UNIT 9 RENEWAL PROCESSES-III

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9.1 INTRODUCTION

In this unit, we shall study a variation of the renewal process that we have been looking at till now. In particular we shall look at the delayed renewal process in Sec. 9.2. This can be thought of as the renewal process which one starts observing not at a renewal time but sometime between two renewals. In Sec. 9.3, we shall derive the renewal function of the delayed renewal process. As a special case we shall also study the equilibrium renewal process in Sec. 9.4. By the end of the unit this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- identify a delayed renewal process and its distributions and to know some other properties;
- find the renewal function M_t^D of the delayed renewal process in terms of the interoccurrence distribution F and G;
- find the relationship between M_t^D and the renewal function of the associated standard renewal process;
- find asymptotic behaviour of the renewal function M_t^D;
- define the equilibrium renewal process and understand its exact renewal function;
- show that the equilibrium renewal process has stationary increments.

9.2 DELAYED RENEWAL PROCESS

In this section, we shall study the delayed renewal process. To understand the difference with what we have studied in the last two units it will be helpful to think of a delayed renewal process as a usual renewal process N_t which is started at an intermediate time or at a time between two renewals. In such a case, the interoccurrence times for the new renewal process will not be i.i.d. In particular, the time upto the first occurrence of the event will not have the same distribution as the other interoccurrence times. We will make this notion more precise in the Example 1 given after formal definitions of delayed renewal sequence and delayed renewal process. First we start by defining the delayed renewal sequence.

Definition 1 (Delayed Renewal Sequence): Suppose $X_1, X_2, \cdots$ is a sequence of nonnegative random variables such that $X_2, X_3, \cdots$ are i.i.d. with common cdf F and where X_1 is a non-negative random variable with cdf G. X_i denotes the time between the $(i-1)$-th and the i-th occurrence of the renewal event. We assume that $F(0)<1$ and $G(0)<1$. Define

$$S_0^D = 0,\ S_n^D = \sum_{i=1}^{n} X_i \tag{1}$$

The sequence $\{S_n^D\}$ is called the **delayed renewal sequence.** As before S_n^D denotes the total time elapsed before the n-th occurrence of the event.

Definition 2 (Delayed Renewal Process): Let $\{N_t^D : t \geq 0\}$ denotes the counting process for the delayed renewal sequence S_n^D, i.e.

$$N_0^D = 0,\ N_t^D = \inf\{k : S_{k+1}^D > t\} \tag{2}$$

This process $\{N_t^D : t \geq 0\}$ is called the delayed renewal process.

Example 1: Let $\{N_t : t \geq 0\}$ be a standard or usual renewal process with interoccurrence time distribution given by the cdf F. Suppose we start observing the process only from time $s > 0$. Then the, new observed process can be written as $N_t^s = N_{t+s} - N_s$ for all $t > 0$. In other words N_t^s is the number of renewals of the original process in the time interval $[s,t]$. Since whatever has happened before time s remains unobserved, we think as if the time s is the starting time point 0 for the new process $\{N_t^s : t \geq 0\}$. Note that N_t^s also denotes the number of renewals of the new process in the time interval $[0,t]$.

Then N_t^s is a delayed renewal process. To see this we argue as follows. Clearly if $X_1, X_2, X_3, \ldots$ denote the interoccurrence times for N_t^s, then $X_2, X_3, \ldots$ are i.i.d. with common cdf F. However, X_1 is exactly the random variable γ_s defined in Eqn.(13) of Sec.8.5 and which denotes the excess life at time s of the original process $\{N_t : t \geq 0\}$. Hence X_1 has the same distribution as that of γ_s which in general will be different from F.

