UNIT 2 MODEL FORMULATION

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2.1 INTRODUCTION

In Unit 1 we introduced you to the need of modelling a real life situation and the role of mathematics in it. Various steps involved in modelling a real life situation were discussed there. We classified the mathematical models into various types and illustrated them through examples from population dynamics, optimization problem, queueing system, etc. In this unit, we shall proceed with the next step in modelling i.e., given a real world problem, how do you convert it to model abstraction leading to a mathematical equation? We shall draw your attention to the various modelling aspects which need to be taken into account while formulating a mathematical model.

In Sec. 2.2, we shall discuss the method of identifying the essentials of a problem which need to be incorporated into the model. In Sec. 2.3, we shall focus on an example from the field of finance, formulate a model and estimate the financial implications and risks associated with reality. Whenever you decide to make some investment in the market, your concern is to choose an investment strategy which maximizes your profit and minimizes the risk involved. But how to select such a strategy? Solution to this problem as provided by Markowitz will be discussed in Sec. 2.3. Sharpe's model which simplifies the rigours involved in various computations using Markowitz's model will also be discussed here.

Objectives

After studying this unit, you should be able to:

- identify the essentials of a given real world problem;
- formulate a model incorporating all the essentials required for the problem;
- obtain a solution of the formulated problem and interpret the result.

In addition to the above objectives, using the discussed model from the field of finance, you should also be able to

i) explain the meaning and calculation of expected return and risk measures for an individual security;
ii) explain the portfolio selection problem of investments;
iii) explain portfolio return and risk measures as formulated by Markowitz;
iv) explain what the efficient frontier is and how important it is to investment analysis.

2.2 ESSENTIALS OF A PROBLEM

Mathematical modelling involves transforming some idealised form of the real-world situation into mathematical terms. It is, therefore, an activity which requires more than the ability to just solve complex sets of equations. Though there is no hard and fast approach to develop a model, you still need to broadly follow the following steps in the beginning.

1) **Establish a main purpose:** Models rarely replicate a system and also they can mean different things to different people. For example, a business man and a biologist have entirely different points of view of looking at the camel as depicted in Fig. 1.

![Diagram of different conceptual views of a camel](image)

Fig. 1: Different conceptual views.

Their conceptual views of the same system or object are rather different since they are both heavily influenced by their own environment, background and objectives. For example, a biologist would be interested in the anatomy and physiology of a camel whereas, a business man’s concern would be the profit he can earn from it. The same is true when we come to the mathematical modelling of any system or process. Although the motivation for building the model is usually to find the means to answering a particular question, the form of that question influences the way in which we build the mathematical model.

In the case of the familiar problem of the motion of a simple pendulum, what is our main purpose? It is to find the period of oscillation, or, angular velocity of the bob, or, we may want to know the tension in the string. Thus, before developing a model we must be clear about the purpose of doing it.
ii) **Sifting essentials from the non-essentials:** Real situations are quite complex. Developing a model which accounts for all the aspects of a real world phenomenon is likely to be difficult, mathematically complex and unmanageable. On the other hand, it may also be possible that a model not accounting for all the aspects of the phenomenon under consideration, is easier to develop and handle and it may still serve our purpose. It is, therefore, necessary to keep a balance between the complexity of the problem and the objectives of our study. For example, while studying the motion of a simple pendulum, if you assume the oscillations to be small and restrict the range of the values of the amplitude $\theta$, to say $|\theta| \leq 30^\circ$, then you only need to study a linear model and hence solve a linear differential equation \[ \frac{d^2\theta}{dt^2} + \frac{g}{\ell} \theta = 0, \] where $\ell$ is the length of the pendulum and $g$ is acceleration due to gravity. On the other hand, if you have a problem in which you cannot assume the oscillations to be small then you need to study a non-linear model resulting in a non-linear differential equation \[ \ell \frac{d^2\theta}{dt^2} + g \sin \theta = 0 \] (ref Unit 3, Block-1 of MTE-14).

Similarly, a realistic study for understanding the population dynamics of an infectious disease is one which differentiates the population by age, gender, geographical location, social stratification and takes into consideration factors like immigration, emigration, birth, death, incubation period of the infection and removal of infectives from circulation. This study will be definitely superior but more complex. The model developed would involve more dependent variables and hence more number of differential equations to be solved as compared to the model where, the population considered is assumed to be closed, homogeneously mixed, with no incubation period of infection and no removal of infectives from circulation.

**The search for essentials of the problem** is related to the main purpose of the model. For instance, the assumption of no removal of infectives in general, is highly restrictive in the context of a human population. However, this might be a reasonable approximation to the early stages of some upper respiratory infections where a long time may elapse before an infective is removed from circulation. A mild cold infection in a classroom may also be considered as an example of no removal in a human population. On the other hand, the assumption of no removal is fairly applicable to epidemics in insect and plant populations. In the case of a common fungal disease of flies, the diseased or dead fly remains attached to a leaf or a blade of grass, creating a situation roughly analogous to no removal. The dead or diseased plants in a forest are rarely removed, and if dead plants are infectious, the assumption of no removal is fulfilled. If you are interested in knowing more about modelling the spread of infectious diseases, you may refer to Unit 10, Block-3 of MTE-14.

iii) **Significance of essentials:** Once the essentials of a problem are identified, they determine the number and type of the variables in the formulation and hence the complexities involved in problem solving. The variables involved may be discrete, continuous or piece-wise
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continuous, deterministic, stochastic, fuzzy or any combination of these. For example, biological population of a given species of large sized animals is an example of deterministic and discrete quantity. Flow of money in a market through organized financial institutions is deterministic and continuous. The sale of long sized items like trucks, car, aeroplanes per year is discrete and stochastic. Flow of water in a canal may be piecewise continuous when the upstream gates are opened or closed according to rainfall in the catchment area. The response level in a human neural network is a fuzzy quantity.

As we have already mentioned in Unit 1, the types of variables involved in a formulation determine the type of model formulated and hence the type of mathematics required to solve the problem. We shall be illustrating this point in the next section through an example. But, before that, you may try the following exercises.

