UNIT 18 EXPLORING MATHEMATICS

Structure **Page Nos.**

18.1 INTRODUCTION

In the previous unit we elaborated on the chief processes involved in mathematical thinking, namely, abstracting and generalising. In this unit we focus on the use of these processes and other processes involved in 'doing mathematics'.

We start with a section in which we discuss the different thought processes involved in exploring mathematical problems. In the next two sections, we carefully observe these processes through exploring some mathematical problems in geometry. Finally, we look at the use of mathematical puzzles for developing these processes.

While you are investigating all these areas, we expect you to focus on the thought processes involved because these are the processes that your learners need to develop. Therefore, while studying this unit, keep thinking about how you can foster these processes in your learners' minds.

What we are doing in this unit, carries on in the next one through a discussion on symmetry of two-dimensional shapes.

Objectives

After reading this unit, you should be able to

- explain the mathematical thinking involved in problem solving and other mathematical explorations;
- suggest ways of generating mathematical thinking in your learners;
- design and carry out activities to help your learners investigate the polyhedra and tilings;
- create mathematical puzzles that challenge, but not over-challenge your learners.

18.2 THE PROCESSES INVOLVED

Let us start this section with a brief look at what 'doing mathematics' means to most children and teachers. The common view is that mathematics is 'done' only in the maths class. During this class, the children are expected to learn to apply the basic operations so that they can solve computations quickly and correctly. Then they are expected to solve word problems related to those operations. The procedure involved is that first the teacher explains 'the way' to solve a particular kind of problem. Following this she gives the children many problems of the same type to solve on exactly the same lines. So, solving a problem is reduced to listening to the teacher, memorising certain solutions and mathematical facts and reproducing them appropriately.

Where is the mathematical thinking being developed in the whole process outlined above? It is certainly important to have basic computational skills, but not by rote. In Block 5 we discussed why it is important for a learner to understand the mathematics involved in an algorithm.

Apart from algorithms, to understand any mathematical concept a child needs to be introduced to it through familiar situations and experiences. In order to make her schema of the concept more elaborate, it is important to use the schema on different occasions in as many ways as is possible. The exposure to a variety of problems related to the concept helps her to deepen her understanding of the concept. For the child (or for us) this helps to interlink the different schemas, which, again, helps to strengthen our conceptual understanding. When the child has to think about what to do and how to do it, she is forced to examine the concept seriously, and hence extend its meaning for her. For those whose concepts are half-formed or are erroneous, solving different problems gives an opportunity to discover the errors and to reach a better understanding of the concept. For example, when helping a child to develop the idea of a triangle, we need to give her an opportunity to identify triangles from non-triangles, use triangles in various ways, allow her a chance to use a variety of triangles, etc. In short, concept formation is linked to the opportunities available to the learner to think, apply her understanding and use her conceptual structures in various ways, finding relationships with other concepts.

Now, try the following exercises.

E1) Give an example of a question given to children which cannot be called a problem. Also give your reasons for your choice.

E2) Explain, with examples from your own learning of mathematics, how solving problems helps in concept formation.

While trying these exercises, you must have focussed on the essential characteristics of doing mathematics — it must be an opportunity for the learner to think mathematically, choosing which step to take, based on what she knows and where she wants to reach. If she is expected to solve a problem, it should not require her to merely reproduce information or mechanically apply algorithms. She needs to, gradually, be exposed to more and more complex problems built around the concept. She needs opportunities and encouragement to **tackle open-ended problems or problems with many solutions related to the concept.** The problems can steadily become more challenging. At each stage, **she should also be encouraged to talk about what she is doing** and her line of reasoning.

Let's see the abilities developed in the process, by doing some problem-solving. (At this point, you may find it useful to **re-read Sec.6.3**, where we have discussed the role of representation in developing mathematical understanding.)

Consider the following problem :

A company makes 100 computers every month. Its employee union accused the company of discriminating against its female employees. The union said that *women were not being given the promotions due to them. The following table gives the data about the promotions in the company.*

Year	*No. of women promoted*	*No. of men promoted*
1996	*5*	*15*
1997	*6*	*16*
1998	*10*	*8*
1999	*8*	*10*
2000	*8*	*10*

If an employee who is promoted during these five years is selected at random, what is the probability that the employee is a woman? Is this data enough for deciding whether female employees are discriminated against?

Now, solve this problem. While doing so, try and separate out the various thought processes you used for doing so.

When I tried the problem, I first tried to understand the situation — **what I knew**, and **what I needed to find out**. Then I needed to think of **the path to use** to move from what I knew to what I needed to find out.

I also needed to know **which information, if any, was extra and not required.** For instance, what the company produced is irrelevant to the problem.

The next step was to write down the mathematical equivalent of the given problem :

Total number of women promoted from 1996 to 2000 = 37
Total number of people promoted in this period = 37 + 59 = 96
To find P(A), where A is the event that a woman was promoted.

