UNIT 19 SYMMETRY AND PATTERNS

Structure **Page Nos.**

19.1 INTRODUCTION

In this unit we continue the trend in Unit 18. Through exploring aspects of symmetry, we expect you to focus on the processes involved in mathematical thinking. Symmetry is, of course, not new to you. Apart from your other exposure to it, in Unit 12 we discussed this concept as part of a look at spatial properties of objects. We mentioned there that such concepts are usually not taught to children in elementary school. However, symmetry is a property commonly found around us — in the shapes we see, in the actions we perform, in the design of the objects we use and in the life forms that surround us. Because of this, children informally observe, use and appreciate symmetry to quite an extent. Using this informal exposure and widespread presence of symmetry, children can be exposed formally to this concept from an early age on. This helps in the gradual development of their formal understanding of the concept. It also helps in relating the mathematics they are taught in school to life around them.

In this unit we will build on ideas about symmetry found in 2D figures, mainly from a geometrical viewpoint. We shall also consider examples of different kinds of symmetry.

As an application of symmetry, we shall consider the variety of patterns found around us. You will see that mathematicians divide patterns into borders and wallpapers. These categories are further broken up, depending on the types of symmetries the pattern shows.

While you are studying the unit, please concentrate on the abilities you are using to comprehend the basic idea underlying symmetry. We hope that this will help you see how children can learn about symmetry. We also offer you suggestions for some activities and exercises, some easy and others not so easy, but all, hopefully, of some interest. Please do these activities yourself, while thinking about and reflecting upon them. Also, think about how you could do these activities (or modified versions of them) with children. Most of the activities suggested are simple enough for children to do on their own, while others will certainly require your help and supervision. The more concrete you can make an activity the more useful it will be to you and your students. So, for example, when considering the symmetries of a rectangle, it is better to cut out a rectangle from a cardboard piece and manipulate it with your hands rather than try and visualise the problem by simply drawing rectangles on paper. The more time you spend **actually doing** what has been suggested, the more you (and hence, your learners) would benefit from it.

Objectives

After reading this unit, you should be able to

- explain what the symmetry of a two-dimensional figure is;
- explain the operations of reflection, rotation, translation and glide symmetries;
- identify the symmetries of simple 2D shapes;
- explain what a pattern, border and wallpaper pattern is;
- explain why any design for a border that you can think of must belong to one of only seven symmetry classes;
- design and carry out activities with children to help them achieve the objectives listed above.

19.2 WHAT IS SYMMETRY?

You must have used the term 'symmetry' in your everyday speech for an object or figure with balanced and pleasing proportions. As with many other words that we use in our everyday conversations, symmetry acquires a somewhat more specific meaning when we use it in a mathematical context. So, for example, while talking with friends we may say that a picture is symmetric if it appears balanced and pleasing. However, if we were to be a little mathematical we would call a picture symmetric if, for example, we could fold it in half in such a way that the left and right halves of the picture matched exactly. If it were possible to do this, then the fold would be called an **axis of symmetry** of the picture. This is still only a rough statement about symmetry, and about only one type of symmetry. We shall make it more precise as we go along.

19.2.1 A Geometric View

To understand what symmetry is, think back to some of the patterns you made when you were a child. Did you put a few drops of ink on a piece of paper, fold it in half and press the two halves together? Try doing this again, and observe the interesting figures you get. You will find that the figures will be divided in two parts about the line along which the paper has been folded. Each part will be the mirror image of the other part, if a mirror were placed along the central line (see Fig.1). In this case we say that the figure is **symmetric about the line**. We call this type of symmetry a reflection symmetry.

Fig.1: Symmetric inkblot patterns

You may find it interesting to know that such inkblots are used by psychologists in what are called Rorschach tests. Their patients are shown such patterns and are asked to say what these patterns remind them of. These answers are then studied to see what they reveal about the intelligence, personality and mental state of the person who made them.

Getting back to symmetry, there is another way you may have used for generating symmetric patterns. Fold a paper in half, and on one half arrange short lengths of string dipped in different coloured inks or paints. Then press the two halves together to get a figure like the one shown in Fig.2.

Fig. 2: Symmetric inked string patterns

Both this and the previous activity can be done with children. The following activity may also help children observe the uses of symmetry. Before doing it, you could have a discussion in the class on symmetry, where they find it around them, is their body symmetric, are parts of their body symmetric, etc. Once they notice these features related to symmetry then you could do the activity with them.

Activity 1 (Making masks) : Give each child a sheet of A4 size paper, and ask her to fold it in half. Then, with a pair of scissors, she should trim the edges in the shape of a quarter oval about the size of her face. Ask her to cut out areas for the mouth, nose and eyes (see Fig.3). Observe her to find how far she uses the concept of symmetry. She can then unfold the paper, glue a small stick to the bottom and hold it before her face as a mask. You could ask the children to think about ways of altering the shape of the areas cut out, to make their masks look sad, or happy, or angry.

Fig. 3: A mask with one axis of symmetry

There are a string of related activities that you can think of. For instance, you could take a sheet of paper in the form of a square and fold it in half. Turn it around by 90°, and fold it in half again. And, yet again, fold it in half, but this time diagonally as shown in Fig.4.

