UNIT 8 RENEWAL PROCESSES-II

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8.1 INTRODUCTION

In this unit, we will study the renewal function M_t in more detail. We will begin the unit by finding the Laplace transform of the function in Sec. 8.2. This will be used to show that the renewal function completely characterises the renewal process. We will then study the elementary and the basic renewal theorems and see some of their applications. We will also define the residual time γ_t and study its asymptotic distribution. By the end of the unit, this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- define the Laplace transform of the renewal function and compute it in some special cases;
- use the renewal argument to evaluate $E[S_{N_t+1}]$ in terms of the renewal function M_t and the mean interoccurrence time μ;
- identify renewal type equations and find the solutions to these equations;
- use the Basic renewal theorem in applications;
- find the asymptotic distributions of the residual time γ_t, the current age δ_t and the total age β_t.

8.2 LAPLACE TRANSFORM OF THE RENEWAL FUNCTION

In Section 7.6, we defined the renewal function and showed that it satisfies the renewal Eqn. (22). It is clear from Theorem 1 in Sec. 7.5 (see also Eqn. (20) in Sec. 7.6) that M_t is completely determined by F, the cumulative distribution function of the interoccurrence times.

In this section, we will study some more properties of the renewal function M_t. We will start by deriving the Laplace transform of M_t. For this purpose we define the Laplace transform of a function in the form needed for our discussion.

Definition 1 (Laplace Transform): Let $h:[0, \infty) \to [0, \infty)$ be a non-decreasing function. The Laplace transform (or Laplace Stiltjes transform) of h, denoted by $\tilde{h}$, is a function of t defined by

$$\tilde{h}(t) = \int_0^\infty e^{-ts} dh(s),$$

for values of t for which the improper integral on the right hand side converges. It is known that the Laplace transform of a function completely characterises the function. In other words the following is true: Let $\tilde{h}_1$ and $\tilde{h}_2$ be the Laplace transforms of h_1 and h_2 respectively. If $\tilde{h}_1 = \tilde{h}_2$ then $h_1 = h_2$.

Example 1: Let X be a non-negative random variable with distribution function H. Then H is clearly a non-negative and non-decreasing function. Thus,

$$\tilde{H}(t) = \int_0^\infty e^{-ts} dH(s) = E[e^{-tX}].$$

Thus, the Laplace transform of the cdf H evaluated at $t > 0$ is just the moment generating function of the random variable X evaluated at $(-t)$.

Thus, $\tilde{H}(t)$ exists if moment generating function $H(s)$ exists.

Example 2: Let Z_1 and Z_2 be two independent random variables with cdf's H_1 and H_2, respectively. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the corresponding Laplace transforms. We have seen that if H is the cdf of $Z_1 + Z_2$ then $H = H_1 * H_2$. Then using Example 1 above the properties of moment generating functions we get

$$\tilde{H}(t) = E\left[e^{-t(Z_1+Z_2)}\right] = E\left[e^{-tZ_1}\right] E\left[e^{-tZ_2}\right] = (\tilde{H}_1)(t)\ (\tilde{H}_2)(t).$$

Iterating the above we can write for $H = H_1 * H_2 * \ldots * H_n$,

$$\tilde{H}(t) = \prod_{j=1}^{n} (\tilde{H}_j)(t)$$

as the Laplace transform of sum of n independent random variables.

In particular if the H_i's are all the same or in other words if the underlying random variables are identically distributed with distribution function H, then $H = [H_1]^{*n}$ and hence

$$\tilde{H}(t) = [(\tilde{H}_1)(t)]^n.$$

We now want to calculate the Laplace transform of the renewal function M_t. Note that since $M_t = E(N_t)$ and N_t counts the number of occurrences of the recurring event, $\tilde{M}_t$ is non-negative non decreasing function. Thus $\tilde{M}_t$, is defined and Definition 1 can be used to calculate it. Recall the definition of F_n, the n-fold convolution of F. Then by using Eqn. (20) of Sec.7.6 and Example 2 above, we get

$$\begin{aligned}
\tilde{M}_t &= \int_0^\infty e^{-ts} dM_s \\
&= \int_0^\infty e^{-ts} d\left(\sum_{k=1}^{\infty} F_k(s)\right) \\
&= \sum_{k=1}^{\infty} \int_0^\infty e^{-ts} dF_k(s) \\
&= \sum_{k=1}^{\infty} (\tilde{F}_k)t \\
&= \sum_{k=1}^{\infty} [\tilde{F}(t)]^k
\end{aligned}$$

$$= \frac{\tilde{F}(t)}{1-\tilde{F}(t)}. \tag{1}$$

Thus, if we know $\tilde{F}(t)$, the transform of F, then Eqn. (1) can be used to calculate $\tilde{M}_t$. The converse is also true which leads us to the following theorem.

