UNIT 17 THE ESSENCE OF MATHEMATICS

Structure **Page Nos.**

17.1 INTRODUCTION

In this unit we try to explain what mathematical reasoning involves, particularly in the context of what abilities, therefore, need to be stressed while teaching it to the children. What we find in the classrooms nowadays is that mathematics is reduced to calculations and applying algorithms mindlessly.

The procedure in the algorithms has become routine without the underlying logic being known or understood by many of us. Therefore, this is what an algorithm is passed on as to our pupils, as we have seen in Block 5. Through this unit, we aim to provoke you to think about the situation, and what aspects of mathematics really need to be stressed when teaching any topic.

We start with bringing out the essence of the abstraction of the objects and relationships in the world of mathematics. Then we talk of the most important processes involved in understanding and developing this world, namely, generalising from observing particular instances and particularising from general statements and conditions. Finally, we discuss what a proof is and why it is so important in the context of mathematics.

Since a lot of what is done here has been briefly discussed in Unit 1, AMT-01, you may find it useful to study that unit again along with this unit.

Objectives

After studying this unit, you should be able to

- explain in what way 'thinking mathematically' requires dealing with abstraction;
- explain how the processes of particularisation and generalisation are essential for doing mathematics;
- describe what a mathematical proof is;
- identify the thought processes that need to be developed in children when teaching them mathematics.

17.2 ABSTRACTION

The other day a teacher trainer, Aruna, was speaking to a group of primary school teachers who had gathered for a training session. They got talking about their views on mathematics teaching. Many of the teachers were complaining that the children don't bother learning mathematics. After a bit, Aruna asked them what mathematics meant to each of them. Let me ask you the same.

? What do you think mathematics is?

Do you agree with most of the teachers present that day, who said that mathematics was numbers and calculations. If so, then does geometry and the study of form and space fit this definition? Aruna asked the teachers this too. So, after a bit of collective thinking, out came a 'definition' that 'mathematics is the study of numbers and space'.

Here Aruna changed track a bit, asking them what numbers are. Again, a lot more discussion took place in which comments like, "Numbers are ... um... for example, 5 is 5 people, 5 chairs, etc" were coming out.

Aruna: So, what you are saying is that all these things have a common property of how many of them there are. You call that 5.
A teacher: Yes, and all numbers are like that.
Aruna: How would you then explain –5?
Another teacher: It would be 5 things missing. For example, if we owe someone 5 rupees.

What is interesting in this conversation is how clear it is that we abstract the notion of numbers from using them as adjectives. When we talk of a number, we are essentially referring to a certain physical property of a set. Thus, when we talk of the number 'two', we could be referring to any collection of objects that can be put in one-to-one correspondence with, for example, the number of sleeves in a shirt. Thus, we say that a coin has *two* sides (each side corresponding to one sleeve), most humans have *two* eyes, a line segment has *two* end points, and there are (usually!) *two* sides to an argument. We abstract a common property of these different concrete objects, namely the number of objects in each of them. This is the number that we call 'two'. As in the previous case, having abstracted the property and understanding what 'two' means, we can now think of the number two without referring to the objects from which we derived the concept. It also has completely abstract and formal relationships with other numbers like 6, $\sqrt{2}$, 2i, etc., and with other abstract mathematical objects (e.g., rectangles).

Fig.1 : Do all of us have two faces?

This leads us to what we said in Sec. 1.3.1, AMT-01. If you read that section again, you would recall that **abstracting a concept** is the ability to look at several particular examples of the concept, find what is common to them, separate that common property from the objects and look at the property as something on its own, having an independent existence. This existence is in the world of mathematics. This world is made up of such abstract objects, which generate further abstract concepts and relations between such objects.

We acquire our understanding of these abstract objects in two ways. One way is the way we develop our concept of number. This consists of a process of careful observation and analysis of different objects, noticing a certain property common to these objects and separating the property from the objects from which it was abstracted. This property, then, becomes an object of study as a concept. This is true of several non-mathematical concepts (like colour) too. In the following exercise we ask you to mull over this process.

