $T: \mathbf{R}^3 \to \mathbf{R}^3$ be defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.

Find R(T) and Ker T.

Solution: To find R(T), we must find conditions on $y_1, y_2, y_3 \in \mathbf{R}$ so that $(y_1, y_2, y_3) \in R(T)$. i.e., we must find some $(x_1, x_2, x_3) \in \mathbf{R}^3$ so that $(y_1, y_2, y_3) = T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.

This means

$x_1 - x_2 + 2x_3 = y_1$ ....(1)

$2x_1 + x_2 = y_2$ ....(2)

$-x_1 - 2x_2 + 2x_3 = y_3$ ....(3)

Subtracting 2 times Equation (1) from Equation (2) and adding Equations (1) and (3) we get

$3x_2 - 4x_3 = y_2 - 2y_1$ ....(4)

and

$-3x_2 + 4x_3 = y_1 + y_3$ ....(5)

Adding Equations (4) and (5) we get

$y_2 - 2y_1 + y_1 + y_3 = 0$, that is, $y_2 + y_3 = y_1$.

Thus, $(y_1, y_2, y_3) \in R(T) \Rightarrow y_2 + y_3 = y_1$.

On the other hand, if $y_2 + y_3 = y_1$. We can choose

$$x_3 = 0,\ x_2 = \frac{y_2 - 2y_1}{3} \text{ and } x_1 = y_1 + \frac{y_2 - 2y_1}{3} = \frac{y_1 + y_2}{3}$$

Then, we see that $T(x_1, x_2, x_3) = (y_1, y_2, y_3)$.

Thus, $y_2 + y_3 = y_1 \Rightarrow (y_1, y_2, y_3) \in R(T)$.

Hence, $R(T) = \{(y_1, y_2, y_3) \in \mathbf{R}^3 \mid y_2 + y_3 = y_1\}$

Now $(x_1, x_2, x_3) \in$ Ker T if and only if the following equations are true:

$x_1 - x_2 + 2x_3 = 0$.

$2x_1 + x_2 = 0$

$-x_1 - 2x_2 + 2x_3 = 0$

Of course $x_1 = 0, x_2 = 0, x_3 = 0$ is a solution. Are there other solutions? To answer this we proceed as in the first part of this example. We see that $3x_2 - 4x_3 = 0$. Hence, $x_3 = (3/4)\, x_2$.

Also, $2x_1 + x_2 = 0 \Rightarrow x_1 = -x_2/2$.

Thus, we can give arbitrary values to x_2 and calculate x_1 and x_3 in terms of x_2. Therefore, Ker T = $\{(-\alpha/2, \alpha, (3/4)\alpha) : \alpha \in \mathbf{R}\}$.

In this example, we see that finding R(T) and Ker T amounts to solving a system of equations. In Unit 9, you will learn a systematic way of solving a system of linear equations by the use of matrices and determinants

The following exercises will help you in getting used to R(T) and Ker T.

E E10) Let T be the zero transformation given in Example 2. Find Ker T and R(T). Does $1 \in R(T)$?

E E11) Find R(T) and Ker T for each of the following operators.

a) $T: \mathbf{R}^3 \to \mathbf{R}^2 : T(x, y, z) = (x, y)$

b) $T: \mathbf{R}^3 \to \mathbf{R} : T(x, y, z) = z$

c) $T: \mathbf{R}^3 \to \mathbf{R}^3 : T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$.

(Note that the operators in (a) and (b) are projections onto the xy-plane and the z-axis, respectively.)

Now that you are familiar with the sets R(T) and Ker T, we will prove that they are vector spaces.

Theorem 4: Let U and V be vector spaces over a field **F**. Let $T:U \to V$ be a linear transformation. Then Ker T is a subspace of U and R(T) is a subspace of V.

Proof: Let $x_1, x_2 \in$ Ker T$\subseteq$ U and $\alpha_1, \alpha_2 \in$ **F**. Now, by definition, $T(x_1) = T(x_2) = \mathbf{0}$.
Therefore, $\alpha_1 T(x_1) + \alpha_2 T(x_2) = \mathbf{0}$
But $\alpha_1 T(x_1) + \alpha_2 T(x_2) = T(\alpha_1 x_1 + \alpha_2 x_2)$.
Hence, $T(\alpha_1 x_1 + \alpha_2 x_2) = \mathbf{0}$.
This means that $\alpha_1 x_1 + \alpha_2 x_2 \in$ Ker T.
Thus, by Theorem 4 of Unit 3, Ker T is a subspace of U.

Let $y_1, y_2 \in R(T) \subseteq V$, and $\alpha_1, \alpha_2 \in$ **F**. Then, by definition of R(T), there exist $x_1, x_2 \in U$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.
So, $\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 T(x_1) + \alpha_2 T(x_2)$
$= T(\alpha_1 x_1 + \alpha_2 x_2)$.
Therefore, $\alpha_1 y_1 + \alpha_2 y_2 \in R(T)$, which proves that R(T) is a subspace of V.

Now that we have proved that R(T) and Ker T are vector spaces, you know, from Unit 4, that they must have a dimension. We will study these dimensions now.

5.3.2 Rank and Nullity

Consider any linear transformation $T:U \to V$, assuming that dim U is finite. Then Ker T, being a subspace of U, has finite dimension and dim (Ker T)$\leq$ dim U. Also note that R(T) = T(U), the image of U under T, a fact you will need to use in solving the following exercise.

E12) Let$\{e_1,....,e_n\}$ be a basis of U. Show that R(T) is generated by $\{T(e_1), ..., T(e_n)\}$.

From E12 it is clear that, if dim U = n, then dim R(T)$\leq$ n.
Thus, dim R(T) is finite, and the following definition is meaningful.

Definition: The **rank** of T is defined to be the dimension of R(T), the range space of T. The **nullity** of T is defined to be the dimension of Ker T, the kernel (or the null space) of T.

Thus, **rank (T) = dim R(T)** and **nullity (T) = dim Ker T.**

We have already seen that rank (T) ≤ dim U and nullity (T) ≤ dim U.

Example 11: Let T:U → V be the zero transformation given in Example 2. What are the rank and nullity of T?

Solution: In E10 you saw that R(T) = {**0**} and Ker T = U. Therefore, rank (T) = 0 and nullity(T) = dim U.
Note that rank (T) + nullity (T) = dim U, in this case.

E E13) If T is the identity operator on V, find rank (T) and nullity (T).

