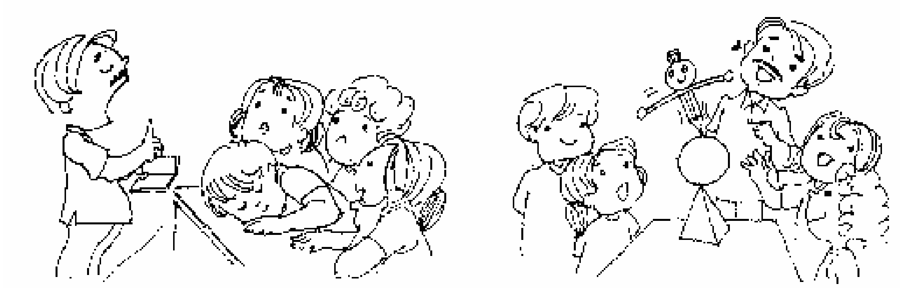
UNIT 1 WHY LEARN MATHEMATICS?

1.1 INTRODUCTION

All of us have encountered mathematics while growing up. Some of us have grown to like it, and therefore, enjoy. doing it. Some others have developed a lukewarm relationship with it, and therefore, face it as a mildly unpleasant necessity. And there are some of us who have been intellectually and emotionally bruised in our encounters with mathematics, and therefore, avoid any further interaction with it.



(a) Maths! Oh, no! **(b) Maths! Oh, great!**

Fig. 1: Attitudes to mathematics.

What is so 'special' about mathematics that it induces such a variety of feelings in people? Is it necessary to expose everybody to it? In this unit we shall reflect on these questions.

Firstly, we shall try to show, through some examples and their analysis, how we use mathematics, how unavoidable it is and what joy and beauty it can offer. Then we shall look at some of its characteristics. It is because of these special features that **mathematics is steadily becoming more important in all walks of life.**

Finally, we shall see how learning mathematics contributes to our way of thinking.

Throughout the unit our aim is to try and convince you about the need for children to learn mathematics.

Now, a word about the way the unit is presented. As you go through it you will find several examples, ideas, opinions and arguments. We have presented them with a view to provoke you to think about the importance and nature of mathematics. Please question everything you read here, and do not hesitate if you come to entirely different conclusions on the issues raised here. If your reasoning is sound, you could probably convince us about your point of view!

Now for the broad objectives of this unit.

Objectives

After studying this unit, you should be able to

- explain how mathematics is useful in our daily lives;
- explain the way mathematical concepts grow;
- identify the special features of mathematics;
- explain what mathematical thinking consists of;
- identify some basic abilities that a child develops while learning mathematics. We start by explaining why we believe mathematics is omnipresent.

1.2 MATHEMATICS IN OUR LIVES

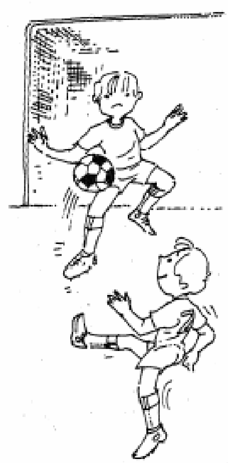


Fig.2: A footballer uses mathematics.

What is the most obvious example of mathematics in your life? To many of us it is the maths that we studied in school. But is that all the mathematics in our lives? Do the people who don't go to school never encounter mathematics? Let's see.

1.2.1 Mathematics Is All Around Us

What is the first thing you do when you get up? Make yourself a nice cup of tea or coffee? If so, then you're using mathematics! Do you agree?

Consider a carpenter making a table. Does he use mathematics in any way? Look at a tailor, a cowherd, a vegetable buyer or a mason. Do they use mathematics in any way?' When we use public transport, or drive our own vehicle, or pay our child's school fees, we are using mathematics. Making a 'charpai', sending a satellite up into orbit, building roads and bridges - can any of these activities be done without using mathematics?

And what about the various sports activities? A cricket captain once said that if he got his field placement right, half the job of getting the other team out would be done. And what floes field placement require? An astute sense of the game and of space. Kho-kho, kabaddi, football, basketball, etc., all require an instinctive awareness and utilisation of space.

What about board games like chess? While playing, you need to think of a winning strategy. For this you need to construct the possible movements at any instant, given the conditions under which the different pieces are allowed to move. In 'Ashta changa', Ludo, 'chaupad', Trade, and other such games, the players use a lot of mathematics.

You can think of many more instances of our contact with mathematics while doing the following exercises.

El) Think of an indoor game and an outdoor sport that you or your children play. Write down what mathematics is used while playing them.

E2) "I do a lot of maths while working in the kitchen", says a friend of mine. List 4 ways in which mathematics is used in the kitchen.