Then I solved this problem using the definition of probability of an event, that is,

$$P(A) = \frac{37}{96}.$$

So, I concluded that approximately 1 in 3 promotions is likely to be that of a female worker. However, this probability gives us no indication of whether women workers are discriminated against. This is because we need some more information. For instance, we need to know how many men and women were eligible for promotion in this period.

Let's see the steps involved in solving this problem and other problems you have worked on.

1. Read the problem carefully to understand what it says — the information and assumptions in it, and what is to be found out, proved or examined.

2. Represent it mathematically, clearly filtering out the irrelevant data in the problem.

3. Gather other relevant information axioms and earlier proved (or known) results.

4. Look for a path for solving the mathematical equivalent of the problem.

5. Interpret the solution in the problem situation.

Why don't you do an exercise related to this?

E3) Give some children a problem to do. After they have solved it, talk to them to find out which of the stages above they went through. Note down what they articulate. If you can get them to discuss the stages, note down what comes out in their discussion.

Problem solving is one important part of doing mathematics. An equally important part is what further questions open up in our minds while solving a problem. To understand this aspect and other processes we have discussed in this section, here is an opportunity for you to investigate some mathematical areas.

18.3 INVESTIGATING PLATONIC SOLIDS

In this section we invite you to explore the processes involved in working mathematically through a study of polygons and polyhedra. Over here, we assume that you are familiar with polygons (see Unit 17). So, let's start with an exercise.

E4) How many different regular polygons are there? How are these polygons related to a circle? Note down the kind of questions that come to your mind while you are trying these questions.

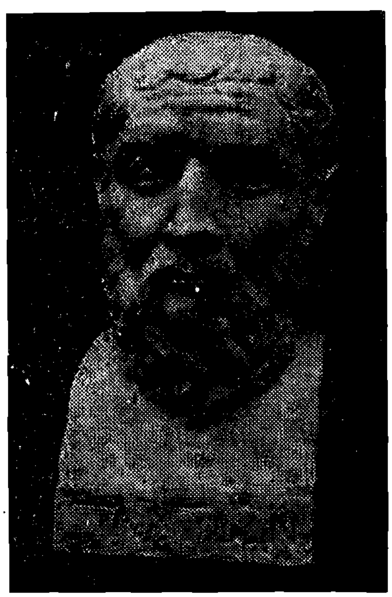

Fig.1 : Bust of Plato (427-347 BC)

While you were doing E4, what did you notice about the way you deal with mathematical problems? Once you have solved it, do you think your schema of 'polygon' has been elaborated? In what way? Did you think about other related mathematics questions that could be explored? One problem that you may have thought of exploring could be : Can what I have found true for 2D be generalised to 3D?

When we go from 2-dimensional figures to 3-dimensional objects, the concept of regular polygons generalises to regular polyhedra (the piural of **polyhedron**). **Regular polyhedra** are solids in which all angles and all sides are equal, for example, a cube.

Now, while doing E4 you must have found that there are infinitely many regular polygons because there is no limit to the number of sides they can have. So, you may expect the same about the regular polyhedra. However, there are **only five different regular polyhedra possible**. These are the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron (see Fig.2). These are also known as the **Platonic solids**, after the Greek philosopher Plato. They have fascinated mathematicians from the time of the ancient Greeks. The faces of the tetrahedron (4 faces, from the Greek word 'tetra', meaning four), the octahedron (8 faces, from 'okto' meaning eight) and the icosahedron (20 faces, from 'eikosi' meaning twenty) are all equilateral triangles. The cube has 6 faces, all of which are squares. The 12 faces of the dodecahedron ('dodeka' meaning twelve) are regular pentagons. It is worth noticing that the faces of all the regular polyhedra are regular polygons. This follows immediately from the requirement that all the sides of a regular polyhedron must have the same length.

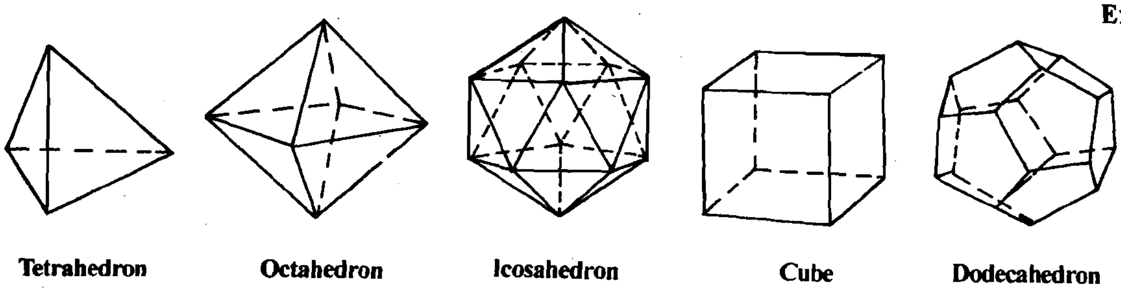

Fig. 2: The five regular polyhedra (the hidden edges are indicated by dashed lines)

What we have just discussed is easy for children to get interested in. To give children a feel for what the five Platonic solids actually look like, there is nothing better than having models of these solids for them to play around with. As solid models are not easy to come by, it is a good idea to get children to make models of these solids from paper. With this in mind, in Fig.3 we have given flat diagrams of the five regular polyhedra.