E E14) Let D be the differentiation operator in E6. Give a basis for the range space of D and for Ker D. What are rank (D) and nullity (D)?

In the above example and exercises you will find that for T: U → V, rank (T) + nullity (T) = dim U. In fact, this is the most important result about rank and nullity of a linear operator. We will now state and prove this result.

This theorem is called the Rank Nullity Theorem.

Theorem 5: Let U and V be vector spaces over a field **F** and dim U = n. Let T:U → V be a linear operator. Then **rank (T) + nullity (T) = n.**

Proof: Let nullity (T) = m, that is, dim Ker T = m. Let $\{e_1,....,e_m\}$ be a basis of Ker T. We know that Ker T is a subspace of U. Thus, by Theorem 11 of Unit 4, we can extend this basis to obtain a basis $\{e_1,.....,e_m, e_{m+1},....., e_n\}$ of U. We shall show that $\{T(e_{m+1}), ..., T(e_n)\}$ is a basis of R(T). Then, our result will follow because dim R(T) will be n − m = n − nullity (T).

Let us first prove that $\{T(e_{m+1}), ..., T(e_n)\}$ spans, or generates, R(T). Let y ∈ R(T). Then, by definition of R(T), there exists x ∈ U such that T(x) = y.

Let $x = c_1e_1 + ... + c_me_m + c_{m+1}e_{m+1} + ... + c_ne_n$, $c_i \in \mathbf{F}\ \forall i$.

Then,

$y = T(x) = c_1T(e_1) + ... + c_mT(e_m) + c_{m+1}T(e_{m+1}) + ... + c_nT(e_n)$

$= c_{m+1}T(e_{m+1}) + ... + c_nT(e_n)$,

because $T(e_1) = .. = T(e_m) = \mathbf{0}$, since $T(e_i) \in$ Ker T $\forall i = 1, ..., m$. ∴ any y ∈ R(T) is a linear combination of $\{T(e_{m+1}), ..., T(e_n)\}$. Hence, R(T) is spanned by $\{T(e_{m+1}), ..., T(e_n)\}$.
It remains to show that the set $\{T(e_{m+1}), ... T(e_n)\}$ is linearly independent. For this, suppose there exist $a_{m+1}, ..., a_n \in \mathbf{F}$ with $a_{m+1}T(e_{m+1}) + ... + a_n T(e_n) = \mathbf{0}$.

Then, $T(a_{m+1} e_{m+1} + ... + a_ne_n) = \mathbf{0}$.

Hence, $a_{m+1} e_{m+1} + ... + a_ne_n \in$ Ker T, which is generated by $\{e_1, ..., e_m\}$.

Therefore, there exist $a_1, ..., a_m \in \mathbf{F}$ such that

$a_{m+1} e_{m+1} + ... + a_ne_n = a_1e_1 + ... + a_me_m$

$\Rightarrow (-a_1) e_1 + ... + (-a_m) e_m + a_{m+1}e_{m+1} + ... + a_ne_n = \mathbf{0}$.

Since $\{e_1, ..., e_n\}$ is a basis of U, it follows that this set is linearly independent. Hence,

$-a_1 = 0, \ldots\ldots, -a_m = 0, a_{m+1} = 0, \ldots, a_n = 0$. In particular, $a_{m+1} = \ldots. = a_n = 0$, which we wanted to prove.

Therefore, dim R(T) = n – m = n – nullity (T), that is, rank (T) + nullity (T) = n.

Let us see how this theorem can be useful.

Example 12: Let $L:\mathbf{R}^3 \to \mathbf{R}$ be the map given by L(x, y, z) = x + y + z. What is nullity (L)?

Solution: In this case it is easier to obtain R(L), rather than Ker L. Since L (1, 0, 0) = 1 ≠ 0, R(L) ≠ {**0**}, and hence dim R(L) ≠ 0. Also, R(L) is a subspace of **R**. Thus, dim R(L) ≤ dim **R** = 1. Therefore, the only possibility for dim R(L) is dim R(L) = 1. By Theorem 5, dim Ker L + dim R(L) = 3.

Hence, dim Ker L = 3–1 = 2. That is, nullity (L) = 2.

E15) Give the rank and nullity of each of the linear transformations in E11.

E16) Let U and V be real vector spaces and $T:U \to V$ be a linear transformation, where dim U = 1. Show that R(T) is either a point or a line.

Before ending this section we will prove a result that links the rank (or nullity) of the composite of two linear operators with the rank (or nullity) of each of them.

Theorem 6: Let V be a vector space over a field **F**. Let S and T be linear operators from V to V. Then

a) rank (ST) ≤ min (rank (S), rank (T))
b) nullity (ST) ≥ max (nullity (S), nullity (T))

Proof: We shall prove (a). Note that (ST) (v) = S(T(v)) for any v ∈ V (you'll study more about compositions in Unit 6).

Now, for any y ∈ R (ST), ∃ v ∈ V such that,

y = (ST) (v) = S (T(v)) ....(1)

Now, (1) ⇒ y ∈ R(S).

Therefore, R(ST) ⊆ R(S). This implies that rank (ST) ≤ rank (S).

Again, (1) ⇒ y ∈ S (R(T)), since T(v) ∈ R(T).

∴ R (ST) ⊆ S (R (T)), so that

dim R (ST) ≤ dim S (R (T)) ≤ dim R (T) (since dim L (U) ≤ U, for any linear operator L).

Therefore, rank (ST) ≤ rank (T).

Thus, rank (ST) ≤ min (rank (S), rank (T)).

The proof of this theorem will be complete, once you solve the following exercise.

E 17) Prove (b) of Theorem 6 using the Rank Nullity Theorem.

We would now like to discuss some linear operators that have special properties.

5.4 SOME TYPES OF LINEAR TRANSFORMATIONS

Let us recall, from Unit 1, that there can be different types of functions, some of which are one-one, onto or invertible. We can also define such types of linear transformations as follows.

Definition: Let $T : U \to V$ be a linear transformation.

a) T is called **one-one** (or **injective**) if, for $u_1, u_2 \in U$ with $u_1 \neq u_2$, we have $T(u_1) \neq T(u_2)$. If T is injective, we also say T is 1 – 1.
Note that **T is 1–1 if $T(u_1) = T(u_2) \Rightarrow u_1 = u_2$.**

b) T is called **onto** (or **surjective**) if, for each $v \in V$, $\exists\, u \in U$ such that $T(u) = v$, that is, $R(T) = V$.