E1) State the type of modelling you will use for the following problems giving reasons for your answers. List the essentials and non-essentials in the problems.

i) The economic viability of an insurance company depends critically on its ability to assess risks and decide on the premium charged to cover risks. If the premium are low, then payouts can exceed revenue collected and the company can go bankrupt. On the other hand, if they are high, the number of customers will go down, thus affecting profitability. The problem is to develop a model to help the insurance company decided the premium it should charge for different risks to ensure economic viability and maximise its profits.

ii) A company manufacturing soft drinks is thinking of expanding its plant capacity so as to meet future demand. The monthly sales for the past 5 years are available. The problem is to develop a model to obtain good estimates for future demand so as to help the company make the right decisions.

E2) Give examples of at least two real life situations where mathematical treatment of the problem is the only approach to find the solution of the problem. Why do you think that there is no other scientific alternative for the treatment of these problems? List the essentials and non-essentials in these problems.

E3) Give at least two examples each of physical situations in which the variables involved are i) continuous and stochastic; ii) piece-wise continuous and stochastic; iii) fuzzy.

Let us now illustrate the mathematical formulation of real world situations from the field of finance.

2.3 MODELS FROM FINANCE

All of us make some or the other kind of investments to secure our future, old age, for our children’s education, their marriage etc. People invest in shares, stocks, bonds, mutual funds, etc. using an investment strategy (called security)
related to the expected market scenario in the future. Our aim is to select a security or a group of securities (i.e., a portfolio) which maximize the expected return and minimize the uncertainty (i.e., risk). But then the problem is how to choose such a security or a portfolio? Markowitz’s model, named after Harry Markowitz, an economist, who is best known for his pioneering work in Modern Portfolio theory, provides a solution to this problem. He studied the effects of asset risk, correlation and diversification on expected investment portfolio return. We shall discuss this model here, but before that let us define some of the terms which are the essentials for the model under consideration.

Securities

When you borrow some money or take a loan from a broker then you have to leave some item of values as security with the broker or sign a piece of paper promising repayment with interest. Failure to repay the loan (plus interest) means that the broker can sell your item to recover the amount of the loan (plus interest) and perhaps make a profit. The terms of agreement are recorded when the deal is made. This piece of paper serving as evidence is called a security. Similarly, you may have some spare money which you would like to lend to earn some profit out of it. You then think of investing your money in Government Bonds, saving certificates, shares, mutual funds, etc. These investment strategies are called the securities. Broadly speaking, a security helps us to save our funds in the event of default (and that is why the name). It may be a simple promissory note, share of the common stock, a bond, etc.

Return

Once the investment has been made, you are interested in knowing how good was your investment strategy. For this, you need to calculate the rate of return on the investment strategy. What is the rate of return? It gives a relation between the initial input and final output of an investment and is calculated as follow.

\[
\text{Return} = \frac{\text{End of period value} - \text{Beginning of period value}}{\text{Beginning of period value}}. \quad (1)
\]

Risk

Risk denotes the probability of specific eventualities which may have both beneficial and adverse consequences. However, in general usage the convention is to focus only on potential negative impact of the investment strategy. Often, it is described as a situation which would lead to negative consequences.

Uncertainty

It is the lack of certainty. You may have limited knowledge and it is impossible for you to exactly describe the existing state or future outcome of an investment strategy. For example, there is always an uncertainty while investing in the stock market or mutual funds.
Portfolio

Suppose you want to invest an amount $W_0$ in $n$ securities (say). Let $w_i$ be the proportion of $W_0$ that you invest in the $i^{th}$ security ($1 \leq i \leq n$). Then the ordered $n$-tuple $P = (w_1, w_2, \ldots, w_n)$, where $\sum_{i=1}^{n} w_i = 1$, is called a portfolio of $n$ securities and $w_i$ is its $i^{th}$ portfolio value (or weight).

For example, $P = \left(0, \frac{1}{3}, \frac{2}{3}\right)$ is a portfolio of three securities where no amount is invested in the first security, $\frac{1}{3}$rd of the total funds are invested in the second and $\frac{2}{3}$rd in the third security. By changing the value of $w_i$ in $P$, subjected to the condition that $\sum_{i=1}^{n} w_i = 1$, all the portfolios of given $n$ securities i.e., a feasible set can be obtained. Formally, we give the following definition.

**Definition:** The set of all the possible portfolios which can be constructed from a given set of securities is called the feasible set or the opportunity set.

Given a portfolio $P = (w_1, w_2, \ldots, w_n)$ of $n$ securities, our main purpose is to see the effect of portfolio values $w_i$ on the terminal value of the return on the portfolio $P$. But each $w_i$ is a certain proportion of the initial funds that are invested in $i^{th}$ security of the portfolio $P$. Thus, for quantifying the return and risk of the portfolio $P$, we have to calculate the return and risk of its constituting $n$ securities. It is therefore, important to select the proportions $w_i$ of the initial funds in such a way that the portfolio $P = (w_1, w_2, \ldots, w_n)$ is optimally good according to our investment objectives. Such a portfolio which provides an investor the maximum level of satisfaction is called an Optimal Portfolio and the problem of selecting such a portfolio is referred to as portfolio selection problem. Thus, as a first step towards modelling we state the formulated problem as follows:

**Portfolio Selection Problem:** From a set of feasible portfolios of risky securities, how can an investor choose a portfolio that gives him/her the maximum level of investment satisfaction (in terms of return and risk)?

Before we take up the problem you may try the following exercises.

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E4) At the end of year 2005, Mohan decided to invest Rs.30,000 in a portfolio of stocks and bonds. Rs.10,000 were put into common stocks and Rs.20,000 into corporate bonds. At the end of 2006, Mohan's stock and bond holdings were worth Rs.13,000 and Rs.16,000, respectively. During 2006, Rs.500 in cash dividends was received on the stocks and Rs.1000 interest payments was received on the bonds. What was the percentage return on
i) Mohan’s stock portfolio during 2006?

ii) Mohan’s bond portfolio during 2006?

iii) Mohan’s total portfolio during 2006?

E5) Given a set of eight securities with portfolio values $w_i's = 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Find a feasible set of portfolio of these securities.

Let us now discuss the Markowitz’s approach to solving the portfolio selection problem.