E1) Identify two other concepts in mathematics and two from non-mathematical areas that arise through a process of abstraction. Explain how this abstraction takes place.

As we have just seen, several mathematical and other concepts are derived by abstracting them from particular instances. Would you be able to abstract the notion of a point or a line by this process? To answer this, let us first consider a point. In school, we are told that a point marks a position in space and that it is dimensionless. How, then, do we represent a point? Even the tiniest dot in space has some dimension

So, we can't abstract the concept from particular concrete instances of the concept, because ideally there cannot be any concrete representation of a point. There is no easy way out of this difficulty. As a result, we simply ignore the problem when dealing with this concept. Thus, on paper, we often mark points like the origin O, while in our minds we know that a point cannot exist in reality. It is an abstract entity present only in our minds. Similar situations arise with many other geometrical concepts as well, such as a line, a segment, or a ray. All these abstract concepts exist because of certain accepted rules and conventions in the world of mathematics. These rules are called **axioms**. And, to be able to deal with such abstract concepts, and related concepts, we choose conventions for representing them symbolically. Once we define one convention, we use it to define conventions for the other objects that exist only in our minds. This is another kind of abstraction. It is by this other form of abstraction that Euclid averred that "a point moves to describe a line". This line, an abstraction itself, moves to generate a surface, and so on.

The essence of mathematics lies in dealing with these forms of abstraction. In the next few sections we shall talk about what we mean by 'dealing'. For now, try this exercise.

E2) Explain what the difference is in the two forms of abstraction we have just discussed, with examples that haven't been given so far.

In this section we have discussed a certain characteristic of mathematical thinking. This thought process moves along a path of generalisation. In fact, generalisation is the way the world of mathematics grows. We discussed this briefly in AMT-01. Let us go into some more details now.

17.3 PARTICULARISING AND GENERALISING

In AMT-01, one of the important mathematical ideas that was discussed was that of generalisation. The process was bought out through several examples. One of them was about the way we formulate the concept of a tail. The process involved observing the tails of some objects, such as a horse or a cow. As children, we notice the tails of different animals are pointed out to us. We also notice that different tails may look different, but all of them are called 'tails'. So, our initial concept of a tail may be that it is that part of an animal that is seen at the back of the rest of the body. Then we extend this concept to the appendage at the rear end of a bird or a fish. We may extend this notion further and modify our image of a tail to include the tails of aeroplanes and kites, thus generalising our notion to living and non-living creatures. As we examine more objects that have a tail, we continue to generalise this notion. Ultimately, we arrive at an image of a tail that may not include some of the specific features of the tails of the different objects that we are considering but will include common features of all of them.

We engage in this kind of generalisation all the time in our daily lives in order to formulate a concept (for example, think of what a face is — the face of a human being or an animal, the face of a clock or the facing page of a book). The process is useful in extending our activities — for example, we can generalise our observations about plant growth in order to grow new plants and we are able to generalise our experiences of a child's mental development in order to construct learning and teaching methodologies. In the study of mathematics, the process of generalisation assumes a special significance. It helps us to understand the structure of specific mathematical objects and to build further knowledge upon existing structures. But what is even more significant is the fact that often such extension of knowledge may become impossible without such generalisation.

In mathematics, we find generalisation occurs in different contexts — we generalise to arrive at definitions of new concepts, as in the case of the definition of quadrilaterals. We generalise procedures, for example, the procedure to add two fractions. We generalise results to new sets of mathematical objects, such as extending the statement 'the sum of the four angles of a square is 360 degrees' to the statement 'the sum of the four angles of a quadrilateral is 360 degrees'. You have also studied the generalisation of arithmetic to algebra in AMT-01, where the use of variables helps us to extend our study and use of numbers in new ways.