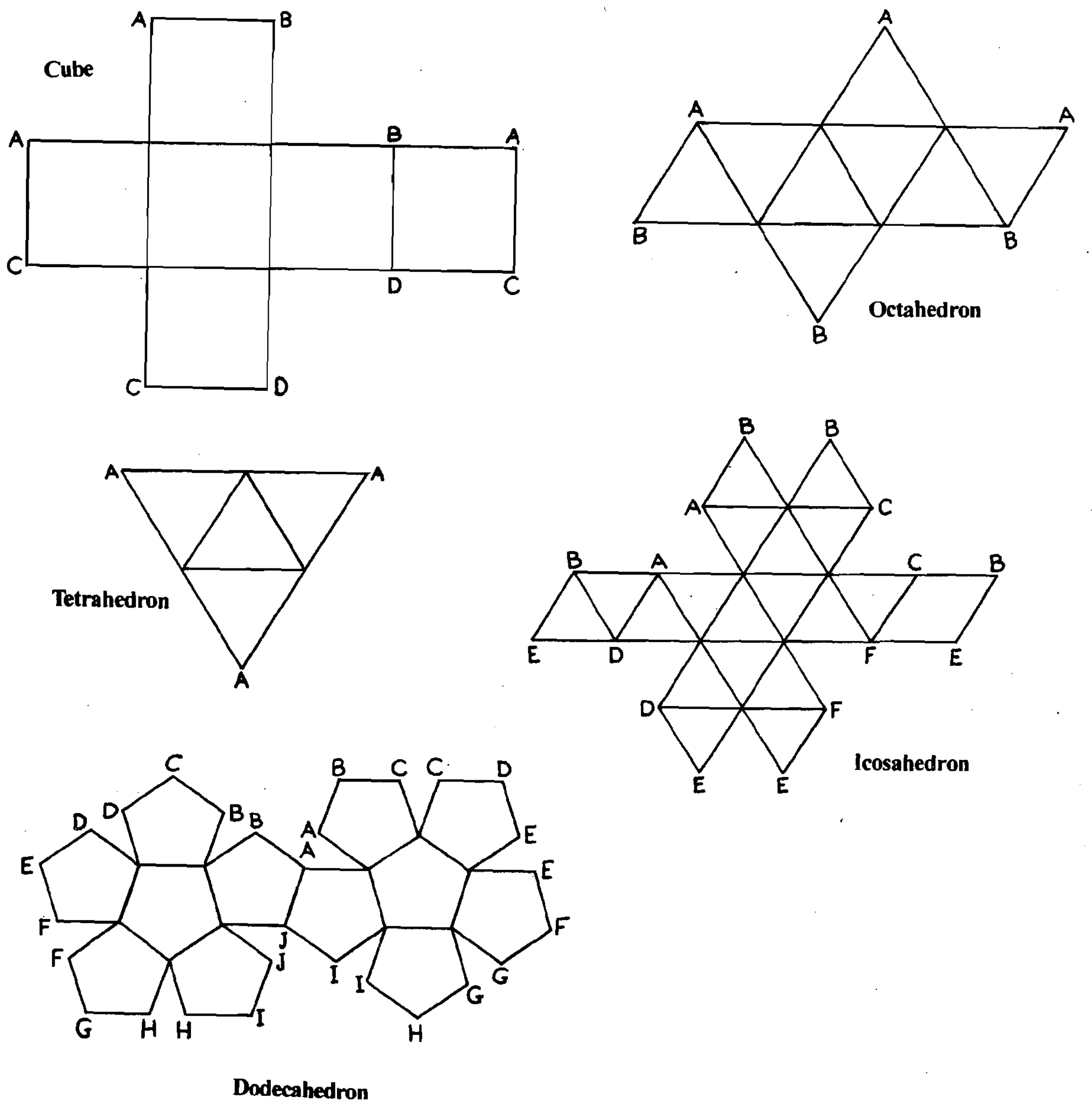

Fig. 3: Cut-outs to make models of the five Platonic solids

These could be copied on to some stiff paper and cut out along the outer edges. These can then be folded along the inner lines and the sides pasted with thin strips of paper to make three-dimensional models. Those corners of the regular polygons that make up the faces of the models and which meet at a common vertex of the polyhedron are labelled with the same letter in our figure. The tetrahedron, the cube and the octahedron are not too difficult to construct, but children may need some help and guidance in making the dodecahedron and the icosahedron.

Now, getting back to exploring mathematics, here is an exercise for you.

E5) Show that there can only be 5 regular polyhedra.

