UNIT 16 THINKING ABOUT NUMBERS

Structure **Page No.**

16.1 INTRODUCTION

In the last two units you have learnt about fractions and decimals. In this unit we will concentrate on whole numbers and discuss different ways of exploring these numbers. In Sec.1 we shall tell you what exploring number means and what is the importance of such an activity. You will see that exploring number means finding different relationships between numbers. Such activity is very useful in developing a deeper conceptual understanding. Then we shall focus on the different ways of exploring numbers. In Sec.16.3 we shall discuss how tasks related to dealing/playing with number patterns help in developing a deeper conceptual understanding of numbers.

Another aspect which is very useful in learning about numbers relates to Games and Puzzles. In Sec.16.4 we have considerd some Games and Puzzles that always fascinte children. We have discussed how exploring games and puzzles helps children in developing their own abilities and sharpen their tools for dealing with numbers as a lot of mathematical activity is involved in finding out why a game or puzzle works.

Objectives

After studying this unit, you should be able to

** expiain what exploring number means

* devise different tasks which enable exploring numbers

* use number patterns to find and create different relationships between numbers

* think of games and puzzles which enable children to deal with numbers more efficiently.

16.2 WHAT 'EXPLORING NUMBER' MEANS

Let us start by looking at the conversation that is going on between a 13-year old girl Suman and a 8-year old boy Rohan.

R:: Didi give me any three digit number.

S: 123

R: Yeah. (after thinking a minute). It can be divided by 3.

S: How did you say so?

R: see, 1+2+3 is 6 and 6 is multiple of 3. Our teacher told us this rule today.

S: Let me give you 4321.
I think it is divisible by 3.

R: (Looks little hesitant). 4+3+2+1 = 10 is not in 3 times table.

Then they divide and check and find that the number does not divide. R is happy and says:

R: See I said it would not divide.

S: But why did this happen?

R: It is a trick. The teacher told this.

What the teacher in the class has done with the children is to tell them a rule, she has not helped them explore numbers and think about different relationship among numbers. There is just a method, just applying a trick (or a rule) for them and no effort at finding the underlying mathematical reason and looking for generalisation based on that.

What we mean by exploring numbers is to look at numbers with children and develop some rules. For example the teacher could have asked children to take some 4,3 and 2 digit numbers, and divide them by three. Once they found which are divisible and which are not they could have been asked to seperate them in to two tables and asked to add digits of these numbers. They could have then be asked to explore what common property could they see in the all the sums that come from numbers divisible by 3. Once the common property is seen they could be asked to check whether any number in the list of numbers not divisible by 3 has that property or not.

Based on what is observed about the rule of 3, the children could be provoked to think about whether it can be generalised and works for 4 or see other patterns emerging. They could be asked to hunt for other kind of divisibility rules for other numbers as well.

Let us now look at how a class 1 teacher devised activities for her children to play with numbers.

Example 1: The children in the class were learning how to add and subtract. She brought two pieces of paper marked as is shown below.

1	2	3	4	5

She used this to depict sums, say 4+3. For adding to 4 then, she put the left-hand end of one ruler against the mark 4 of the other as given below:

				1	2	3	4
1	2	3	4	5	6	7	

She then pointed out that the mark 3 on the second ruler, touched the mark 7 of the first ruler. She also asked them to take 4 stones and add to 3 stones and find out the sum. Noticing that 4+3 was 7 and 7 came against 3 in this situation, she then asked them to check for other numbers. This way children understood and realized that by using numbers written in this way they could add any two numbers to give the sum. Thereafter, she asked children to try with numbers other than 3, like 1,2,4,5 etc. and she started noting down the results like this.

4 + 1 = 5   3 + 1 = 4

$$\begin{aligned} 4+2 &= 6 \qquad 3+3 = 6\text{etc.}\\ 4+3 &= 7\\ 4+4 &= 8 \text{ and so on}\end{aligned}$$

After some time children themselves were very excited to find that they could write the sum without adding by merely using numbers written in this format.

As they worked on this they got even more excited. One of them said "each time we increase the number that we are adding to 4, our answer also increases".

—x—

The fact that if the number is increased by 1 the sum is also increased by 1 may seem simple to us but it is not so simple for children. Children might know very well, for example, that 6+6=12, but might struggle hard to "remember" what 6+7 equals. But if a child discovers that when you add 1 to one of the two numbers being added together, you make your answer bigger by 1. She can deduce that 6+7 is 13, knowing that 6+6 is 12. Later on the child might also discover that when you add 2 to one of the two numbers you are adding together, it makes your answer 2 bigger and the same is true for 3, or 4 and so on. All this leads to a greater confidence with numbers and reduces the need for rote learning for addtion. At a later stage the child may be able to abstract the rule and understand in an intutive way why it works. The algebraic proof using symbols may come much later.

Note that in the above situation the child observed a pattern for 4 and later she generalised it to find out about other numbers. This is what exploring means, seeing something and then finding out whether it works for other situations too. If so it can be taken to be a general rule. This way other relationships between the numbers are also discovered. It is quite possible that children may notice many more things even in the above example, such as the following sequence

$$\begin{aligned} 4+3 &= 7\\ 14+3 &= 17\\ 24+3 &= 27\\ 34+3 &= 37\text{and so on}\end{aligned}$$

For many children, the sums, 4+3 = 7, 14+3 = 17 are independent. They might say that 4+3 = 7 and then turn round and say that 24 + 3 = 29, or something equally surprising. This happens because they have been taught these as a pile of disconnected facts to be memorized. Children have no sense of the logic or order of numbers against which they can check their memory, or that they can use if their memory is uncertain. Addition is not the only operation with which we can explore, let us look at some other possibilities.