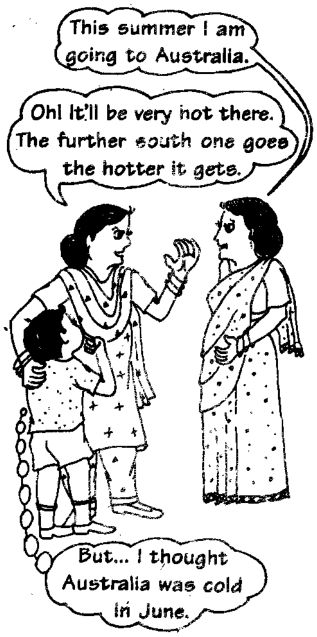

Fig.2

In this section, we study generalisation in different mathematical contexts. For instance, think about the way most of us develop the general concept of a polygon. We get to know triangles of various shapes and sizes. We get to know rectangles, squares and other quadrilaterals. We look around us and see patterns having pentagons (i.e., 5-sided closed figures). We wonder — can we have figures having 20 sides, 50 sides, 77 sides, and so on? If so, what would their properties be? Is anything common to all these figures? In this way we develop our concept of a polygon — a closed figure having three or more sides. This is an example of generalisation. With such generalisation we also generalise related notions like those of area, perimeter and other concepts associated with polygons.

Usually, to understand what the general concept is, we begin learning about it by observing and studying properties of particular cases. For instance, by studying the areas of squares, parallelograms or triangles, we may naturally acquire the general concept of 'area of a polygon'.

For another example, we may notice that the sum of all the internal angles of a square is 4 right angles and the sum of all the internal angles of a rectangle is also 4 right angles. Then we may find that the sum of all the internal angles of a parallelogram is also 4 right angles. So, we begin to wonder : Is the sum of the internal angles of any quadrilateral 4 right angles as well? This is an example of generalising from particular examples.

Then we prove that the sum of the internal angles of a quadrilateral is equal to 4 right angles. So, our general question, or **conjecture** has been proved. Now, suppose we are asked to solve, for example, the problem : If one internal angle of a parallelogram is 40°, find the other three. Then we use the general statement that we know is true for any quadrilateral (as well as other results) to find the angles as 40°, 140°, 140°. What we are now doing is going from the general to a particular example. We call this process **particularisation**.

You could do this whole exercise of generalising and particularising for triangles also. Such examples can be used to help your learners understand the processes of generalisation and particularisation while studying properties of closed figures. In this way, children will realise that while understanding or creating mathematics, we are constantly going **from particular to general and from general to particular**.

In fact, to understand a concept, it helps the child to gradually construct it in her mind. This is done gradually through experiencing concrete examples and studying particular cases. Though many of us accept this fact in theory, how often do we find this happening in our classrooms? Not commonly. In fact, it is more common to find teachers giving their students various definitions in all generality, and expecting the children to remember them. Even when examples to illustrate the definition are given, they are not varied enough. Also, the teachers don't realise the importance of asking the children to give examples on their own.

Sometimes, the children are given some particular examples in the textbook or on the board, quickly followed by the general definition. Neither kind of teaching helps the young minds in acquiring the concept because they require more opportunities to think about and use the concept concerned. This is perhaps most evident in geometry where students learn about different polygons without building any links among them. This is one reason why so many people wrongly believe, for instance, that a square is not a rectangle.

The point we are emphasising here is that, in most cases, the move from particular to general cases represents a move towards a higher cognitive plane. The children need to, first, become somewhat familiar with a concept in particular cases by dealing with plenty of concrete examples. They need to build links between these specific cases and the essence that they have abstracted. Only then can they move towards understanding the concept in all its generality. We teachers need to understand this if we don't want concepts to be reduced to mere definitions, which are rote learnt.

Why don't you try an exercise now?

E3) Are only concepts generalised? Can procedures be generalised? How?

While thinking about E3, you may have thought about the discussion on algorithms in Block 5. For example, the algorithm for adding fractions is nothing but a **generalised procedure** for adding **any** two fractions. (In fact, we can identify two levels of generalisation in this process. At one level, we have evolved a method that works for **all** fractions. At a different level, we are also generalising the idea of addition — we are now adding not only whole numbers but also parts of wholes.) Think of any algorithm in mathematics — may be one for finding the least common multiple of two or more numbers, or that of finding the square root of a number, or that of constructing the angle bisector of an angle using only a ruler and a compass. Each of these algorithms is **a generalised step-by-step procedure**. Each such algorithm has an underlying logic. What we mean by generalisation in this case is that the logic of the algorithm is not restricted to just a few particular cases. It works in exactly the same way for any member of the class. You have already seen this in the case of the algorithm for addition of fractions. For another example, consider the method that helps us to construct the angle bisector of an angle of 90°. This method is independent of the degree measure of that angle. Therefore, it can be used to construct the bisector of any angle.