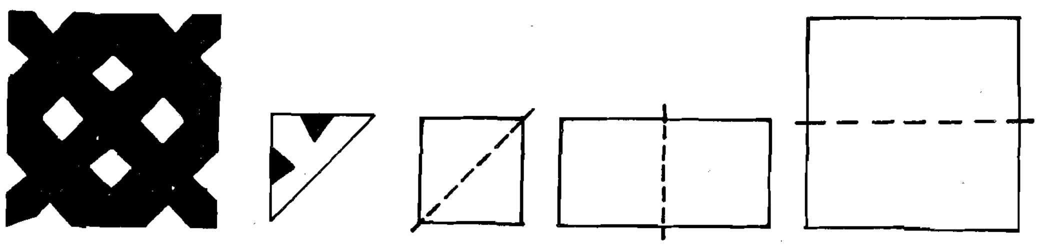

Fig. 4: A cut-out paper shape with 4 axes of symmetry

Cut the folded square into any shape you want, taking care not to cut out any fold completely. This is to ensure that the paper remains joined together. Now open out the folds in the paper. Do you see how **the number of axes of symmetry of the final shape doubles with each fold** you make?

Children can use thin, coloured paper to cut some quite intricate patterns in this way (see Fig.5). In fact, they can create many paper decorations for festive occasions in this way.

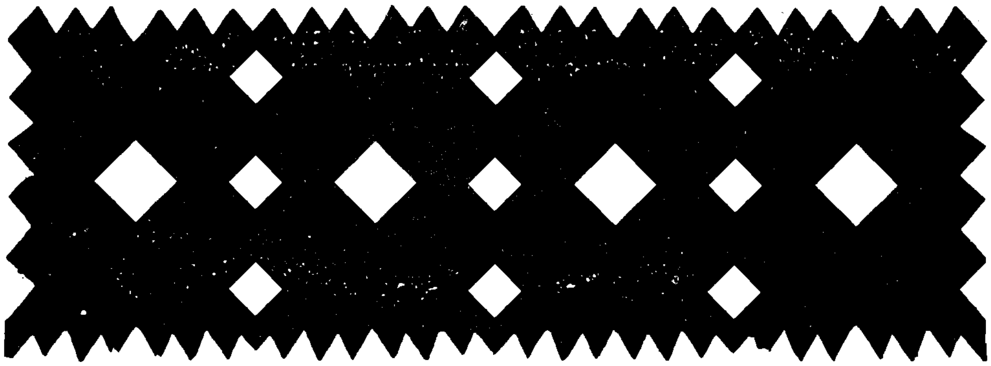

Fig. 5 : A decorative paper cut-out

Try the following exercise now.

E1) Ask some children around you to make patterns with inkblots. Ask them to articulate what symmetry, if any, they find in those patterns.

E2) Divide the class into groups. Give them several patterns, some of which show symmetry and some that don't. Ask the groups to discuss the symmetry in them, and separate out the two kinds of patterns.

So far we have considered several examples of symmetry. All these were geometric in nature. What was happening in them? We had divided the figure in two in such a way that when the two shapes were folded on top of each other they coincided exactly. If you remember your studies of geometry, you would know that if we can make two planar shapes **coincide exactly** (point by point everywhere) by placing them on top of each other, then we say we have brought them into **congruence**. This concept of congruence can be applied to two figures or two parts of the same figure. In fact, in all the examples of symmetry in this unit so far, we found that the figure was made up of two congruent parts, each lying on either side of a line. In this case we say that the figure is **symmetric about that line** (or **with respect to that line**).

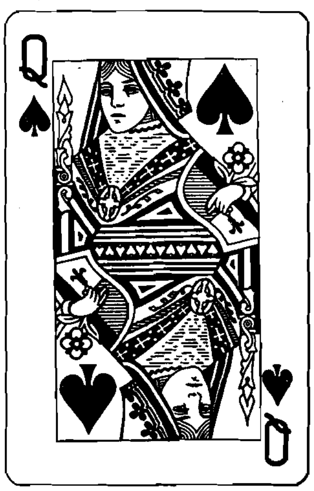

Fig.6 : A figure with rotational symmetry

Now consider a card like the Queen of Spades (Fig.6). If you rotate it through 180° without moving its centre, you will find that it coincides with itself. So, this card is also symmetric. But in this case the symmetry is obtained by rotating the object. So it is a **rotational symmetry**. Is there any line about which this card can be symmetric? Think about this, and we will discuss these different kinds of symmetry further in Sec.19.3.

Now, think about a really irregularly shaped planar figure, like a triangle of sides 3 cm, 4 cm and 5 cm. Does this have any symmetries? Think about this while trying the following exercise.

E3) Give one example each of a letter of the alphabet with no symmetry, and one with one or more rotational symmetries.

So far we have looked at symmetry, strictly from a visual geometric point of view. As you may know, there is also an algebraic point of view. Let us consider that now.