E3) In a conversation with a friend I said, "Maths is in virtually everything around us. Even activities like making 'rangoli' patterns, making designs and prints for clothes, dancing, feeding one's children and catching a train involve mathematics." He disagreed violently and said, "Some of the things you mention can perhaps have some elements of mathematics. But all of them don't involve mathematics." Would you agree with my friend or with me? Why?

Aren't you convinced by now that mathematics permeates all the areas of your life that interest and excite you? It is another matter that you may not be aware of it. To add more substance to our point, consider another situation.

Lata wants to put up a swing on the tree in front of her house, and wants to enjoy swinging on it. Does she need or use any mathematics for this? To put up the swing she needs a rope and an appropriate branch of the tree. There are several questions that she needs to ask, like:

Fig.3 : Lata's unconscious use of mathematics.

1. How high should the branch be?
2. How sturdy should it be?
3. Will any other branches obstruct the ropes when someone uses the swing?
4. How long should the rope be? Does the length of rope have any relationship with the height of the branch?
5. How thick should the rope be?
6. Can Lata and her friend swing together?

To answer all these questions Lata has to have a sense of mathematics. For example, to answer Question 3, she needs to have a feel for geometry! She has to mentally visualise the sweep of the ropes when the swing is going up or down, and decide if any branch of the tree would interfere with the swing.

Suppose Lata succeeds in putting up her swing. With no one to push her, she can move the swing by pulling and pushing the ropes and shifting her weight on the swing rhythmically.. As the swing starts going up and down, she has to match the rhythm and pattern of the movements of the swing, and has to apply force in a way that makes the swing go higher and higher.

So, in putting up and using the swing Lata is using her experience to estimate a lot of **mathematically calculable quantities.** She works through the estimates, checks them and decides to use them or take up another set of estimates. She does all this mathematics without realising that she's doing it. And therefore, she does it without stress, boredom and frustration!

The following exercises give you a chance to look at this and other examples more closely.

E4) Which areas of mathematics does Lata need to know to be able to answer the questions given above? Explain your choice of area.

E5) List specific examples of

(i) estimates of quantity, and/or

(ii) seeing relationships, and/or

(iii) grasping patterns

in the following situations.

a) Hari lives in Jaipur, and has to reach Delhi on 23-d March by 10:00 A.M.

b) Suresh wants to knit a sweater.

By now you must have realised that mathematics is not limited to the school textbook. In fact, we can see mathematics all around us. But, is it in everything we do? Let's find out.

1.2.2 Do All Our Activities Involve Mathematics?

The answer to this is `yes' and `no'. For those who look for mathematics and know where to look for it, it is `yes'. For those who do not look for it, mathematics is only what they do in school, having no relationship to their real world. In other words, mathematics is not like pebbles or leaves that are lying around on the ground, waiting to be picked up. One has to dig below the surface to find it.

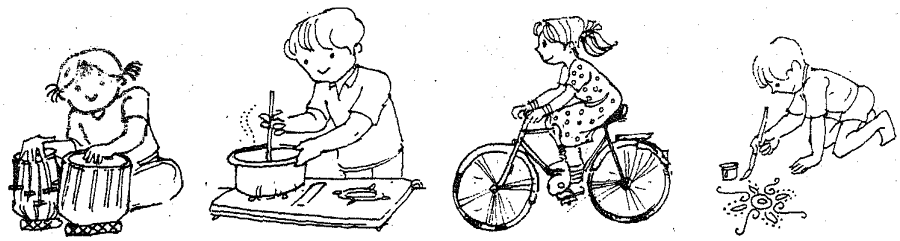


***Fig.4: Maths* is everywhere.**

To understand this let us look at an example about making 'chapati'.

A friend of mine, Prakash, says that there is a lot of chemistry involved in cooking 'chapatis'. What does he mean? According to him, when a `chapati' is being cooked, it is undergoing chemical changes, like the ones he studied about in school in the "chemistry class".

However, another friend of mine says that when she makes a `chapati', she is more interested in the various shapes the dough takes under the rolling pin. She also likes to discover the relationships between the movement of the pin and these shapes. So, she sees mathematics in the very same process that Prakash sees chemistry in.

What this example shows is that an activity, incident or phenomenon can be looked upon and studied from several angles. If we look for the mathematics in it, we will find it. Once we have our `mathematical eyes' open and we start noticing rules and patterns, we can see mathematics in almost everything-a song, a story book, the path taken by a fly while buzzing around, the shape of a ¯match-box and the number of its surfaces, the proportion of paper required to cover these surfaces, electric wiring in your house, patterns in leaves, and so on.

In particular, all children's activities and experiences are steeped in mathematics. **These experiences should be related to the mathematical concepts and ideas that we teach them.** Only then will these ideas appear relevant to the children, and be absorbed by them easily.

E6) Think of at least one real-life experience each of children that would relate to the following mathematical concepts: Addition, volume, symmetry, probability.

