UNIT 12 NON-POISSON QUEUES

Structure

12.1 INTRODUCTION

In this unit, we shall discuss the queueing system in which the arrivals follow Poisson distribution and the service times have a general or arbitrary distribution. This type of queue is a Non-Markovian queue. The probability of a service completion in time interval $(t, t+h)$ depends upon the time for which the service has already been given. To be specific, here we study the system at discrete time points at which customers depart after service and not in continuous time. These departure instants are called regeneration points. Then the arrivals which are Markovian between any two consecutive departures are needed to be considered only. The difference in the system $M/M/1$ and $M/G/1$ is that in $M/M/1$ system all the time points are regeneration points and the whole process in continuous time is Markovian, but in $M/G/1$, the queue length at the departure epochs constitutes an Imbedded Markov chain in the Non-Markovian queue process is continuous time.

Objectives

After studying this unit, you should be able to

- distinguish a Markov chain and a non-Markov chain;
- apply the general distribution in the queueing models;
- formulate and solve the $M/G/1$ queue;
- find various operational characteristics of $M/G/1$ system;
- apply $M/G/1$ queue to various situations.

12.2 THE M/G/1 SYSTEM

In the previous unit, we discussed the queueing processes, in which both arrival and departures follow Markovian property. In this section, we shall consider a queue following the general service-time distribution.

To formulate the model, let us state various assumptions underlying this model.

i) M : The queueing system is with Poisson arrival at a constant mean rate λ.

ii) G : Service times follow general distribution with mean $1/\mu$.

iii) Single server is considered.

iv) The number of customers allowed in the queue can be infinite.

v) FCFS discipline is considered.

vi) All customers wait until served.

Let us further assume the following notations:

i) $F(t)$ be the distribution function of the service time v and $f(t)$ be its density function, given by $f(t) = F'(t)$. Also

$$\tilde{F}(s) = \text{L.T. of } F(t) = \int_0^\infty e^{-st} dF(t).$$

ii) $N(t)$ be the number of customers in the queue at time t.

iii) t_n be the time instant of the n^{th} departure from the system, i.e. on the time at which n^{th} customer leaves the system after completing service, where $n = 1, 2, 3, \ldots$ and $t_0 = 0$.

iv) A_n is the number of arrivals that take place during the service time of the n^{th} customer.

The points t_n are called the regeneration points of the process $N(t)$ and the sequence of these points $\{t_n\}$ forms a renewal process.

Let $N(t_n + 0) = N_n$. The discrete time process $\{N_n\}$ constitutes a Markov chain imbedded in the continuous time process $N(t)$.

The transition probabilities

$$P_{ij} = \text{Prob } \{N_{n+1} = j / N_n = i\};\ i, j \ge 0, n \ge 1 \tag{1}$$

are given by

$$\begin{aligned} P_{0j} &= \alpha_j;\ j \ge 0 \\ P_{ij} &= \alpha_{j-i+1};\ i \ge 1, j \ge i-1 \\ &= 0;\ \text{otherwise}, \end{aligned} \tag{2}$$

where α_k is the probability of k arrivals take place during a service time of random duration of a customer. As you see, these probabilities are independent of n. Since the Markov Chain is time-homogeneous.

Now we can readily of write α_k, which is given by

$$\alpha_k = \int_0^\infty \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v)\, dv; k = 0, 1, 2, \cdots$$

and the generating function of α_k is given by

$$\begin{aligned} \alpha(s) &= \sum_{k=0}^\infty \alpha_k s^k \\ &= \sum_{k=0}^\infty \int_0^\infty \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v)\, dv.\ s^k \\ &= \int_0^\infty e^{-\lambda v}\left(\sum_{k=0}^\infty \frac{(\lambda v)^k s^k}{k!}\right) dF(v) \\ &= \int_0^\infty e^{-\lambda v}\left(\sum_{k=0}^\infty \frac{(\lambda s v)^k}{k!}\right) dF(v) \\ &= \int_0^\infty e^{-\lambda v} e^{\lambda s v}\, d F(v) \\ &= \bar{F}(\lambda - \lambda s)\ (|s| < 1) \end{aligned} \tag{3}$$