Example 2: Let $\{N_t^D : t \geq 0\}$ be a delayed renewal process as defined in definitions 1 and 2. Let $X_1, X_2, \ldots$ be the corresponding interoccurrence times. Let $N_t = N_{t+X_1}^D - 1$ for all $t \geq 0$. Then N_t is just the number of renewals in the time interval $[X_1, t+X_1]$. Clearly the corresponding interoccurrence times are $X_2, X_3, \cdots$ respectively. But now that sequence of interoccurrence times is i.i.d by definition. Thus, the process N_t is a (standard) renewal process. Thus, the delayed renewal process behaves like a standard renewal process after a delay of time X_1. This is also the reason for the name **delayed renewal process**.

Let us try the following exercise.

E1) Let $\{N_t : t \geq 0\}$ be a Poisson process. Then show that for every $s > 0$ the delayed renewal process $\{N_t^s : t \geq 0\}$ defined in Example 1 above is also a Poisson process and hence is a (standard) renewal process.

Example 2 suggests that the delayed renewal process will inherit a lot of the properties of the standard renewal process. We have the first theorem. We will once again use the convolution notation introduced earlier. We also recall that for every $n, F_n = F^{*n}$ (see Eqn. (11) of Unit 7).

Theorem 1: Let $\{N_t^D : t \geq 0\}$ be a delayed renewal process with the distribution of the first occurrence time given by cdf G and the remaining interoccurrence times being i.i.d. with cdf F. Then

$$P(N_t^D = 0) = 1 - G(t) \tag{3}$$

$$P(N_t^D = k) = (G * F_{k-1})_t - (G * F_k)_t \qquad k \geq 1. \tag{4}$$

Proof: Note that

$$P(N_t^D = 0) = P(X_1 > t) = 1 - G(t)$$

and the first assertion is proved. Now

$$P(N_t^D = k) = P(N_t^D \geq k) - P(N_t^D \geq k+1) = P(S_k^D \leq t) - P(S_{k+1}^D \leq t) \tag{5}$$

Further

$$\begin{aligned} P(S_2^D \leq t) &= P(X_1 + X_2 \leq t) \\ &= \int_0^\infty P(X_1 \leq t - s)dF(s) \\ &= \int_0^t G(t-s)dF(s) \\ &= (G * F)_t. \end{aligned}$$

Using an induction argument, we get that

$$P(S_k^D \leq t) = (G * F_{k-1})_t. \tag{6}$$

Assertion (4) now follows from (5).

In the next section, we shall study more properties of the delayed renewal process.

9.3 RENEWAL FUNCTION OF THE DELAYED RENEWAL PROCESS

As in the previous section, we will denote a delayed renewal process by $\{N_t^D : t \geq 0\}$. We want to study more properties of this process. Let $X_1, X_2, \ldots$ be the interoccurrence times of this process. By definition the X_i's form an independent sequence. Throughout the remainder of the unit G will denote the cdf of X_1 while the common cdf of $X_2, X_3, \ldots$ will be denoted by F. Further we will denote by $\{N_t : t \geq 0\}$ the (standard) renewal process with common interoccurrence distribution F.
Let

$$M_t^D = E(N_t^D) \tag{7}$$

be the renewal function of the delayed renewal process. As before M_t will denote the renewal function of the renewal process N_t.

Theorem 2: The renewal functions M_t^D and M_t are related by

$$M_t^D = G(t) + (G * M)_t. \tag{8}$$

Proof: Using the usual renewal argument, we get

$$E(N_t^D / X_1 = s) = \begin{cases} 0 & \text{if } s > t \\ 1 + M_{t-s} & \text{if } s \le t \end{cases}.$$

Integrating with respect to the distribution of X_1, we get

$$\begin{aligned} M_t^D &= \int_0^\infty E(N_t^D | X_1 = s) dG(s) \\ &= \int_0^t [1 + M_{t-s}]\, dG(s) \\ &= G(t) + \int_0^t M_{t-s} dG(s) \\ &= G(t) + (G * M)_t . \end{aligned}$$

This proves the result.

As an immediate consequence we have the result as in the next exercise. You can try to prove it yourself.