Can you think of examples of such functions?
The identity operator is both one-one and onto. Why is this so? Well, $I: V \to V$ is an operator such that, if $v_1, v_2 \in V$ with $v_1 \neq v_2$ then $I(v_1) \neq I(v_2)$. Also, $R(I) = V$, so that I is onto.

E 18) Show that the zero operator $\mathbf{0}: \mathbf{R} \to \mathbf{R}$ is not one-one.

An important result that characterises injectivity is the following:

Theorem 7: $T : U \to V$ is one-one if and only if Ker $T = \{\mathbf{0}\}$.

Proof: First assume T is one-one. Let $u \in$ Ker T. Then $T(u) = \mathbf{0} = T(\mathbf{0})$. This means that $u = \mathbf{0}$. Thus, Ker $T = \{\mathbf{0}\}$. Conversely, let Ker $T = \{\mathbf{0}\}$. Suppose $u_1, u_2 \in U$ with $T(u_1) = T(u_2) \Rightarrow T(u_1 - u_2) = \mathbf{0} \Rightarrow u_1 - u_2 \in$ Ker $T \Rightarrow u_1 - u_2 = \mathbf{0} \Rightarrow u_1 = u_2$. Therefore, T is 1 – 1.

Suppose now that T is a one-one and onto linear transformation from a vector space U to a vector space V. Then, from Unit 1 (Theorem 4), we know that T^{-1} exists.

But is T^{-1} linear? The answer to this question is 'yes', as is shown in the following theorem.

Theorem 8: Let U and V be vector spaces over a field F. Let $T : U \to V$ be a one-one and onto linear transformation. Then $T^{-1}: V \to U$ is a linear transformation.

In fact, T^{-1} is also 1–1 and onto.

$T^{-1}(y) = x \Leftrightarrow T(x) = y$

Proof: Let $y_1, y_2 \in V$ and $\alpha_1, \alpha_2 \in F$. Suppose $T^{-1}(y_1) = x_1$ and $T^{-1}(y_2) = x_2$. Then, by definition, $y_1 = T(x_1)$ and $y_2 = T(x_2)$.
Now, $\alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 T(x_1) + \alpha_2 T(x_2) = T(\alpha_1 x_1 + \alpha_2 x_2)$
Hence, $T^{-1}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 x_1 + \alpha_2 x_2$
$= \alpha_1 T^{-1}(y_1) + \alpha_2 T^{-1}(y_2)$

This shows that T^{-1} is a linear transformation.

We will now show that T^{-1} is $1-1$. For this, suppose $y_1, y_2 \in V$ such that $T^{-1}(y_1) = T^{-1}(y_2)$. Let $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$.

Then $T(x_1) = y_1$ and $T(x_2) = y_2$. We know that $x_1 = x_2$. Therefore, $T(x_1) = T(x_2)$, that is, $y_1 = y_2$. Thus, we have shown that $T^{-1}(y_1) = T^{-1}(y_2) \Rightarrow y_1 = y_2$, proving that T^{-1} is $1-1$. T^{-1} is also surjective because, for any $u \in U$, $\exists\, T(u) = v \in V$ such that $T^{-1}(v) = u$.

Theorem 8 says that a one-one and onto linear transformation is **invertible,** and the inverse is also a one-one and onto linear transformation.

This theorem immediately leads us to the following definition.

Definition: Let U and V be vector spaces over a field F, and let $T: U \to V$ be a one-one and onto linear transformation. Then T is called an **isomorphism** between U and V.

In this case we say that U and V are **isomorphic vector spaces.** This is denoted by $U \simeq V$.

An obvious example of an isomorphism is the identity operator. Can you think of any other? The following exercise may help.

E19) Let $T: \mathbf{R}^3 \to \mathbf{R}^3 : T(x, y, z) = (x + y, y, z)$. Is T an isomorphism? Why? Define T^{-1}, if it exists.

E20) Let $T: \mathbf{R}^3 \to \mathbf{R}^2 : T(x, y, z) = (x + y, y + z)$. Is T an isomorphism?

In all these exercises and examples, have you noticed that if T is an isomorphism between U and V then T^{-1} is an isomorphism between V and U?

Using these properties of an isomorphism we can get some useful results, like the following.

Theorem 9: Let $T: U \to V$ be an isomorphism. Suppose $\{e_1, \ldots, e_n\}$ is a basis of U. Then $\{T(e_1), \ldots, T(e_n)\}$ is a basis of V.

Proof: First we show that the set $\{T(e_1), \ldots, T(e_n)\}$ spans V. Since T is onto, $R(T) = V$. Thus, from E12 you know that $\{T(e_1), \ldots, T(e_n)\}$ spans V.

Let us now show that $\{T(e_1), \ldots, T(e_n)\}$ is linearly independent. Suppose there exist scalars $c_1, \ldots, c_n$, such that $c_1 T(e_1) + \ldots + c_n T(e_n) = \mathbf{0}$. .... (1)

We must show that $c_1 = \ldots = c_n = 0$.

Now, (1) implies that

$T(c_1 e_1 + \ldots + c_n e_n) = \mathbf{0}$.

Since T is one-one and $T(\mathbf{0}) = \mathbf{0}$, we conclude that

$$c_1 e_1 + \ldots + c_n e_n = \mathbf{0}.$$

But $\{e_1, \ldots, e_n\}$ is linearly independent. Therefore,

$$c_1 = \ldots = c_n = 0.$$

Thus, we have shown that $\{T(e_1), \ldots, T(e_n)\}$ is a basis of V.

Remark: The argument showing the linear independence of $\{T(e_1), \ldots, T(e_n)\}$ in the above theorem can be used to prove that **any one-one linear transformation $T: U \to V$ maps any linearly independent subset of U onto a linearly independent subset of V** (see E22).

We now give an important result equating 'isomorphism' with '$1-1$' and with 'onto' in the finite-dimensional case.

Theorem 10: Let $T : U \to V$ be a linear transformation where U, V are of the same finite dimension. Then the following statements are equivalent.

a) T is 1 – 1.

b) T is onto.

c) T is an isomorphism.

Proof: To prove the result we will prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). Let dim U = dim V = n.

Now (a) implies that Ker T = {**0**} (from Theorem 7). Hence, nullity (T) = 0. Therefore, by Theorem 5, rank (T) = n, that is, dim R (T) = n = dim V. But R (T) is a subspace of V. Thus by the remark following Theorem 12 of Unit 4, we get R (T) = V, i.e., T is onto, i.e., (b) is true. So (a) $\Rightarrow$ (b).