Transition probability matrix $\mathbf{P} = (\mathbf{P}_{ij})$ of Markov chain $\{N_n\}$ is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & \cdots \end{array} \\ \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \end{array} \qquad (4)$$

Let us assume that $0 < \alpha_0 < 1$ and $\alpha_0 + \alpha_1 < 1$. Then in the Markov Chain N_n, every state can be reached from every other state, therefore it is irreducible. Since all $P_{00} > 0$, therefore the chain is aperiodic. The transition probability diagram is given below:

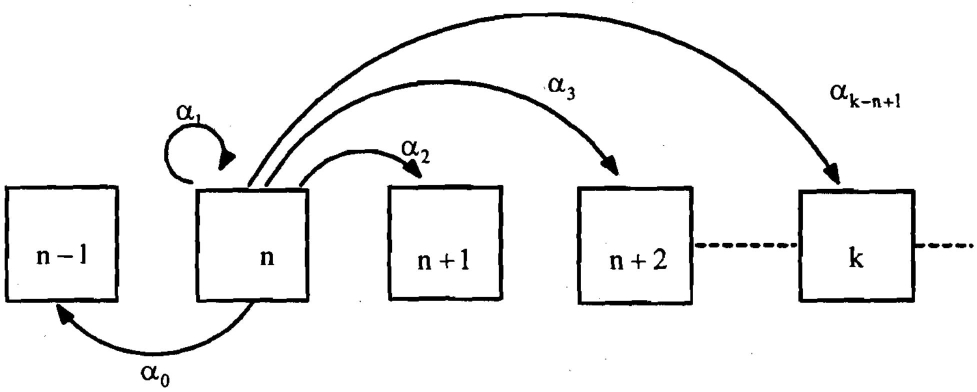

Fig. 1: Transition state diagram for the M/G/1 system

Differentiating Eqn. (3) with respect to s, we get

$$\alpha'(s) = -\lambda \bar{F}'(\lambda - \lambda s)$$

At $s = 1$, we have

$$\alpha'(1) = -\lambda \bar{F}'(0) \qquad (5)$$

Let us find $\bar{F}'(0)$, for this, using the definition of $\bar{F}(s)$, we have

$$\bar{F}(s) = \int_0^\infty e^{-sv} f(v)\, dv$$

$$\bar{F}'(s) = \int_0^\infty -v e^{-sv} f(v)\, dv$$

and $$\bar{F}'(0) = \int_0^\infty -v f(v)\, dv = \frac{-1}{\mu} \quad \left(\frac{1}{\mu} = \text{mean service time}\right) \qquad (6)$$

From Eqn. (5) and Eqn. (6), we can write

$$\alpha'(1) = \frac{\lambda}{\mu} = \rho \text{ (traffic intensity)} \qquad (7)$$

From the definition of transition probability matrix, we know that the elements of **P** give one-step transition probabilities.

Now, let us assume the probability vector which gives the probability distribution of N_n, be

$$\pi^0 = (\pi_0^{(0)}, \pi_1^{(0)}, \pi_2^{(0)}, \ldots)$$

Then, we can easily write the probability distribution of N_{n+1}, which is given by

$$\pi^{(1)} = \pi^{(0)} \mathbf{P} \qquad (8)$$

Also, in general

$$\pi^{(m)} = \pi^{(m-1)}\mathbf{P} = \pi^{(0)}\mathbf{P}^m \tag{9}$$

where $\pi^{(m)}$ is the probability vector for N_{n+m} $(m = 1, 2, 3, \ldots)$.

Hence, $\mathbf{P}^m$ is the matrix which gives n-step transition probabilities. Let us assume that the probability of i j-th element of $P^{(m)}$ is $P_{ij}^{(m)}$, then we can write

$$P_{ij}^{(m)} = P\{N_{n+m} = j \mid N_n = i\} \tag{10}$$

For the steady state, if $\pi = \lim_{n\to\infty} \pi^{(n)}$, where

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots) \tag{11}$$

we have $\pi = \pi\, P$, (12)

which can also be written as

$$\pi_j = \pi_0\alpha_j + \sum_{i=1}^{j+1} \pi_i\, \alpha_{j-i+1} \quad (j \ge 0) \tag{13}$$