How did you go about solving E5? Of course, you know that there are **at least** 5 regular polyhedra, the ones made in Fig.3. But, how do you know that **these are all**? That is, how did you go about showing that any regular polyhedron is forced to be one of the five you know? Did you try to use anything you already know or have observed? For instance, did you notice that at any vertex of a polyhedron there cannot be less than three faces? One face is clearly not enough and two would only give rise to an edge.

Secondly, did you observe that the sum of the angles of all the faces at each vertex must be less than 360°? If the sum were exactly 360°, then all the faces would lie in one plane and there would be no corner of a 3D solid. Also note that each face has to be a regular polygon.

Can **any** regular polygon be a face of a polyhedron? Since the sum of the angles of the faces at each vertex of the polyhedron has to be less than 360°, and since there must be at least three faces at each vertex, the angle of a face at the vertex must be less than 120°. This immediately restricts the regular polygons that can form faces of the regular polyhedra to be equilateral triangles, squares or regular pentagons (see E7 in Unit 17). Therefore, a regular polyhedron can only have these 3 types of polygons as faces.

Now that you have reached this stage, you are probably thinking about the various possibilities for the regular polyhedra. The simplest case is that of a regular polyhedron whose faces are equilateral triangles. We have already used the fact that the number of faces at each vertex must be more than two. They must also be less than 6, since each angle of the face is equal to 60°. The number of faces at each vertex of a regular polyhedron whose faces are equilateral triangles can therefore only be 3, 4 or 5. These correspond to the regular tetrahedron, octahedron and icosahedron, respectively.

Now consider the case of regular polyhedra whose faces are squares. The number of faces at each vertex can only be 3. (Why?) It has to be more than two and less than four since each angle of a square is equal to 90°. The corresponding solid is, of course, the cube.

By exactly the same arguments the number of faces at each vertex of a regular polyhedron whose faces are regular pentagons can only be 3. The corresponding solid is the regular dodecahedron.

You should verify that there are no other possibilities. Once this is done, you have proved that there can only be 5 kinds of regular polyhedra.

Now try these exercises.

E6) Go back to the discussion on ' proof ' in Unit 17. Then, note down the mathematical thought processes and the kinds of statements used in the proof above. Under which of the four stages of a proof listed on P.12 of Unit 17 do they come? Are there any other stages or categories in the proof above that are not mentioned in Unit 17?

E7) We list some of the properties of the five regular solids in the table below. Use the paper models you have made to verify the entries in the table for each of the five regular solids.

Table 1: Properties of the regular polyhedra

Type of polyhedron	Faces are *n*-gons *n*	Number of faces F	Number of vertices V	Number of edges E	Number of faces at each vertex
Tetrahedron	3	4	4	6	3
Cube	4	6	8	12	3
Octahedron	3	8	6	12	4
Dodecahedron	5	12	20	30	3
Icosahedron	3	20	12	30	5

There is a relationship between F, V and E. Find it.

E8) Try the activity in E7 with your learners (of Class 4 or 5) and also with children of Class 8. What were their responses, mathematical and otherwise, to it? How, and why, did the responses of the two sets of children differ?

Let us now explore another area of spatial mathematics. While you are investigating it, keep thinking about the same broad questions that you kept in mind in the previous section.

18.4 STUDYING TILINGS

'Tiling' is the study of shapes that can be placed alongside each other to fill space completely **without leaving any gaps**, like the tiles covering your floor. If you look around you, you will see a variety of tilings — on floors, on walls, decoration pieces, etc. An example is given in Fig.4.

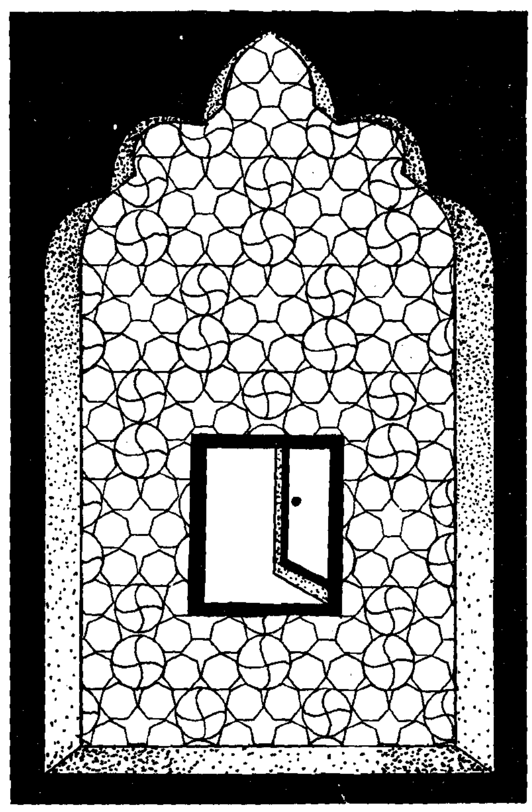

Fig.4

What are the shapes that are usually used as tiles to fill the tilings? In two dimensions, we usually find squares or rectangles used as tiles. If a tiling is done by one kind of regular polygon of the same shape and size, it is called a **regular tiling**. Do you see regular tilings around you? The most common kind is the one made by squares.