By now you must have realised how much mathematics you use, consciously and unconsciously. From the following sub-section, you may get an inkling about how we can apply it to stimulate our minds for recreational purposes.

1.2.3 Maths For Fun

Often, when I have time on my hands, I try to solve interesting mathematical questions of the following kind. Sometimes my friends and I create the problems, and sometimes we find them in `fun books'. Why don't you try these problems? Take your time!

1. Suppose I offer you a loan to start a safety matchstick production unit on the following terms:

   - I shall first advance you Rs.50,000/- to set up your unit, and wait for 3 months so that you can buy the equipment and start production. (Assume that Rs.50,000/- is enough for this.)

   - After the three months are over and you have started production, I shall give you Rs.50,000/- every day, for 60 days.

   - In return, I ask you for very little. You shall give me one matchstick on the first day, double that (i.e.,2) on the second day, double that (i.e., 2x2=4) on the third day, and so on, for 60 days.

   - On the 60th day, after the transactions are over, our agreement shall end, and we shall owe each other nothing. In exchange for the huge amount of money that I will have given you, I would be satisfied with the matchsticks you will have given me in 60 days.

   a) Would you accept the offer!

   b) Who will gain from this deal, and by how much?

2. A village has only one barber. He shaves all those men in the village who don't shave themselves. Will the barber shave himself?

The barber's problem is one of Sir Bertrand Russell's paradoxes.

3. You are given a wooden plank with three holes (see Fig.5). The holes have the shapes of a square, a triangle and a circle. Can you make one plug from a wooden block that would exactly fit each of the three holes?

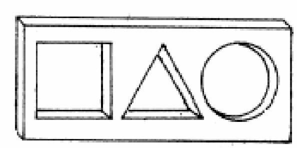


Fig. 5.

4. Consider the "number square",

8	1	6
3	5	7
4	9	2

If you add the numbers in any row, or column, or diagonal, the sum will be the same, 15 in this case. Because of this property, the square is called a **magic square**. Can you make a 4 x 4 magic square with the numbers 1 to 16?

5. What is the biggest number that can be written using only four ones? (You are allowed to use any number operation, but the only digit you can use is 1, and that too 4 times. The answer is much larger than 1111.)

6. Arrange 24 persons in 6 rows, so that each row has 5 people.

Didn't you enjoy doing them? Whenever I'm travelling, or I'm not busy at work, I find that doing such problems are great fun. Similarly, children enjoy such mental activity, as long as the puzzles are within their mental reach.

Here's a chance for you to think up examples of 'recreational maths'!

E7) Create three problems/riddles/puzzles for children to help them feel that 'maths is fun'. Try them out on the children around you. Find out which ones they enjoyed, and why.

So far we have only spoken about one aspect of mathematics-its utility in our daily lives. But is this why mathematics is called "the queen of the sciences"? No. It is because of the beauty and power of mathematical concepts, which is due to the special features of mathematics that we will now discuss.

1.3 HOW MATHEMATICAL IDEAS GROW

In this section we shall consider three aspects of the nature of mathematical ideas, namely, that they progress from concrete to abstract, from particular to general, and they form hierarchical structures. You may say that these three aspects can be seen in all human pursuit of knowledge. We shall examine them in the context of mathematics, and try and see how basic, crucial and important they are to mathematics.

1.3.1 Concrete to Abstract

Mathematics, like all human knowledge, grows out of our concrete experiences. Let us take the example of three-dimensional shapes. Think about how you came to understand the concept of "roundness" and of a sphere. Was your mental process something like the following?

We see all sorts of objects around us. While dealing with them, we notice that some of these things, like a ball, an orange, a water melon, a 'laddu', have the same kind of regularity, namely, a roundness. And so, the notion of `roundness' gradually develops in our mind. We can separate the objects that are round from those that aren't. We also realise that the property of roundness, common to all the round objects, has nothing to do with the other specific attributes of these objects, like the substance they are made of, their size, or their colour. We gradually separate the idea of `roundness' from the many concrete things it is abstracted from. On the basis of the essential property of `roundness', we develop the concept of a sphere. Once we have formed this concept, we don't need to think of a particular round object when we're talking of a sphere. We have successfully abstracted the concept from our concrete experiences.

In a similar way, we learn to abstract the concept of `redness', say. But there's a major difference between this concept, and mathematical concepts. Firstly**, every mathematical concept gives rise to more mathematical concepts**. For example, related to the concept of a sphere we generate the concepts of radius, centre, surface area and volume of a sphere.

Secondly, we can think of various **purely abstract and formal relationships** between the related concepts. For instance, examine the relationship between a sphere and its volume. Irrespective of the size of a sphere or the material it is made of, the relationship is the same. The volume of a sphere depends on its radius in a certain way, regardless of how big or small the sphere is.