Theorem 1: The renewal function $\{M_t : t \geq 0\}$ completely characterizes the renewal process $\{N_t : t \geq 0\}$.

Proof: Clearly by Eqn. (1) if the renewal function is completely known then so is $\{\tilde{M}_t : t \geq 0\}$. Further it follows from Eqn. (1) that

$$\tilde{F}(t) = \frac{\tilde{M}_t}{1+\tilde{M}_t}. \tag{2}$$

Thus, $\{\tilde{F}(t) : t \geq 0\}$ is completely known. Since Laplace transform of a function completely determines the function, we get that $\{F(t) : t \geq 0\}$ is completely determine. Theorem of Sec.7.5 now implies that the distribution of $\{N_t : t \geq 0\}$ is completely determined. This completes the proof.

We recall the renewal Eqn. (22) of Sec. 7.6, which is

$$M_t = F(t) + \int_0^t M_{t-s} dF(s).$$

We can rewrite this using the convolution notation as

$$M_t = F(t) + (M * F)_t. \tag{3}$$

Remark: Note that if the distribution function F has a density f then

$$\int_0^t M_{t-s}\, dF(s) = \int_0^t M_{t-s}\, f(s)\, ds = (M * f)_t$$

where the convolution appearing in the last term above is as in Definition 4 of Sec. 7.5. However we will also continue to use the convolution notation for distribution functions as in Eqn. (3).

If we take the Laplace transform on both sides of Eqn. (3), we get

$$\tilde{M}_t = \tilde{F}(t) + \tilde{M}_t \tilde{F}(t). \tag{4}$$

Note that this is same as Eqn. (2).

We will now calculate the Laplace transform of M_t in some special cases.

Example 3: Let the interoccurrence times be exponentially distributed with rate $\lambda > 0$. Then we know that $\{N_t : t \geq 0\}$ is a Poisson process with rate λ and hence $M_t = \lambda t$. Thus,

$$\tilde{M}_t = \int_0^\infty e^{-ts} d(\lambda s) = \int_0^\infty e^{-ts} \lambda d(s) = \frac{\lambda}{t}.$$

Note that one can also use Eqn. (1) to compute the above. We leave this as an exercise.

Example 4: Let N_t be the binomial process introduced in Example 3 and E6) of Unit 7. Then $N_t \equiv N_{[t]}$ is a Binomial random variable with parameters $[t]$ and $(1-p)$. Here $[t]$ denotes the largest integer smaller than or equal to t. Thus, the

Laplace transform of M_t is the moment generating function (m.g.f.) (evaluated at $(-t)$) of a binomial random variable and, thus, we get

$$\tilde{M}_t = [1+(1-p)\,(e^{-t}-1)]^{[t]} = [p+(1-p)\,e^{-t}]^{[t]}$$

Example 5: Let N_t be the negative binomial process introduced in Example 2 and E5 of Unit 7. Then N_t is negative binomial random variable with parameters $[t]$ and p. As in Example 4 we can find the m.g.f. of the random variable in order to find the Laplace transform of M_t. However, in this case the interoccurrence distribution is very simple with the corresponding random variable taking only two values 0 and 1 with probabilities $1-p$ and p respectively. Thus, $\tilde{F}(t)$ is just the m.g.f. of this simple random variable and equals $(1-p)+pe^{-t}$. Thus, using Eqn. (1) we get

$$\tilde{M}_t = \frac{\tilde{F}(t)}{1-\tilde{F}(t)} = \begin{cases} \dfrac{(1-p)+pe^{-t}}{p-pe^{-t}} & \text{if } t>0 \\ 1 & \text{if } t=0 \end{cases}.$$

Let us now try to solve the following exercises.

E1) For Example 3 compute $\tilde{F}(t)$ and use this to calculate $\tilde{M}_t$.

E2) Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of interoccurrence times with common probability mass function given by

$$P(X_n = 0) = 0.2,\ P(X_n = 1) = 0.3,\ P(X_n = 2) = 0.5.$$

Let $N_t,\ t \geq 0$ be the corresponding renewal process. Find the Laplace transform $\tilde{M}_t$ of the renewal function M_t.