What are the other kinds possible? Here is an exercise about this now.

E9) Prove that the only regular tilings are those made up of the equilateral triangle, the square and the regular hexagon. Further, note down the points you reflect on, the questions you ask yourself and the different routes you may follow while finding the proof.

How did you go about trying the exercise above? Did you physically take several equilateral triangles, say, and try to cover a surface with them? While doing so, did you notice that at any intersection in a tiling there must always be more than two tiles meeting? This concrete example may have also helped you realise that the sum of the angles of all the vertices meeting at an edge must be 180°. The sum of the vertices of the polygons meeting at other points will be 360°. This means that we can only have three equilateral triangles (or two squares) meeting at an edge. Also, we can have 6 equilateral triangles, 4 squares or 3 regular hexagons meeting vertex to vertex. This exhausts all possibilities for regular tilings. Therefore, there are only three regular tilings, all of which are shown in Fig.5.

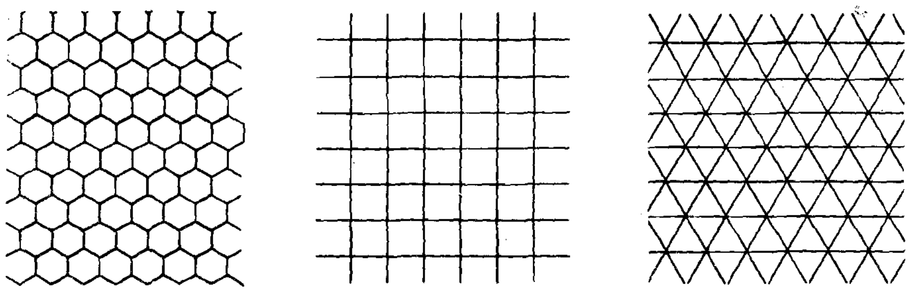

Fig. 5 : The regular tilings

If non-regular polygons are used in any tiling, then of course the possibilities are endless. The same is true if a mixture of regular polygons of different sizes are used. In particular the plane can be tiled completely by triangles or quadrilaterals of arbitrary shape.

A **tessellation** is another name for a tiling, which is used by artists more than mathematicians. Tessellations use either a single shape which may or may not be regular, or at most a few shapes, to cover the plane. The emphasis is on using shapes which look natural like birds, fish, horses, people, etc., rather than pure geometric forms. Through the following activity, you can pick up some basic principles involved in creating tessellations and make some of your own tilings.

Activity 1 (Making tessellations) : You need to start by establishing a regular grid on the plane. You can use triangles, squares, rectangles, parallelograms, hexagons, etc., to create a grid which covers the whole area you wish to work with. Suppose you start with a grid of squares. You can choose as your unit a 3×3 square. We know that periodic repetitions of this unit will tile the plane. (Why?)

Now, the secret of a tessellation is to remove parts of this square from one side and add it in a corresponding position on the opposite side of the square. In this way, although the shape of the unit changes, its total area remains the same. In the process you create cuts and wedges that fit into each other. (Why does this happen?)

So, suppose you remove a small square from the top left-hand corner of the unit figure (see Fig. 6(a)) and add it to the top right-hand corner. Similarly, remove another small square from the middle of the bottom of the figure and add it to the middle of the top. This, then, produces your basic motif shown on the left-hand size of Fig. 6(b).

Consider your original grid to be tiled by a set of the basic 3 × 3 squares and replace each such square by the motif you have just created. This will produce the pattern shown on the right of Fig. 6(b). Stretch your imagination a little, and you can consider this to be a tessellation of a horse and rider!

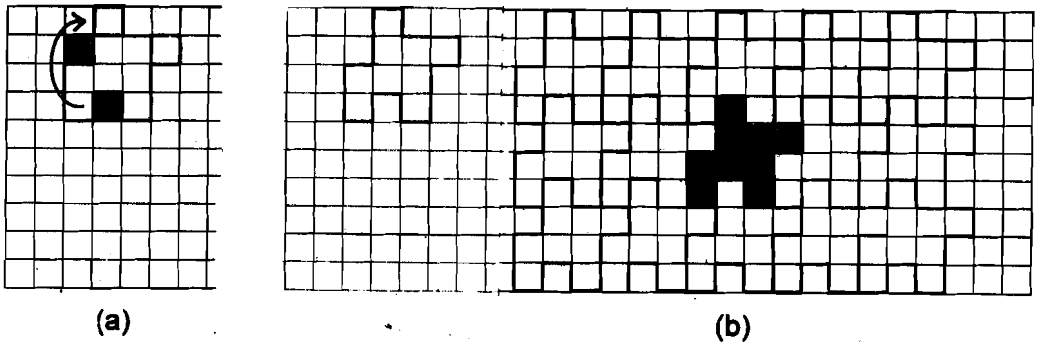

Fig. 6 : (a) Creating the motif,
(b) Tesselleting the plane with the motif to get a tessellation of a horse and rider.