By about fifth standard children have collected many facts connected with numbers. For instance with the number 6 they have already come across the following and more.

$$\begin{aligned} 2\times 3 &= 6\\ 3\times 2 &= 6\\ 6\div 2 &= 3\\ 6\div 3 &= 2\end{aligned}$$

These, rather than being seperate multiplication or division facts are properties of the number 6. They are suggested by the nature of the number system and children can find these and many more for themselves. For example if you have as many objects as shown below.

******

We can arrange these

as

***
**

or
as

**
*
*

All those are simply different ways of writing down and talking about this one number. So anyone, having discovered this property (or 1'fact') about 6, and having been told the different ways in which we write and talk about this, could look for and find similar properties about other numbers, and then use those to write them.

One of my friends says explorations with numbers will enable the child to see that different ideas about numbers are connected. This, in turn will enable them to construct relationships between numberss

E1) What do you think are the other advantages of letting children explore numbers?

Most of the time it happens that we don't bother to discuss mathematical ideas or try to give children the opportunity to find the logic behind these ideas. We just state certain "facts" (with numbers), show children how to do problems, give some home work, correct it, drill the same facts, and test them.

We do not recognise that children have their private secret world of games with a number of mathematical and logical tricks and games floating round the school as well. This is certainly not encouraged by the teachers, and perhaps works without their even knowing about it (may be we may even discourage it). But if we look at some of these, we may find them to be extremely useful. For example one such game is illustrated in the following example.

Example 2: Student A would come up to student B, preferably with students C,D, and E nearby. The game is as follows.

A: Think of a number. Don't tell me what it is, but be sure to remember it.

B: OK, I've thought of a number.

A: Make sure you don't forget it! You can write it on a piece of paper.

B: Don't worry, I won't! I have written it.

A: Now add three to it — and don't tell me the answer.

B: Have done it.

A: Now add ten to it.

B: Done

A: Now take away seven from it.

B: You mean "subtract" O.K.

A: Now add five to it.

B: Done

A: Now take away the number you started with.

B: O.K, done it.

A: (Triumphantly) The number you have got is eleven!

In normal situations B,C and D would challenge A to do the trick again. And A would happily do it again. The testing would go on and on and it might take A several successful performances to convince them that she really knew how to do the trick, and could do it as many times as she wanted. At which point they would walk away, shaking their heads and wondering. Or maybe they would beg him to show them how to do the trick.

No child I knew ever showed another child how to do this trick. Yet every year a gang of us would figure it out and learn to do it, while a new bunch of recruits would come into the school, ready to be tricked and mystified in their turn.

—x—

Now try this exercise based on the above example.

E2) How do you think A knows the answer? Can the conversation be changed to get another number?

E3) Is there any exploration involved while children attempting this trick? If so what is the exploration?

E4) Would this help them to learn more about numbers? Give reasons for your answer.

E5) List 5 different kinds of exercises that can be given to children which helps them to explore numbers. In each case specify
  i) What are they exploring
  ii) What are they learning while doing the task?

By now you must have got some idea what exploration of numbers means. You must have also noticed that exploration helps in developing a deeper conceptual understanding because this task helps the child to look into the logic behind a fact and therefore the learning becomes more sensible and meaningful. This point will be more clear to you when we look at different means of exploring. In the next section we shall discuss this.

16.3 NUMBER PATTERNS

We shall now discuss how recognising different patterns helps in exploring numbers.

One day when I was visiting my cousin, Sujaya's house, I found that her 10-year old child and two of her slightly older friends were playing with the following pattern of numbers. (See Fig.1)

1 1 1 1 1 1 1 1 1 1 1
1 ? ? ? ? ! ? ? ! . .
1 ? ? ? ! ? ? ! ! .
1 ? ? ? ! ? ? ? ! . .
1 ? ? ? ! ? ? . ! .
1 ? ? ! ? ? ? ? ! .
1 1 1 1 1 1 1 1 1

Fig.1

Then I also joined them in their play. I found that they were busy filling the '?' given the patterns.

I asked them how they were deciding the number to be filled? Then they replied that it is given that each numbers of the form given by $\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$ satisfies ad = bc + 1, using this property, they were filling '?' one by one.

After completing the table, I asked them 'if instead of the property ad = bc + 1, I consider the property a + d = (b + c) + 1, what will be the resulting pattern? They tried to work it out and show it to me. After that I asked them to find a pattern when we replace the starting '1's by 0's. They, very happily did that also.

E6) What are children learning while they are going through this task.

E7) Try it with a group of children aged between 10-12 and see how they proceed. Could they do it on their own or you had to give them an example to start with.

As we have seen in the above example, children could be fascinated by number patterns and opportunities to exercise their minds and solve problems. Number patterns and relationships can be source of fun not only to children but to adults provided the task is set up in an interesting way. Before we take up more ideas and explore the ways 'playing with numbers can be presented' so as to involve learners, let us think about whether all number puzzles/patterns would fascinate children or whether there are certain features that would ensure the fascination.