2.3.1 Markowitz Model

Markowitz’s approach to portfolio selection begins by assuming that an investor has a given sum of money to invest at the present time. This money will be invested for a particular length of time, known as the investor’s holding period. Markowitz’s approach is a single period approach where at the beginning of the period say $t = 0$, the investor must make decision on what particular securities to purchase and hold until the end of the period say $t = 1$. Since a portfolio is a collection of securities, this decision is equivalent to selecting an optimal portfolio from a set of possible portfolios. Typically it is seen that an investor wants the returns to be high as well as, as certain as possible. This means that the investor, looking forward to maximize expected return and minimize risk has two conflicting objectives that must be balanced against each other when making the purchase decision at $t = 0$. The Markowitz approach for how the investor should go about making this decision gives full consideration to both these objectives. He used certain concepts from probability theory to solve this problem. According to Markowitz, the investor should view the rate of return associated with any of these portfolios to be what is known in statistics as a random variable. Expected rate of return can be viewed as a measure of the potential reward associated with any portfolio and standard deviations can be viewed as a measure of risk associated with any portfolio. Thus, once each portfolio has been examined in terms of its potential rewards and risks, the investor is in a position to identify the portfolio that appears most desirable to him or her.

Thus, as a second step in the model building process, we can say that the Markowitz model is based on the following assumptions:

- Investor invest money for a particular length of time, called the holding period.
- Investors are rational and behave in a manner as to maximize their utility with a given level of income or money.
- Investors are risk averse and try to minimize the risk and maximize return.
- Investors prefer higher returns to lower returns for a given level of risk.
- The evaluation of portfolios is carried out in terms of returns and the risk associated with the constituting securities, over a given holding period.
The obvious question which must be occurring in your mind is – How to quantify the expected return and risk of a security or a portfolio? Let us try to do that as a next step in the modelling process.

**Expected Return for a Security**

Some investments, like bank certificates of deposit, have guaranteed rates of return. Investments like stocks are a bit more complicated. If you ask an investor how much she expects to earn on the stock of a pharmaceutical company or high-tech firm, she will answer in terms of a range like “I expect to earn somewhere between 10 and 20 percent on these shares”. Whenever an investor describes future returns in terms of a range like this, you can be sure that there is risk involved. In the investment context, risk is the uncertainty that a given investment will earn its anticipated rate of return.

To calculate the expected rate of return, you have to first enumerate all the possible rates of return that an investment could have. For simplicity’s sake, let’s imagine an investment with four possible rates of return, -10%, -5%, 10% and 20%.

The first two rates of return indicate a loss while the last two indicate a gain. The next step is to assign probabilities to each rate of return. How do you assign these probabilities? It requires to make some educated guesses based on past performance of the investment itself, and the demonstrated performance of similar investments. General market and economic factors should also be taken into account, while assigning these probabilities. Let the probabilities 0.1, 0.1, 0.5 and 0.3 are assigned respectively to the above four rates of return.

Then the expected rate of return or simply expected return $E(R)$ of the given investment (security) is calculated as follows:

$$E(R) = \sum_{i=1}^{n} (\text{Possible Return} \times \text{Probability})$$

$$= \left[ (-0.10) (0.10) + (-0.05) (0.10) + (0.10) (0.50) + (0.20) (0.30) \right]$$

$$= 0.095.$$ 

The expected return is 0.095. This means that the investor can expect a return of 9.5% on her investment.

Thus, for any security, the expected return is the weighted average all possible outcomes, where each outcome is weighted by its respective probability of occurrence. It is calculated as

$$E(R) = \sum_{j=1}^{n} R_j \ p_{ij}$$

where

$E(R) = \text{the expected return on a security,}$

$R_j = \text{the } j^{th} \text{ possible return,}$

$p_{ij} = \text{the probability of the } i^{th} \text{ return } R_j,$

and

$n = \text{the number of possible returns or outcomes.}$
You may now use the above result to do the following exercise.

E6) Let the return distribution on a security A be given as follows:

<table>
<thead>
<tr>
<th>Possible rate of return (in percent) (R$_j$)</th>
<th>-8</th>
<th>-6</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associated probabilities (p$_{ij}$)</td>
<td>0.04</td>
<td>0.06</td>
<td>0.2</td>
<td>0.3</td>
<td>0.25</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Find the expected return of the security A.

Risk for a Security

In Markowitz sense, the variability or risk of a security is measured by its standard deviation. To calculate the risk associated with the expected return, the variance or standard deviation is used. As you already know standard deviation is a measure of the spread or dispersion in the probability distribution, that is, it measures the dispersion of a random variable around its mean. The larger this dispersion, the larger the variance or standard deviation and hence larger is the risk for a security. Knowing the returns associated with the securities the variance of the returns is calculated as

\[
\sigma^2 = \sum_{j=1}^{n} (R_j - E(R))^2 p_{ij}
\]  

(4)

where

- \(\sigma^2\) = the variance of returns,
- \(E(R)\) = the expected return on a security,
- \(R_j\) = the \(j^{th}\) possible return,
- \(p_{ij}\) = the probability of the \(i^{th}\) return \(R_j\), and
- \(n\) = the number of possible returns.

\(\sigma = (\sigma^2)^{1/2}\), the positive square root of the variance is the standard deviation of the returns.

Let us consider the following example.

Example 1: Assume that the return distribution of a security is as given follows:

<table>
<thead>
<tr>
<th>Possible Return</th>
<th>0.01</th>
<th>0.07</th>
<th>0.08</th>
<th>0.1</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associated Probability</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td>0.1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Find the standard deviation of the security.

Solution: The calculations for the standard deviation are as shown in Table 1.
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Table 1

<table>
<thead>
<tr>
<th>Possible Return (R_j)</th>
<th>Associated Probability (P_{ij})</th>
<th>R_j \times P_{ij}</th>
<th>(R_j - E(R))^2</th>
<th>(R_j - E(R))^2 P_{ij}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.2</td>
<td>0.002</td>
<td>0.0049</td>
<td>0.00098</td>
</tr>
<tr>
<td>0.07</td>
<td>0.2</td>
<td>0.014</td>
<td>0.0001</td>
<td>0.00002</td>
</tr>
<tr>
<td>0.08</td>
<td>0.3</td>
<td>0.024</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.10</td>
<td>0.1</td>
<td>0.010</td>
<td>0.0004</td>
<td>0.00004</td>
</tr>
<tr>
<td>0.15</td>
<td>0.2</td>
<td>0.030</td>
<td>0.0049</td>
<td>0.00098</td>
</tr>
</tbody>
</table>

\[ \sum P_{ij} = 1 \quad \sum R_j P_{ij} = 0.08 \quad \sum (R_j - E(R))^2 P_{ij} = 0.00202 \]

\[ \sigma = (0.00202)^{1/2} = 0.0449 = 4.49\% \]

You may now try the following exercises.