Try an exercise now.

E4) Give an example of movement 'from general to particular' taken from your daily life. Also explain why you chose that example.

E5) Not all generalisations related to mathematical objects are valid. Give an example to show this, taken from the primary school level mathematics.

We have seen that the world of mathematics grows through the process of generalisation — of concepts and relations between them. When we are generalising concepts or algorithms, we need to ensure that the generalisation is valid. There are broadly two forms of reasoning we use for this purpose, which we shall discuss next.

17.4 WHAT IS A PROOF?

In the previous section we noted that doing mathematics involves generalising on the basis of observations of particular cases. Once we have noticed patterns in these instances, we make inferences based on these patterns. Thus, you may infer that June is the hottest month of the year (if you live in Punjab, say). Or you may infer what a one-year-old child will look like based on your observations of several children of that age. You may see a cow eating grass, then another one doing the same thing and infer that all cows feed on grass. This form of drawing inferences based on repeated similar experiences is called **inductive logic**. There is a form of this logic that we use in mathematics, and call it **mathematical induction**. We have discussed this in Block 1, AMT-01, also. Over there, you have seen how this principle **uses inductive logic to formulate a conjecture** based on observed patterns. For instance, you may observe that $1^3 + 2^3 = 9 = 3^2$, $1^3 + 2^3 + 3^3 = 36 = 6^2$, and so on. You may also notice that $3 = 1+2$, $6 = 1 + 2 + 3$, and so on. Based on these particular cases, you may conjecture that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

The next step is to prove that your conjecture is true in all generality. How would you do this?

In AMT-01, Unit 1, we discussed the fact that in mathematics when we claim that a statement is true in general, what we really mean is that **it holds true, without exception**, in all cases in which the conditions of the statement are satisfied. This means that mathematically speaking, it is not enough to show that the particular statement is true in several different cases (even if the number of such cases is very large); what we must be able to do is to actually show, through a process of inductive and/or deductive reasoning, that the statement is valid in all the cases where the conditions of the statement are true. And, what is **deductive reasoning?** The following discussion between Reena and Vijay may help you answer this.

Fig.3

Example 1 : Reena and Vijay study in Class 7, with eighteen other students. They were having an argument about the class average. Reena told Vijay that if each of their class test marks were increased by 5, the class average would also go up by 5. Vijay tried it out, and found that Reena was right. Then Reena said that, in fact, this is not true only for 5 — if the marks were increased by any number, the average would increase by that number. Vijay didn't believe this. Reena tried to convince him in different ways.

First, she wrote down the marks of each of the twenty students and then added 6 to each of them. She calculated the average before and after this addition. She showed him that the result was true. Vijay, being a hard nut to crack, was still not convinced. He argued that it worked in this case by some coincidence, but how does he know that it will work in other cases. Reena thought about this for a bit, and finally said, "Let's do it for any number." And then, she did it for several numbers but she couldn't convince Vijay. Then, suddenly, she thought of using algebra!

She decided to show that the result would be true if n marks were added to each student's marks. She wrote down the marks of each of the twenty students, and then added n to each score. Then, to get the average, she added all these up and divided by 20. She found that this number was the average of the

$\left[\text{original marks} + \underbrace{n + n + \cdots n}_{20\,\text{times}}\right] \div 20$, that is, n. Now Vijay was finally convinced that what Reena had said before was actually true!

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What Reena did in Example 1 was to use her knowledge of algebra, together with a known result for calculating the average of a set of numbers. From this, she deduced her statement through a form of general reasoning. This is an example of the method of deduction. So, in applying deductive logic, we use known results, definitions, accepted axioms and rules of inference to prove that a statement is true or false.