To make a tessellation, we can add and remove any shape from the basic unit we choose. For example, starting with the same basic 3 × 3 tile, we can add/remove shapes as shown in Fig. 7(a). Then we get a basic motif that gives us the tessellation in Fig. 7(b).

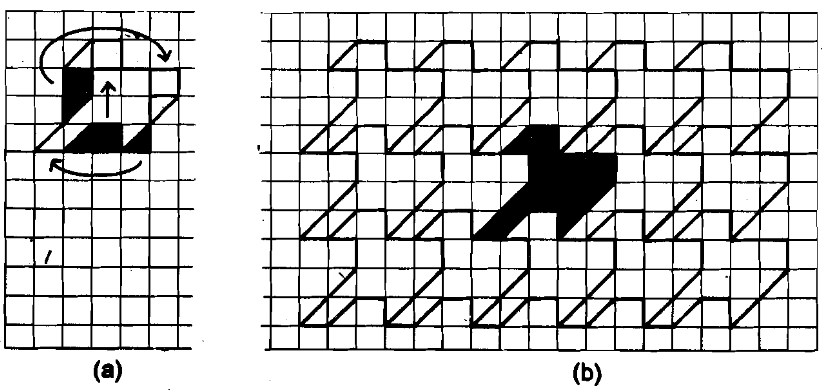

Fig. 7 : (a) A basic motif for a tessellation, (b) The tessellation of horses

The important thing to remember about creating a motif is

i) decide on the grid and the basic unit,
ii) you can remove any shape from the basic unit **provided** you add it back to the unit at the corresponding place on the opposite side to give rise to the new shape.

This can be done as many times as you please. The skill lies in creating a natural looking shape at the end. For example, in Fig.8 on the next page we show how, starting from a grid of parallelograms, you can proceed step by step to create a tessellation of roosting birds.

Why don't you try an exercise now?

E10) Create at least two tessellations using the steps we have just discussed. Also try out what we have said in this section with children of an appropriate age group. What were their reactions?

E11) Ask children of Classes 5 or 6 to tile a plane with squares and regular pentagons, respectively. Note down the discussions that take place amongst them during this activity. What understanding does this give you of their mathematical thought processes?

In the following example we have added/removed curved portions to get a tessellation.

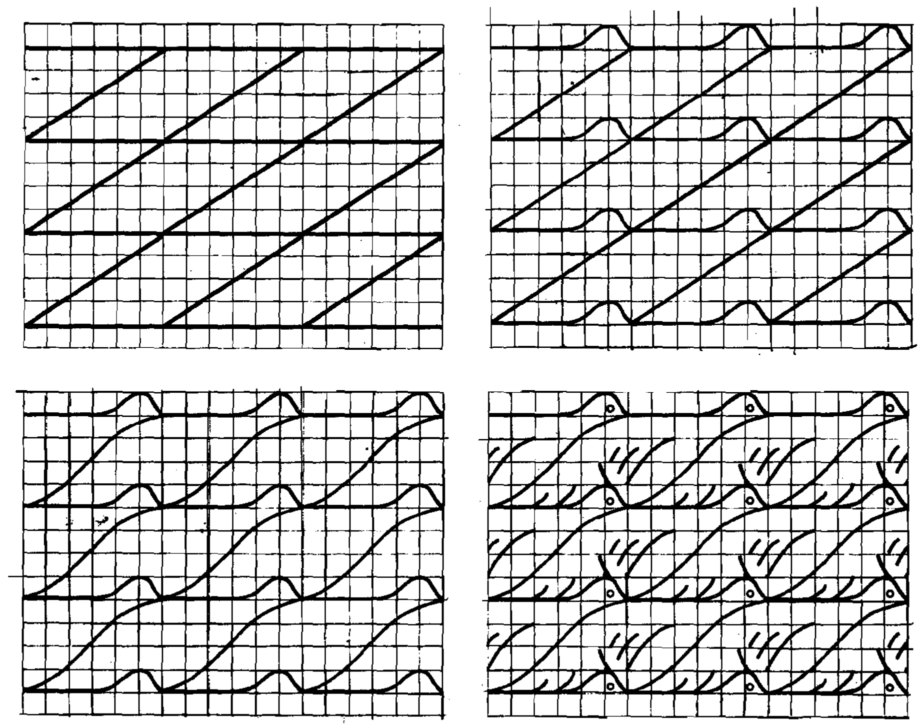

Fig. 8 : A tessellation of roosting birds

The tessellations that we have considered so far make repeated use of just one basic form. This need not be so. We can always take our repeating motif and divide it into two parts such that each part looks like a separate natural shape. Escher, who was an acknowledged master, has used many basic shapes to give all kinds of tessellations of the plane (see Fig.9).