E7) Calculate the expected return and risk of a security given the following information

<table>
<thead>
<tr>
<th>Probabilities</th>
<th>0.15</th>
<th>0.20</th>
<th>0.40</th>
<th>0.10</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Possible returns</td>
<td>0.20</td>
<td>0.16</td>
<td>0.12</td>
<td>0.05</td>
<td>−0.05</td>
</tr>
</tbody>
</table>

E8) Compare the risk of the two securities 1 and 2 whose return distributions are as given in Table 2 below:

Table 2

<table>
<thead>
<tr>
<th>Possible rate of return for security</th>
<th>Associated probabilities P_{1j} = P_{2j}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0.19</td>
<td>0.09</td>
</tr>
<tr>
<td>0.17</td>
<td>0.16</td>
</tr>
<tr>
<td>0.11</td>
<td>0.18</td>
</tr>
<tr>
<td>0.10</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td></td>
</tr>
</tbody>
</table>

When analysing the returns and risks on an investment made by an investor we must be concerned with the total portfolio held by an investor. Individual security returns and risks are important, but it is the return and risk to the investor's total portfolio that ultimately matters, because investment opportunities can be enhanced by packaging them together to form portfolios.

**Portfolio Expected Return**

The expected return on any portfolio is easily calculated as a weighted average of the individual securities' expected returns. For a portfolio \( P = (w_1, w_2, \ldots, w_n) \) of \( n \) securities, where \( w_i \)'s are the portfolio values (or weights), the expected return can be calculated as

\[
E(R_p) = \sum_{j=1}^{n} w_j E(R_j)
\]
where

\[ E(R_p) = \text{the expected return on the portfolio} \]

\[ w_j = \text{the portfolio weight for the } j^{th} \text{ security,} \]

\[ \sum_{j=1}^{n} w_j = 1, \]

\[ E(R_j) = \text{the expected return on the } j^{th} \text{ security, and} \]

\[ n = \text{the number of different securities in a portfolio.} \]

Consider for example a portfolio consisting of three stocks X, Y and Z with expected returns of 12%, 20% and 17% respectively. Assume that 50% of the funds is invested in security X, 30% in Y and 20% in Z. Then the expected return on this portfolio is

\[ E(R_p) = 0.5 \times 12\% + 0.3 \times 20\% + 0.2 \times 17\% = 15.4\%. \]

It may be noted that regardless of the number of assets (securities) held in a portfolio, or the proportion of total investable funds placed in each asset, the expected return on the portfolio is always a weighted average of the expected returns of individual asset in the portfolio.

**Portfolio Risk**

Portfolio risk is measured by the variance (or standard deviation) of the portfolio’s return, exactly as in the case of each individual security. Although the expected return of a portfolio is a weighted average of its expected returns, portfolio risk is not a weighted average of the risk of the individual securities in a portfolio. Mathematically,

\[ E(R_p) = \sum_{j=1}^{n} w_j E(R_j). \]

But,

\[ \sigma_p^2 \neq \sum_{j=1}^{n} w_j \sigma_j^2. \]

Portfolio risk is a unique characteristic and not simply the sum of individual security risk. This is because a security may have a large risk if it is held by itself but risk may be small when held in a portfolio of securities. But then the question is how a portfolio of assets can reduce risk and how the risk is measured? Let us analyze the portfolio risk.

**Analyzing Risk of a Portfolio**

Let us first assume that rates of return on individual securities are statistically independent such that rate of return on any one security is unaffected by the rate of return on another security. In this situation, the standard deviation of the portfolio is given by

\[ \sigma_p = \frac{\sigma_j}{\sqrt{n}}. \]

Thus, \( \sigma_p \) decreases as \( n \) increases i.e., the risk of the portfolio will be reduced as more securities are added to the portfolio. You do not have to take decision about which security to add, as all of them have identical properties. The only concern is how many securities are to be added? However, in the real world, the assumption of statistically independent returns on stocks is unrealistic.
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Most stocks are positively correlated with each other, that is, the movements in their returns are related. For example, a rise in interest rates will adversely affect most of the firms, because most of the firms borrow funds to finance part of their operations. Therefore, there is a need for diversification i.e., putting small fractions of the total funds in as many securities as are found suitable after their evaluation is done.

**Diversification**

Diversification is the key to the management of portfolio risk. It allows the investors to reduce portfolio risk without affecting returns. Thus, there is a need to quantify diversification. Markowitz was the first one to quantify the concept of diversification. He showed quantitatively why and how portfolio diversification helps to reduce the risk of an investor's portfolio. Markowitz showed that to measure and reduce portfolio risk to its minimum level for any given level of return, we must measure the interrelationships among security returns. The reason being that in any one time period, poor performance by some securities may be offset by strong performance in other securities. Let us now consider how to measure these interrelationships, or co-movements or covariance, among security returns.

**Covariance in Security Returns**

Covariance is an absolute measure of the co-movements between security returns used for calculating portfolio risk. Therefore, in order to calculate portfolio variance or standard deviation we need to calculate covariance between securities in a portfolio. The formula for calculating covariance between two securities A and B is given as follows:

$$\sigma_{AB} = \frac{\sum_{j=1}^{n} [(R_{A,j} - E(R_A)) \cdot (R_{B,j} - E(R_B)) \cdot p_{ij}]}{n}$$  \hspace{1cm} (8)

where

- $\sigma_{AB}$ = the covariance between securities A and B,
- $R_{A,j}$ = $j^{th}$ possible return on security A,
- $E(R_A)$ = the expected value of the return on security A, and
- $n$ = the number of likely outcomes for a security for the period.

Eqn. (8) gives the covariance as the expected value of the product of deviations from the mean. Covariance can be positive, negative or zero.

