UNIT 10 RENEWAL PROCESSES-IV

10.1 INTRODUCTION

In this unit, we will apply the theory of renewal processes that we have studied so far to the replacement problem which can be described as follows. A component in a system, e.g. a bulb, a transistor, a machine, etc. fails at some random time. The component is replaced by another of the same type. We will assume that the replacement is done immediately or at the same instant that the component fails. This process is repeated as long as the system is in operation.

The lifetimes of the components can be considered to be an i.i.d. sequence of nonnegative random variables $X_1, X_2, \ldots$ with cdf $F(x) = P(X \leq x)$. We will assume throughout this unit that the mean lifetime of a component, $\mu = E(X_1)$, is finite.

We will look at the following three replacement policies and will use renewal theory to find the average number of renewals per unit time in the long run. There is a cost involved with each of these policies. We can use the results obtained to compare the policies and find an optimal one.

1. **Individual replacement policy**: The component is replaced immediately on failure.

2. **Age replacement policy**: The component is replaced immediately on failure or when it attains a fixed age $T > 0$, whichever occurs earlier.

3. **Block replacement policy**: The component is replaced immediately on failure and also at fixed time points $T, 2T, 3T, \ldots$.

We shall discuss the three replacement policies given above in Sec. 10.2, 10.3 and 10.4, respectively.

By the end of this unit, this is what you should be able to do.

**Objectives**

After studying this unit, you should be able to:

- understand individual replacement policy, age replacement policy and block replacement policy and identify the renewal processes associated with these policies;
- find the mean number of replacements per unit time in the long run for each of the policies;
- find the average long run cost per unit time for each of these policies;
- compare the policies.
Consider a component which is a part of some system. The component may fail after some random time and is immediately replaced by another identical component. Such a replacement policy is one which is usually employed and is called the individual replacement policy.

Let $X_1, X_2, \ldots$ denote the lifetimes of successive components which are put in service. By this we mean that the $i^{th}$ component fails after being in service for time $X_i$. We assume that the lifetimes are independent and identically distributed nonnegative random variables with the common distribution function $F$. We will denote the common distribution function by $F$. To avoid trivialities we assume that $F(0) < 1$. Further the mean lifetime $\mu$ is assumed to be finite, i.e.

$$\mu = \int_0^\infty s dF(s) = E(X_k) < \infty.$$ 

We are interested in the number of components which fail during a time interval $[0, t]$. For the individual replacement policy this also coincides with the number of components replaced in the time interval $[0, t]$. Let $N_t^i$ denotes this process. In other words

$$N_t^i = \text{number of failures in the time interval } [0, t] = \text{number of replacements in the time interval } [0, t] \text{ under the individual replacement policy.} \quad (1)$$

Then clearly $(N_t^i : t \geq 0)$ is a renewal process with $X_1, X_2, \ldots$ being the corresponding interoccurrence times. In this case the renewal event is the event that a component fails and is replaced immediately. Let $M_t^i$ be the renewal function. Then

$$M_t^i = E(N_t^i). \quad (2)$$

As seen in Unit 7, finding the explicit value of this function at any time $t$ is not always possible. In any case it will depend on the cdf $F$. However, the elementary renewal theorem (Theorem 3 of Unit 8) implies that

$$\lim_{t \to \infty} \frac{M_t^i}{t} = \frac{1}{\mu}. \quad (3)$$

Thus, one can expect to replace the components at a mean rate of $1/\mu$ components per unit time in the long run. Typically, there is cost involved with every replacement. Let us assume that in the individual replacement policy this cost is $c$ units per replacement. Thus, the expected cost of replacement in the individual replacement policy in the time interval $[0, t]$ will be $cM_t^i$. Then Eqn.(2) implies that the average long run cost will be $c/\mu$ per unit time.

In the long run, any replacement policy which replaces components before they actually fail will end up using more than $1/\mu$ components per unit time. However, sometimes such a policy has its advantages. This is particularly so, if the failure of a component not only stops the system but also adversely affects other components of the system which in turn leads to added costs. In the next section, we will look at one such policy, the age replacement policy.

### 10.3 AGE REPLACEMENT POLICY

Consider a situation in which failure of a component leads to some unwanted side effects whereby the other components of the systems also get affected. It is natural to
assume that the chance of failure of a component increases with its age. In such a case it might be advantageous to replace the component after a certain time or age.