19.2.2 An Algebraic View

From your studies of equations, you would remember that the equation $y = x^2$ represents a parabola, and $y = x$ represents a straight line. What happens to the first equation when x is replaced by $-x$? Since $x^2 = x.x$ becomes $(-x).(-x)$, which is the same as x^2, the equation $y = x^2$ remains the same by such an interchange. So, this equation is symmetric under the interchange of x with $-x$. In fact, any expression which involves combinations of terms with x raised to an even power and constants (for instance, $1 + 3x^2 - 2x^4$) has this property. It may not be immediately clear what this behaviour of expressions involving x^2, under the interchange x going to $-x$, has to do with the ideas of symmetry that we introduced in the earlier sub-section. To see the connection, you could do the following task.

Take a sheet of graph paper. On it plot the function x^2 for x lying in the range -5 to $+5$. To do this, you could make a table of values with x equal to $-5, -4, \ldots, +4, +5$ and the values of x^2 corresponding to these values of x in the second row (see Table 1).

Table 1: Table of values of the function x^2

x	–5	–4	–3	–2	–1	0	1	2	3	4	5
x^2	25	16	9	4	1	0	1	4	9	16	25

To plot the curve $y = x^2$, draw a horizontal line at the bottom of the sheet of graph paper. This is your **x-axis**. Choose a suitable scale and mark off values of x from -5 to $+5$ at unit intervals along this axis as in Fig. 7(a). Then draw a vertical line through the point $x = 0$ on the horizontal axis (the point $x = 0$ is called the **origin** and is usually denoted by O). This vertical line is your **y-axis** (or the axis for x^2). We now choose a suitable scale on this axis so that all values between

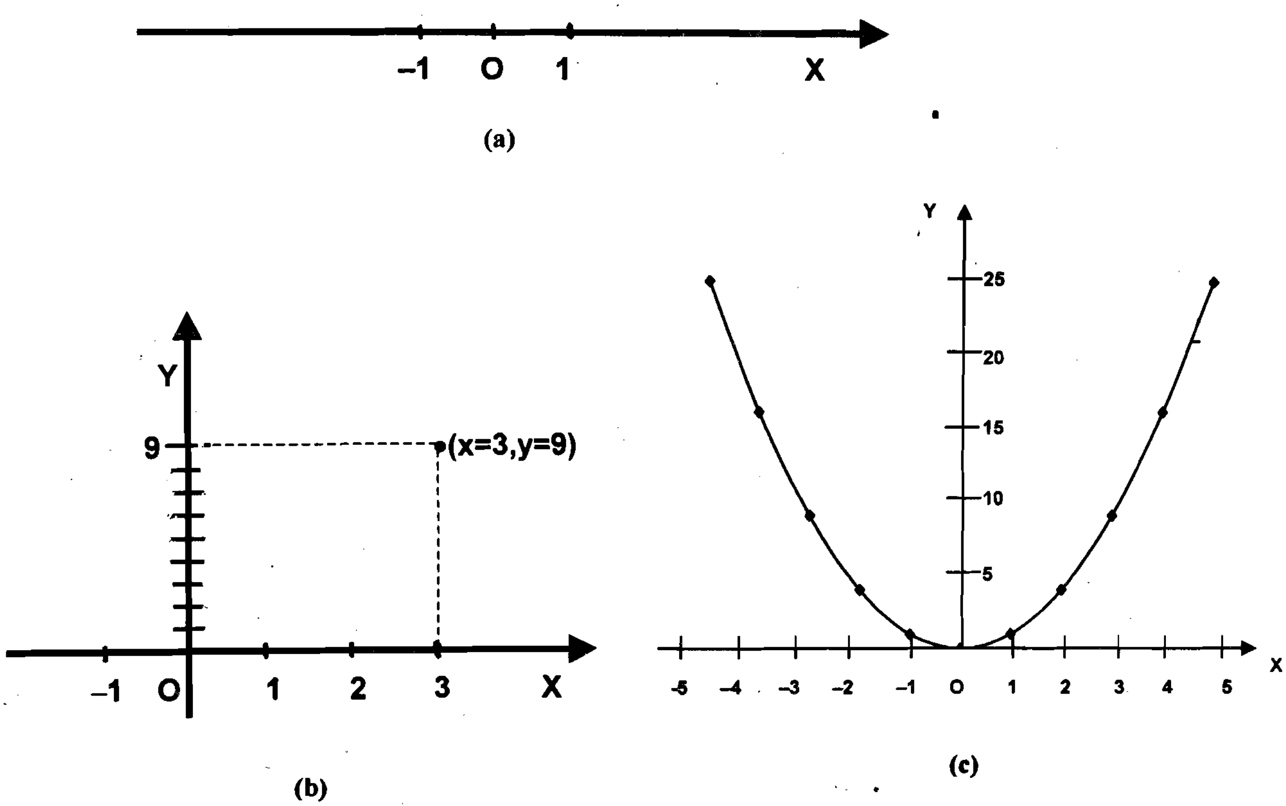

Fig.7: (a) The x-axis, (b) plotting the points, (c) graph of $y = x^2$ for x lying between –5 and +5

0 and 25 (the range of values of x^2 for values of x lying between –5 and +5) can be accommodated on the graph paper in the vertical direction. Now for each value of x find the value of x^2 from your table and mark it on your graph paper. In Fig. 7(b) this is shown for $x = 3$ when $x^2 = 9$. Draw a smooth curve through all the points you have plotted and you will have made a graph of the expression x^2 as a function of x, for x lying between –5 and +5 (see Fig.7(c)). This will be a parabola.