Let us see another situation. Consider the multiplication table of 9. Is there anything interesting here that we don't see in, say, the table of 3, 4 etc.? We write the table as in Table 1.

Table-1

1 × 9	=	09
2 × 9	=	18
3 × 9	=	27
4 × 9	=	36
5 × 9	=	45
6 × 9	=	54
7 × 9	=	63
8 × 9	=	72
9 × 9	=	81
10 × 9	=	90

Do you see any pattern for numbers written on the right hand side of the equality sign? You must have observed that if we write the digits 9 to 0 in a decreasing order in the unit's place and 0 to 9 in an increasing order in the ten's place as shown, we get all the products on the right hand side.

E8) Why this pattern in the mutiplication Table of 9.

E9) If you look at other multiplication tables, would you see some pattern in them? Would all tables have some patterns? Why?

While answering E8, you must have understood that this happens because of the base 10 system of the representing numbers? Is this true in any other system, for instance in base 5 system. Why don't you try this for yourself.

E10) Write the multiplication Table of 9 in base 5 system and see whether this pattern holds there?

E11) Find out a pattern in multiplication of 19,29,39 and 79 and 99? What is the logical relation that is the source of these patterns.

Patterns are very important from the point of learning. Apart from what we have said above that patterns are aesthetically (artistically), appealing, students like them. They form the basis of developing ability to generalise and abstract. Patterns have an element of systematic repetition, because of which we understand and remember patterns much more easily than, any rule. Often patterns are particular examples with the help of which we build up a general rule. In this sense, patterns correspond to and mark the path of going from **particular** to **general**. The students learning mathematics, especially at the primary stage, find working with patterns much more concrete than working with a rule. It is consistent with their natural way of thinking which is **inductive** (going from particular to general) rather than **deductive** (going from general to particular). We should, therefore, use patterns while building up rules, concepts and formulas.

Now let us see how we can help students recognise patterns for learning multiplication rules.

Let us ask ourselves what does "becomming familiar with the table" means. What does the table represent, how do we learn and understand the table? In what way is the table used; as a chant or in some other way? Would it feel better if the child was to discover some patterns on the board or what if she can recite the tables but cannot use them? Is there a need for the child to recognize the system and be able to use it and talk about it then merely rote learn it as was done earlier. The way to develop this ability in children is to help them recognize the underlying pattern in all the tables. For example if we ask students to make a grid of numbers on their note books as shown in Fig.2, then we can ask them to color numbers in a pattern as per the instructions given. For example you could ask them to colour every alternate number starting from 2 and write down the numbers that have been coloured. Then you could ask children to talk about what was common between these numbers. Once it was articulated that they are all multiples of 2, you could ask them to think about pattern for 3 and then get them to fill the pattern.

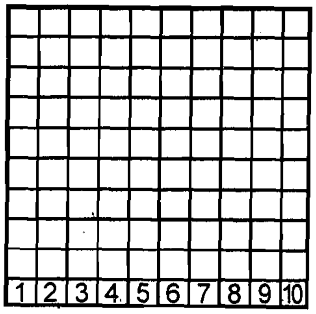

Fig.2: Square grid

After this you could go on to colour numbers after every 4th,5th,6th number starting from 4,5,6 etc. Then you could ask them to list numbers that have not been coloured and talk about what these numbers that are not coloured have in common and what is their special property that this exercise has shown.

You could also make children identify those numbers that have been coloured twice thrice or even more. For example list out those numbers that have been coloured both as multiples of 2 as well as of 3? Which have been coloured as multiples of all three 2,3,5 etc.?

Many more exercises can be built for children to identify various common numbers that are divisible wholly by a set of these numbers. Children could enlist all numbers having these properties and try to think about other common factors of these chosen numbers.

Ask children while colouring multiples of 4 or 6 if they have to colour any numbers that are already not coloured. Why is it that they do not have a new box to be coloured? so? What numbers get coloured as multiples of 7? If you are repeating this task with Class V or VI children, you could also ask them to go beyond 7 and see if there are any numbers upto 100 that do not get coloured as a multiple. For example if we take 11 or 13 or 17 and colour every eleventh or thirteenth or seventeenth number, is there any number reached that is not coloured?

What does this tell you? You could discuss this observation with children and help them realise that for number below 100 if the chosen number is not a multiple of 2,3,5,7, then it is a prime number. Can you think why it is so? Think about it and we would come back to it later.

E12) What other patterns you can have or children to look for, using arrangement of numbers?

E13) How will we use the grid of numbers to introduce HCF and LCM? Does this activity help them to understand the reason behind finding HCF and LCM?

Coming back to the question about identifying primes. How do we decide whether a number is prime or not? Which numbers do we take to check as factors? For example to check whether 127 is prime or not what do we do? Do we keep trying all numbers till 126 or till 63?

In order to understand this let us continue the discussion about finding prime numbers upto 100.

Let us explore and look for numbers not divisible by any number except 1 and by the number itself. 2,3,5,7 are such examples but 4,6,8,9 are not and 11 is. In identifying these prime numbers, we must realise that from the number grid we have tried to colour all numbers that are multiples of a number other than 1 and themselves. Therefore, to confirm whether 47 is a prime number or to test 99 for being a prime number, up to which number do we need to check? If 47, 99 etc. are not multiples of 2,3 or 5 can we say they are prime numbers or we have to go upto 7 or is it that we have to go up to 17 and beyond? Do we need to worry about numbers like 15,20,25 etc. Why?