Fix a \( T > 0 \). Then in an **age replacement policy** a component is replaced upon its failure or upon its reaching age \( T \), whichever happens earlier. We once again use the same set up as before. In other words the lifetimes of the components, denoted by \( X_1, X_2, \ldots \) are assumed to be i.i.d. each with distribution function \( F \).

Let \( \{N_t^{A,t}: t \geq 0\} \) be the counting process defined by

\[
N_t^{A,t} = \text{number of replacements in the time interval } [0, t] \text{ under the age replacement policy.} \quad (4)
\]

We reemphasize that replacement of a component does not mean the failure of the component in service. Thus, the above process is different from \( \{N_t^{A,f}: t \geq 0\} \) defined below by

\[
N_t^{A,f} = \text{number of failures in the time interval } [0, t] \text{ under the age replacement policy.} \quad (5)
\]

Note that whatever happens between the \((i-1)\)th and the \(i\)th replacement depends only on the life length of the \(i\)th component and hence only on \( X_i \). Let \( Y_i \) denotes the time for which the \(i\)th component is in service. Then the sequence \( Y_1, Y_2, \ldots \) is an i.i.d. sequence. These are exactly the interoccurrence times for the process \( \{N_t^{A,f}: t \geq 0\} \). Thus, we get that \( \{N_t^{A,f}: t \geq 0\} \) is a renewal process.

Let us denote the common distribution function of \( Y_i \)'s by \( F^T(x) \). Note that the component will certainly be replaced at age \( T \) if it has not failed till that age. However, it will be replaced on failure before age \( T \). Thus,

\[
F^T(x) = P(Y_i \leq x) = \begin{cases} F(x) & \text{for } x < T \\ 1 & \text{for } x \geq T \end{cases} \quad (6)
\]

Further the mean renewal time, denoted by \( \mu^T \) is given by

\[
\mu^T = \int_0^\infty P(Y_i > x) \, dx = \int_0^T (1 - F^T(x)) \, dx = \int_0^T (1 - F(x)) \, dx. \quad (7)
\]

This clearly implies that \( \mu^T \leq \mu \). Let the renewal function be denoted by

\[
M_t^{A,f} = E(N_t^{A,f}). \quad (8)
\]

Then the elementary renewal theorem (See Theorem 3 of Unit 8) implies that

\[
\lim_{t \to \infty} \frac{M_t^{A,f}}{t} = \frac{1}{\mu^T}. \quad (9)
\]

Thus, one can expect to replace the components at a mean rate of \( 1/\mu^T \) components per unit time in the long run. In general, this will be greater than \( 1/\mu \), the long run mean rate of replacements in the individual replacement policy.

We can now do the same analysis for the process \( \{N_t^{A,f}: t \geq 0\} \) defined in Eqn.(5). Let \( Z_1, Z_2, \ldots \) denote the sequence of random variables which denotes the time between two actual successive failures. Clearly this sequence will also be an i.i.d. sequence which implies that \( \{N_t^{A,f}: t \geq 0\} \) is a renewal process. Note that the random time \( Z_i \) consists of a sum of random number of time periods, say \( R_i \), each of length
T, corresponding to the components which do not fail by the age T plus the last time period of length \( W_i \) corresponding to the age of the component that fails before time \( T \). We can write

\[ Z_i = R_i T + W_i. \]  

(10)

The distribution of \( W_i \) is that of time to failure of a component conditioned on the event that its failure occurs before age \( T \). Thus, for any \( i \)

\[ P(W_i \leq w) = P(X_i \leq w | X_i \leq T) = \begin{cases} \frac{F(w)}{F(T)} & 0 \leq w \leq T \\ 1 & w > T \end{cases}. \]  

(11)

Also \( R_i \) takes the value \( k \) if and only if, in the \( i^{th} \) renewal interval, the first \( k \) components do not fail till age \( T \) and the \((K+1)^{th}\) component is the first component to fail before time \( T \). For each component, the probability of survival till time \( T \) is \((1 - F(T))\). Thus,

\[ P(R_i = k) = (1 - F(T))^k F(T), \quad k = 0, 1, 2, \ldots \]  

(12)

We want to apply the elementary renewal theorem to find the renewal function

\[ M_i^{A,f} = E(N_i^{A,f}). \]  

(13)