Notice that the scale along the y-axis is much smaller than the scale along the x-axis in this case. This is done only for convenience, to accommodate the graph in a reasonable space.

Now, do you see any symmetry in the graph in Fig.7(c)? What happens if you fold it about the y-axis? The two halves of the graph coincide. The vertical axis of the graph is, therefore, an axis of symmetry. What does this mean in terms of the values of x? When we fold the paper, the positive values of x geometrically coincide with the negative values of x. So, for a given value of x as well as for the negative of this value, the corresponding values of x^2 are the same. So, the algebraic operation of replacing x by –x is equivalent to folding the geometric representation of the equation about the vertical axis. Thus, the symmetry of x^2 in algebra is directly related to the geometrical symmetry of the corresponding graph.

Try an exercise now.

E4) Do the exercise you did with $y = x^2$, but now apply it to $y = x$. What is the graph you get? Is it symmetric with respect to the vertical or the horizontal axes? Why?

In this section we have seen several examples of symmetric figures. We shall now develop a more formal view of symmetry.

19.3 KINDS OF SYMMETRY

We have seen earlier that a plane figure is symmetric if we can shift it or reflect it in a mirror in such a way that it still looks no different from itself in its former position, i.e., it leaves the figure congruent to itself. This is one kind of symmetry. Let us consider this and other kinds of symmetry, one by one.

19.3.1 Reflection Symmetry

As we have pointed out, a plane figure is said to have **reflection symmetry** if we can place a plane mirror in such a position that the part of the figure which is hidden by the mirror and the reflection of the rest of the figure coincide exactly. A figure may have several reflection symmetries. For instance, try the following activity.

Activity 2 : Consider an equilateral triangle. Draw such a triangle on a sheet of paper, or cut out an equilateral triangle from a card. Take a small plane mirror. Hold it perpendicular to the plane of the paper in such a manner that the portion of the triangle that is not hidden by the mirror, and its reflection in the mirror, together give the appearance of the whole triangle. In such a position the plane of the mirror is called **a plane of reflection symmetry**, and the bottom edge of the mirror is called **an axis of reflection symmetry** of the triangle. You can check that there are actually three such axes for an equilateral triangle. One such position is shown in Fig.8 along with a plane drawing in which these three axes are indicated by dashed lines.

How did you like the triangle activity? Try it with some children around you, and get them to talk about what they understand about reflection symmetries from it.

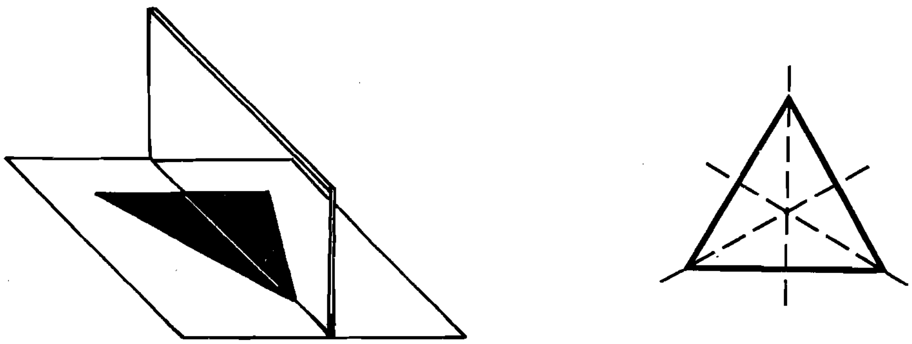

Fig.8: The reflection symmetries of an equilateral triangle

Notice that all points that lie along the axis of reflection, remain unchanged whereas every other point is reflected to a point that is an equal distance behind the axis of reflection. Notice, also, that as a result of a reflection the relative distance between any two points of the figure remains unaffected, i.e., the shape of the figure remains unchanged.

Now try the following exercise yourself and with children of Classes 5 and 6.

E5) How many axes of reflection symmetry does an isosceles triangle have?

E6) Using a mirror, check that a square has 4 axes of reflection symmetry. What about a rectangle and a circle?

E7) Consider the letters in Fig.9.

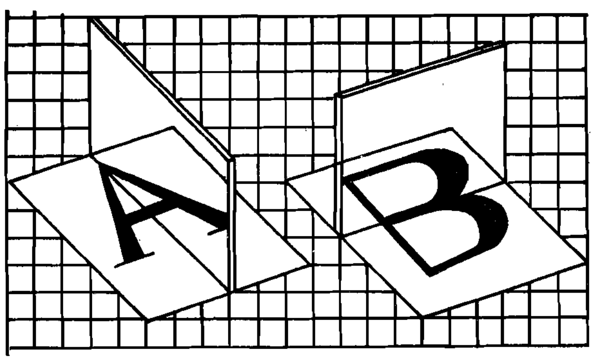

Fig.9: Reflection symmetries of some of the letters of the English alphabet

The letter A is clearly symmetric with respect to reflection in a mirror placed along the **vertical axis** of the letter. The letter B is also clearly symmetric with respect to reflection in a mirror placed along a **horizontal axis** passing through its middle.

i) List two other letters in the alphabet which share this property of A. Which letters share the property of B? Which letters have both these symmetries?

ii) Give an example of a letter that is neither symmetric under reflections about a vertical axis nor about a horizontal axis, but is symmetric if these transformations are applied one after the other.

iii)

iii) Find the two letters that transform into each other when reflected about a horizontal axis.