Thus, not only can we abstract a mathematical idea from concrete instances, we can also generate more related abstract ideas and study relationships between them in an abstract manner. These abstract mathematical ideas exist in our minds, independent of our concrete experiences that they grew out of. They can generate many more related abstract concepts and relationships amongst themselves. The edifice of ideas and relationships keeps growing, making our world of abstractions larger and larger.

You may like to think of another example of this aspect of the nature of mathematics.

E8) Would you say that the number system developed in this way? If so, how? Let us now consider another way in which mathematics grows. This is closely related to what we have just been discussing.

1.3.2 Particular to General

When I say `tail', what do you think of? Do you think of the tail of a horse, or of a monkey? Or do you think of the tail of your pet dog?

The tail of a particular animal has many features that are not part of the concept of `tail'. For instance, my horse has a dark tail, two feet long. I could describe the thickness of its hair, its colour, at what angle it is inclined to the body, and so on. But would this description fit the tail of any horse? Wouldn't some of these features need to be changed from horse to horse? So, if I want to apply this concept to all horses, I need to form an image of a tail which is not bound by the particular properties of my horse's tail.

Fig. 6: A tail, is a tail!

Now, I notice that cows and dogs also have similar things attached to their bodies. So, I further generalise my concept of tail to include the tails of all animals.

So, while generalising from a particular case to cover more and more cases, we leave out some features of the specific example, and pick out what is common to the various examples. We abstract their common aspects and form a general concept.

Isn't this the same way we form the concept of a quadrilateral? This concept is the result of examining squares, rectangles, trapezia, etc., and picking out their common properties, namely, that they are all closed figures with four sides. So, we form a general concept of a closed four-sided figure, and call any such figure a quadrilateral.

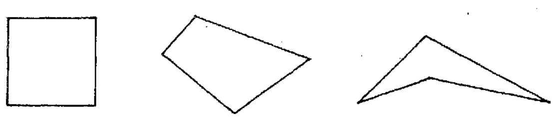


Fig.7: Various quadrilaterals,

You must have noticed by now that as we **generalise ideas we are moving towards more and more abstraction.**

You may like to try these exercises now.

E9) Write down an example each, related to natural numbers and fractions, to, show the movement from particular to general.

E10)) Is ‘moving from particular to general’ the same as ‘moving from concrete to abstract'? Why?

And now let us consider another important aspect of mathematics.

An hierarchy of ideas is a system of ideas in which they are organised into different grades, ranked one above another.

1.3.3 Hierarchical Structures

As the abstractions from concrete objects and materials become more and more general, they represent wider and wider ideas. If we put down each step of the process of generalisation, we would have a series of ideas, each contained in the generalised idea following it.

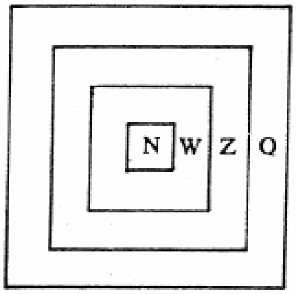


Fig. 8 : The hierarchy in the number system.

For example, consider the number system.

i) From the counting of concrete objects, we abstract the set of natural **numbers,** namely, N = {1,2,3,4...}.

ii) If, in this set, we include zero, we get the set of **whole numbers,** namely, W={0,1,2,3,4...}.

iii) This set can be further enlarged to include negative numbers, and we get **Z** = {...-3, -2, -1, 0, 1; 2, 3...} as the set of **integers.**

iv) To the set of integers, we can add positive and negative fractions to get the set of **rational numbers, Q,** and so on. We have pictorially shown this in Fig. 8.

Now, if I don't understand what the natural numbers mean, I will certainly not be comfortable with whole numbers. Similarly, if I don't grasp what negative numbers mean, I doubt if I will understand what a rational number is. So, to understand each of these abstract concepts, I need to understand every concept that comes before it in the stepwise build-u; , or hierarchy, of ideas.

Now, look closely at the mathematical ideas that you are familiar with, and try the following exercise.

E11) Write down three hierarchical chains in mathematics. You can look for examples related to number operations, geometry and algebra.

The hierarchy of concepts also has implications for the way concepts are learnt. If you look at the historical development of any concept, hierarchically lower concepts usually came before hierarchically higher concepts. The learning of concepts by children is also, broadly, on the same pattern. Therefore**, it is usually better to introduce a child to an hierarchy of ideas the way they developed.** Unfortunately, this does not always happen.

For example, a square is a particular type of rectangle, and a rectangle is a particular case of a parallelogram. But many children in Class 2 are taught these concepts at the same time, without even relating them to each other. What is the result? Even two years later many of them will say that a square is not a parallelogram.

So, to really understand a new mathematical idea a person requires a proper understanding of mathematical concepts that come before it. This is what we mean when we say mathematics is **an hierarchically structured** discipline. In the following exercise we ask you to consider the implications of this fact for teaching.