E2) Show that $M_t^D < \infty$ for all $t \ge 0$.

Note that Eqn. (8) of the present unit is different from the renewal type Eqn. (8) of Unit 8, since now the function M_t^D appears on the left hand side while M_t appears on the right hand side. We nevertheless take Laplace transform (as defined in Unit 8) on both sides to get

$$\tilde{M}_t^D = \tilde{G}(t) + \tilde{G}(t)\tilde{M}_t \tag{9}$$

Substituting for $\tilde{M}_t$ using Eqn. (1) of Unit 8, we get

$$\begin{aligned} \tilde{M}_t^D &= \tilde{G}(t) + \tilde{G}(t)\frac{\tilde{F}(t)}{1 - \tilde{F}(t)} \\ &= \tilde{G}(t)\left[1 + \frac{\tilde{F}(t)}{1 - \tilde{F}(t)}\right] \\ &= \frac{\tilde{G}(t)}{1 - \tilde{F}(t)} . \end{aligned} \tag{10}$$

As was the case with M_t, exact evaluation of M_t^D is not always possible. We have the following theorem which gives the asymptotic behaviour of M_t^D. We will state the theorem without proof. You can try the proof which is exactly along the lines of the proof of Theorem 3 of Unit 8. We will, however, prove the theorem in a special case a little later.

Theorem 3: Let $\{N_t^D : t \ge 0\}$ be a delayed renewal process such that the mean $\mu = E(X_2) < \infty$. Then

$$\lim_{t \to \infty} \frac{M_t^D}{t} = \frac{1}{\mu}. \tag{11}$$

We will prove the following limit theorem as an application of Theorem 2 above. Recall the Definition 2 of Unit 8 of an arithmetic (and non-arithmetic) function.

Theorem 4: Let $h > 0$ be fixed. If F is not an arithmetic function then

$$\lim_{t \to \infty} [M_t^D - M_{t-h}^D] = \frac{h}{\mu}.$$

Proof: Since Eqn. (8) holds for all t, and since $M * G = G * M$, we get

$$M_t^D = G(t) + \int_0^t M_{t-s} dG(s)$$

and for $t > h$

$$M_{t-h}^D = G(t-h) + \int_0^{t-h} M_{t-h-s} dG(s).$$

We can put $M_s \equiv 0$ for $s < 0$, then we get

$$\begin{aligned} M_t^D - M_{t-h}^D &= G(t) - G(t-h) + \int_0^t (M_{t-s} - M_{t-h-s}) dG(s) \\ &= G(t) - G(t-h) + \int_0^{t/2} (M_{t-s} - M_{t-h-s}) dG(s) \\ &\quad + \int_{t/2}^t (M_{t-s} - M_{t-h-s}) dG(s). \end{aligned}$$

Note that the integrand $(M_{t-s} - M_{t-h-s})$ is bounded for all $s \in [0,t]$ by

$$M_{t-s} - M_{t-h-s} \le M_h - 0 = M_h.$$

Further, Blackwell's renewal theorem (Corollary 1 of Sec. 8.4) implies that

$$\lim_{t \to \infty} [M_t - M_{t-h}] = \frac{h}{\mu}.$$

Since $t - h - s \ge 0$ for all $s \in [0, t-h]$ we can use this in the first integral above to get that

$$\begin{aligned} \lim_{t\to\infty} \int_0^{t/2} (M_{t-s} - M_{t-h-s})\, dG(s) &= \lim_{t\to\infty} \int_0^{\infty} I_{\{s \le t/2\}} (M_{t-s} - M_{t-h-s})\, dG(s) \\ &= \frac{h}{\mu} \int_0^{\infty} dG(s) = \frac{h}{\mu}. \end{aligned}$$

The interchange of limit and integral in the expression above is justified since the integrand remains bounded independent of t. Same is the case for the second integral and we get

$$\lim_{t\to\infty} \int_{t/2}^t (M_{t-s} - M_{t-h-s}) dG(s) = \frac{h}{\mu} \lim_{t\to\infty} (G(t) - G(t/2)) = 0.$$

Similarly $\lim_{t\to\infty} (G(t) - G(t-h)) = 0$. Combining all these we get the result.