- **Positive covariance** indicates that the returns on the two securities tend to move in the same direction at the same time. When one increases (decreases), the other also do the same.
- **Negative covariance** indicates that the returns on the two securities move inversely. When one increases (decreases), the other tends to decrease (increase).
- **Zero covariance** indicates that the returns on two securities are independent and do not move together in the same or opposite directions.

The covariance $\sigma_{AB}$ when divided by the standard deviations of securities A and B gives
\[ \rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} \]  

(9)

where \( \rho_{AB} \) is called the correlation coefficient of securities A and B.

From Eqn. (9) we can write

\[ \sigma_{AB} = \rho_{AB} \sigma_A \sigma_B. \]  

(10)

The correlation coefficient \( \rho_{AB} \) is a statistical measure of relative movements between security returns. It measures the extent to which the returns on any two securities are related and is bounded by \(+1.0\) and \(-1.0\).

When \( \rho_{AB} = +1.0 \), the returns are perfectly positively correlated, the risk of a portfolio is simply a weighted average of the individual risks of the securities.

When \( \rho_{AB} = -1.0 \), the returns are perfectly negatively correlated. The returns of the securities have a perfect inverse linear relationship to each other. When the returns of one security is high that of other is low.

With zero correlation, there is no linear relationship between the returns on the two securities. The risk of the portfolio reduces when two securities with zero correlation are combined with each other.

Let us now consider an example to understand how \( \rho_{AB} \) and \( \sigma_{AB} \) is calculated between two securities A and B.

**Example 2:** Assume that the return distribution on the two securities X and Y be as given in Table 3 below.

<table>
<thead>
<tr>
<th>Possible rates of returns for security</th>
<th>Associate probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>.19</td>
<td>.18</td>
</tr>
<tr>
<td>.17</td>
<td>.16</td>
</tr>
<tr>
<td>.11</td>
<td>.11</td>
</tr>
<tr>
<td>.10</td>
<td>.9</td>
</tr>
</tbody>
</table>

Find \( \sigma_{XY} \) and \( \rho_{XY} \).

**Solution:** Calculations for \( \sigma_{XY} \) are shown in Table 4 below:

<table>
<thead>
<tr>
<th>Possible Returns</th>
<th>Associated Probabilities</th>
<th>Expected Returns</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{Xj} )</td>
<td>( R_{Yj} )</td>
<td>( P_{Xj} = P_{Yj} = P_j )</td>
<td>( R_{Xj} \times P_j )</td>
</tr>
<tr>
<td>.19</td>
<td>.18</td>
<td>0.33</td>
<td>0.0627</td>
</tr>
<tr>
<td>.17</td>
<td>.16</td>
<td>0.25</td>
<td>0.0425</td>
</tr>
<tr>
<td>.11</td>
<td>.11</td>
<td>0.22</td>
<td>0.0242</td>
</tr>
<tr>
<td>.10</td>
<td>.09</td>
<td>0.20</td>
<td>0.0200</td>
</tr>
</tbody>
</table>

\[ \sum R_{Xj} P_j = 0.1494 \]
\[ \sum R_{Yj} P_j = 0.1416 \]

\[ E(X) \]
\[ E(Y) \]
You can check that
\[
\sigma_X = \left\{ 0.33 \times (0.0406)^2 + 0.25 \times (0.0206)^2 + 0.22 \times (-0.0394)^2 + 0.2 \times (-0.0494)^2 \right\}^{1/2}
= (0.0014)^{1/2} = 0.038
\]
\[
\sigma_Y = \left\{ 0.33 \times (0.0384)^2 + 0.25 \times (0.0184)^2 + 0.22 \times (-0.0316)^2 + 0.2 \times (-0.0516)^2 \right\}^{1/2}
= (0.0013)^{1/2} = 0.036
\]
then \( \sigma_{XY} \) can be calculated using Formula (1) as
\[
\sigma_{XY} = 0.33 \times 0.0406 \times 0.0384 + 0.25 \times 0.0206 \times 0.0184
+ 0.22 \times (-0.0394) \times (-0.0316) + 0.20 \times (-0.0536) \times (-0.0516)
= 0.0013.
\]
Hence, by Formula (2), we have
\[
\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.0013}{0.038 \times 0.036} = 0.95 < 1.
\]

You may now try the following exercise.

E9) For the data given in E8), find the covariance \( \sigma_{12} \) and correlation coefficient \( \rho_{12} \) between two securities 1 and 2.

Knowing the correlation and covariance that gives the comovement in security returns, we are now ready to calculate portfolio risk. We start with the case of two securities and then generalise it to \( n \) securities.

If \( P = (w_1, w_2) \) be a portfolio of two securities 1 and 2 with \( w_1 + w_2 = 1 \), then the risk of the portfolio, as measured by the standard deviations of returns is given by
\[
\sigma_P = \left[ w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \rho_{12} \sigma_1 \sigma_2 \right]^{1/2}
\]
where
\( \sigma_P \) = the standard deviation of the portfolio \( P \),
\( \sigma_1 \) = the standard deviation of security 1,
\( \sigma_2 \) = the standard deviation of security 2,
\( w_1 \) = the portfolio weight of security 1, and
\( w_2 \) = the portfolio weight of security 2.
\( \rho_{12} \) = the correlation coefficient between security 1 and 2.

Portfolio risk is effected both by the correlation between assets and by the percentage of funds invested in each asset. We shall now illustrate it through an example.

Example 3: Consider a portfolio \( P \) of two securities \( X \) and \( Y \) whose return distributions are as given in Example 2. Let 50% of investable funds is to be placed in each security. Find the risk for the portfolio \( P \).
Solution: We have the following data from Example 2.

\[ \sigma_X = 0.0038, \sigma_Y = 0.036, \rho_{XY} = 0.95. \]

Also we have \( w_1 = 0.5, w_2 = 0.5 \).