E12) The hierarchical structure of mathematics is one reason that it is considered a difficult subject to learn/teach. Do you agree? Why?

Now that we have seen some ways in which mathematical ideas grow, and are acquired, let us consider the special nature of mathematics.

1.4 THE NATURE OF MATHEMATICS

In this section we will see what makes mathematics such a powerful means of communication. We will look into three reasons for this, in the following subsections.

1.4.1 Mathematical Statements Are Unambiguous

Consider any mathematical concept that you're familiar with, say, a sphere. The definition of a sphere is clear and precise. Given any object, you can very definitely say whether It is a sphere or not. Similarly, the definition of any concept in mathematics, or any mathematical statement, is absolutely unambiguous, leaving no place for doubt. This is because we formally construct this abstract world of mathematics by first accepting a certain set of axioms which are consistent. Then, on the basis of these axioms, we formally define certain concepts **clearly and precisely.** Once the axioms are chosen and the situation defined, there is no scope for ambiguity.

Very often, of course, when we apply mathematical terms to real-life situations, we use the terms loosely, and not according to their. mathematical definition. That is when the term can become imprecise. A good example is the usage of `half in day-to-day conversation, which we shall talk more about in Block 4. I'm sure you can think of several other examples.

E13) List some more examples of a mathematical term being used 'imprecisely in day-to-day situations. Give reasons for your choice.

So far, we have talked about various aspects of mathematics that can often be found in other disciplines too. But what we are going to talk about now is peculiar to mathematics.

1.4.2 Truth Criteria

Consider the following statements:

i) Peahens (i.e., female peacocks) lay eggs around September.

ii) Water boils at 100°C.

iii) 5 divides 15 without leaving any remainder.

iv) If you add two odd numbers, the result is always an even number.

If we want to see whether these statements are valid or not, how would we go about checking them? 'Will the same method work for **checking** all the statements?

Consider the first hypothesis. It can only be tested empirically. That is, we would have to observe peahens and find out when they lay eggs. If we observe a large number of peahens, and find that nearly all of them lay eggs around September, then we can say that peahens lay eggs around September. Note that this empirical rule would be accepted, even if there are rare instances that do not follow the rule.

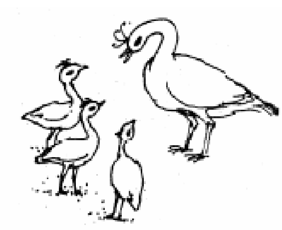


Fig. 9: Remember, remember! Lay eggs in September!

By **actual** experimentation, you can check that the second statement is true, only under certain conditions. By heating water under pressure, it can be shown to boil at a temperature much above 100°C. (It is this principle that is used in the pressure cooker.)

The third statement is a mathematical statement. Can you prove that it is true by observation? You may take several examples of sets of 15 objects and divide them into 5 equal parts. But you need to prove it for **all sets** of 15 objects. So, how would you prove it? If you understand what 5, 15, division and remainder mean, you can prove it mathematically.

What I am saying is **that in mathematics, truth is only a matter of consistency and logic.** The proof of a mathematical statement consists of a series of logical arguments, applied according to certain accepted rules, definitions and assumptions.

Let's go further now. Consider Statement (iv). Can you prove it empirically? If so, how would you exhaust all the possible pairs of odd numbers? Until you have checked all the possible odd number pairs, your hypothesis would not be accepted as a mathematical rule. You may verify it for a large number of cases, but then someone may ask if it is true for 35678947321987 and 10000420001293? So, unless the result is proved formally for a general pair of odd numbers, the result is not mathematically acceptable.

One way of proving the result is:

Any odd number can be written as 2n+1, where n is some whole number. So, take two odd numbers, $2n_1+1$ and $2n_2+1$, where n_1 and n_2 are whole numbers. The sum of these numbers is $(2n_1+1) + (2n_2+1)$-= $2(n_1+n_2+1) = 2m$, say, where $m = n_1+n_2+1$ is a whole number.

So, $(2n_1+1) + (2n_2+1) = 2m$, which is divisible by 2, and hence is an even number. Thus, the sum of two odd numbers is an even number. Q.E.D.!

Here we have made use of definitions (odd, even, whole numbers), previously drawn results (the sum of whole numbers is always a whole number) and logic. No observations. If someone does not know these definitions and previously drawn results, it is not possible for her or him to understand this proof.

This kind of logic, which uses known results, definitions and rules of inference to prove something, is called **deductive logic.** This kind of logic starts from a general statement or definition, which is accepted beyond doubt, and deduces the next step in the proof. For example, suppose we accept the statements

(A) "All men are mortal", and

(B) "Raghav is a man." Then, from (A) and (B) we are forced to deduce that "Raghav is mortal".

Another kind of logic used in mathematics is **inductive** logic. To try and understand what it means, let us look at a non-mathematical example first.