In the case when F in not arithmetic, Theorem 4 can be used to prove Eqn. (11) given in Theorem-3 as follows. We need to use the following real analytic lemma which we state without proof.

Lemma 1: Let a_n be a sequence of real numbers such that $a_n \to a$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = \lim_{n\to\infty} a_n = a.$$

Now assume that F is non-arithmetic. For $n \ge 0$, setting $a_n = (M_{n+1}^D - M_n^D)$ and using Lemma 1, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left[M_{k+1}^D - M_k^D \right] = \lim_{n\to\infty} (M_{n+1}^D - M_n^D) = \frac{1}{\mu}.$$

The last equality follows from Theorem 4 above with $h = 1$. Moreover, the sum appearing on the left hand side above is a telescopic sum and equates to $M_n^D - M_0^D$. Thus, we get

$$\lim_{n\to\infty} \frac{1}{n} M_n^D = \lim_{n\to\infty} \frac{1}{n} M_0^D + \frac{1}{\mu} = \frac{1}{\mu} \qquad (12)$$

Thus, Eqn. (11) is proved for all integer time points. Now let $t > 0$ be arbitrary. Let $[t]$ denotes the integer part of t. Since the numbers renewals are increasing in t, we get

$$\frac{M^D_{[t]}}{t} \leq \frac{M^D_t}{t} \leq \frac{M^D_{[t]+1}}{t}$$

Therefore

$$\frac{[t]}{t}\frac{M^D_{[t]}}{[t]} \leq \frac{M^D_t}{t} \leq \frac{M^D_{[t]+1}}{[t]+1}\frac{[t]+1}{t}.$$

Taking limit at $t \to \infty$ and using Eqn.(12), we get

$$\frac{1}{\mu} \leq \lim_{t \to \infty} \frac{M^D_t}{t} \leq \frac{1}{\mu}.$$

This proves Eqn. (11).

In the next section, we will study a special case of the delayed renewal process called the *equilibrium renewal process*.

9.4 EQUILIBRIUM RENEWAL PROCESS

In this section we will discuss a special case of the delayed renewal process namely the *equilibrium renewal process*. Recall that a delayed renewal process is one in which the interoccurrence times $X_2, X_3, \ldots$ are i.i.d. with cdf F, while time until the first renewal X_1, independent of the other X_i's possibly has a different cdf, say G. We also recall that μ denotes the mean of $X_j, j \geq 2$. Thus,

$$\mu = E(X_2) = \int_0^\infty (1 - F(s)), ds \tag{13}$$

We assume that $\mu < \infty$.

Suppose that while observing a renewal process, the last renewal happened so far back in the past that nobody remembers it. Then one can think of the renewal process as a delayed renewal process where the distribution function G of X_1 is same as the asymptotic distribution function of the excess life γ_t of the original renewal process. Recall that we had defined γ_t in Eqn. (13) of Unit 8 and its asymptotic distribution was given in Lemma 2 of that unit. In other words, we shall take

$$G(x) = \lim_{t \to \infty} P(\gamma_t \leq x) = \frac{1}{\mu}\int_0^x (1 - F(s)), ds \tag{14}$$

Note that Eqn. (13) implies that G defined in Eqn. (14) is actually a valid cumulative distribution function.

We can now give the following definition.

Definition 3 (Equilibrium Renewal Process): Let $\{N^e_t : t \geq 0\}$ be a delayed renewal process with cdf G given by Eqn. (14). Then $\{N^e_t : t \geq 0\}$ is called an equilibrium renewal process.