Multiplying Eqn. (13) by s^j both the sides and taking summation from $j = 0$ to ∞, we get

$$\sum_{j=0}^{\infty} \pi_j s^j = \pi_0 \sum_{j=0}^{\infty} \alpha_j s^j + \sum_{j=0}^{\infty}\sum_{i=1}^{j+1} \pi_i\, \alpha_{j-i+1} s^j$$

$$\Pi(s) = \pi_0\alpha(s) + \sum_{i=1}^{\infty}\sum_{j=i-1}^{\infty} \pi_i\, \alpha_{j-i+1} s^j \text{ (changing the order of summation)}$$

$$\Pi(s) = \pi_0\alpha(s) + \sum_{i=1}^{\infty} \pi_i s^{i-1} \left(\sum_{j=i-1}^{\infty} \alpha_{j-i+1}\, s^{j-i+1} \right)$$

$$= \pi_0\alpha(s) + \sum_{i=1}^{\infty} \pi_i\, s^{i-1} \sum_{k=0}^{\infty} \alpha_k\, s^k$$

$$= \pi_0\alpha(s) + \frac{1}{s}\sum_{i=1}^{\infty} \pi_i\, s^i \alpha(s)$$

$$\Pi(s) = \pi_0\alpha(s) + \frac{\alpha(s)}{s}[\Pi(s) - \Pi(0)]$$

$$\Pi(s)\left(1 - \frac{\alpha(s)}{s}\right) = \pi_0\, \alpha(s) - \frac{\alpha(s)\, \Pi(0)}{s}$$

$$= \frac{\pi_0\, \alpha(s)\, (s-1)}{s}$$

or
$$\Pi(s) = \frac{\pi_0\, \alpha(s)\, (1-s)}{(\alpha(s) - s))} \tag{14}$$

Let us find the value for π_0, for this we assume that $\lim_{s\to 1} \Pi(s) = 1$, therefore on solving Eqn. (14), we get

$$\lim_{s\to 1} \Pi(s) = \pi_0\alpha(1) \lim_{s\to 1} \frac{1-s}{\alpha(s) - s}$$

$$1 = \pi_0 \frac{-1}{\alpha'(1) - 1}$$

or $\pi_0 = 1 - \rho$. (using Eqn. (7))

and hence, Eqn. (14) is

$$\Pi(s) = \frac{(1-\rho)\,(1-s)\,\alpha(s)}{\alpha(s) - s} \tag{15}$$

If $\alpha'(1)=1$, then $\Pi=0$ and steady state distribution Π will not exist. It can be proved that for existence of steady state distribution the Markov chain should be positive recurrent and this happens when $\alpha'(1)<1$.

Now, let us illustrate the following example for clear understanding.

Example 1: A company manufactures a very specialized single product needing a large number of machines. To manufacture the product all the machines must operate. The company has only one repair facility for the machines in the state of complete breakdown. Suppose the breakdown of machines is according to a Poisson process with a mean rate of 3/hr. The repair facility is provided for minor repair which takes 5 min. to repair and the repair facility is utilized for major repair which takes 9 min. to repair the machine. It is also observed that the number of major repairs needed is one fourth of the minor repairs. Find the probability that no machine will breakdown at any time.

Solution: Let us now find P_0,

Given is $\lambda = 3/\text{hr}$.

$$\frac{1}{\mu} = 5\times\frac{3}{4} + 9\times\frac{1}{4} = 6\,\text{min}.$$

or $\mu = 10/\text{hr}$.

Here, $\rho = \frac{3}{10}$ and $P_0 = 1-\rho = \frac{7}{10} = 0.7$

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Let us try the following exercises.

E1) Give any real life situation of $M/G/1$ queue.

E2) Derive the generating function for steady state distribution of number in the $M/M/1$ queue.

E3) Consider the situation given in Example 1. Find the probability that more than two machines are down at any time.

Now, let us find the operational measures of this model in the following section.

12.3 OPERATIONAL MEASURES OF M/G/1

First of all let us find W, the average time spent by a customer in the system. We have

$$W = W_q + \frac{1}{\mu}$$

where $W_q = E(w_q)$, and $W = E(w)$.

w_q being the total time a customer spends in queue and w that the customer spends in the system.

So, let us first get expression for $E(w_q)$. Now assume that g_q is the p.d.f. of w_q and g that of w. As before v be the service time of the customer while in steady state.