I see a dog and find that it has a tail. I see a second dog and find that this one also has a tail. This is true for the third and fourth dogs too, and so on, for a large number of dogs. So, I conclude that all dogs have a tail. This is like the proof for the statement about peahens laying eggs in September.

This kind of logic only proves that the statement has a very high probability of being true. But it does not prove that we shall **never** find a dog without a tail, or we shall **never** find a peahen which might lay eggs in March.

But mathematicians need to use a form of logic that will prove a statement for **all** the cases. Therefore, they use a particular form of inductive logic. It is known as **mathematical induction.** Let us consider an example to see how it works.

Suppose we want to prove that the sum of the first n natural numbers is where n is any natural number, that is, $\frac{n(n+1)}{2}$, where n is any natural number, that is,

$$1+2+3+...+n=\frac{n(n+1)}{2}$$

(i) First we prove that **the statement is true for n = 1.**

(ii) Then we prove that **if** the statement is assumed to be true for m, then it must be true for (m+1), where m is any natural number. Notice that here **we are not saying that it is true for m** or that it *is true* for (m+l). We are saying that **if** it is true for m, **then** it is true for (m+l) also.

Now, if both (i) and (ii) are satisfied, then we can say that it is true for 1, and whenever it's true for in, it is also true for m+1.

Now, since it is true for m = 1, it is true for m+l = 1 + 1 = 2.

Since it is true for 2, it is true for 2+1 = 3, and so on.

Since there is no limit on the value of m (as long as it is a natural number); we have proved the statement for all natural numbers.

This is the structure of mathematical induction.

Let us now apply it to prove that the sum of the first n natural numbers is n(n+1)/2.

(i) For n=1, the statement $1+2+3+...+n=\frac{n(n+1)}{2}$ will become

$1=\frac{1(1+1)}{2}=\frac{1\times 2}{2}=1$, which is true.

Therefore, for n=1, the statement is true.

(ii) We now assume it to be true for n=m, where m is a natural number. That is, we assume that

$$1+2+3+...+m+(m+1)=\frac{m(m+1)}{2}+(m+1)$$

$$=\frac{m(m+1)+2(m+1)}{2}$$

$$=\frac{(m+1)\ (m+1)}{2}$$

Thus, $1+2+3+...+(m+1) = =\frac{(m+1)\ (m+1)}{2}$

That is, the statement is true for (m+1) as well.

So, we have shown that if it is true for m, it is true for (m+1) too.

Therefore, by mathematical induction, we conclude that $1+2+3+...+n=\frac{n(n+1)}{2}$ for all natural numbers n.

This is how inductive logic is used to prove mathematical results.

E14) Which kind of logic is used for proving Pythagoras's theorem?

E15) From your experience of mathematics, give at least one example each of the use of inductive and deductive logic to prove mathematical statements.

Now consider the statement:

"The square of a prime number has at least 4 factors".

How would you prove or disprove it?

Since the statement is supposed to be true for all prime numbers, it must be true for any particular one also. Let us see if it is true for 7.

7 x 7 is 49, and the factors of 49 are 1, 7, and 49. Hence, the statement is false for

7. So, 7 is a **counter-example**, and shows that the statement is false.

Notice the difference between our attitude to proving and disproving statements by examples. When we were **proving a statement,** we wanted it shown to be true **for all cases. But to disprove a statement, just one** example is **enough.**

Why don't you try an exercise now?

E16) Prove or disprove the statements that

i) the sum of the interior angles of a quadrilateral is 360°,

ii) the sum of the interior angles of a pentagon is 450°.

Let us now see what helps in making mathematical statements brief and clear.

1.4.3 Use of Symbols

"Multiply seven thousand six hundred and fifty three by four thousand nine hundred and eighty one."

Try doing this without writing the numbers in the form of numerals, that is, without using the digits 0 to 9. Even understanding the problem is a problem! On the other hand, if I write "7653 x 4981 = 7", isn't the question easier to understand? Could it be the use of symbols that brought this ease?

Take another example. Read the following statement:

"The sum of seven and eight multiplied by itself is equal to the sum of eight multiplied by itself and seven multiplied by itself with twice the product of seven and eight."

Does this make any sense to you?

Now look at this: $(7 + 8)^2 = 7^2+8^2+(2x7x8)$

This is the same statement, written with symbols. This symbolic representation makes the statement brief and clear, provided I understand the symbols and how to read statements formed by using them.

You know that mathematics deals with abstract ideas, which are precise and unambiguous. For working with these concepts, and for communicating them efficiently, we need to use common systems of notation with rules for manipulation. These systems are what add to the power of mathematics, and allow us to easily visualise whether a mathematical argument is valid or not.

Algorithm : A series of steps in an order, to be followed for solving problems.