As before we define the renewal function as the mean number of renewals, i.e.

$$M^e_t = E(N^e_t) \quad t \geq 0 \tag{15}$$

Then, we have the following theorem.

Theorem 5: The renewal function M_t^e Satisfies

$$M_t^e = \frac{t}{\mu} \text{ for all } t \geq 0. \tag{16}$$

Proof: Taking Laplace transform on both sides of Eqn.(14), we get

$$\begin{aligned}
\tilde{G}(t) &= \int_0^\infty e^{-ts} dG(s) \\
&= \frac{1}{\mu}\int_0^\infty e^{-ts}(1-F(s), ds \\
&= \frac{1}{\mu}\int_0^\infty e^{-ts}\int_s^\infty dF(r) ds \\
&= \frac{1}{\mu}\int_0^\infty \int_0^r e^{-ts} ds dF(r) \\
&= \frac{1}{\mu}\int_0^\infty \frac{1-e^{-tr}}{t} dF(r) \\
&= \left(\frac{1}{\mu}\right)\left(\frac{1-\tilde{F}(t)}{t}\right).
\end{aligned}$$

Since an equilibrium renewal process is also a delayed renewal process Eqn. (10) implies that

$$\tilde{M}_t^e = \frac{\tilde{G}(t)}{1-\tilde{F}(t)} = \frac{1}{\mu t}.$$

Note that this coincides with the Laplace transform of the function $g(t) = t/\mu$. Since Laplace transforms characterise non-negative functions Eqn. (16) holds and the proof is complete.

Note that for the equilibrium renewal process Eqn. (16) holds for every time point t which is not the case in general for the standard or delayed renewal process for which the equality holds only asymptotically. Now we will study the distribution of the excess lifetime random variable γ_t^e for the equilibrium process $\{N_t^e : t \geq 0\}$. (Recall Eqn. (13) of Unit 8).

Theorem 6: For the equilibrium renewal process, the residual lifetime random variable γ_t^e satisfies for all $t, x \geq 0$

$$P(\gamma_t^e \leq x) = G(x) \tag{17}$$

where G is given by Eqn. (14).

Proof: Let $\{N_t : t \geq 0\}$ be the standard renewal process. Let γ_t be the residual lifetime random variable for this process as defined in Eqn. (13) of Unit 8. For every $x \geq 0$ define functions $A_t^{e,x}$ and A_t^x by

$$A_t^{e,x} = P(\gamma_t^e > x), \; A_t^x = P(\gamma_t > x).$$

Because of Eqn. (16) of Unit 8, we know that

$$A_t^x = 1 - F(t+x) + \int_0^t (1-F(t+x-s)\, dM_s.$$

Setting $a_t^x = (1-F(t+x))$ the above renewal type equation can be rewritten in convolution notation as

$$A_t^x = a_t^x + (M * a^x)_t \tag{18}$$

We will once again use the familiar conditioning argument. Note that

$$P(\gamma_t^e > x | X_1 = s) = \begin{cases} 1 & \text{if} \quad s > t + x \\ 0 & \text{if} \quad t + x \geq s > t \\ A_{t-s}^x & \text{if} \quad t \geq s. \end{cases}$$

Unconditioning with respect to X_1 (whose cdf is G), using Eqn. (18) and Eqn. (8), we get

$$\begin{aligned} A_t^{e,x} &= 1 - G(t+x) + \int_0^t A_{t-s}^x dG(s) \\ &= 1 - G(t+x) + (G * A^x)_t \\ &= 1 - G(t+x) + (G * a^x)_t + (G * M * a^x)_t \\ &= 1 - G(t+x) + (M^e * a^x)_t \\ &= 1 - G(t+x) + \int_0^t (1 - F(t+x-s)) dM_s^e. \end{aligned}$$

But by Theorem 5 $M_t^e = t/\mu$ and hence

$$\begin{aligned} A_t^{e,x} &= 1 - G(t+x) + \frac{1}{\mu}\int_0^t (1 - F(t+x-s)) ds \\ &= 1 - G(t+x) + \frac{1}{\mu}\int_x^{t+x} (1 - F(u)) du \\ &= 1 - G(t+x) + G(t+x) - G(x) \\ &= 1 - G(x). \end{aligned}$$

This completes the proof.