Suppose I was to tell you that it can be shown that a number, that is not 1, is below hundred and is not a multiple of 2,3,5,7 is a prime number. You can check that all prime numbers upto 100 come within this. Can you think how it can be proved? We have checked it from the numbers not coloured in the table or chart that we made but this is not enough. We may have missed out something. Is there a way we can show it logically.

This is what we would mean by understanding the algorithm that if you want to look for primes below 100 cross out all the multiples of 2,3,5,7 and you would only get prime number left. Think about how you may prove it logically.

Here is a proof. The problem to prove is that a number below 100 that is not a multiple of 2,3,5,7 is either 1 or is a prime number. Consider a number 'p' which is not 1 but is between 1 and 100. In order to justify the algorithm that we have stated we have to show that if p is a number between 1 and 100 and p is not a multiple of either 2,3,5 or 7, then p is a prime number.

Let us suppose that p is not a prime number. If we work with this assumption and if our algorithm is correct then we should arrive at a contradiction. Let us see if that happens and we do reach a contradiction. Let us write $p = p_1p_2$. where p_1 and p_2 are proper factors of p. That is neither p_1 nor p_2 is 1. We may note that $p_1, p \neq 2, 3, 5, 7$. Then either $p_1 < p_2$ or $p_2 < p_1$. Let us consider that $p_1 < p_2$. Then I can write

$$p_1^2 < p_1p_2 = p$$

i.e.$p_1 < (+\sqrt{p}) < \sqrt{100} = 10$

Can you see how it works? $p_2 > p_1$ means that when $p_1 = 10, p_2$ has to be at least 11 and then $p = 110$ which is not possible since our number p lies between 1 and 100. Therefore when $p_1 = 10$ we have $p_1 = p_2$ and $p_1p_2 = 100$, leading to p being a multiple of 2 and 5. So we cannot have $p_1 \geq 10$, of course p_2 can be greater than 10 but we are looking at the smaller factor only at the moment. For p_1, the other possibility is 9 but p_1 cannot be 9. Why not? The answer is simple and you may have found it already.

p_1 cannot be 9 because in that case 3 will be a factor of p_1 and therefore 3 will be a factor of p also. But if p is a multiple of 3 then our assumption that p is not a multiple of 3 is contradicted. Hence $p_1 \neq 9$.

And of course as you can see p_1 cannot be 8,6 and 4 also. What is the reason for it?

8 and 4 are multiples of 2 and 6 is a multiple of both 2 and 3 and therefore with $p_1 = 4$ or 8 p would be a multiple of 2 or of 2 and 3. So since $p_1 < 10$ and it is not 9,8,6,4 then what can it be! The only numbers that remain since $p_1 \neq 1$ are 2,3,5,7. So we have reached a situation that $p_1 \neq 1$, and p_1 cannot be 2,3,4,5,6,7,8 or 9. What it means is that for a number $p < 100$, if $p = p_1p_2$ and $p_2 > p_1$ then $p_1 \neq 1$ is not possible. This mean that for $p = p_1p_2$, if p_1 is not 2,3,5,7 then $p_1 = 1$ and $p_2 = p$ which shows that p is a prime number with the only factors of p being one and itself. Thus we have shown that all the numbers which are not coloured are prime numbers and these are all the prime numbers between 1 and 100.

Now we shall make a note.

Note: A caution about inferences: Often patterns are used to build rules. In doing so, however you must exercise caution. The process of inference of a general rule from particular examples that form a pattern is essentially inductive. It is not unique. It may be most resonable and most probable, but it may not be the only rule that can be formed on the basis of a few particular examples. For example, suppose, we are, given the four numbers in a sequence as

$$2, 9, 28, 65, \ldots$$

and are asked what the fifth number would be in the chain. Most people would reply it is 126, since each of the four given numbers is a cube plus on.

$$2 = 1^3 + 1, 9 = 2^3 + 1, 28 = 3^3 + 1, 65 = 4^3 + 1$$

the answer would indeed be reasonable. But, suppose, the sequence under consideration is,

$$2, 9, 28, 65, 2, 9, 28, 65, 2, 9, 28, 65, \ldots$$

and hence the fifth number is 2, would this not be justified? This would technically be, because our inference about forming a rule from the four numbers is not unique and we do have patterns of this kind too. We must give additional information if the inference is to be unique.

On the other hand if the rule is given, then the particular examples are deducted from it, there is no **non-uniqueness** about them. In the **case above** we may say that the nth term of the sequence is $a_n = n^3 + 1$. The first five terms, then, are 2,9,28,65,126. The fifth term is 126. We can prove that it is so from the given rule. A proof is deductive in nature and inference is inductive. In the example of finding prime numbers also once we have shown that logically for $p < 400$ if 2,3,5,7 are not its factors, then p is a prime number. We can accept it as a rule. Without showing it logically we cannot take it to be correct. Let us extend this to 'what numbers do we need to check before we can decide that a number < 200 is prime?' Following the method given, try and find out the rule. Why don't you try these exercises now.