REGULAR DIVISION OF THE PLANE I (DETAIL), 1957; WOODCUT, 9½ X 7⅞ IN.

Fig. 9 : A tessellation by M.C.Escher

While creating tessellations, there is a notion of symmetry that is involved. We shall study this notion in detail in the next unit.

So far, in this unit we have carefully looked at the thought processes we go through while learning mathematics. Both the mathematical areas we have looked at in detail in this unit have been spatial, that is, space related. Now, let us look at the processes involved in solving algebraic and logical puzzles.

18.5 WORKING OUT PUZZLES

If you are given a problem like 12345 – 3249, you are likely to do it in a flash. This is because you have acquired the ability to apply a subtraction algorithm. Now, see how long you take to do the following subtraction :

In the problem below, each letter has been assigned one digit from 0 to 9. Find the numbers involved in the subtraction

$$\begin{array}{r} ABCD \\ -\ \underline{EBEB} \\ EDEB \end{array}$$

How have you gone about finding the digits involved? For instance, you would know that one possibility is that D = 6, B = 3. Then D–B = B. But remember that D and B are occurring again in the 'hundreds' column. And $3 - 3 \neq 6$. So, you would need to try another possibility for D and B.

In this way, what solution did you get? Note that there may be more than one solution to this problem. One is (A, B, C, D, E) = (2, 5, 3, 0, 1).

Try these exercises now.

E12) i) While solving the problem above, what were the different aspects of mathematical thinking you applied?

ii) Think of two such puzzles you can give to your learners. Try out these puzzles with them. What problem solving abilities were the children using in the process, and how did you find out?

E13) Find the operation used and the digits represented by the letters in

$$\begin{array}{r} AB \\ \underline{AB} \\ ACC \end{array} \quad \text{and} \quad \begin{array}{r} FG \\ \underline{FH} \\ DE \end{array}$$

Now consider the following problem.

There is a sequence of 16 numbers which reads the same from left to right as well as from right to left. Also, the sum of any 7 consecutive numbers in the sequence is –1, and the sum of any 11 consecutive terms is +1. Find the numbers.

How are you going about this one? I started by trying out the sequence

1, –1, 1, –1, ...

This met the second condition, but not the first or the third. In this way, I tried a few more sequences till I decided to use algebra for dealing with this problem. So, using the first condition, my sequence became

a, b, c, d, e, f, g, h, h, g, f, e, d, c, b, a.

Then I used the second and third conditions to reduce the sequence to

a, a, c, a, a, a, c, a, a, c, a, a, a, c, a, a

Now, can you guess how I got a and c? Why don't you try and find the sequence? Maybe your solution agrees with mine. An answer I got was a=5, b= −13. Are there any other possibilities?

What were the mental processes that you went through while reaching a solution? Would you be going through the same stages while solving the following problem?

Ashrafi was convinced that her key had been hidden by one of her friends — Aarti, Birla, Kalyan or Megha. Each of these friends made a statement about this matter. But only one of these four statements was true.

Aarti said, "I didn't take it."
Birla said, "Aarti is lying."
Kalyan said, "Birla is lying."
Megha said, "Birla has taken it."

Who told the truth?

This problem can be solved in various ways. Of course, each way requires the use of mathematical logic.

So, let me begin by assuming that Aarti is telling the truth. Then Birla's statement is false, so that Kalyan's statement is true. But we have assumed that both Aarti and Kalyan can't give true statements. So, Aarti must be lying.

Now, let me assume that Birla is telling the truth. See if you find any contradictions with this assumption.

In this way, checking the various possibilities, moving logically step by step, I arrived at the solution. Can you see the mathematical thinking involved in this problem?

Once you have thought this out, you may like to try the following mental exercises.

E14) In a sports club, there are 20 people who play badminton and 25 who play tennis. 10 members play both sports. How many club members play at least one of these sports?

E15) A milk tank contains 1000 litres of milk. It fills packets at the rate of 50 litres per minute. Every five minutes another 50 litres is poured into the tank. How long will it be before the tank is empty?

E16) What do you think was practised by you as you tried to solve these problems?

E17) Do you think such tasks are useful for helping children learn and acquire these ideas? Give reasons for your answers.

The purpose of asking you to engage with the problems above was two-fold. Firstly, we wanted you to have fun. We also wanted to help you focus on the processes that are used for doing so. If you are aware of these abilities being used, then you would agree that these are the abilities to be fostered in your learners. One way is to give them problems that they would enjoy and that would challenge them a bit. We end this unit with leaving you to think of various ways in which this can be done.

But first, let us see what we have covered in this unit.

18.6 SUMMARY

In this unit we have focussed on the thought processes involved in learning and doing mathematics, particularly while solving and creating problems. More specifically, we covered the following points.