Then the risk \( \sigma_P \) of the portfolio \( P \) is obtained as

\[
\sigma_P = \left[ (0.5)^2 (0.038)^2 + (0.5)^2 (0.036)^2 + 2(0.5)(0.5)(0.038)(0.036)(0.95) \right]^{1/2} \\
= [0.0007 + 0.0006 + 0.0006 \times 0.95]^{1/2} \\
= 0.0436
\]

You may note that the risk of the portfolio depends heavily on the value of the correlation coefficient between the returns of the two securities as is evident from the following values

\[
\begin{align*}
\rho &= +1.0; \quad \sigma_P = 0.0447 \\
\rho &= +0.5; \quad \sigma_P = 0.04 \\
\rho &= +0.15; \quad \sigma_P = 0.0373 \\
\rho &= 0.0; \quad \sigma_P = 0.036 \\
\rho &= -0.5; \quad \sigma_P = 0.0316 \\
\rho &= -1.0; \quad \sigma_P = 0.0265
\end{align*}
\]

The risk of the portfolio steadily decreases from 0.0447 to 0.0265 as the correlation coefficient declines from +1.0 to -1.0. In the same way it can be seen by holding the correlation coefficient constant that the size of the portfolio weights assigned to each security has an effect on portfolio risk. You can check it yourself in the above example by assigning different values to \( w_1 \) and \( w_2 \) instead of assuming \( w_1 = w_2 = 0.5 \).

***

The two-security case for calculating portfolio risk can be generalised to the n-security case. Portfolio risk in the case of \( n \) securities can be calculated as follows:

\[
\sigma_P^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}
\]

where

- \( \sigma_i^2 \) = the variance of the return on the portfolio,
- \( \sigma_i^2 \) = the variance of return for security \( i \),
- \( \sigma_{ij} \) = the covariance between the returns for securities \( i \) and \( j \), and
- \( w_i \) = the portfolio weights or percentage of investable funds invested in security \( i \).

You may now try the following exercises.

E10) Four securities have the following expected returns:

\[ A = 15\%, \ B = 12\%, \ C = 30\% \] and \[ D = 22\% \]

Calculate the expected returns for a portfolio consisting of all four securities under the following conditions.

a) The portfolio weights are 25% each.
b) The portfolio weights are 10% in A and the remaining is equally divided among the other three securities.

c) The portfolio weights are 10% each in A and B and 40% each in C and D.

E11) Let the returns on three securities 1, 2, 3 be 25%, 28% and 14% respectively with \( \sigma_1 = 4, \sigma_2 = 5, \sigma_3 = 6, \sigma_{12} = \sigma_{13} = 15 \) and \( \sigma_{12} = -10 \).

Find the standard deviation \( \sigma_p \) of the portfolio \( P = (0.5, 0.3, 0.2) \).

You have learnt to evaluate portfolios on the basis of their expected returns and risk as measured by the standard deviation. The evaluation of the risk of a portfolio involves the evaluation of the following three parameters:

i) Standard deviation of each security.

ii) Covariance between pair of securities

iii) Proportion of funds invested in each security.

Once the risk-return opportunities available to an investor is determined it is seen that a large number of possible portfolios exist when the percentage of the investors' wealth to be invested in each security is varied.

Does the investor need to evaluate all these portfolios? The answer to this question is no. The reason being that an investor needs to look at only a subset of the available portfolios meeting the following two conditions.

1. Portfolios that offer maximum expected returns for varying level of risk.
2. Portfolios that offer maximum risk for varying levels of expected returns.

The set of portfolios meeting these two conditions is known as the efficient set or efficient frontier. As a next step in the modelling process, we now try to get new information about the problem being studied. From a large number of possible portfolios, we try to locate the efficient frontier. The efficient set can be located from the feasible set, also known as the opportunity set. We shall now see how this is done.

**Efficient Frontier**

Let us consider Fig. 2 illustrating the location of the feasible set.

![Fig. 2: The feasible set of portfolios.](image-url)
You know that the feasible set represents the set of all portfolios that could be formed from a group of n-securities. That is, all possible portfolios that could be formed from the n-securities lie either on or within the boundary of the feasible set. The points denoted by A, B, C and D in Fig. 2 are examples of such portfolios. In general this set will have an umbrella type shape similar to the one shown in Fig. 2. The efficient set can then be located by applying the two conditions of the efficient set. Let us first, identify the set of portfolios meeting the first condition.

If you look at Fig. 2, there is no portfolio offering less risk than that of portfolio A. This is because if a vertical line is drawn through A, there will no point in the feasible set to the left of the line. Also there is no portfolio offering more risk than that of portfolio C. Thus, the set of portfolios offering maximum expected return for varying levels of risk is the set of portfolios lying on the ‘northern’ boundary of the feasible set between points A and C.

Let us now consider the second condition. The portfolio offering maximum expected return is D. If you draw a horizontal line through D then there is no point in the feasible set that lies above this line. Similarly, there is no portfolio offering a lower expected return then portfolio B. Thus, the set of portfolios offering minimum risk for varying levels of expected return is the set of portfolios lying on the ‘western’ boundary of the feasible set between points B and D. Now since both the conditions are to be fulfilled while identifying the efficient set, only the portfolios lying on the northwest boundary between points A and D need to be considered. Accordingly, these portfolios form the efficient set, and the investor will have to find his or her optimal portfolio from this set of efficient portfolios. All other portfolios are inefficient and can be safely ignored.

As a final step in the portfolio selection problem, let us now see how an optimal portfolio can be obtained from the set of efficient portfolios.

Selecting an Optimal Portfolio

To select an optimal portfolio of assets using the Markowitz analysis, investor should

1. Identify optimal risk-return combinations available from the set of risky assets being considered by using the Markowitz efficient frontier analysis.

2. Choose the final portfolio from among those in the efficient set based on an investor’s preferences.

To select the expected return-risk combination that satisfy an individual investor’s personal preferences, indifference curves are used. Investors are assume to know their indifference curve. These curves representing an investor’s preferences for risk and return can be drawn on a two-dimensional figure, where the horizontal axis indicates risk as measured by standard deviation (\(\sigma_p\)) and vertical axis indicates reward as measured by expected return (\(E(R_p)\)).
Indifference curves for a hypothetical investor are shown in Fig. 3.