You will study more about deduction and induction in Block 6

E14) Co-prime numbers are those which have no common factors except 1. Test the following hypothesis: If n_1 and n_2 are co-prime numbers $n_1 > n_2, n_1 \neq 1$ and $n_2 \neq 1$). Then $(n_1 - n_2)$ is coprime to n_1 or n_2 and $(n_1 + n_2)$ is co-prime to n_1 or n_2.

If this hypothesis is true then how would you show it logically.

Is $n_1 - n_2$ coprime to $n_1 + n_2$ as well?

Next we shall look at another situation where a Class 5 teacher helps her students to find different patterns.

Example 3: The teacher Jaya divided her class (Class 5) into 6 groups. She had displayed a table (as given in table below) on the board.

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

Then she asked the students to observe the table and find as many rules or patterns as they can among the numbers. She interacted with each group and helped them to look row-wise and column-wise. Some of the patterns the groups came up were the following:

i) Every row ends with 1

ii) First column contains only 1's

iii) The second column contains natural numbers in increasing order.
Among the other observations the most significant observation were the following.

iv) Every row has a symmetric arrangement and that if alternate numbers in a row are given negative sign and all numbers in the row are added, the sum is zero.

The teacher was delighted by this observation and gave special credit to the group that came up with it. Like this she encouraged them to find more and more patterns. Children came up with many different patterns. The teacher found some patterns they discovered inadequate. The teacher showed them this by asking them to check the pattern for consistency over all numbers in her presence and waited patiently for the children to say "oh! no this does not satisfy the pattern" and go back to look for more. She encouraged them and provoced them to think by saying there are so many more patterns that are interesting" but restrained from giving them any patterns. However, for children who were not able to take off, she helped by making them to see a couple of patterns and then left them to let them, find out different patterns by themselves. She felt that this exercise would help students to think about numbers and their relationships and gradually make them understand numbers better and help them analyse and build on what they knew. Her argument was that giving children rules and patterns only increased the load of facts to be remembered and did not enable them to be more comfortable with numbers. Thinking in open ended way not only helps creativity but also deepens the concept.

To analyse what she has said and to experience it a bit why don't you try this exercise.

E15) Observe the number table given below. Find as many relationships as possible among the numbers in the table by studying the arrangement of numbers including along the diagonals.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

E16) Try the question in E12) on your students and list five different observations they have come up.

E17) What do you think about the arguments of the teacher in the above example? Where do you agree with her and where do you disagree? Give reasons for your answer.

So far we have been discussing how patterns provides a powerful tool for children to explore the number system. Sometimes properties of geometrical shapes also becomes a tool to explore numbers. Let us see such a situation.

If you are asked "can $(n, n+1, n+2)$, where n is whole number, be the sides of a triangle or of a right-angle triangle? And if so for which values can this happen? What aspects of mathematical reasoning would you use while exploring the properties of these number triplets?

For example, would you try putting n=1 or n=2 or n=3. Think about it, will this make a triangle? Note that a triangle has the property that the sum of any two sides is always greater than the third side. And why is that? For any n, we should have

$$n + n + 1 > n + 2 \text{ therefore } n > 1.$$

Hence we cannot have $n = 1$ in the above triplet. Is there any other condition on n?

Now let us look at the next idea. In this we are talking about a right angled triangle. If you will think about what you were taught as a child you will recall the rule that a right-angled triangle has the property that the square of the hypotenuse is equal to the sum of the squares of the other two sides. For what values of n, can the triplets $(n, n+1, n+2)$ form the sides of the right-angle triangle. For this we would need

$$\begin{aligned} n^2 + (n+1)^2 &= (n+2)^2 \\ \text{i.e.} 2n^2 + 2n + 1 &= n^2 + 4n + 4 \\ \text{i.e.} n^2 - 2n - 3 &= 0 \\ \text{therefore } n &= \frac{2 \pm \sqrt{4+12}}{2} \\ &= 3, -1 \end{aligned}$$

$n = -1$ is impossible (why?). Therefore (3,4,5) is one possibility.

Is there any other possibility? If we think about looking for other triplet like $(n, n+m, n+2m)$ then do such triplets exist? For example can (6,8,10) be a possibility? Is there any other? Is there a pattern that you can extract from this?

Consider this for any whole number 'n' the triplete (3n, 4n, 5n) forms the sides of a right-angled triangle. In this case the triplet (3,4,5) is sometimes called a basic triplet because we can generate a family of triplets satisfying the property. Can you find any other basic triplets? Try the following exercise now.

E18) Find 3 other distinct basic triplets.

E19) Can 1 be a side of a right-angle triangle? Give reasons for your answer. What different types of number relations or properties are used in finding the answer?

In the next section we shall consider few more tools for exploring numbers.

16.4 GAMES AND PUZZLES

An important aspect of learning mathematics is its ability to keep one's mind occupied with interesting, challenging and mind stretching leisure-time activities like games and puzzles with numbers.

Here we start with a puzzle.

Puzzle

Take any three digit number. Now write the same number either after or before it, so that you get a six digit number. For example, if your three digit number is 432, the six digit number will be 432,432. Divide this number by 7. You will find it to be exactly divisible. Now divide the quotient by 11. This division, too, will give a whole number. Divide this number by 13. Not only will you find the division to be exact, but the result may surprise you. You should get your original 3-digit number back.

Here is an exercise for you.