For this we need to find \( E(Z_i) \). Using Eqn.(10)-(12) and the definition of \( \mu^T \) in Eqn.(7), we get

\[ E(Z_i) = T E(R_i) + E(W_i) \]

\[ = T \sum_{k=0}^{\infty} k P(R_i = k) + \int_0^{\infty} P(W_i > w) \]

\[ = T \sum_{k=0}^{\infty} k F(T)(1 - F(T))^k + \int_0^T \left( 1 - \frac{F(w)}{F(T)} \right) dw \]

\[ = T \frac{(1 - F(T))}{F(T)} + 1 \frac{F(T)}{F(T)} \int_0^T (F(T) - F(w)) dw \]

\[ = \frac{1}{F(T)} \int_0^T (1 - F(w)) dw \]

\[ = \frac{\mu^T}{F(T)}. \]  

(14)

The elementary renewal theorem now says that

\[ \lim_{t \to \infty} \frac{M_i^{A,f}}{t} = \frac{1}{E(Z_i)} = \frac{F(T)}{\mu^T}. \]  

(15)

Thus, depending of the cdf \( F \), the modified failure rate in the age replacement policy might give a possibly lower failure rate then \( 1/\mu \).

Now suppose that each replacement costs \( c_i \) units of money. This is so whether or not the replacement is planned or due to failure of the component. However, whenever a failure occurs, then some extra cost \( c_f \) is incurred. Define \( C(T) \) to be the long run average cost per unit time for an age replacement policy as a function of \( T \). Since, by Eqn.(9) and Eqn.(15) there are \( 1/\mu^T \) replacements on an average per unit time and \( F(T)/\mu^T \) failures on an average per unit time in the long run, we get
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\[ C(t) = c_1 \frac{1}{\mu_T} + c_2 \frac{F(T)}{\mu_T} = \frac{c_1 + c_2 F(T)}{\int_0^T (1-F(w)) dw}, \quad (16) \]

For a given \( F \) and known values \( c_1 \) and \( c_2 \), one can find the optimal value of \( T \) which will minimize Eqn.(16).

Further the long range average costs per unit time for the individual and age replacement policies can be compared to choose one which is optimal.

**Example 1:** Suppose the lifetimes \( X_1, X_2, \ldots \) are i.i.d. exponential random variables with parameter \( \lambda > 0 \). Let \( T > 0 \) and age replacement policy is to be employed.

a) Find \( \mu_T \).

b) Let \( c_1 = 2 \) and \( c_2 = 3 \). Find the long run average cost per unit time.

**Solution:**

a) We have \( F(x) = 1 - e^{-\lambda x} \) for \( x \geq 0 \). Using Eqn.(7), we get

\[ \mu_T = \int_0^T (1-F(x)) dx = \int_0^T e^{-\lambda x} dx = \frac{1-e^{-\lambda T}}{\lambda}. \]

b) Using Eqn.(16), we get

\[ C(T) = \frac{c_1 + c_2 F(T)}{\mu_T} = \frac{\lambda(2 + 3(1-e^{-\lambda T}))}{1-e^{-\lambda T}} = 3\lambda + \frac{2\lambda}{1-e^{-\lambda T}}. \]

Clearly the function \( C(T) \) is minimized for \( T = \infty \) when the second term above becomes zero. This suggests that when the lifetime distributions is exponential, the optimal policy is to replace only upon failure of a component.

You can now do the following exercise.

E1) Repeat Example 1 when the lifetimes are uniformly distributed on \( [0, 1] \) and \( c_1 = 2 \), \( c_2 = 8 \). Further, find a \( T \) that minimizes \( C(T) \). For this case, which is a better policy in the long run in terms of cost – the individual replacement policy or the age replacement policy?

So far, we discussed the individual replacement policy and the age replacement policy. In this section, we shall discuss the block replacement policy.

### 10.4 BLOCK REPLACEMENT POLICY

Another replacement policy which implements planned replacement is the block replacement policy. In this for a fixed length of time \( T > 0 \) planned replacements take place at each of the times \( T, 2T, 3T, \ldots \). In addition, of course a component gets replaced upon failure.

Once again, let \( X_1, X_2, \ldots \) denote the lifetimes of the successive components with common distribution function \( F \) and with finite mean \( \mu \). As in the previous section, we consider two counting processes which are defined as follows.

\[ N_t^{b,r} = \text{number of replacements in the time interval } [0, t] \text{ under the block replacement policy}. \quad (17) \]
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\( N_{t}^{B,f} \) = number of failures in the time interval \([0, t]\) under the block replacement policy. 