E3) Let $\{X_n : n \geq 1\}$ be an i.i.d. sequence of interoccurrence times with common probability density function given by

$$f(x) = \begin{cases} e^{-(x-1)} & \text{if } x>1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $N_t,\ t \geq 0$ be the corresponding renewal process. Find the Laplace transform $\tilde{M}_t$ of the renewal function M_t.

Though we have calculated the renewal function and its Laplace transform in a few cases, it is not always easy to compute these for all the renewal processes. Thus, we study the asymptotic behaviour of M_t as $t \to \infty$ which we will do in the next section.

8.3 ELEMENTARY RENEWAL THEOREM

At the end of the last subsection we remarked that it is not always easy to compute the renewal function M_t for all finite t. However, in this subsection, we will see that by studying the asymptotic properties of the function we can approximately calculate the function for large values of t. We begin with a couple of exercises which will be used in the proof of Theorem 2 below.

E4) Show that for every fixed $t>0,\ F_n(t)$ is monotonically decreasing in n. Use this to show that for all nonnegative integers n, r, k with $0 \leq k \leq r-1$

$$F_{nr+k}(t) \leq [F_r(t)]^n F_k(t),$$

where we define $F_0(t)=1$.

E5) Let $F(0)<1$. Use E4) to show that for every $t>0$ there exists an r such that $F_r(t)<1$.

Now we can prove the following result. We assume that $F(0)<1$.

Theorem 2: $M_t<\infty$ for all $t\geq 0$.

Proof: Fix $t>0$. By E5) there exists a r such that $F_r(t)<1$. Then for this r, using Eqn. (20), we get

$$\begin{aligned} M_t &= \sum_{j=1}^{\infty} F_j(t) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} F_{nr+k}(t)-F_0(t) \\ &\leq \sum_{n=0}^{\infty}\sum_{k=0}^{r-1} [F_r(t)]^n F_k(t)-F_0(t) \\ &= \frac{1}{1-F_r(t)}\sum_{k=0}^{r-1} F_k(t)-F_0(t) \\ &< \infty, \text{ where we define } F_0(t)=1 \end{aligned}$$

This proves the theorem.

We will now prove an auxiliary lemma.

Lemma 1: Assume that $E(X_1)<\infty$. Then

$$E(S_{N_t+1}]=E(X_1)\,(M_t+1).$$

Proof: Note that $S_{N_t+1}>t$ denotes the time of first occurrence of the event after time t. We will prove the lemma using the renewal argument. Let $A_t=E[S_{N_t+1}]$. The conditioning on X_1, the time of occurrence of the first event, we get

$$E[S_{N_t+1}\mid X_1=x]=\begin{cases} x & \text{if } x>t \\ x+A_{t-x} & \text{if } x\leq t \end{cases}.$$

Then

$$\begin{aligned} A_t &= E[S_{N_t+1}] \\ &= \int_0^{\infty} E[S_{N_t+1}\mid X_1=x]\,dF(x) \\ &= \int_0^{t} [x+A_{t-x}]\,dF(x)+\int_t^{\infty} x\,dF(x) \\ &= \int_0^{\infty} x\,dF(x)+\int_0^{t} A_{t-x}\,dF(x) \\ &= E(X_1)+\int_0^{t} A_{t-x}\,dF(x) \\ &= E(X_1)+(A*F)_t. \end{aligned}$$

Now taking Laplace transforms on both sides and noting that the Laplace transform of a constant function is itself we get

$$\tilde{A}_t = E(X_1) + \tilde{A}_t \tilde{F}(t).$$

Thus,

$$\tilde{A}_t = \frac{E(X_1)}{1-\tilde{F}(t)}$$

$$= E(X_1)\left[1+\frac{\tilde{F}(t)}{1-\tilde{F}(t)}\right]$$

$$= E(X_1)\,[1+\tilde{M}_t].$$

Since Laplace transforms of non-negative functions characterise the functions and since the right hand side in the above equation is the Laplace transform of the function $g(t) = E(X_1)\,[1+M_t]$, we get

$$A_t = E(X_1)[1+M_t], \quad t \geq 0$$

and this proves the result.

Alternatively, this lemma can be proved by the following argument $S_{N_t+1} = \sum_{i=1}^{N_t+1} X_i$,

where X_i being i.i.d.

$$E(S_{N_t+1}) = E\,E(S_{N_t+1}/N_t)$$

$$= E(N_t+1)\,E(X_i)$$

$$= E(X_i)\,(M_t+1).$$

Now we will state and prove our first asymptotic result, the Elementary Renewal Theorem. We restate all the assumptions in the statement of the theorem.