Thus, we have

$$\Pi_i = P(N = i)$$

$$= \int_0^{\infty} \frac{e^{-\lambda w} (\lambda w)^i}{i!} . g(w)\, dw \qquad (16)$$

Using Eqn. (3), the generating function of Π_i is given by

$$\Pi(s) = \bar{g}(\lambda - s\lambda) \text{ [where } \bar{g}s \text{ is L.T. of g]} \qquad (17)$$

If w is the total time of a customer spent in the system, then

$$w = w_q + v \qquad (18)$$

and hence $g(w) = g_q(w) * f(v)$ [* is convolution] (19)

Taking the Laplace transform both the sides and applying convolution theorem for L.T., we can easily write that

$$\bar{g}(s) = \bar{g}_q(s)\, \bar{F}(s) \qquad (20)$$

where bars denote corresponding L.T.'s.

Using Eqn. (17), Eqn. (20) can be rewritten as

$$\Pi(s) = \bar{g}_q(\lambda - s\lambda)\, \bar{F}(\lambda - s\lambda)$$

$$\Pi(s) = \bar{g}_q(\lambda - s\lambda)\, \alpha(s) \qquad \text{[using Eqn. (3)]}$$

$$\bar{g}_q(\lambda - s\lambda) = \frac{\Pi(s)}{\alpha(s)}$$

or,

$$\bar{g}_q(s) = \frac{\Pi\left(1 - \frac{s}{\lambda}\right)}{\alpha\left(1 - \frac{s}{\lambda}\right)} \qquad (21)$$

$$= \frac{(1-\rho)\left(1 - 1 - \frac{s}{\lambda}\right)}{\alpha\left(1 - \frac{s}{\lambda}\right) - 1 + \frac{s}{\lambda}} \qquad \text{[using Eqn. (14)]}$$

$$= \frac{(1-\rho)s}{\lambda\left[1 - \frac{s}{\lambda} - \alpha\left(1 - \frac{s}{\lambda}\right)\right]}$$

$$\bar{g}_q(s) = \frac{(1-\rho)s}{s - \lambda - \lambda\bar{F}(s)} \qquad (22)$$

The average number in the system is

$$E(N) = \Pi'(1) = \lim_{s \to 1} \frac{d\Pi(s)}{ds}$$

The average queueing time is

$$E(w_q) = -\bar{g}_q'(0)$$

The following measures can be analytically derived

L = The average number in the system

$$= \frac{\lambda^2 E(v^2)}{2[1 - \lambda E(v)]} + \lambda E(v)$$

L_q = Average number of waiting in queue

$$= \frac{\lambda^2 E(v^2)}{2[1 - \lambda E(v)]}$$

$$W = \frac{\lambda E(v^2)}{2[1-\lambda E(v)]} + \lambda E(v)$$

And $W_q = \dfrac{\lambda E(v^2)}{2[1-\lambda E(v)]}$

Let us now summarise this unit.

12.4 SUMMARY

In this unit, we have discussed

1. The queueing system in which arrival follows poisson distribution and the service times a general distribution.

2. The $M/G/1$ system in detail with the solution.

3. The operational measures of $M/G/1$ system like the waiting length, average queueing time, etc.

12.5 SOLUTIONS/ANSWERS

E2) For $M/M/1$

$$P_n = (1-\rho)\rho^n \left(\text{where } \rho = \frac{\lambda}{\mu} < 1\right)$$

$$P_{n+1} = (\rho+1)P_n - \rho P_{n-1} \qquad (n \geq 1)$$

and $\quad P_1 = \rho P_0$

Multiplying P_{n+1} by s^n and taking summative from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} P_{n+1}\ s^n = (\rho+1)\sum_{n=1}^{\infty} P_n s^n - \rho\sum_{n=1}^{\infty} P_{n-1} s^n$$