As in the case of age replacement policy the two processes \( \{N_{t}^{B,f} : t \geq 0\} \) and \( \{N_{t}^{B,f} : t \geq 0\} \) are different. Let the corresponding renewal functions be denoted by \( M_{t}^{B,f} = E(N_{t}^{B,f}) \), \( M_{t}^{B,f} = E(N_{t}^{B,f}) \)

respectively. We will first analyze the failure renewal process. Note that immediately after every planned replacement (at time \( T, 2T, \ldots \)), the process starts anew. Moreover, in every interval of the type \(((j-1)T, jT]\), \( j = 1, 2, \ldots \) replacements will take place if and only if there is a failure. But this is exactly like the renewal process in the individual replacement policy.

Let \( \{N_{j}^{i} : t \geq 0\}, j = 1, 2, \ldots \) denote this process which counts the number of failures in the interval \(((j-1)T, jT]\). Since the process starts afresh at time point \( jT \) due to planned replacements taking place at that time, the processes \( \{N_{j}^{i} : t \geq 0\}, j = 1, 2, \ldots \) are mutually independent. Note that the interoccurrence times for these processes are i.i.d. with cdf \( F \), the lifetime distribution function. This also implies that

\[ E(N_{j}^{i}) = M_{j}, \quad j = 1, 2, \ldots \]  

where \( M_{j} \) is exactly the same renewal function as defined in Eqn.(2).

We now want to find an expression for \( N_{t}^{B,f} \) for \( t > 0 \). Note that \( t \) can be written as \( t = nT + s \) for some \( n = 0, 1, 2, \ldots, 0 \leq s < T \).

Then \( N_{t}^{B,f} \), the number of failures in the time interval \([0, t]\), coincides with the sum of the number of failures in each of the intervals \(((j-1)T, jT]\), \( j = 1, 2, \ldots, n \) and the number of failures in the interval \((nT, t]\) = \((nT, nT + s]\). i.e.

\[ N_{t}^{B,f} = \sum_{j=1}^{n} N_{j}^{i} + N_{s}^{i+1}. \]  

Hence

\[ M_{t}^{B,f} = E(N_{t}^{B,f}) = \sum_{j=1}^{n} E(N_{j}^{i}) + E(N_{s}^{i+1}) \]

\[ = \sum_{j=1}^{n} M_{T} + M_{s} \]

\[ = nM_{T} + M_{s}. \]  

Further, this implies

\[ \lim_{t \to \infty} \frac{M_{t}^{B,f}}{t} = \lim_{t \to \infty} \frac{nM_{T}}{t} + \lim_{t \to \infty} \frac{M_{s}}{t} \]

\[ = \lim_{n \to \infty} \frac{nM_{T}}{nT + s} + \lim_{n \to \infty} \frac{M_{s}}{nT + s} \]

\[ = \frac{M_{T}}{T}. \]  

Thus, the long run mean number of failures per unit time under the block replacement policy equals \( M_{T} / T \), when planned replacements take place after every \( T \) units of time. The total number of replacements under this policy for \( t = nT + s \) clearly equals the number of failures up to time \( t \) plus the number of planned replacements up to time \( t \). Thus, we have the relation
It now follows directly from Eqn.(23) and Eqn.(24) that

\[ N^{B,T}_t = N^{B,f}_t + \left[ \frac{t}{T} \right] \]  

(24)

It now follows directly from Eqn.(23) and Eqn.(24) that

\[
\lim_{{t \to \infty}} \frac{M^{B,T}_t}{{t}} = \lim_{{t \to \infty}} \frac{M^{B,f}_t}{{t}} + \lim_{{t \to \infty}} \left[ \frac{t}{T} \right]
\]

\[ M^{B,T}_t = \frac{M^{B,f}_t + 1}{T} \]

(25)

As we had done earlier the long run average cost per unit time can now be computed. Once again, suppose that the cost of any replacement is \( c_1 \) and there is an additional cost \( c_2 \) whenever there is a failure. Then the long run average cost per unit time in the block replacement policy is

\[ C(T) = (c_1 + c_2) \frac{M^{B,T}_1}{T} + c_1 \]

(26)

For a fixed \( c_1, c_2 \) one can find a \( T \) which will minimize \( C(T) \) given in Eqn.(26). Further the optimal cost in the age replacement and block replacement policies can be compared with the cost of individual replacement policy to find a minimal cost policy.

**Example 2:** Let the lifetimes \( X_1, X_2, \ldots \) are i.i.d. exponential random variables with parameters \( \lambda > 0 \). Find the long run average rate of replacements per unit time under the block replacement policy for a given \( T > 0 \).