Similarly, if (b) holds then rank (T) = n, and hence, nullity (T) = 0. Consequently, Ker T = {**0**}, and T is one-one. Hence, T is one-one and onto, i.e., T is an isomorphism. Therefore, (b) implies (c).

That (a) follows from (c) is immediate from the definition of an isomorphism.

Hence, our result is proved.

Caution: Theorem 10 is true for **finite**-dimensional spaces U and V, of the **same dimension**. It is not true, otherwise. Consider the following counter-example.

Example 13 (To show that the spaces have to be finite-dimensional): Let V be the real vector space of all polynomials. Let D: V $\to$ V be defined by D $(a_0 + a_1 x + \ldots + a_r x^r) = a_1 + 2a_2 x + \ldots + r a_r x^{r-1}$. Then show that D is onto but not 1 – 1.

Solution: Note that V has infinite dimension, a basis being $\{1, x, x^2, \ldots\}$. D is onto because any element of V is of the form $a_0 + a_1 x + \ldots + a_n x^n = D\left(a_0 x + \frac{a_1}{2}x^2 + \cdots + \frac{a_n}{n+1}x^{n+1}\right)$

D is not 1 – 1 because, for example, $1 \neq 0$ but D (1) = D (0) = 0.

The following exercise shows that the statement of Theorem 10 is false if dim U $\neq$ dim V.

E E21) Define a linear operator $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that T is onto but T is not 1 – 1. Note that dim $\mathbf{R}^3 \neq$ dim $\mathbf{R}^2$.

Let us use Theorems 9 and 10 to prove our next result.

Theorem 11: Let $T : V \to V$ be a linear transformation and let $\{e_1, \ldots, e_n\}$ be a basis of V. Then T is one-one and onto if and only if $\{T(e_1), \ldots, T(e_n)\}$ is linearly independent.

Proof: Suppose T is one-one and onto. Then T is an isomorphism. Hence, by Theorem 9, $\{T(e_1), \ldots, T(e_n)\}$ is a basis. Therefore, $\{T(e_1), \ldots, T(e_n)\}$ is linearly independent.

Conversely, suppose $\{T(e_1), \ldots, T(e_n)\}$ is linearly independent. Since $\{e_1, \ldots, e_n\}$ is a basis of V, dim V = n. Therefore, any linearly independent subset of n vectors is a basis of V (by Unit 4, Theorem 5, Cor.1). Hence, $\{T(e_1), \ldots, T(e_n)\}$ is a basis of V. Then, any element v of V is of the form $v = \sum_{i=1}^{n} c_i T(e_i) = T\left(\sum_{i=1}^{n} c_i e_i\right)$, where $c_1, \ldots, c_n$ are scalars. Thus, T is onto, and we can use Theorem 10 to say that T is an isomorphism.

Here are some exercises now.

E E 22) a) Let T: U $\to$ V be a one-one linear transformation and let $\{u_1, \ldots, u_k\}$ be a linearly independent subset of U. Show that the set $\{T(u_1), \ldots, T(u_k)\}$ is linearly independent.

b) Is it true that every linear transformation maps every linearly independent set of vectors into a linearly independent set?

c) Show that every linear transformation maps a linearly dependent set of vectors onto a linearly dependent set of vectors.

E

E23) Let $T: \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3, x_1 + x_2)$. Is T invertible? If yes, find a rule for T^{-1} like the one which defines T.

We have seen, in Theorem 9, that if $T: U \to V$ is an isomorphism, then T maps a basis of U onto a basis of V. Therefore, dim U = dim V. In other words, if U and V are isomorphic then dim U = dim V. The natural question arises whether the converse is also true. That is, if dim U = dim V, both being finite, can we say that U and V are isomorphic? The following theorem shows that this is indeed the case.

Theorem 12: Let U and V be finite-dimensional vector spaces over **F**. Then U and V are isomorphic if and only if dim U = dim V.

Proof: We have already seen that if U and V are isomorphic then dim U = dim V. Conversely, suppose dim U = dim V = n. We shall show that U and V are isomorphic. Let $\{e_1,\ldots,e_n\}$ be a basis of U and $\{f_1,\ldots,f_n\}$ be a basis of V. By Theorem 3, there exists a linear transformation $T: U \to V$ such that $T(e_i) = f_i$, $i = 1,\ldots,n$.

We shall show that T is 1 − 1.

Let $u = c_1 e_1 + \ldots + c_n e_n$ be such that $T(u) = \mathbf{0}$.

Then $\mathbf{0} = T(u) = c_1 T(e_1) + \ldots + c_n T(e_n)$

$= c_1 f_1 + \ldots + c_n f_n$.

Since $\{f_1, ..., f_n\}$ is a basis of V, we conclude that $c_1 = c_2 = ... = c_n = 0$. Hence, $u = \mathbf{0}$. Thus, Ker T = $\{\mathbf{0}\}$ and, by Theorem 7, T is one-one.

Therefore, by Theorem 10, T is an isomorphism, and $U \simeq V$.

An immediate consequence of this theorem follows.

Corollary: Let V be a real (or complex) vector space of dimension n. Then V is isomorphic to $\mathbf{R}^n$ (or $\mathbf{C}^n$), respectively.

Proof: Since dim $\mathbf{R}^n = n = \dim_{\mathbf{R}} V$, we get $V \simeq \mathbf{R}^n$. Similarly, if $\dim_{\mathbf{C}} V = n$, then $V \simeq \mathbf{C}^n$.

We generalise this corollary in the following remark.

Remark: Let V be a vector space over **F** and let $B = \{e_1, ..., e_n\}$ be a basis of V. Each $v \in V$ can be uniquely expressed as $v = \sum_{i=1}^{n} \alpha_i e_i$. Recall that $\alpha_1, ...\alpha_n$ are called the coordinates of v with respect to B (refer to Sec. 4.4.1).

Define $\theta : V \rightarrow \mathbf{F}^n : \theta(v) = (\alpha_1, ...\alpha_n)$. Then θ is an isomorphism from V to $\mathbf{F}^n$. This is because θ is 1 – 1, since the coordinates of v with respect to B are uniquely determined. **Thus,** $V \simeq \mathbf{F}^n$.

We end this section with an exercise.