Let us now consider another type of symmetry.

19.3.2 Rotational Symmetry

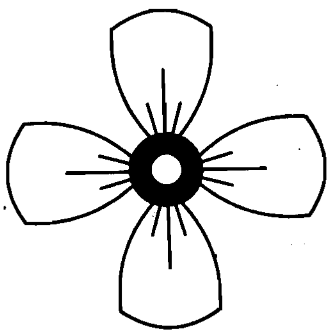

Fig.10 : How many rotational symmetries does this figure have?

Let us start by considering the 4-petalled flower shown in Fig.10. If you rotate it through 90°, without shifting its centre, what happens? It looks exactly the same as before. So, after this rotation, we get the same figure, as if it had never been moved. This is an example of a rotational symmetry of the figure. Can you find any more rotational symmetries of this figure? What about rotation through 180°, 270° and 360°? From this example you may have already gauged what a rotational symmetry is.

To put it formally, a plane figure is said to possess **rotational symmetry** if rotation through some angle about an axis perpendicular to the plane of the figure leaves the figure congruent to itself.

In the example in Fig.10, we found that rotating through 270° gives us a symmetry. Why didn't we say – 90°? Merely convention! To avoid any confusion, the angle of rotation is always measured in the anti-clockwise direction. Also, you would have found that a rotation through 360° would be the same as a rotation through 0°. Such a rotation is of course the same as not moving the figure at all. We call this the 'identity' or the 'do-nothing' transformation. **Every figure, irrespective of its shape, is symmetric under a rotation of 360° (or 0°).**

You may like to try finding rotational symmetries of some objects now. You may also like to do these tasks with your pupils.

An equilateral triangle has three-fold rotational symmetries.

E8) i) Use the equilateral triangle you had cut out in Activity 2 to verify that it has 3 rotational symmetries about an axis passing through its centre and perpendicular to its surface. How would you show these in a diagram?

ii) Check that a rectangle (which is not a square) is symmetric only under rotations through 0° and 180°, i.e., it has a two-fold symmetry under such rotations.

E9) How many rotational symmetries does a circle have about an axis perpendicular to its plane and passing through its centre?

E10) i) Which letters of the alphabet are symmetric with respect to rotations through 180° about an axis perpendicular to its plane?

ii) Are these the same letters which have reflection symmetries only when both vertical and horizontal reflections are applied one after the other? (See your response to E7(ii).)

iii) What does your response to (i) and (ii) above suggest? Explore this further.

So far we have considered two kinds of symmetry. These are the main kinds of symmetry for 2-dimensional figures. There is a third kind, which we shall now look at.

19.3.3 Glide Symmetry

Consider the strip showing footprints in Fig.11.

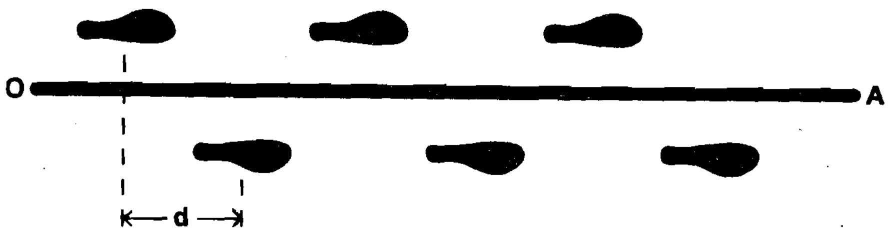

Fig.11 : An example of glide symmetry

If you shift it horizontally through a distance d and reflect it in the central line OA shown, you will get the same strip. This is an example of a **glide symmetry**. In this example we have combined a translation (a shift in one direction) along an axis followed by a reflection about the same axis. Notice that, whenever a translation is involved, the pattern or planar figure has to be infinite. (Why?)

Try some exercises now.

E11) In Fig.12 below, we show two examples of patterns with glide symmetry. Please make two more designs of your own that show glide symmetry. **Also** give two examples of patterns that **don't possess** glide symmetry.

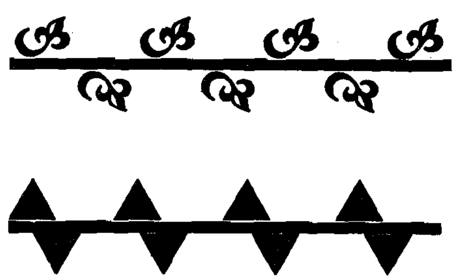

Fig.12

E12) Can a translation along a line be a symmetry of a planar figure? If so, what kind of figure would have such a symmetry, called a **translation symmetry**?

Let us now use our knowledge of symmetries of plane figures to explore the various types of patterns we see around us.