Using symbolic notation for various operations makes it easy to apply the **algorithms** for solving problems involving the operations. However, a **note** of **warning** here! Although notations, symbols and algorithms make operations simple and fast, they also make them mechanical. While teaching/learning mathematics, it is very easy to fall into the trap of developing the ability to do the operations mechanically without knowing what is being done and why. To give just one example; consider the following problem: 5132 _ 5. Most of us would usually do it as shown in Fig. 10.

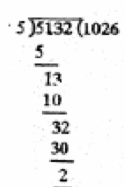


Fig. 10.

But how many of us ask

a) why is 5 written below 5132 in that manner only?

b) how did I get 13 in the third line, and why?)

c) while writing 13 there, why do I need to write zero in the dividend?

I may not be able to answer these questions, or many others of this nature. And yet, I can do the sum correctly.

This poses one of the most serious problems in teaching mathematics. Most teachers will be happy if the child has mastered the algorithm, even if she has no idea why the algorithm produces the correct answer. But this approach to teaching/learning makes it more difficult for the child to acquire mathematical concepts later on, and sometimes it may block further learning completely.

Learning algorithms is not learning matheatics.

Here's an exercise for you now.

E17) Which of the following procedures do you believe in, and why?

i) Before a child learns an algorithm, she must understand the mathematics behind it.

ii) It is always better to let a child use an algorithm extensively, without explaining the mathematics involved in it.

iii) In some cases (i) works, and in some cases (ii).

So far we have been trying to understand the nature of mathematics. Let us now see how our thought processes are altered by doing mathematics.

1.5 THINKING MATHEMATICALLY

Have you ever thought of what mental processes you are going through when you are solving a mathematical problem? Why don't you try the following problem? While doing it, carefully monitor the mathematical processes you are undergoing.

For positive numbers m and n, their AM is $\frac{m+n}{2}$, and their GM is $\sqrt{mn}$.

The problem is to find out what the relationship is between the arithmetic mean (AM) and the geometric mean (GM) of any two positive numbers.

How would you tackle this? Would you start by looking at a few specific pairs of numbers? If so, you are specialising.

Now, suppose you take, say 1 and 3. The AM of 1 and 3 is $\frac{1+3}{2}$. The GM of 1 and 3 is $\sqrt{1\times 3} = \sqrt{3}$. By taking several pairs, suppose you get the following chart: $\sqrt{3}$

Number pair	AM	GM
(1, 3)	2	$\sqrt{3}$
(2, 2)	2	2
(3, 27)	15	9
(20, 25)	$\frac{45}{2}$	$10\sqrt{5}$

Do you start noticing a **pattern?** Does this make you **conjecture** a rule? What is **the** general rule? Is it that AM $\geq$ GM? You need to check if your **generalisation** is right. This means that you need to **prove your conjecture.** This means that you need to **start from certain assumptions,** and arrive at your result by a series of steps, **each** following logically from the previous one.

There are **several ways of proving** it. One way is that you can take any two positive numbers x and y. Now, you want to see whether

$$\frac{x+y}{2} \geq \sqrt{xy}$$

This will he true if and only if

$(x+y) \geq 2\sqrt{xy}$, which is true if and only if

$(x+y)^2 \geq 4xy$, which is true if and only if

$x^2 + y^2 + 2xy \geq 4xy$, which is true if and only if

$x^2 + y^2 - 2xy \geq 0$, which is true if and Only if

$(x-y)^2 \geq 0$,

and this is always true, since the square of a number is always non-negative.

So, you have proved the general rule that the AM of any two positive numbers is greater than or equal to their GM.

But, may be your curiosity has been provoked. Are you wondering if a similar statement is true for 3 positive numbers? Or for negative numbers? In this case, you are posing a problem. Of course, once you pose it, I'm sure you'll test your conjecture, and prove or disprove it. And, carrying on in this manner, you may generalise your statement to n numbers, and prove it.

Remember that, **without** a **proof your conjecture** is not acceptable as a **true mathematical statement.**

Sometimes, of course, you may make a conjecture which is not right. For example, suppose that you had initially found the values of the AM and GM for the pairs (1,1), (2,2), (3,3), and so on. Then you could have conjectured that AM = GM. But then, to test this, you may have tried it out for (1,3), and discovered that your conjecture isn't correct. So, you would need to **modify** it, and then develop your mathematical argument again.

So, what have you been doing in the process of **problem-posing** and problem-solving? Weren't you thinking mathematically along the following lines?

E18) Several circles can be drawn through a point. How many can be drawn through two points, or three points,...?

a) Work on this problem and note down the processes you use.

b) Did the properties of mathematics, mentioned in Sec. 1.4, show up while you were developing your arguments? If so, in what way?

If you've done E18, you must have realised that trying mathematical problems improves one's abilities to

- **think precisely**
- **articulate clearly**
- **think logically and systematically**
- **look for patterns and relationships**

These abilities, if well developed, can help us greatly in other real-life situations Therefore, these mental abilities should be developed right from childhood on.