E E 24) Let T : U → V be a one-one linear mapping. Show that T is onto if and only if dim U = dim V. (Of course, you must assume that U and V are finite-dimensional spaces.)

Now let us look at isomorphisms between quotient spaces.

5.5 HOMOMORPHISM THEOREMS

Linear transformations are also called **vector space homomorphisms**. There is a basic theorem which uses the properties of homomorphisms to show the isomorphism of certain quotient spaces (ref. Unit 3). It is simple to prove, but is very important because it is always being used to prove more advanced theorems on vector spaces. (In the Abstract Algebra course we will prove this theorem in the setting of groups and rings.)

This theorem is called the Fundamental Theorem of Homomorphism.

Theorem 13: Let V and W be vector spaces over a field **F** and T: V → W be a linear transformation. Then **V/Ker T** $\simeq$ **R (T)**.

Proof: You know that Ker T is a subspace of V, so that V/Ker T is a well defined vector space over **F**. Also $R(T) = \{T(v) \mid v \in V\}$. To prove the theorem let us define θ: **V/ Ker** T → R (T) by θ (v + Ker T) = T (v).

Firstly, we must show that θ is a well defined function, that is, if v + Ker T = v′ + Ker T then θ (v + Ker T) = θ (v′ + Ker T), i.e., T (v) = T (v′).

Now, v + Ker T = v′ + Ker T ⇒ (v – v′) ∈ Ker T (see Unit 3, E23)

⇒ T (v – v′) = **0** ⇒ T (v) = T (v′), and hence, θ is well defined.

Next, we check that θ is a linear transformation. For this, let a, b ∈ **F** and v, v′ ∈ V. Then
θ {a (v + Ker T) + b (v′ + Ker T)}
= θ (av + bv′ + Ker T) (ref. Unit 3)
= T (av + bv′)
= a T (v) + bT (v′), since T is linear.
= aθ (v + Ker T) + b θ (v′ + Ker T).

Thus, θ is a linear transformation.

We end the proof by showing that θ is an isomorphism. θ is 1 – 1 (because θ (v + Ker T) = **0** ⇒ T (v) = **0** ⇒ v ∈ Ker T ⇒ v + Ker T = **0** (in V/Ker T)

Thus, Ker θ = {**0**}

θ is onto (because any element of R (T) is T (v) = θ (v + Ker **T**)).

So we have proved that θ is an isomorphism. This proves that V/Ker T $\simeq$ **R (T)**.

Let us consider an immediate useful application of Theorem 13.

Example 14: Let V be a finite-dimensional space and let S and T be linear transformations from V to V. Show that

rank (ST) = rank (T) – dim (R (T) ∩ Ker S).

Solution: We have $V \xrightarrow{T} V \xrightarrow{S} V$. ST is the composition of the operators S and T, which you have studied in Unit 1, and will also study in Unit 6. Now, we apply Theorem 13 to the homomorphism θ : T (V) → ST(V) : θ (T(v)) = (ST) (v).

Now, Ker θ = {x ∈ T (V)| S (x) = **0**} = Ker S ∩ T (V) = Ker S ∩ R (T).

Also R(θ) = ST (V), since any element of ST (V) is (ST) (v) = θ (T (v)). Thus,

$$\frac{T(V)}{\text{Ker } S \cap T(V)} \simeq ST(V)$$

Therefore,

$$\dim \frac{T(V)}{\text{Ker } S \cap T(V)} = \dim ST(V)$$

That is, dim T (V) – dim (Ker S ∩ T (V)) = dim ST (V), which is what we had to show.

E E25) Using Example 14 and the Rank Nullity Theorem, show that
nullity (ST) = nullity (T) + dim (R (T) ∩ Ker S).

Now let us see another application of Theorem 13.

Example 15: Show that $\mathbf{R}^3/\mathbf{R} \simeq \mathbf{R}^2$.

Solution : Note that we can consider **R** as a subspace of $\mathbf{R}^3$ for the following reason: any element α of **R** is equated with the element (α, 0, 0) of $\mathbf{R}^3$. Now, we define a function f: $\mathbf{R}^3 \to \mathbf{R}^2$: f (α, β, γ)= (β, γ). Then f is a linear transformation and Ker f = {(α, 0, 0)| α ∈ **R**} $\simeq$ **R**. Also f is onto, since any element (α, β) of $\mathbf{R}^2$ is f (0, α, β). Thus, by Theorem 13, $\mathbf{R}^3/\mathbf{R} \simeq \mathbf{R}^2$.

Note : In general, for any n ≥ m, $\mathbf{R}^n/\mathbf{R}^m \simeq \mathbf{R}^{n-m}$. Similarly, $\mathbf{C}^{n-m} \simeq \mathbf{C}^n/\mathbf{C}^m$ for n ≥ m.

The next result is a corollary to the Fundamental Theorem of Homomorphism.

But, before studying it, read Unit 3 for the definition of the sum of spaces.

Corollary 1: Let A and B be subspaces of a vector space V. Then A + B/B $\simeq$ A/A ∩ B.

Proof: We define a linear function T : A → $\frac{A+B}{B}$ by T (a) = a + B.

T is well defined because a + B is an element of $\frac{A+B}{B}$ (since a = a + **0** ∈ A + B).

T is a linear transformation because, for α_1, α_2 in **F** and a_1, a_2 in A, we have

$$T(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 a_1 + \alpha_2 a_2 + B = \alpha_1 (a_1 + B) + \alpha_2 (a_2 + B)$$
$$= \alpha_1 T(a_1) + \alpha_2 T(a_2).$$

Now we will show that T is surjective. Any element of $\frac{A+B}{B}$ is of the form a + b + B, where a ∈ A and b ∈ B.

Now a + b + B = a + B + b + B = a + B + B, since b ∈ B.

= a + B, since B is the zero element of $\frac{A+B}{B}$

= T (a), proving that T is surjective.

$$\therefore R(T) = \frac{A+B}{B}$$

We will now prove that Ker T = A ∩ B.

If $a \in$ Ker T, then $a \in$ A and T (a) = **0**. This means that a + B = B, the zero element of $\frac{A+B}{B}$.
Hence, $a \in B$ (by Unit 3, E23). Therefore, $a \in A \cap B$. Thus, Ker T $\subseteq A \cap B$. On the other hand, $a \in A \cap B \Rightarrow a \in A$ and $a \in B \Rightarrow a \in A$ and $a + B = B \Rightarrow a \in A$ and
T (a) = T (**0**) = **0**

$\Rightarrow a \in$ Ker T.