E20) Try the example given above. Does it work for all the 3 digit numbers? Can you find out why this works?

You must have enjoyed doing this exercise. Without the attempt to unravel why this works the fact that this happens is very fascinating but this is more like magic. The important thing is to be able to understand the logic behind these examples and explore to what extent they work and why? For example a good exercise for class VII or VIII children could be to find out if it works similarly for 2 digit and 4 digit numbers and if it does why and if it does not why not?

Puzzles of this kind interest not only adults but also children. Along with mathematical games they form what is often called recreational mathematics. Calling it by this name however should not reduce its importance in your mind or make you feel that this is optional. It is an essential aspect to enable the learner to form independent concepts and consolidate her constructs. It is another issue as to how it would be enmeshed with the curricula. Its importance is that children find it interesting and engaging. They learn through these without being conscious of the effort or feeling the pressure and fear said to be associated with a mathematics class. By skillfully setting up puzzles and games in the context of relevant curricula, we can make the mathematics class an enjoyable learning experience for our students. Let us look at a few examples.

We shall start with a task where a friend of mine, who is a class teacher of Class 3, uses a game to practise multiplication.

Example 4: The teacher divides the whole class into 4 groups of 10 students each. To each group she gives two tables of numbers, a 6 × 6 square and a 3 × 3 squares like the one given in Table A and Table B.

Table A

145	407	667	1769	3233	265
1073	799	55	1961	253	517
583	305	391	671	493	319
85	187	1219	851	115	1081
1537	2491	185	629	1739	235
901	2867	1363	2257	1037	1403

Table B

5	23	47
11	29	53
17	37	61

The game is played by dividing each group of 10 into two teams of 5 students each. Both the teams have their own set of markers. Each team turn by turn, selects a pair of numbers from Table B and multiplies them and looks for the product in Table A. If the result of the multiplication exists in the table then they can put their markers on the answer in Table A. Obviously the marker can be put only if the number has not been covered till then.

Among the two teams the first to occupy four squares in a line, either in a row or a column or a diagonal, wins the game.

After they have played for sometime she selects from each group the winning team. Then the game is played with the winning team with the rest of the children watching. They act as referees and check if the selection and multiplication is happening properly. The children are then asked to sit in groups and make another board with Table A and Table B. These boards are exchanged between groups and used to play the game.

—x—

Try these exercises now.

E21) Explain what is the learning involved in the game tried in Example 4? How would the children choose the numbers to pick up?

E22) Can this game be used for addition and subtraction also. How? What is the exploration involoved in this kind of activity?

Do you think the above kind of tasks would be interesting for children? My friend who teaches in a school claims that in her experience the games as shown in the example always fascinate children. By this the teacher can make her class more interactive. Once students realise that you are not simply asking for a memorised routine or for the results of uninteresting calculation, they will loosen up and get on with the calculation process in a more relaxed yet serious manner. She argues that such processes involve lively discussions, intellectual arguments, and a greater degree of student involvement.

E23) What do you think about this view? Do you agree with it? Give reason for your answer.

Next we shall talk about another game which another friend of mine, who is a class teacher of Class 5 used to practise in her class V.

Example 5: The teacher was trying different strategies for teaching children how to find factors. One of the strategies she used was a game. In this game the teacher divided her class into two teams A and B. Then she explained the game as follows:

"A number like 24 which has several factors is given. This is called the 'Key number'. One team has to choose and declare a number less than 24. This number belongs to that team and one is added to their score. All the factors of this number, other than 1 and itself however belong to the rival team and as many points are added to their score as the number of factors.

The winning strategy, is therefore, to choose numbers in such a way so as to maximise one's team's score and minimise the other team's score. Once a number is used for score (i.e. a quoted number and its factors), it cannot be used again.

To start the game the teacher tossed a coin and the toss was won by team A. She told Team A to choose a number less than 24. The team selected the number 22. They got one point. Then the teacher asked Team B to find factors of this number. They chose numbers 11 and 2. Then Team B got 2 points. She wrote the points separately on the board.

Table 1

Team A	Team B
1	2
0	1
1	0

Table 2

The numbers chosen	Factors
22	11 × 2
23	-
13	-

Then she asked Team B to choose a number other than 2, 11, 22 less than 24. They selected 23. So they got a point. Then Team A was asked to find factors of it. Team A found that 23 did not have factors other than 1 and itself. Therefore they did not get any point. At this point they realised their mistake of choosing the number 22. They were careful afterwards. Next time they chose the number as 13 and they got one point for that, whereas their rival team, Team B did not get any score. To make note of the numbers already used, the teacher made another table in which she noted down the numbers and its factors. The game continued till all the numbers from 1 to 23 were used up. Team A scored more than Team B and Team A won the game.

The teacher told us that her students used to enjoy this game greatly. She said that it helped her students improve their understanding of 'finding factors'. Do you agree with her? Did children also learn something else?

—x—

You can devise similar games with your students and try to make them think and analyse. What are the other games you can think of?

E24) Devise a game for children near you. Explain your interaction with the children including - how it was presented and what did the children do with the task?

You must have come across several occasions where a group of children keep themselves busy by asking each other puzzles. Here we give a puzzle which can be used in classroom teaching.