E6) Would Reena's statement still be valid if there were 25 students in the class instead of 20? Or if the number of students was 40? or 1111? Could you show, through a deductive process similar to the one above, that the result is true for any given number of students?

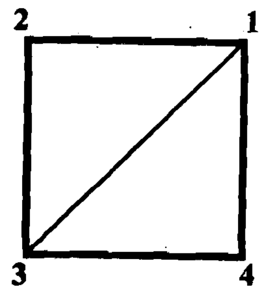
Let us go back to talking about a ' proof '. A **mathematical proof** of a statement consists of one or more steps which make up **mathematically acceptable** evidence to support that statement.

It is no exaggeration to say that **the idea of proof is the single most important idea in all of mathematics.** Let us look at an example to see how inductive and deductive logic go hand in hand to give a proof in mathematics.

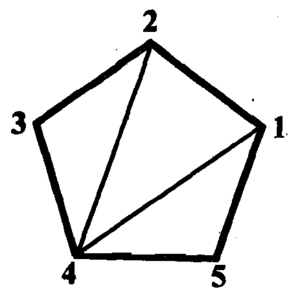
Making charts is often a good way of looking for patterns

Suppose I ask you to find the sum of the interior angles of any convex polygon. How do you go about trying to answer this question? You may already know that the sum of the interior angles is related to the number of sides of a polygon in some way. You would probably begin by looking at special cases. You already know that this sum is 180° for a triangle and 360° for a quadrilateral. Suppose you also know that for a pentagon this sum is 540° and for a hexagon it is 720°. (Remember similar exercises that you did in AMT-01?) You could try drawing a chart like the following:

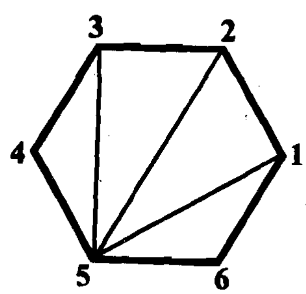
Number of sides of polygon	3	4	5	6
Sum of the interior angles (in degrees)	180	360	540	720



Square



Regular pentagon



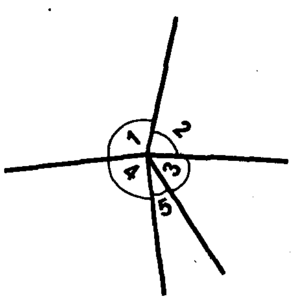
Regular hexagon

After a little thought, you may notice that each number in the second row is a multiple of 180. You may then decide to write each number down as a multiple of 180. Thus, you will get: $180 = 1 \times 180$, $360 = 2 \times 180$, $540 = 3 \times 180$, $720 = 4 \times 180$. Are these numbers related to the number of sides in each case? In other words, is there a common rule relating 3 to 1, 4 to 2, 5 to 3, and so on? Some reflection on this question may lead you to infer that the sum is $[(n-2) \times 180]^\circ$, where n is the number of sides of the polygon. But how would you check whether your guess (or **conjecture**) is right? After all, it may happen that this result may not hold if you take a 20-sided polygon, or one with 62537 sides. You would need to find a proof to show that the statement 'the sum of the interior angles of an n-sided polygon is $(n-2) \times 180$ degrees for any $n \geq 3$' is valid. You would do so **through a series of steps, each of which is deduced logically from the previous one.** This would constitute the **proof of the statement**. Let us see one way of providing such evidence in this case.

As you may remember, to logically derive a result we must accept certain definitions and/or axioms and/or earlier proven statements. In this case, two statements that we shall assume are

' The sum of the interior angles of a triangle is 180°', and

'The sum of the angles around a point is 360°' (as illustrated for one case in Fig.4).



$$\angle 1 + \angle 2 + \ldots + \angle 5 = 360^\circ$$

Fig.4

Consider any n-sided polygon and take any point, say O, inside it. Join this point to each of the vertices of the polygon. As there are n vertices, the interior of the polygon gets divided into n triangles. (In order to understand this picture more clearly, we could even draw a polygon and make the necessary construction as in Fig.5. However, we must **remember that this picture (or any) is only an aid to see the logic of the proof.** Sometimes a picture could give you an incomplete or wrong understanding of the general situation.)