This proves that $A \cap B$ = Ker T.

Now using Theorem 13, we get

A/Ker T $\simeq$ R (T).

That is, $A/(A \cap B) \simeq (A + B)/B$

E E 26) Using the corollary above, show that $A \oplus B/B \simeq A$ ($\oplus$ denotes the direct sum defined in Sec. 3.6).

There is yet another interesting corollary to the Fundamental Theorem of Homomorphism.

Corollary 2 : Let W be a subspace of a vector space V. Then, for any subspace U of V containing W,

$$\frac{V/W}{U/W} \simeq V/U$$

Proof: This time we shall prove the theorem with you. To start with let us define a function T : V/W $\rightarrow$ V/U : T (v + W) = v + U. Now try E 27.

E E 27) a) Check that T is well defined.

b) Prove that T is a linear transformation.

c) What are the spaces Ker T and R (T)?

So, is the theorem proved? Yes; apply Theorem 13 to T.

We end the unit by summarising what we have done in it.

5.6 SUMMARY

In this unit we have covered the following points.

1) A linear transformation from a vector space U over F to a vector space V over F is a function T: U $\rightarrow$ V such that,

LT1) $T(u_1 + u_2) = T(u_1) + T(u_2) \forall u_1, u_2 \in U$, and

LT2) $T(\alpha u) = \alpha T(u)$, for $\alpha \in \mathbf{F}$ and $u \in U$.

These conditions are equivalent to the single condition

LT3) $T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ for $\alpha, \beta \in \mathbf{F}$ and $u_1, u_2 \in U$.

2) Given a linear transformation $T : U \to V$,
   i) the kernel of T is the vector space $\{u \in U \mid T(u) = \mathbf{0}\}$, denoted by Ker T.
   ii) the range of T is the vector space $\{T(u) \mid u \in U\}$, denoted by R(T).
   iii) The rank of T = $\dim_F R(T)$.
   iv) The nullity of T = $\dim_F \text{Ker } T$.

3) Let U and V be finite-dimensional vector spaces over **F** and $T: U \to V$ be a linear transformation. Then rank (T) + nullity (T) = dim U.

4) Let $T: U \to V$ be a linear transformation. Then
   i) T is one-one if $T(u_1) = T(u_2) \Rightarrow u_1 = u_2 \ \forall u_1, u_2 \in U$.
   ii) T is onto if, for any $v \in V \ \exists \ u \in U$ such that $T(u) = v$.
   iii) T is an isomorphism (or, is invertible) if it is one-one and onto, and then U and V are called isomorphic spaces. This is denoted by $U \simeq V$.

5) $T : U \to V$ is
   i) one-one if and only if Ker T = $\{\mathbf{0}\}$
   ii) onto if and only if R (T) = V.

6) Let U and V be **finite**-dimensional vector spaces with the **same dimension**. Then $T: U \to V$ is $1 - 1$ iff T is onto iff T is an isomorphism.

7) Two finite-dimensional vector spaces U and V are isomorphic if and only if dim U = dim V.

8) Let V and W be vector spaces over a field **F**, and $T : V \to W$ be a linear transformation. Then $V/\text{Ker } T \simeq R(T)$.

5.7 SOLUTIONS/ANSWERS

E1) For any $\alpha_1, \alpha_2 \in \mathbf{F}$ and $u_1, u_2 \in U$, we know that $\alpha_1 u_1 \in U$ and $\alpha_2 u_2 \in U$. Therefore, by LT1,

$$T(\alpha_1 u_1 + \alpha_2 u_2) = T(\alpha_1 u_1) + T(\alpha_2 u_2)$$
$$= \alpha_1 T(u_1) + \alpha_2 T(u_2), \text{ by LT2.}$$

Thus, LT3 is true.

E 2) By LT2, $T(0.u) = 0.T(u)$ for any $u \in U$. Thus, $T(\mathbf{0}) = \mathbf{0}$. Similarly, for any $u \in U$, $T(-u) = T((-1)u) = (-1)T(u) = -T(u)$.

E 3) $T(x, y) = (-x, y) \forall (x, y) \in \mathbf{R}^2$. (See the geometric view in Fig. 4.) T is a linear operator. This can be proved the same way as we did in Example 4.

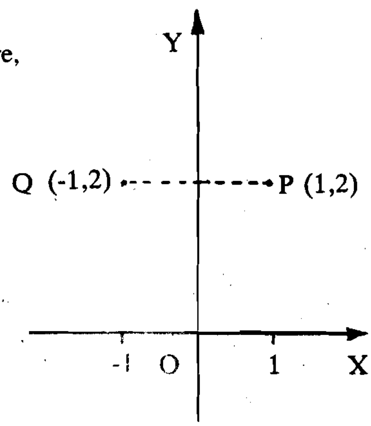

Fig. 4: Q is the reflection of P in the y-axis.

E 4) $T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$

$= (a_1 x_1 + a_2 x_2 + a_3 x_3) + (a_1 y_1 + a_2 y_2 + a_3 y_3)$

$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$

Also, for any $\alpha \in \mathbf{R}$,

$T(\alpha(x_1, x_2, x_3)) = a_1 \alpha x_1 + a_2 \alpha x_2 + a_3 \alpha x_3$

$= \alpha(a_1 x_1 + a_2 x_2 + a_3 x_3) = \alpha T(x_1, x_2, x_3)$.

Thus, LT1 and LT2 hold for T.

E 5) We will check if LT1 and LT2 hold. Firstly,

$T((x_1, x_2, x_3) + (y_1, y_2, y_3)) = T(x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$= (x_1 + y_1 + x_2 + y_2 - x_3 - y_3, 2x_1 + 2y_1 - x_2 - y_2, x_2 + y_2 + 2x_3 + 2y_3)$

$= (x_1 + x_2 - x_3, 2x_1 - x_2, x_2 + 2x_3) + (y_1 + y_2 - y_3, 2y_1 - y_2, y_2 + 2y_3)$

$= T(x_1, x_2, x_3) + T(y_1, y_2, y_3)$, showing that LT1 holds.

Also, for any $\alpha \in \mathbf{R}$,

$T(\alpha(x_1, x_2, x_3)) = T(\alpha x_1, \alpha x_2, \alpha x_3)$

$= (\alpha x_1 + \alpha x_2 - \alpha x_3, 2\alpha x_1 - \alpha x_2, \alpha x_2 + 2\alpha x_3)$

$= \alpha(x_1 + x_2 - x_3, 2x_1 - x_2, x_2 + 2x_3) = \alpha T(x_1, x_2, x_3)$, showing that LT2 holds.