or

$$\frac{1}{s}[P(s) - (\rho s+1)P_0] = (\rho+1)[P(s) - P_0 - \rho s P(s)]$$

or

$$P(s) = \frac{P_0}{1-s\rho}$$

Also $P(1) = 1 = \dfrac{P_0}{1-\rho} \Rightarrow P_0 = 1-\rho$

Therefore $P(s) = \dfrac{1-\rho}{1-s\rho}$

E3) The required probability is

$$P[\text{more than 2 machines are down}] = \sum_{n=3}^{\infty} P_n$$

$$= 1 - \sum_{n=0}^{2} P_n$$

$$= 1 - (P_0 + P_1 + P_2)$$

From Example 1, $P_0 = \dfrac{7}{10}$. Now let us follows

Repair	Time	Density function	CDF
Minor	5 min.	¾	¾
Major	9 min.	¼	1

Now $\alpha_n = \frac{1}{n!}\left\{e^{-3(5/60)}\left[3\left(\frac{5}{60}\right)\right]^n \cdot \left(\frac{3}{4}\right) + e^{-3(9/60)}\left[3\left(\frac{9}{60}\right)\right]^n \cdot \frac{1}{4}\right\}$

$$\alpha_n = \frac{1}{n!}\left[e^{-1/4}\left(\frac{1}{4}\right)^n \cdot \frac{3}{4} + e^{-\frac{9}{20}}\left(\frac{9}{20}\right)^n \cdot \frac{1}{4}\right]$$

$$= \frac{3}{4n!}e^{-1/4}\left(\frac{1}{4}\right)^n + \frac{1}{4n!}e^{-\frac{9}{20}}\left(\frac{9}{20}\right)^n$$

or $\alpha(s) = \sum_{n=0}^{\infty}\frac{3}{4n!}e^{-1/4}\left(\frac{1}{4}\right)^n s^n + \sum_{n=0}^{\infty}\frac{1}{4n!}e^{-9/20}\left(\frac{9}{20}\right)^n s^n$

$$= \sum_{n=0}^{\infty}\left[\frac{3}{4}e^{-1/4}\left(1/4\right)^n + \frac{1}{4}e^{-9/20}(9/20)^n\right]\frac{s^n}{n!}$$

Using Eqn. (15), we get

$$\Pi(s) = \frac{(1-\rho)(1-s)\left[\sum_{n=0}^{\infty}\left\{\frac{3}{4}e^{-1/4}\left(1/4\right)^n + \frac{1}{4}e^{-9/20}\left(9/20\right)^n\right\}\frac{s^n}{n!}\right]}{\left[\sum_{n=0}^{\infty}\left\{\frac{3}{4}e^{-1/4}\left(1/4\right)^n + \frac{1}{4}e^{-9/20}\left(9/20\right)^n\right\}\frac{s^n}{n!}\right] - s}$$

$$= \frac{(1-\rho)(1-s)\sum_{n=0}^{\infty}a_n\frac{s^n}{n!}}{\sum_{n=0}^{\infty}a_n\frac{s^n}{n!} - s} \text{ where } a_n = \frac{3}{4}e^{-1/4}\left(\frac{1}{4}\right)^n + \frac{1}{4}e^{-9/20}\left(\frac{9}{20}\right)^n$$

$$= \frac{(1-\rho)\left[\sum_{n=0}^{\infty}a_n\frac{s^n}{n!} - \sum_{n=0}^{\infty}\frac{a_n s^{n+1}}{n!}\right]}{a_0 + (a_1 - 1)s + \sum_{n=z}^{\infty}\frac{a_n s^n}{n!}}$$

$$= (1-\rho)\left[1 + \sum_{n=1}^{\infty}\left(\frac{a_n}{a_0 n!} - \frac{a_{n-1}}{a_0(n-1)!}\right)s^n\right]\left[1 + \frac{(a_1-1)}{a_0}s + \sum_{n=2}^{\infty}\frac{a_n s^n}{a_0 n!}\right]^{-1}$$

Collecting the coefficients of s and s^2, we get

$$P_1 = (1-\rho)\frac{1-a_0}{a_0}$$

$$p_2 = (1-\rho)\left[\frac{1-a_0-a_1}{a_0^{\;2}}\right]$$

Here,

$$a_0 = \frac{3}{4}e^{-1/4} + \frac{1}{4}e^{-9/20} = 0.744$$

and $a_1 = \frac{3}{16}e^{-1/4} + \frac{9}{80}e^{-9/20} = 0.218$

Corresponding

$P_1 = 0.241$

$P_2 = 0.048$

Hence the req. probability is 0.011.

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