Each curved line in Fig. 3 indicates one indifference curve for the investor and represents all combinations of portfolios that the investor would find equally desirable. For example, the investor with the indifference curves as shown in Fig. 3, would find portfolios A and B equally desirable, since they lie on the same indifference curve $I_2$, even though they have different expected returns and standard deviations. Portfolio B has higher standard deviation than portfolio A so it is less desirable on that dimension but on the other hand, it is preferred as it provides higher expected return. Thus, all portfolios that lie on a given indifference curve are equally desirable to the investor. Also, indifference curves cannot intersect, since they represent different levels of desirability.

As an investor, we would always love to have portfolios with more return and less risk, our indifference curves will always show some inclination towards the return line. That is, we choose portfolios in such a way that the corresponding curve heads towards the northwest direction. In other words, farther an indifference curve is from the horizontal axis, the greater is the utility. Let us now see how do we use indifference curves in conjunction with the efficient frontier to find which feasible portfolio is optimal.
To start with, an investor should plot his or her indifference curves on the same figure as the efficient set and then choose the portfolio that is on the indifference curve and is furthest northwest. This portfolio will correspond to the point where an indifference curve is just tangent to the efficient set. In Fig. 4, this is portfolio $P^*$ on indifference curve $I_2$.

In Fig. 4, the curve $I_3$ represents the best level of satisfaction but it is of no use as it does not meet the feasible region at all. In regard to $I_1$, there are several portfolios that an investor could choose, say for example $P$, but portfolio $P^*$ dominates these portfolios, since it is on an indifference curve that is further northwest. Also, you may note that this point of intersection will be unique because of the convexity of the efficient frontier. Hence $P^*$ is our optimal portfolio. With the selection of $P^*$ our problem of selecting an optimal portfolio is solved completely. However, the model has certain limitations which we shall list now.

**Limitations of Markowitz Model**

Markowitz model presents a systematic procedure for the selection of the optimal portfolio of risky securities but is cumbersome to work with. It requires a full set of covariance between the returns of all securities to be considered in order to calculate portfolio variance. This leads to gathering an enormous amount of input data while dealing with the portfolios having large number of securities. Evaluation of a single portfolio of $n$-securities requires $2n$ quantities for estimating the expected return and standard deviation of these $n$-securities. There are $\frac{n(n-1)}{2}$ covariances to be estimated for these $n$-securities. Thus to evaluate a portfolio of $n$ (risky) securities we need to estimate in all a total of $2n + \frac{n(n-1)}{2} = \frac{n(n+3)}{2}$ quantities. A lot of energy and time is needed to carry out the computations for the selection of an optimal portfolio. However, with the availability of computer packages, things can be managed for large values of $n$.

You may now try the following exercises.

**E12)** How many portfolios are on an efficient frontier? What is the Markowitz efficient set?

**E13)** How is an investor's risk aversion indicated in an indifference curve? Are all indifference curves upward sloping?

**E14)** Why indifference curves of an investor cannot intersect?

The limitations of the Markowitz model raise two issues:

i) Are there simpler methods for computing the efficient frontier?

ii) Can the Markowitz analysis be used to optimize asset classes rather than individual assets?
As mentioned in Unit 1, the model-building process proceeds through several 
iterations, each a refinement of the preceding model. The Sharpe model 
developed by William Sharpe has simplified the rigours involved in computing 
the efficient frontier using the Markowitz approach. We shall now discuss this 
model in the next sub-section.

2.3.2 Sharpe Model

The Sharpe model commonly known as the single-index model, provides an 
alternative expression for portfolio variance, which is easier to calculate than in the case of the Markowitz analysis. This alternative approach can be used to 
solve the portfolio problem as formulated by Markowitz for determining the efficient set of portfolios. The single-index model, relates return on each security to the returns on a common index. Generally a broad market index of common stock returns is used for this purpose.

The single-index model can be expressed by the following equation

\[ R_j = \alpha_j + \beta_j R_M + e_j \]  \hspace{1cm} (13)

where

- \( R_j \) = the return on j\textsuperscript{th} security,
- \( R_M \) = the return on the market index,
- \( \alpha_j \) = part of j\textsuperscript{th} security return independent of market performance,
- \( \beta_j \) = a constant measuring the expected change in the dependent variable, \( R_j \), given a change in the independent variable, \( R_M \), and
- \( e_j \) = the random residual error.

The single index model divides a security's return into two components \( \alpha_i \) and a market related part \( \beta_j R_M \). Given these values, the error term is the difference between the return on j\textsuperscript{th} security and the sum of two components of return, i.e.,

\[ e_j = R_j - (\alpha_j + \beta_j R_M) \]  \hspace{1cm} (14)

For example, let us assume the return for the market index for period t is 10\%, \( \alpha_j = 4\% \) and \( \beta_j = 1.5 \). Then the estimate for stock j is

\[ R_j = 4\% + 1.5R_M + e_j \]

or, \[ R_j = 4\% + (1.5) (10\%) = 19\% \]

This shows that if the market index return is 10\%, then the likely return for your stock is 19\%. Further, if the actual return on j\textsuperscript{th} stock for period t is 16\%, the error term is 16\% - 19\% = -3\%.

Note that \( R_M \) and \( e_j \) are random variables and the \( \beta \) term, or beta, is important as it measures the sensitivity of a stock to market movements. To use our model, we need to know for each stock we consider, the estimates of beta. Previously available data may be used to estimate future beta or analysts could also give some subjective estimate of beta.
The single-index model uses two simplifying assumptions. The first assumption used in the single-index model is that the random error term and the market index are uncorrected, meaning that the outcome of the market index has no bearing on the outcome of the random error term. The second assumption is that the random error terms of any two securities are uncorrelated, meaning that the outcome of the random error term of \( i \)th security has no bearing on the outcome of the random error term of \( j \)th (\( i \neq j \)) security. This can be expressed as \( \text{cov}(e_i, e_j) = 0 \). In other words, the returns of the two securities will be correlated (i.e., move together) only in their common response to the return on the market. This mean that stocks covary together only because of their common relationship to the market index. If either of these two assumptions are invalid i.e., they are not good description of reality, then model will be an approximation and alternately, more than one index model may be useful in such cases.