Make 9 squares boards with the writings as shown in the figure. (see Fig.3)

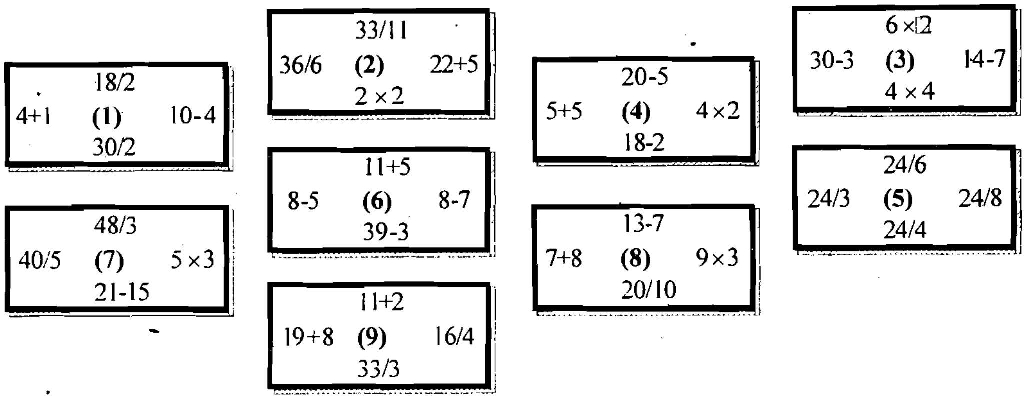

Fig.3

Along the four edges of each of the 9 pieces expressions, like 22+5, 36/6, etc. are written. The child has to arrange (or fit) these 9 squares in such a way that the value of the expression written on an edge is the same as the value of the expression written on an adjacent edge. Like this the child has to arrange all the pieces in a 3 × 3 square as shown in Fig.4

18/2 4+1 **(1)** 10-4 30/2	33/11 36/6 **(2)** 22+5 2×2	6×2 30-3 **(3)** 14-7 4×4
20-5 5+5 **(4)** 4×2 18-2	24/6 24/3 **(5)** 24/8 24/4	11+5 8-5 **(6)** 8-7 39-3
48/3 40/5 **(7)** 5×3 21-15	13-7 7+8 **(8)** 9×3 20/10	11+2 19+8 **(9)** 16/4 33/3

Fig.4

The above puzzle enables children to match all four edges or prepare cards in which only two sides need to be matched with adjacent neighbours. The cards can be made with varying degree of complexity for children of different classes and age. For example matching all four sides can be used in class V while matching 2 sides can begin from class 3 itself. In a discussion with teachers, where this puzzle was shown teachers gave many other possible ideas that could be used. One of the teachers suggested that we get the children to make such puzzles. This would not only be more enjoyable and more challenging but something that would remain outside the classroom with the children when they are on there own.

Now try these exercises concerning the above Puzzle.

E25) What are the children doing when they are solving the above puzzle? What could they be learning?

E26) Do you think such explorations and puzzles can be set up for numbers only? What else can they be used for? Make a similar Puzzle using names of geometrical figures and their properties e.g. rhombus and using the property that it has all sides equal.

Some of you may be familiar with the following kind of number puzzles which appear in daily news papers.

The number Puzzle presented here is called a 'challenger'.

Challenger

Directions

Fill each square with a number from 1 to 9 such that

- horizontal squares should add to totals on right.
- vertical squares should add to totals on bottom
- diagonal squares through centre should add to total in upper and lower right.

There may be more than one solution.

				14
			4	10
4				10
	5			10
		3		9
11	11	8	9	6

Were you able to do that quickly? Do you think children can do it too? Think about these and try these exercises.

E27) Find two solutions of the above puzzle.

E28) Do you think such activity helps in exploring numbers. Justify your answer.

We see that **games** and **puzzles** have both a recreational aspect as well as a mathematical (learning) aspect. They are challenging, stretching our imagination and making learning enjoyable.It is important for us as teachers to develop, modify and use these activities with children. This way we can help them develop a better conceptual understanding of numbers and further develop the ability to generalise, test, build logical constructs and proofs etc. . The importance of puzzles and other explorations as we have seen is manifold. W we need to consider it our task to construct such puzzles and puzzle making and solving situations.

With this we come to an end of the unit and block.

16.5 SUMMARY

In this unit we have covered the following points.

1) We have explained what exploring numbers means and what is the importance of such an activity in learning mathematics. We have also considered different tasks which enable us to (or children) to explore numbers.
2) We have discussed some specific means of exploring numbers like playing/dealing with number patterns. We have also shown how such activities help to develop a deeper conceptual understanding of numbers.
3) We have considered different puzzles and games. We have shown that they have both recreational aspect as well as mathematical learning aspect. We have explained how different games and puzzles can be constructed and used to develop in children different mathematical aspects like generalisation, building logical constructs, proofs etc.

16.6 COMMENTS ON EXERCISES

E1) Bulding confidence for learning mathematics. You can think of more advantages.

E2) In this case the answer is always 11. To see that, assume that the number is N. Then, following his teps, we get
$N + 3 + 10 - 7 + 5 - N$,
which is always 11.
The conversation can be changed by changing either 3, 7, 10,7 or 5.

E3) Yes. Exploration is involved when the child finds the expression
$N + 3 + 10 - 7 + 5 - N$.