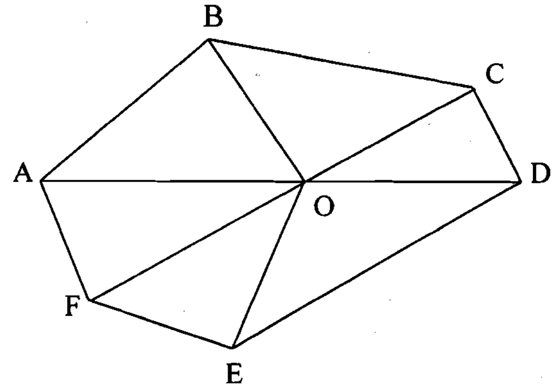


Fig.5

Now, for each of the n triangles, the sum of the angles in it is $180°$. Since there are n triangles, the total sum of all the angles inside this polygon is $(n \times 180)°$.
But the total sum of the angles inside the polygon is the sum of the interior angles of the polygon plus the angles around the point O. Since the sum of the angles around O is $360°$, the sum of the interior angles of the polygon is $[(n \times 180) - 360]° = [(n - 2) \times 180]°$. (Remember, in the picture $n = 6$, but we are actually dealing with any $n \geq 3$.)

The series of statements above constitutes a ' mathematical proof ' for the stated result. In it, each step follows logically from the preceding step and/or one of the results that we assumed before we began this proof. This method of reasoning is what is called 'deductive logic'. Thus, here, by a piece of deductive logic, we have actually shown that what we had **inferred** through inductive logic is indeed true in each and every case.

Now, let us go back to the conjecture made earlier, that $1^3 + 2^3 \ldots + n^3 = (1+2+\ldots+n)^2$ for every $n \geq 1$. Let us try and see if a proof by the **principle of induction** can be worked out. As you know from AMT-01, we need to check that the statement is true for the first case in our statement, which in this case is $n = 1$. Since $1=1$, the statement is true for $n = 1$. Now let us assume it for a case $n = k$, and then use the method of deduction to prove that the statement is true for $n = k + 1$.

So, we assume that $1^3 + 2^3 \ldots + k^3 = (1+2+\ldots+k)^2$. We now look at $1^3 + 2^3 + \ldots + (k+1)^3$.

$$\begin{aligned}\text{Since } 1^3 + 2^3 + \ldots + (k+1)^3 &= 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 \\ &= (1+2+\ldots+k)^2 + (k+1)^3 \\ &= (1+2+\ldots+k)^2 + (k+1)(k+1)^2 \\ &= [1+2+\ldots+(k+1)]^2 - 2(1+2+\ldots+k)(k+1) + k(k+1)^2 \text{ (why?)} \\ &= [1+2+\ldots+(k+1)]^2 \text{ since } 1+2+\ldots+k = \frac{k(k+1)}{2} \text{ (see E7).}\end{aligned}$$

So, given that the statement is true for n=k, we have shown its validity for n=k+1. Therefore, by the principle of induction it is true for every $n \geq 1$.

Here's an opportunity for you to use both the routes to prove mathematical statements.

E7) a) Use the principle of mathematical induction to prove that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \quad \text{for every } n \geq 1, n \in \mathbf{N}.$$

b) Prove that each internal angle of an n-sided polygon whose sides are all equal is $\left[180 - \frac{360}{n}\right]^\circ$ for $n \geq 3$.

Let us, now, take a brief look at what we said about proofs just now, namely, proving a mathematical statement involves the following:

- A general statement about a certain class of objects that satisfy a set of conditions. This statement may be formulated on the basis of observation of patterns found in particular cases, or on the basis of mathematical intuition, or on some other basis.
- The objective is to show, through deductive reasoning, that the given statement is true in all cases where the conditions of the statement are valid.
- What we could use to achieve our objectives are one or more statements, which we call **premises**. These premises can be of four types :
  i) a statement that has been proved earlier;
  ii) a statement that follows logically from the earlier statements given in the proof;
  iii) a mathematical fact that has never been proved, but is universally accepted as true, that is, an axiom;
  iv) the definition of a mathematical term.
- The proof of the statement, then, consists of these premises.