Thus, at any time point t, the distribution of the excess time random variable in an equilibrium renewal process coincides with the *limiting distribution* of the excess time random variable in standard renewal process. The last two theorems also explain why this particular delayed renewal process (with G as in Eqn. (14)) is called an equilibrium renewal process. We end this section with an important result whose proof is left as an exercise.

E3) Show that the equilibrium renewal process $\{N_t^e : t \geq 0\}$ has stationary increments.

Let us now summarize this unit.

9.5 SUMMARY

In this unit, we studied the following:

1. We defined a delayed renewal process and also saw its connections with the standard renewal process.

2. We identified the distribution of the delayed renewal process in terms of the cdf's F and G.

3. We defined the renewal function M_t^D for the delayed renewal process. We also derived an equation satisfied by M_t^D and M_t, the renewal function for the associated standard renewal process.

4. We found the Laplace transform of M_t^D. We also studied some asymptotic properties of M_t^D.

5. We defined the equilibrium renewal process and found the exact expression for its renewal function. We also found the exact distribution of the excess lifetime random variable for the equilibrium renewal process. Finally, we saw that the equilibrium renewal process has stationary increments.

9.6 SOLUTIONS/ANSWERS

E1) We have seen in Example 1 that the distribution of X_1 is the same as the distribution of the excess time γ_s. Moreover, we saw in Example 6 that when $\{N_t : t \geq 0\}$ is a Poisson process with parameter λ, then γ_s is an exponential random variable with parameter λ. But in this case, this is exactly same as the common distribution of the interoccurrence times. Thus, in this case the interoccurrence times of the delayed renewal process are i.i.d exponential with parameter λ which means that the delayed renewal process is actually a standard renewal process. Moreover, we have already identified this process in Example 1 of Unit 7 to be a Poisson process with parameter λ.

E2) We already know that $M_t < \infty$ for all $t \geq 0$. Then using Eqn. (8) and the fact that $G(t) \leq 1$ for all t, we get

$$M_t^D = G(t) + (G * M)_t \leq 1 + M_t < \infty.$$

E3) Since N_t^e is also a delayed renewal process, Theorem 1 implies that the distribution is given by Eqn. (3) and Eqn. (4). In particular,

$$P(N_t^e = 0) = 1 - G(t), \quad P(N_t^e = k) = G * F_{k-1}(t) - G * F_k(t) \qquad k \geq 1$$

where G is now given by Eqn. (14). Now fix an $s > 0$ and consider the process

$$N_t^{e,s} = N_{s+t}^e - N_s^e$$

which counts the number of renewals after time s. Then as seen earlier in Example 1, $N_t^{e,s}$ is a delayed renewal process. Let Y_1 be the time of first occurrence of the event for this new process. Then Y_1 is exactly γ_s^e, the excess time random variable at time s for the equilibrium renewal process.
Now Theorem 6 implies that the cdf of γ_s is G given by Eqn. (14). Hence the distribution of Y_1 is G. This implies (by definition) that the delayed renewal process $N_t^{e,s}$ is also an equilibrium renewal process. In particular Eqn. (3) and Eqn. (4) hold for this process as well. Elaborating on this point, we get

$$P(N_{s+t}^e - N_s^e = 0) = 1 - G(t) = P(N_t^e = 0),$$

and

$$P(N_{s+t}^e - N_s^e = k) = G * F_{k-1}(t) - G * F_k(t) = P(N_t^e = 0) \qquad k \geq 1.$$

This shows that the equilibrium process has stationary increments.

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