**Solution:** The required rate is given by Eqn.(25) to be \( (M^{B,T}_1 + 1)/T \). For the given lifetime distribution, we had identified in Unit 7 the renewal process to be a Poisson process with parameter \( \lambda \) in Example 1. We had then derived the renewal function \( M^{B,T}_1 \) in Example 10. We know that \( M^{B,T}_1 \) is a Poisson process with parameter \( \lambda T \). Thus, the required rate is \( \lambda + (1/T) \).

We end the unit with the following exercises.

1.2) A machine shop needs a certain kind of machine continuously. Whenever the machine fails it is replaced instantaneously. Assume that successive machine lifetimes are uniformly distributed over the interval \([2, 5]\) years. Find the long-term rate of replacements.

1.3) In the above exercise, assume that planned replacements take place every 3 years. So that a machine is replaced on failure or at the end of 3 years. Compute
   a) long-term rate of replacements
   b) long-term rate of failures
   c) long-term rate of planned replacements.

Let us now summarize this unit.

**10.5 SUMMARY**

In this unit, we studied the following.

1. We studied the individual replacement policy and used the elementary renewal theorem to find the long run mean number of renewals per unit time. We also looked at the long run mean average cost per unit time under this policy.

2. We studied the replacement renewal process and the failure renewal process under the age replacement policy and the corresponding renewal functions in terms of the lifetime distribution \( F \). We also looked at the long run average cost per unit time under this policy.

3. We saw how to use these results to compare the two policies of individual replacement and age replacement.
4. We studied the block replacement policy and found the long run mean number of replacements and failures respectively per unit time under this scheme.

**10.6 SOLUTIONS/ANSWERS**

E1) a) We have \( F(x) = x \) for \( 0 \leq x \leq 1 \). Then, for \( T \in [0, 1] \), Eqn.(7) implies

\[
\mu^T = \int_0^T (1 - F(x)) \, dx = \int_0^T (1 - x) \, dx = T - \frac{T^2}{2}.
\]

b) Using Eqn.(16), we get

\[
C(T) = \frac{c_1 + c_2 F(T)}{\mu^T} = \frac{2 + 8T}{T - T^2 / 2}.
\]

To find \( T \) which minimizes the above we find the derivative of \( C(T) \) and equate it to zero to get

\[
C'(T) = \frac{8(T - T^2 / 2) - (2 + 8T)(1 - T)}{(T - T^2 / 2)^2} = 0.
\]

This leads to the quadratic equation

\[4T^2 + 2T - 2 = 0\]

whose two roots are given \( T = 1/2 \) and \( T = -1 \). Further one can check that \( C''(1/2) > 0 \) and hence we get that \( T = 1/2 \) minimizes \( C(T) \). Thus, the minimum long range average cost per unit time in the age replacement policy is

\[C(1/2) = 16.\]

In the individual replacement policy, since a replacement takes place only upon failure, the cost of each replacement is \( c_1 + c_2 = 2 + 8 = 10 \). The long range mean number of replacements for this policy is given by Eqn.(3) and equals \( 1/\mu \) where \( \mu \) is the mean for the distribution function \( F \). In this example this equals \( 1/2 \). Thus, the long range average cost per unit time is \( 10/(1/2) = 20 \). Clearly the age replacement policy with \( T = 1/2 \) will be preferable.

E2) Since the mean interoccurrence time = mean of uniform \([2, 5]\) = \(7/2\), by elementary renewal theorem the long-term rate of replacements is \(2/7\).

E3) (a) We are considering the age-replacement policy with \( T = 3 \). The life-time distribution is uniform, its cdf is given by

\[
F(x) = \begin{cases} \frac{x - 2}{3} & \text{for } 2 \leq x \leq 3 \\ 1 & \text{for } 3 < x \\ \end{cases}
\]

We use Eqn. (7) of this unit to calculate its mean. Thus,

\[
\mu^T = \int_0^3 x dF(x) + 3P(Y_i = 3) = \int_0^3 \frac{5-x}{3} \, dx + \frac{2}{3} = \frac{17}{6}.
\]

Now the long-term rate of replacements is given by the elementary renewal theorem to be \(6/17\).

b) The long-term rate of failures is given by Eqn. (15) and equals

\[F(3)/\mu^T = (1/3)(6/17) = 2/17.\]

c) The long-term rate of planned replacements can be calculated from the above and equals \((6 - 2)/17 = 4/17\)

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