Once we successfully show that the given statement is valid, we say that our statement has been proved.

As we see above, proving any statement about a given collection of objects mathematically involves proving it for **each and every object** in the collection. This means that a statement about a collection of objects is false if it does not hold for even one case in the collection. So, one way to **disprove** a mathematical statement (i.e., prove that it is false) is to find one example of an object that satisfies the hypotheses but for which the statement is not true.

For an example, consider the statement: Every prime number is odd. To disprove this statement, we only have to find one example of a number that is prime but not odd. It is easy to see that 2 is a prime number that is even. The existence of 2, therefore, is enough to show that the given statement is a false one.

Similarly, consider the statement : Every odd number is prime. Can you prove or disprove it? There are so many counter examples — 9, 15, ... But only one is enough to disprove it.

Try these exercises now.

E8) Give one example each of a true mathematical statement and a false one. also prove that these statements are true and false, respectively.

E9) Consider the following "proof" of the statement 2 = 1, given to me by my cousin.

Step 1: Assume that a and b are two numbers such that a = b.

Step 2: Multiplying both sides by a, we get $a^2 = ab$.

Step 3: Subtracting b^2 from both sides, we get $a^2 - b^2 = ab - b^2$

Step 4: Factorising both sides, we get $(a + b)(a - b) = b(a - b)$

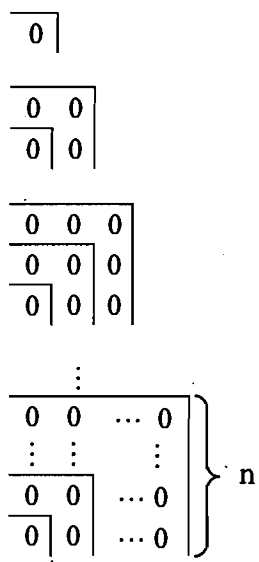
Step 5: Dividing both sides by a – b, we get a + b = b.

Step 6: Now take a = b = 1. This gives 1 + 1 = 1, i. e, 2 = 1.

Do you accept this proof ? If not, where is the catch?

The point of E9 is to demonstrate that we must be very careful about applying even well known results while proving further mathematical results. We may feel that we know all our basic mathematical facts and rules very well, and we can even apply them, proverbially speaking, "with our eyes closed". But, it is easy to make a mistake unless we are mathematically alert.

You may wonder whether the process of proving a statement that we have outlined above is the only way of doing so. How about "visual proofs"? For example, consider the following proof for the statement $1 + 3 + 5 + ... + (2n - 1) = n^2$, where n is a natural number.



$1 = 1^2$

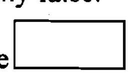
$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

⋮

$1 + 3 + 5 + \cdots + (2n - 1) = n^2$

The truth is that though such visual evidence can be **useful as an aid** for proving the relevant statement rigorously, mathematicians do not accept it as proof. This is because we have to remember that in mathematics, what we demand of a proof is that **it should be valid in all situations where the conditions of the statement we are proving are valid.** It would often be quite impossible to visually consider all the possible situations in which the conditions of a statement are true. In fact, what is even worse is that we may draw a diagram in which a particular statement is true and not even realise that there are other possible situations where all the conditions of the statement are satisfied and yet the statement is actually false.

For example, many of us find areas of rectangles like . When we draw more rectangles with increased length, we tend to increase the breadth too. So, we find that as the length increases so does the area. This leads many of us to the generalisation that the greater the length of a rectangle, the greater its area. But, is this a true statement? Why don't you drew rectangles of varying lengths and breadths and see? The area depends on both the length and the breadth. You can easily find examples to show that the generalisation is wrong. This false generalisation has come about entirely because of the examples through which our mathematical intuition was built up.