19.4 PATTERNS

Consider any pleasing design that is made by repeating the same motif in different positions on a plane surface. What kind of symmetries does it have? Does it show a translation symmetry, glide symmetry, or any other symmetry? Based on the answer to this, the pattern can be classified as a **border or a wallpaper**. Let us see how.

19.4.1 Borders

Consider the pattern given in Fig.13. It consists of a design repeated at regular intervals along a line and **extended indefinitely** on both sides. Any such pattern is called a **border** (or a **frieze pattern**). A border can be on the floor, on a wall, or on the edge of a sari. Every border has to have translation symmetry along its axis — its translation axis. The design whose repetition gives rise to a border is called the **motif** of the border.

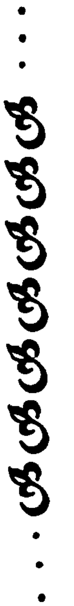

Fig.13 : A border whose axis of translation symmetry is vertical.

Now, think of all the borders you have seen. They may seem to be thousands of different kinds of borders. In fact, an infinite number of borders are possible even when working with a single motif. However, **all borders fall into only one of seven symmetry classes**. Do you accept this statement? Not without valid proof, I hope! Let us spend some time on trying to prove or disprove this statement.

We have just said that every border must possess an axis of translation symmetry. This translation can be combined with a reflection or rotational symmetry in some cases. However, not every combination is allowed. Suppose we wish to combine a reflection with the translation, then only those reflections are allowed that leave the axis of translation unchanged. Therefore, the axis of reflection can only be either parallel or perpendicular to the axis of translation. Reflection about any other axis would change the direction of the original axis of translation. For instance, consider the pattern in Fig.14. This has a reflection symmetry with respect to OB, but not with respect to OA.

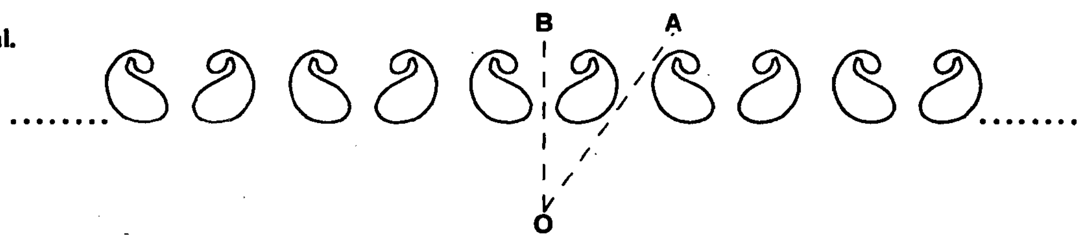

Fig.14

Similarly, only those rotations are permitted which leave the axis of translation coincident with itself after the rotation. Thus, only an axis of rotation that is perpendicular to the plane of the border and passes through the axis of translation is allowed. The further restriction is that all rotations about such an axis can only be through an angle of 180°, because after rotation the direction of the axis of translation has to remain unchanged.

For the same reason, borders can possess only such glide symmetries as lie along the axis of translation.

So, in all, we can only have borders that possess the following symmetries:

(1) only translation symmetry,
(2) only glide symmetry along the axis of the border,
(3) translation with a reflection whose axis is along the axis of the border,
(4) translation with a reflection whose axis is perpendicular to the axis of the border,
(5) translation with a rotation through 180° of the kind described above,
(6) translation plus a rotation as in (5), along with a reflection along the axis of the border,
(7) translation plus a rotation as in (5), along with a reflection perpendicular to the axis of the border.

Check whether we have exhausted all the possibilities. Can there be any other combination of symmetries? Try some out. You will find that any such combination will be equivalent to one of the seven we have listed above. Please remember that when we talk of the symmetry of the border, it is the symmetry of the whole border. Thus, when we say that a border belongs to a type that combines translation with a reflection of the kind in (3) above, the meaning is that the whole border must remain unchanged if we were to carry out such a reflection. The seven types of borders possible are shown in Fig. 15 using a simple wave-like motif.

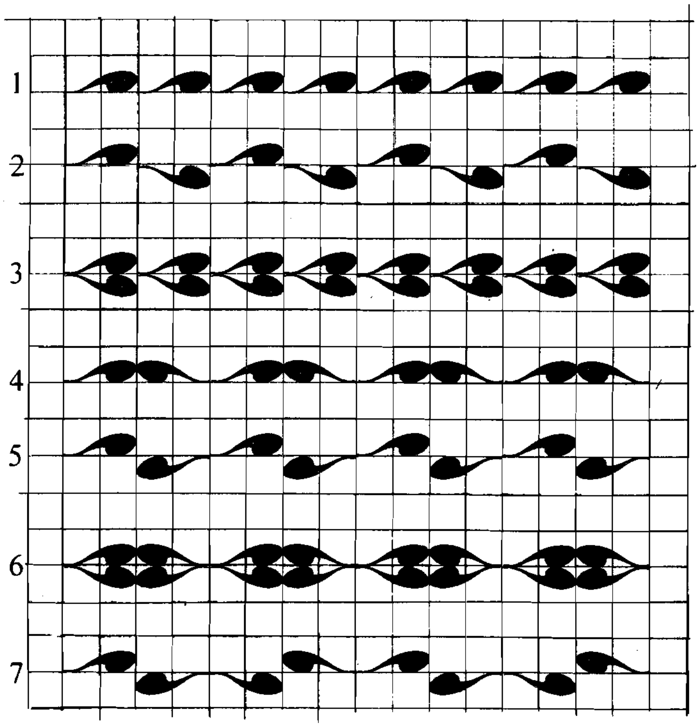

Fig.15 : The seven possible types of borders

You can see how this single motif when repeated in different ways gives rise to borders that look so different from each other.