In the single-index model, all the covariance terms can be accounted for stocks being related only through common reactions to the market index, that is, covariance depends only on the market risk. Therefore, the covariance between any two securities can be written as

\[
\sigma_{ij} = \beta_i \beta_j \sigma_M^2
\]

where \( \sigma_M^2 \) is the variance of the market index. In the Markowitz model, we need to consider all the covariance terms in the variance-covariance matrix. In the single-index model, just as the security return is split into two components, the risk of an individual security is also divided into two components. This simplifies the covariance and also the calculations for the total risk for a security and for a portfolio are simplified to a large extent. The total risk of a security, as measured by its variances, consists of two components, market risk, which is common to all companies, and the company specific risk. Thus, we can write

\[
\sigma_j^2 = \beta_j^2 \sigma_M^2 + \sigma_{e_j}^2
\]

(Market risk) + (company specific risk)

This simplification also holds for portfolios. It provides an alternative expression for finding the minimum variance set of portfolios in the form

\[
\sigma_p^2 = \beta_P^2 [\sigma_M^2] + \sigma_{e_p}^2
\]

The single-index model greatly simplifies the calculation of the portfolio variance and therefore, the calculation of efficient portfolios. In the case of the Markowitz analysis, 200 stock require 19,900 covariances and 200 variances. Whereas, we would require \( 3n + 2 \) or 602 estimates for 200 securities while using single index model.

As in the case of Markowitz analysis, in order to determine the composition of the tangency portfolio, the investor needs to estimate all the expected returns, variances and covariances. In the case of single-index model, this can be done by estimating \( \alpha_j, \beta_j \) and \( \sigma_{e_j} \) for each of the \( n \) risky securities. Also needed
are the values of $R_M$ and its variance $\sigma^2_M$. One way to estimate the parameters is with time series regression. With these estimates Eqns. (13), (16) and (17) can be used to calculate the returns, variances and covariances for the securities. Using these values, the curved efficient set of portfolios can be derived, from which the tangency portfolio can be determined.

**Limitations of the Sharpe Model**

The single-index model is a valuable simplification of the full variance-covariance matrix needed for the Markowitz model. It reduces by a large amount the number of estimates needed for a portfolio of securities. However, the model makes an assumption – the residuals for different securities are uncorrelated. The accuracy of the estimate of the portfolio variance thus depends on the accuracy of this assumption. For example, if the covariance between the residuals for different securities is positive, not zero as assumed, the true residual variance of the portfolio will be underestimated and the model will be in accurate. In such cases models with more than one index are available and can be used. We shall not be discussing such models here.

Before we end this unit you may try the following exercises.

---

E15) How is the covariance between any two securities calculated with the single-index model?

E16) How would you compare the Markowitz model with the Sharpe model?

---

We now end this unit by giving a summary of what we have covered in it.

### 2.4 SUMMARY

In this unit, we have covered the following points:

1. Once the essential characteristics of the real world problem are identified its conversion into a mathematical description in terms of equations can be done in different ways according to the objectives of the study as illustrated in the case of motion of a simple pendulum.

2. Portfolio selection problem: How can an investor choose an optimal portfolio, from a feasible set of risky securities i.e., choose a portfolio that provides him/her the maximum level of satisfaction in terms of return and risk.

3. Markowitz portfolio theory provides the way to select an optimal portfolio based on using the full information about securities.

4. An efficient portfolio has the highest expected return for a given level of risk or the lowest level of risk for a given level of expected return.

5. The Markowitz analysis determines the efficient set of portfolios, all of which are equally desirable. The efficient set in an expected return-standard deviation space is a curve which is upward concave.

6. The efficient frontier gives the investment possibilities that exist from a given set of securities. Indifference curves express investor preferences.
7. The optimal portfolio for a risk-averse investor occurs at the point of tangency between the investor’s highest (northwest) indifference curve and the efficient set of portfolios.

8. The Sharpe model relates returns on each security to the returns on a common index.

9. The Sharpe model provides an alternative expression for portfolio variance, which is easier to calculate in comparison to the case of the Markowitz analysis.

### 2.5 SOLUTIONS/ANSWERS

E1) i) Dynamic, discrete, probabilistic
Essentials: risk, premium, number of customers, profit, etc.
Non-essentials: religion and gender of the people working in the company.

ii) Discrete, dynamic, probabilistic
Essentials: price, advertising, competition, population changes, seasonal variations, etc.
Non-essentials: name, religion and gender of the people working in the company.

E2) State the problem giving reasons for why there is no other treatment of the problem possible.

E3) i) The meteorological quantities like rainfall, atmospheric temperature, pollution are all continuous and stochastic. Similarly give examples for ii) and iii).

E4) i) 20%
ii) −7.5%
iii) 1.67%

E5) \[ P_1 = \left( \begin{array}{c} 0 \\ \frac{1}{3} \end{array} \right), P_2 = \left( \begin{array}{c} 0 \\ \frac{1}{4} \end{array} \right), P_3 = \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right) \]. Similarly find others.

E6) \[ E(R) = 0.0832 \].

E7) \[ E(R) = 0.1075, \ \sigma = 7.75\% \].

E8) \[ \sigma_1 = 0.0337, \ \sigma_2 = 0.0362 \]. Security 2 is more risky than security 1.

E9) \[ E(I) = 0.1264, E(2) = 0.1443, \ \sigma_1 = 0.0337, \ \sigma_2 = 0.0362 \]
\[ \sigma_{12} = 0.33 \times (0.0636) \times (-0.0543) + (0.25) \times (0.0436) \times (0.0157) + 0.22 \times (-0.0164) \times (0.0357) + 0.2 \times (-0.0264) \times (-0.0357) \]

E10) a) 0.1975  b) 0.207  c) 0.235

E11) \[ \sigma_p = 0.0371 \].

E12) There are large number of portfolios on an efficient frontier. Markowitz efficient set is the set of portfolios having highest expected return for a
given level of risk or the lowest level of risk for a given level of expected returns.

E13) Yes, all indifference curves are upward sloping.

E14) Let three be a point $X$ of intersection of two indifference curves $I_1$ and $I_2$. Show that all portfolios on $I_1$ must be as desirable as those on $I_2$ and then reach a contradiction since $I_1$ and $I_2$ are two curves that are supposed to represent different levels of desirability.

E15) Explain Eqn. (15).

E16) Comparison between the two models can be made in terms of risk and return evaluation, in terms of number of estimates required, etc.