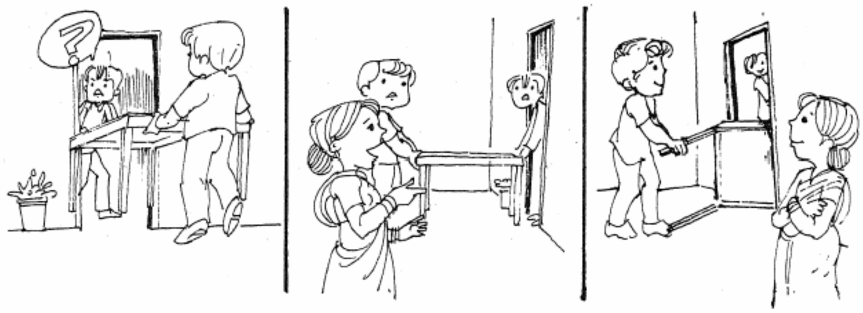


Fig.11: The ability to think mathematically helps us to deal with our real-life problems

In the next three units we will look broadly at different aspects of developing mathematical thinking in children.

Let us now briefly see what we have covered in this unit.

1.6 SUMMARY

In this unit we have considered he following points.

1. Mathematics is omnipresent, powerful and beautiful.
2. Mathematics is useful in all spheres of life.
3. Mathematics can also be used to provide stimulating leisure time activity.
4. Mathematical ideas usually grow from concrete situations to abstract concepts, and from particular cases to general notions.
5. The body of mathematical knowledge is hierarchically constructed, and similarly acquired, in general.
6. Mathematical statements and definitions are precise, clear and unambiguous.
7. To verify a mathematical statement, you need to prove it for all cases. If it is not true for even one case, then it is not true at all.
8. The extensive use of symbols is what makes mathematics a brief, clear, and hence, strong means of communication.
9. Before accepting and using an algorithm in mathematics, you need to understand the reasoning behind it.
10. Mathematical thinking consists of solving problems and posing new ones. Problem-solving requires the skills of precise thinking and logical reasoning.
11. The skills developed in a child by exposing her to problem-solving enables her to think rationally in real world situations too.

And now, you may like to go through the unit once more to check whether you have done all the exercises. Once you are sure you've tried them, you may like to look at our comments on them, which follow.

1.7 COMMENTS ON EXERCISES

El) For instance, while doing crosswords, I need to see the length of the words I fill in, the matching of the common letters, and so on.

E2) For example, ratio and proportion, estimation, counting, etc. In what way are these skills applied?

E4) i) Measuring length

ii) Measuring weight and estimating the strength and tensility of the branch

iv) Measuring and comparing lengths.

v) Relating the thickness to the strength, and to the weight that it can lift.

vi) As in (v), and relating weight on swing to the geometry of its movements.

E5) For example, Hari has to estimate the distance, speed of movement, etc., and relate the condition of the road to the speed. You can think of many more examples.

E6) To give just a few examples :

We see so much of symmetry-around us and have a deep sense of awareness and appreciation of patterns. Flower arrangements, folk drawing and cloth designs, etc., all use symmetry and pattern. In plants there are innumerable examples of symmetry, shapes, patterns, etc. Such examples exist in animals, in objects, in pictures and other things. But we do not register all this as related to mathematics. Thus, when done in maths class, these concepts are totally disjoint from our life experience.

We also use the concept of probability so many times in casual conversation. For example,

- The Indore-Bilaspur train is late many times. It would be late today also!
- There is no possibility of rain today.
- He would certainly come. He always keeps his word.

E7) For an example, you can listen to our audio programme, "Learning Mathematics Can Be Fun".

E8) Read Sec. 1.3.3.

E9) For instance, the sum of two natural numbers is a natural number; 1/n is greater than 1/n+1 for any natural number n. You can think of many more examples.

E10) Concrete to abstract' is certainly a move towards generalisation. But `particular to general' could be from an abstract situation towards further abstraction, like squares generalised to quadrilaterals.

Ell) For example, addition generalises to multiplication in N, which generalises to multiplication in Q. In geometry, an example is the hierarchy of n-sided polygons. An example from algebra could be the algorithms to find the square root of a number, generalised to find the cube root of a number, generalised to find the fourth root, and so on. There are many many more examples.

E13) For example, when people say that the world is round, it is not so because the earth is not a precise sphere.

E14) Deductive logic

E15) We have given you examples in the text above.

E16) i) Try proving it by assuming that the sum of the interior angles of a triangle is 180°.

ii) You can either prove that the sum is 540°, or give an example in which it is riot 450°.

E17) As far as possible, I feel that (i) should be used. But sometimes the child may not be developed enough to understand the mathematics behind the algorithm. In this case, the teacher can try to explain it later.