The other point that comes out from this example is that if we find that we are arriving at a result that appears to be going against our common (and mathematical) sense, then we probably need to pause and re-examine our work carefully. But, **sometimes** our intuition or common sense may be **wrong** because this may be limited by what one sees in a few particular instances. That is, our generalisation from particular instances may be wrong.

We shall look into other aspects of mathematical thinking in some more detail in the next unit. For now, let us summarise what we have done in this unit so far.

17.5 SUMMARY

In this unit we have covered the following points.

1. We have seen that the world of mathematics is made up of abstract objects, and relations between these objects. There are certain rules and conventions that we agree that these objects will follow.

2. The essence of mathematical reasoning is generalising on the basis of patterns observed in particular instances. These generalisations should be valid. Validity means that they should hold true **for every case** that fits the conditions under which the generalisation is made.

3. We have seen what a proof is in mathematics and what disproving a statement involves.

17.6 COMMENTS ON EXERCISES

E1) For example, mathematical concepts could be 'closed and open figures'. Examples of non-mathematical ideas could be 'hot' or sharpness.

Think about how each of these concepts develop in the child's mind. Some examples have been given in AMT-01. A very small child does not know what 'hot' is. What kind of experiences gradually make her understand this concept? How does she learn to compare and find out which is hotter? Give details of possible experiences and how they will help her abstract the idea of 'hot'.

E2) One form of abstraction is abstracting from the concrete. For example, from pens, ballpens, pencils, etc, the idea of things that write is abstracted. Give more examples of this kind of abstraction.

The second kind of abstraction is abstraction from ideas that are not actually available in reality. For example, abstracting the concept of negative numbers.

E3) Procedures are also generalisable. For example, if one knows how to make one kind of 'daal', the same procedure of using a pressure cooker to cook, can be used for cooking other 'daals' as well. The procedure for cooking a vegetable can also be, similarly, generally developed. Give more examples of procedures that are generalised, including some from mathematics.

E4) We know that, in general, people smile when they are pleased or when they want to be friends. Therefore, in particular, when we meet a new person and he/she smiles at us, we assume that the person wants to be friends. This is an example of particularisation.

E5) There could be many examples in which the generalisation could be wrong. For example, we know that when two natural numbers are multiplied, we get a number larger than or at least equal to the larger of the two numbers. This is not true for fractions, and a generalisation of this kind would be wrong. Give other examples of possible generalisations that are erroneous.

E6) Suppose there are m students, and the marks obtained by them are $n_1, n_2, \ldots, n_m$. Add a number p to each of them. Find averages in both cases — with and without p added.

E7) a) Check that the statement is true for n=1. Assume it is true for n=k.

Then $1 + 2 + \ldots + k = \frac{k(k+1)}{2}$

Now $1 + 2 + \ldots + (k+1) = 1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{(k+1)(k+2)}{2}$$

$\therefore$ the statement is true $\forall\ n \geq 1$.

b) We will assume that you know that the internal angles of a triangle add up to 180°. In an equilateral triangle, there are 3 internal angles and they are all equal. Each angle of an equilateral triangle must therefore be 60°, which is $[180 - \frac{360}{4}]^\circ$.

Now, use what you have proved earlier about the sum of the interior angles of a polygon. Also, use the definition of a regular polygon — all its sides are of the same length, and hence, all its internal angles have the same degree measure. Then see if you can prove the given statement.

E8) For example, consider the statement : If m_1 and m_2 are two consecutive even numbers, then $m_1^2 + m_2^2$ is also the square of an even number. Prove or disprove it. Note that it is true for m_1=6 and m_2=8.

Another statement is 'the sum of two odd numbers is even'. See if it is true or false. You need to consider the most general odd numbers. So, let them be $2n_1+1$ and $2n_2+1$, where n_1 and n_2 are natural numbers. Now check whether the statement is true or false.

E9) The problem lies in Step 5, where we have divided both sides by a – b. In Step 1, you know that we had assumed that a = b. Dividing by a – b therefore implies that in Step 5, we are dividing both sides by 0, which you know is not mathematically valid.