All this analysis should help you to spot the basic symmetry of any border that you come across and also help you in designing borders of your own using different motifs. This is what the following exercise is about, which you should also do with children of Class 3 or older children.

E13) i) Look around you for at least six different examples of borders. Copy them down in your notebook. Try and determine the symmetry type of each of the borders you have copied. You may need to consult Fig.15 repeatedly in order to do this.

ii) Design the seven types of borders corresponding to a motif different from the one given in Fig.15.

Let us now consider the other kind of pattern that is possible.

19.4.2 Wallpapers

Consider the pattern given in Fig.16.

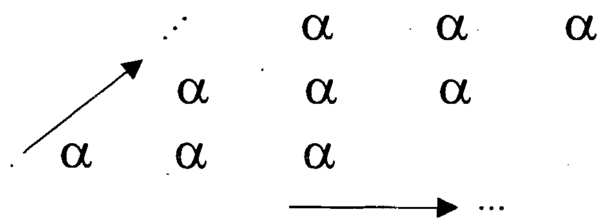

Fig.16

The arrows in Fig.16 indicate that the basic motif α is continuing in the two directions shown by the arrows. Since this pattern continues in more than one direction, it is not a border. We call such a pattern a **wallpaper pattern**. The name comes from the practice in western countries of covering walls with paper that usually has patterns on it.

It has been proved that there can only be 17 different types of wallpaper patterns, depending on the symmetry they exhibit. We shall not be presenting this proof here, but you may like to try your hand at making different patterns of your own. See the variety that you can manage. In Fig. 17 we show a number of patterns made from different repeated arrangements of a triangle. See if you can use repetitions of a triangle to generate some other patterns.

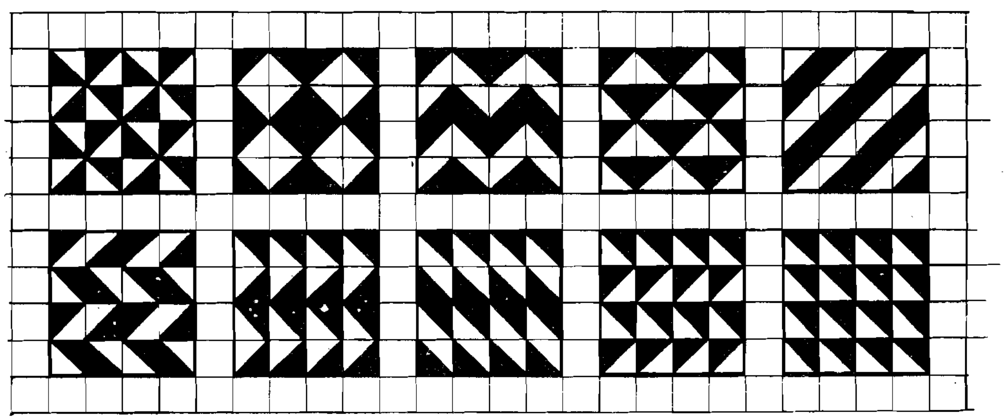

Fig.17

If we replace the basic motif of a wallpaper pattern (the triangle in Fig.17), by a dot, we get rows of dots repeated in parallel rows. This is called a **net**. A net, in fact, is the framework on which any wallpaper pattern can be built. It represents the mode of repetition. For instance, the underlying net of the wallpaper pattern in Fig.16 is given in Fig.18.

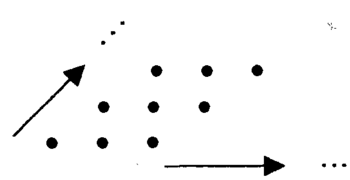

Fig.18 : A net corresponding to the wallpaper pattern in Fig.16

Note that the pattern extends to infinity in the directions shown, though we are only showing a finite portion.

Why don't you do an exercise now?

E14) What are the different kinds of symmetry you see in the different wallpaper patterns given here and in those created by you?

It is time now to end this unit on symmetry here. We have come a long way from where we started, and I hope your learners do the same. Let us take a brief look at what we have covered in this journey.

19.5 SUMMARY

In this unit we have covered the following points.

1. We have discussed what symmetry means, both from a geometric and algebraic point of view.

2. We have considered the four basic types of symmetry that exist. All other symmetries are combinations of these. Through many examples, we have looked at the symmetries of 2D objects.

3. We have explained a mathematical understanding of pattern. Based on the way the motif of a pattern is repeated, it is either a border or a wallpaper pattern. We have seen why there are only 7 types of borders. We have also mentioned that there are 17 types of wallpaper patterns based on the types of symmetry that the pattern displays.