E 6) We want to show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$, for any $\alpha, \beta \in \mathbf{R}$ and $f, g \in \mathbf{P}_n$.

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + .. + a_n x^n$ and $g(x) = b_0 + b_1 x + ... + b_n x^n$.

Then $(\alpha f + \beta g)(x) = (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1) x + ... + (\alpha a_n + \beta b_n) x^n$.

$\therefore [D(\alpha f + \beta g)](x) = (\alpha a_1 + \beta b_1) + 2(\alpha a_2 + \beta b_2) x + ... + n(\alpha a_n + \beta b_n) x^{n-1}$

$= \alpha(a_1 + 2a_2 x + ... + n a_n x^{n-1}) + \beta(b_1 + 2b_2 x + ... + n b_n x^{n-1})$

$= \alpha(Df)(x) + \beta(Dg)(x) = (\alpha Df + \beta Dg)(x)$

Thus, $D(\alpha f + \beta g) = \alpha Df + \beta Dg$, showing that D is a linear map.

E 7) No. Because, if T exists, then

$T(2u_1 + u_2) = 2T(u_1) + T(u_2)$.

But $2u_1 + u_2 = u_3$. $\therefore T(2u_1 + u_2) = T(u_3) = v_3 = (1, 1)$.

On the other hand, $2T(u_1) + T(u_2) = 2v_1 + v_2 = (2,0) + (0,1)$

$= (2,1) \neq v_3$.

Therefore, LT3 is violated. Therefore, no such T exists.

E 8) Note that $\{(1,0), (0,5)\}$ is a basis for $\mathbf{R}^2$.

Now $(3,5) = 3(1,0) + (0,5)$.

Therefore, $T(3,5) = 3T(1,0) + T(0,5) = 3(0,1) + (1,0) = (1,3)$.

Similarly, $(5,3) = 5(1,0) + 3/5\,(0,5)$.

Therefore, $T(5,3) = 5(0,1) + 3/5\,(1,0) = (3/5, 5)$.

Note that $T(5,3) \neq T(3,5)$

E 9) a) $\dim_{\mathbf{R}} \mathbf{C} = 2$, a basis being $\{1, i\}$, $i = \sqrt{-1}$.

b) Let $T: \mathbf{C} \to \mathbf{R}$ be such that $T(1) = \alpha$, $T(i) = \beta$.

Then, for any element $x + iy \in \mathbf{C}$ $(x, y \in \mathbf{R})$, we have $T(x + iy) = xT(1) + yT(i) = x\alpha + y\beta$. Thus, T is defined by $T(x + iy) = x\alpha + y\beta \; \forall x + iy \in \mathbf{C}$.

E 10) $T: U \to V : T(u) = 0 \; \forall u \in U$.

$\therefore$, $\text{Ker } T = \{u \in U \mid T(u) = 0\} = U$

$R(T) = \{T(u) \mid u \in U\} = \{0\}$. $\therefore 1 \notin R(T)$.

E11) a) $R(T) = \{T(x, y, z) \mid (x, y, z) \in \mathbf{R}^3\} = \{(x, y) \mid (x,y,z) \in \mathbf{R}^3\} = \mathbf{R}^2$.

$\text{Ker } T = \{(x, y, z) \mid T(x,y,z) = 0\} = \{(x, y, z) \mid (x, y) = (0, 0)\}$

$= \{(0,0,z) \mid z \in \mathbf{R}\}$

$\therefore$, Ker T is the z-axis.

b) $R(T) = \{z \mid (x, y, z) \in \mathbf{R}^3\} = \mathbf{R}$.

$\text{Ker } T = \{(x,y,0) \mid x, y \in \mathbf{R}\}$ = xy-plane in $\mathbf{R}^3$.

c) $R(T) = \{(x, y, z) \in \mathbf{R}^3 \mid \exists x_1, x_2, x_3 \in \mathbf{R}$ such that $x = x_1 + x_2 + x_3 = y = z\}$

$= \{(x,x,x) \in \mathbf{R}^3 \mid x = x_1 + x_2 + x_3$ for some $x_1, x_2, x_3 \in \mathbf{R}\}$

$= \{(x,x,x) \in \mathbf{R}^3 \mid x \in \mathbf{R}\}$

because, for any $x \in \mathbf{R}$, $(x, x, x) = T(x, 0, 0)$

$\therefore$, R(T) is generated by $\{(1,1,1)\}$.

$\text{Ker } T = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$, which is the plane $x_1 + x_2 + x_3 = 0$, in $\mathbf{R}^3$

E 12) Any element of R(T) is of the form T(u), $u \in U$. Since $\{e_1, ..., e_n\}$ generates U, $\exists$ scalars $\alpha_1, ..., \alpha_n$ such that $u = \alpha_1 e_1 + ... + \alpha_n e_n$.

Then $T(u) = \alpha_1 T(e_1) + ... + \alpha_n T(e_n)$, that is, T (u) is in the linear span of $\{T(e_1), ..., T(e_n)\}$.

$\therefore \{T(e_1), ..., T(e_n)\}$ generates R(T).

E13) T: V → V : T (v)= v. Since R (T) = V and Ker T = {**0**}, we see that rank (T) = dim V, nullity (T) = 0.

E14) R (D) = $\{a_1 + 2a_2x + ... + na_nx^{n-1} \mid a_1,.., a_n \in \mathbf{R}\}$

Thus, R (D) $\subseteq \mathbf{P}_{n-1}$. But any element $b_0 + b_1 x + ... + b_{n-1}x^{n-1}$, in

$\mathbf{P}_{n-1}$ is $D\left(b_0 x + \frac{b_1}{2} x^2 + \cdots + \frac{b_{n-1}}{n} x^n\right) \in$ R (D).

Therefore, R(D) = P_{n-1}.

∴, a basis for R (D) is $\{1, x,...., x^{n-1}\}$, and rank (D) = n.

Ker D = $\{a_0 + a_1x + ... + a_n x^n \mid a_1 + 2a_2x + ... + na_n x^{n-1} = 0, a_i \in \mathbf{R} \forall i\}$

$= \{a_0 + a_1x + ... + a_n x^n \mid a_1 = 0, a_2 = 0,..., a_n = 0, a_i \in \mathbf{R} \forall i\}$

$= \{a_0 \mid a_0 \in \mathbf{R}\} = \mathbf{R}$.