E4) Yes. Because in this way they learn different relationship with numbers. For example, by understanding whey this trick works, the child learns that the number which is assumed is added and subtracted, therefore becomes immeterial and many more can be learned from this.

E8) Because of the base 10 number system and 9 = 10-1.

E9) Look at multiplication table of 3 and see that if you add the digits in each product you will get either 3,6 or 9.
Like that you can try in other tables. It is not necessary that there is a striking pattern in a multiplication table.

E10)

$$\begin{aligned}
9 \times 1 &= 9 = 1 \times 5 + 4 = 14\\
9 \times 2 &= 18 = 3 \times 5 + 3 = 33\\
9 \times 3 &= 27 = 5 \times 5 + 2 = 52\\
9 \times 4 &= 36 = 7 \times 5 + 1 = 71\\
9 \times 5 &= 45 = 9 \times 5 + 0 = 90\\
9 \times 6 &= 54 = 10 \times 5^2 + 4 = 204\\
9 \times 7 &= 63 = 12 \times 5^2 + 2 \times 5 + 3 = 14\\
9 \times 8 &= 72 = 2 \times 5^2 + 4 \times 5 + 2 = 242\\
9 \times 9 &= 81 = 3 \times 5^2 + 1 \times 5 + 1 = 311\\
9 \times 10 &= 90 = 3 \times 5^2 + 3 \times 5 = 330
\end{aligned}$$

The pattern does not hold in this sytem.

E11)

$$\begin{aligned}
1 \times 29 &= 29\\
2 \times 29 &= = 58\\
3 \times 29 &= 87\\
4 \times 29 &= 116\\
5 \times 29 &= 145\\
6 \times 29 &= 174\\
7 \times 29 &= 203\\
8 \times 29 &= 232\\
9 \times 29 &= 261\\
10 \times 29 &= 290
\end{aligned}$$

Here we observe that the digits in the unit's place are numbers 0 to 9 in decreasing order.
It is because of the base 10 number system.

E13)Ask children to mark every multiple of a certain number (say every 6th number starting from 6). Given the numbers for which LCM has to be found, look for all other numbers whoseLCM is to be found and mark the multiples of all these similarly. One number can be marked as a multiple of many different numbers. The number which is marked in both (or all) is a common multiple. You could do it for as many numbers for which you have to find the lowest common multiple. The numbers marked for all are the commomultiples. The least them is the LCM. For HCF, from the patterns filled up for multiples of 2,3,5,7,11 etc. pick out the numbers of which you need the HCF. Find out all those numbers that are factors of ech of these numbers and write them down for all the numbers. Look for common factors between the numbers - separate the common factors and multiply them to get the highest common factor. Ensure that while multiplying if the chosen numbers have common factors then retain the common factor only once. for example 72, 81, 144 have common factors 1,3,9. The HCF is not 9x3 but 9 since the common factor 3 is not to be counted again. Analyse both these and see if you can evolve a common marking technique for both.

In finding HCF and LCM this way one gets to use the definition or rather conception of these terms again and again. While getting children to do this task or while doing it yourself think about what this makes you do. Also think about whether this has helped you formulate a clearer understanding of LCM and HCF? It, yes, how did it do it? In what way did it help? For example you could say it helps you to see the process of finding multiples and also factors.

E14)You could check it by taking for example 15 and 8

$n_1 - n_2 = 7$

and 7 is coprime to 8 as well as 15.

Try it for other numbers as well then you have to show it logically.

For example you could assume that $n_1 - n_2$ is not coprime to either n_1 or n_2 and then see if it gives a problem. Let us assume that $n_1 - n_2$ is not coprime to n_1 and both have a common factor. You can write then $n_2 = n_1 - (n_1 - n_2)$

Let $n_1 = x_1, x_2, \ldots.$ Since $(n_1 - n_2)$ is not coprime to n_1 therefore they have at least one common factor other than one, say x_1. Let us write $n_1 - n_2 = x_1y_1y_2\ldots$

As is obvious from here

$$\begin{aligned} n_2 &= x_1x_2x_3\ldots - x_1y_1y_2y_3\ldots \\ &= x_1(x_2x_3.. - y_1y_2y_5\ldots) \end{aligned}$$

Therefore n_2 and n_1 have atleast one common factor except 1.

Others can be proved similarly.

E19)No, because if $N^2 + 1 = M^2$ for some positive integers M and N, then M=1 and N=0, which is not possible. Similarly, '1' cannot be the hypotenuse also.

E20)It is true for any three digit number. To see this suppose that the three digit number is $n_1n_2n_3$. Then we have

$$n_1n_2n_3, n_1n_2n_3 = 1001 \times n_1n_2n_3$$

and the number 1001 is divisible by 7,11 and 13. Also we have

$$7 \times 11 \times 13 = 1001$$

Therefore, after all the computations we get the number $n_1n_2n_3$.

21) 1) The process of multiplication of two numbers

2) Factorising into two factors: checking whether a number is a factor or not

If children find those factors of numbers given in Table A which are in Table B and then choose those numbers from Table B, they can finish the game fast.

E22) Yes.

Make corresponding tables for addition/subtraction. The exploration involved in this case is in observing that a number in Table A can be written as a sum/difference of two numbers in Table B.

E27)

				14
2	3	1	4	10
4	1	2	3	10
2	5	2	1	10
3	2	3	1	10
11	11	8	9	6

NOTE