1. Exploring any mathematical concept involves considering it in different ways. Solving a variety of problems related to this concept helps to build and consolidate one's understanding of the concept.

2. We looked at why there are only 5 types of regular polyhedra though there are infinitely many regular polygons.

3. We discussed what a tiling is, how many regular tilings there can be and how to create tessellations.

4. We looked at interesting non-routine mathematics problems that entertain us and keep the brain ticking. The idea was to focus on the directions in which the thought processes were moving.

5. We asked you to work with your learners on the same lines and analyse their reactions. Through this, you would be able to gauge their use and understanding of mathematical thought processes.

18.7 COMMENTS ON EXERCISES

E1) While thinking of an example, remember the essence of a problem — it should force the learner to think mathematically. This does not mean merely retrieving learnt facts or applying algorithms unthinkingly.

E2) Pick an example of a concept or process you learnt, for example, the concept of LCM. Note down what kind of exercises, activities and problems helped you to elaborate your schema of the concept, and in what way. Does LCM mean more to you than merely applying the algorithm for finding it? Which practical situations require you to use it? Did finding the answer to this question help you understand this concept better? In what way?

Similarly, how would solving real-life and other problems related to a concept help your students learn the concept?

E3) Pick up a problem that requires plenty of thinking. Do not hassle the children when you ask them how they have done the problem. They may not be very clear in remembering or explaining how they solved it. You could also have them sit in a group and try to work out a solution together. Their conversation while they think about the question would help you to understand the processes that they are going through.

If you discuss the stages with them, be sure to use simple language and small logical steps so that they can understand what you are talking about.

E4) For each $n \geq 3$, we can define a regular n-gon. Try drawing these. See what happens to the shape of the polygons as n becomes larger and larger. As $n \to \infty$, the n-gon becomes a circle.

What questions regarding relationships between different mathematical objects, pattern finding and generalisation did you think of in the process?

E5) Here you have to think of 5 regular polyhedra. Then you need to prove that these are the only ones. Think of what a proof involves — first gather together what is known and what is assumed. Then see how you can use this to prove your result. The discussion following E5 will, of course, lead you there.

It may be useful, while you think, to try to actually make these polyhedra and see the implications of this concrete activity.

E6) For instance, there have to be 3 or more faces at any vertex of a polyhedron. Is this statement an axiom? Or is it based on assumptions? This statement follows logically from the definition.

Each face has to be a regular polygon follows from our definition.

In this way, consider all the other steps in the proof.

E7) The number of faces F, the number of vertices V and the number of edges E of a regular polyhedron are connected by Euler's famous formula $F + V - E = 2$.

E8) If the children are not able to make the paper models, you could make these available to them. Then ask them to study these models and fill in the table. Were they able to do it? What did they say about this exercise, while discussing amongst themselves as well as with you? What differences showed up in the quality of remarks and discussion among the older children vis-à-vis the younger ones?

E9) One route is given in the discussion following E9. Think of other routes. Compare the thought processes and steps in the different solutions. In fact, think of all the regular tilings. Is the list very long? Try covering a book (any surface) with all these tilings one by one. Did you find a problem with some of them? Remember you cannot change a shape of the tile in between.

E11) Divide the children into groups of 6-8, depending upon the space available and the number of children. Explain to them what tiling means and let them proceed with the tiling exercises. Do not interfere in their thinking. Observe the discussions among them as they do this activity.

Analyse the discussions for their notions regarding symmetry, angles, vertex, etc. What other mathematical thought process can you study in this exercise?

E12) i) You probably first assembled various single digit subtraction facts. Then , from them you chose the ones that may fit. Then, moving step by step you would eliminate the non-possibilities, based on contradictions you got.

ii) Give simple puzzles to start with, like A + B = A, A – B = C (and get them to come out with many different solutions).

E13) Consider the first problem. Note down why the operation can't be subtraction. If it is addition, what value of A would give you A in the hundreds place in the answer?

If the operation is multiplication, what could A, B and C be? One solution is A = 1, B = 2, C = 4. Think of others.

Try the second problem similarly.

E14) The answer is 20+25–10. Why is this the answer? Can you make any generalisation on the basis of this?

E15) You may have first tried to solve a simpler version — when more is not added every 5 minutes. In this case the tank would be empty in 20 minutes (Why?).

Now, let's try the more complicated version. After 20 minutes, the tank has emptied 1000 litres, but has added 20/5 × 50 litres = 200 litres. These 200 litres require another 4 minutes. Fortunately, not another 5 minutes! (Why?) So, the tank empties in 24 minutes.

E16) For example, could you say that you practised some multiplication and division? This practice is trivial. We are asking you for the **mathematical thinking** that you practised while doing the problem.

E17) Based on your answer to E16, you can now consider the usefulness of such tasks for children. List out the abilities of a child that are developed in the process of problem solving.

Note that it is through such problems that children's interest is generated and maintained in learning mathematics. Using such devices is a very good way for helping a learner develop and practise her mathematical thinking.