∴, a basis for Ker D is {1}.

⇒ nullity (D) = 1.

E 15) a) We have shown that R (T) = $\mathbf{R}^2$. ∴ rank (T) = 2.

Therefore, nullity (T) = dim $\mathbf{R}^3$ – 2 = 1.

b) rank (T) = 1, nullity (T) = 2.

c) R (T) is generated by {(1, 1, 1)}. ∴ rank (T) = 1.

∴ nullity (T) = 2.

E 16) Now rank (T) + nullity (T) = dim U = 1.

Also rank (T) ≥ 0, nullity (T) ≥ 0.

∴, the only values rank (T) can take are 0 and 1. If rank (T) = 0, then dim R(T) = 0. Thus, R (T) = {**0**}, that is, R (T) is a point.

If rank (T) = 1, then dim R(T) = 1. That is, R(T) is a vector space over **R** generated by a single element, v, say. Then R(T) is the line $\mathbf{R}_v = \{\alpha v \mid \alpha \in \mathbf{R}\}$.

E 17) By Theorem 5, nullity (ST) = dim V– rank (ST). By (a) of Theorem 6, we know that – rank (ST) ≥ – rank (S) and – rank (ST) ≥ – rank (T).

∴, nullity (ST) ≥ dim V– rank (S) and nullity (ST) ≥ dim V– rank (T).

Thus, nullity (ST) ≥ nullity (S) and nullity (ST) ≥ nullity (T). That is, nullity (ST) ≥ max {nullity (S), nullity (T)}.

E18) Since 1 ≠ 2, but **0** (1) = **0** (2) = **0**, we find that **0** is not 1 – 1.

E19) Firstly note that T is a linear transformation. Secondly, T is 1 – 1 because T (x, y, z) = T (x′, y′, z′) ⇒ (x, y, z) = (x′, y′, z′)

Thirdly, T is onto because any (x,y,z) ∈ $\mathbf{R}^3$ can be written as T (x – y, y, z)

∴, T is an isomorphism. ∴ $T^{-1}: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ exists and is defined by T^{-1} (x, y, z) = (x – y, y, z).

E 20) T is not an isomorphism because T is not 1 – 1, since (1, – 1, 1) ∈ Ker T.

E 21) The linear operator in E11) (a) suffices.

E 22) a) Let $\alpha_1,..., \alpha_k \in \mathbf{F}$ such that $\alpha_1 T(u_1) + .. + \alpha_k T(u_k) = \mathbf{0}$.

$\Rightarrow T(\alpha_1 u_1 + ... + \alpha_k u_k) = \mathbf{0} = T(\mathbf{0})$

$\Rightarrow \alpha_1 u_1 + ... + \alpha_k u_k = \mathbf{0}$, since T is 1 – 1.

$\Rightarrow \alpha_1 = 0,..., \alpha_k = 0$, since $\{u_1,....u_k\}$ is linearly independent

∴ $\{T(u_1),..., T(u_k)\}$ is linearly independent.

b) No. For example, the zero operator maps every linearly independent set to {**0**}, which is not linearly independent.

c) Let T: U → V be a linear operator, and $\{u_1,...,u_n\}$ be a linearly dependent set of vectors in U. We have to show that $\{T(u_1),..., T(u_n)\}$ is linearly dependent. Since $\{u_1,...,u_n\}$ is linearly dependent, ∃ scalars $a_1,...,a_n$, not all zero, such that

$a_1 u_1 + ... + a_n u_n = \mathbf{0}$.

Then $a_1 T(u_1) + ... + a_n T(u_n) = T(\mathbf{0}) = \mathbf{0}$, so that $\{T(u_1),...., T(u_n)\}$ is linearly dependent.

E23) T is a linear transformation. Now, if $(x, y, z) \in$ Ker T, then $T(x, y, z) = (0, 0, 0)$.

$\therefore$, $x + y = 0 = y + z = x + z \Rightarrow x = 0 = y = z$

$\Rightarrow$ Ker T $= \{(0, 0, 0)\}$

$\Rightarrow$ T is 1 − 1.

$\therefore$, by Theorem 10, T is invertible.

To define $T^{-1}: \mathbf{R}^3 \to \mathbf{R}^3$ suppose $T^{-1}(x, y, z) = (a, b, c)$.
Then $T(a, b, c) = (x, y, z)$

$\Rightarrow (a + b, b + c, a + c) = (x, y, z)$

$\Rightarrow a + b = x, b + c = y, a + c = z$

$\Rightarrow a = \frac{x + z - y}{2}, \; b = \frac{x + y - z}{2}, \; c = \frac{y + z - x}{2}$

$\therefore T^{-1}(x, y, z) = \left(\frac{x + z - y}{2}, \frac{x + y - z}{2}, \frac{y + z - x}{2}\right)$ for any $(x, y, z) \in \mathbf{R}^3$.

E24) $T : U \to V$ is 1 − 1. **Suppose T is onto. Then T is an isomorphism and dim U = dim V,** by Theorem 12. Conversely, suppose dim U = dim V. Then T is onto by Theorem 10.

E25) The Rank Nullity Theorem and Example 14 give

dim V − nullity (ST) = dim V − nullity (T) − dim (R (T) $\cap$ Ker S)

$\Rightarrow$ nullity (ST) = nullity (T) + dim (R (T) $\cap$ Ker S)

E26) In the case of the direct sum $A \oplus B$, we have $A \cap B = \{\mathbf{0}\}$.

$\therefore \frac{A \oplus B}{B} \simeq A$

E27) a) $v + \mathbf{W} = v' + \mathbf{W} \Rightarrow v - v' \in W \subseteq U \Rightarrow v - v' \in U \Rightarrow v + U = v' + U$

$\Rightarrow T(v+W) = T(v' + W)$
$\therefore$ T is well defined.

b) For any $v + W$, $v' + W$ in V/W and scalars a, b, we have
$T(a(v + W) + b(v' + W)) = T(av + bv' + W) = av + bv' + U$
$= a(v + U) + b(v' + U) = aT(v + W) + bT(v' + W)$.
$\therefore$, T is a linear operator.

c) Ker T $= \{v + W \mid v + U = U\}$, since U is the "zero" for V/U.
$= \{v + W \mid v \in U\} = U/W$.
$R(T) = \{v + U \mid v \in V\} = V/U$.