4. We have stressed again and again that you should carry out the activities and exercises suggested here with your learners. This will help them develop their mathematical understanding.

19.6 COMMENTS ON EXERCISES

E1) What was their understanding of symmetry? If they were from different age groups, how did their comments and observations differ? What else did they comment about the patterns?

E2) It will be interesting to note the kind of errors that show up in the categorisation. Talk to the children to find out from them the understanding that led to these errors.

E3) What about the letters Y and Z?

E4) The graph you get now is given in Fig. 19, a straight line. Give reasons to explain why it is not symmetric with respect to either of the coördinate axes. Does it have any rotational symmetries?

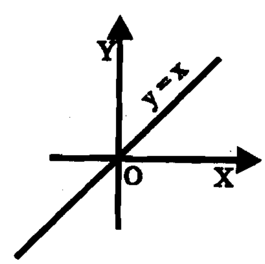

Fig. 19

E5) It has one axis. Show it in a diagram.

E6) Draw a square on a sheet of paper. Using a plane mirror, locate its axes of reflection symmetry. Verify that in addition to the two diagonals, there are two other such axes that pass through the centre of the square with the mirror parallel to a side of the square. Two of the positions in which the mirror can be placed are shown in Fig. 20 together with a plane drawing in which all four axes are indicated by dashed lines.

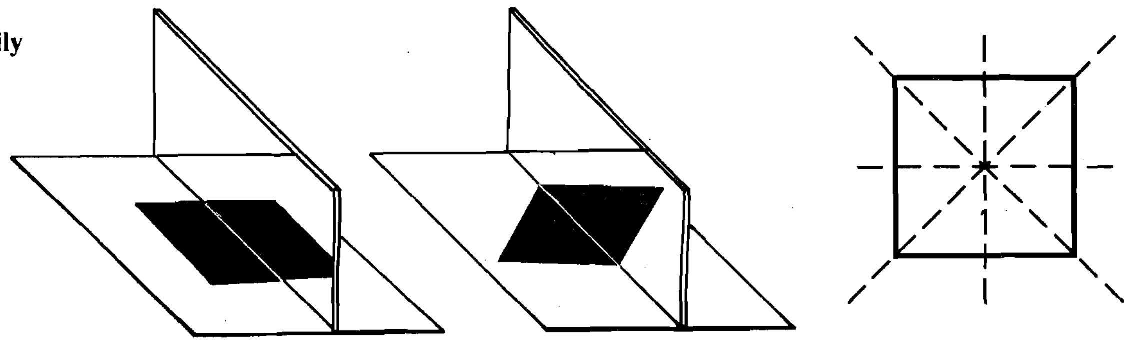

Fig. 20 : Reflection symmetries of a square

E7) i) O shares both the symmetries. What about C and H?
ii) What about N?
iii) W and M, for one.

E8) i) Check rotations through 120°, 240° and 360°.
ii) What about rotation through 90° and 270° ? Will these give symmetries of the figure?

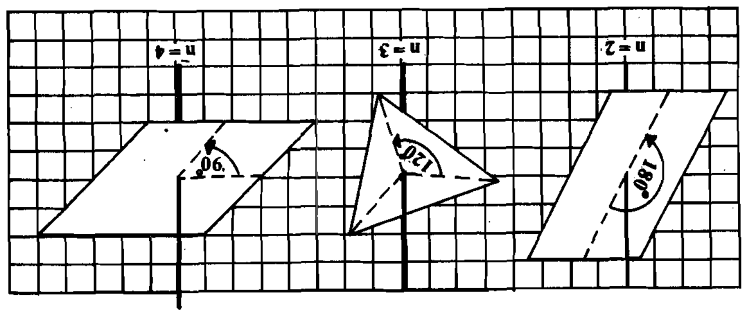

Fig. 21 : Some rotational symmetries of a rectangle, an equilateral triangle and a square.

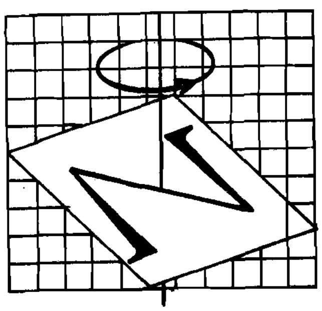

Fig. 22

E9) Why are these infinitely many? Also find the number of reflection symmetries a circle has.

E10) i) O certainly is. List the others.
iii) Are there any letters that satisfy (ii) and not (i)? Apart from letters, can you generalise your observations to any 2D figures satisfying (i) and/or (ii)?

E12) Suppose you take an infinite pattern, as shown in Fig. 11. Shift it through a certain distance horizontally so as to get the same pattern. This shows that this infinite strip has **translation symmetry along the horizontal axis**.

Can any finite figure have such symmetry?

E13) Before giving these exercises to the children to do, give them an appropriate background to be able to do them.

The use of colour as part of the design can, in fact, enhance the beauty of the border and the children's interest. Using some of the borders that they have designed, ask the children to colour them in different colours, using some symmetrical arrangement in the way they apply colour to their design.