Theorem 3: Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with common distribution function F. Let $E(X_1) = \mu < \infty$. Let S_n be the corresponding renewal sequence and let N_t be the associated renewal process. Then for the renewal function $M_t = E(N_t)$ the following holds.

$$\lim_{t\to\infty} \frac{M_t}{t} = \frac{1}{\mu}.$$

Proof: Since by definition, S_{N_t+1} is the time instant of the first occurrence of the event after time t we have $S_{N_t+1} > t$. Now using Lemma 1, we get

$$t < E[S_{N_t+1}] = \mu[1+M_t].$$

Dividing both side by μt and rearranging terms, we get

$$\frac{M_t}{t} > \frac{1}{\mu} - \frac{1}{t}.$$

Taking lim inf as $t \to \infty$ on both sides, we get

$$\liminf_{t\to\infty} \frac{M_t}{t} \geq \frac{1}{\mu}. \qquad (5)$$

Now fix $c > 0$. Let X_i^c be the truncated random variables defined by

$$X_i^c = \begin{cases} X_i & \text{if } X_i \leq c \\ c & \text{if } X_i > c \end{cases}.$$

Then note that the random variables $X_1^c, X_2^c, \ldots$ form another i.i.d. sequence with $E(X_1^c) = \mu^c$ (say). Let the associated renewal sequence, renewal process and renewal function be S_n^c, N_t^c and M_t^c respectively. Since $S_{N_t^c}^c \leq t$ and $X_i^c \leq c$ for all i, we get

$$S^c_{N^c_t+1} = S^c_{N^c_t} + X^c_{N^c_t+1} \le t + c.$$

Note that since $X^c_i \le X_i$ we get that $N^c_t \ge N_t$ which implies $M^c_t \ge M_t$. Then once again using Lemma 1 we have

$$t + c \ge E[S^c_{N^c_t+1}] = \mu^c[1 + M^c_t] \ge \mu^c[1 + M_t].$$

Thus, dividing by $t\mu^c$

$$\frac{M_t}{t} \le \frac{1}{\mu^c} + \frac{c}{t\mu^c} - \frac{1}{t}.$$

Taking limsup as $t \to \infty$ on both sides, we get for all $c > 0$

$$\limsup_{t\to\infty} \frac{M_t}{t} \le \frac{1}{\mu^c}. \tag{6}$$

Finally since μ^c is the mean of the non-negative random variable X^c_1 and hence can be got by

$$\mu^c = \int_0^c P(X^c_1 > x)\,dx = \int_0^c (1 - F(x))\,dx,$$

Therefore

$$\lim_{c\to\infty} \mu^c = \int_0^\infty (1 - F(x))\,dx = \mu.$$

Thus, taking limit as $c \to \infty$ in Eqn. (6) we get

$$\limsup_{t\to\infty} \frac{M_t}{t} \le \frac{1}{\mu}. \tag{7}$$

Now Eqn. (5) and Eqn. (7) together imply

$$\lim_{t\to\infty} \frac{M_t}{t} = \frac{1}{\mu}$$

and this completes the proof.

If follows directly from the elementary renewal theorem that the average number of renewals upto time t, namely M_t increases to ∞ as $t \to \infty$. It also gives the rate at which M_t increases.

In the next section, we will state a more general version of the above theorem called the Basic Renewal Theorem.

8.4 BASIC RENEWAL THEOREM

In the previous section, we studied the Elementary renewal theorem. The same type of arguments give a more general theorem which we will study in this section. We recall the definition of the convolution $g_1 * g_2$ of two functions g_1 and g_2 given in Unit 7.

Let a_t be a bounded function. As usual $F(t)$ will denote the cdf of the interoccurrence time and M_t the renewal function. We look at the following **renewal-type equation** (compare with the renewal equation, Eqn. (3)).

$$A_t = a_t + (A * F)_t. \tag{8}$$

We are interested in finding solution A_t of Eqn. (8). The following theorem identifies such solution.

Theorem 4: Suppose a_t is a bounded function. There exists one and only one function A_t which is bounded on all finite intervals and which satisfies Eqn. (8). The unique solution is given by

$$A_t = a_t + \int_0^t a_{t-r} dM_r = a_t + (a * M)_t \tag{9}$$

where M_t is the renewal function.

Proof: Let A_t be defined by Eqn. (9). We will show that this is a solution of Eqn. (8). Using Eqn. (20) of Sec. 7.6 and Eqn. (9), we get

$$\begin{aligned}
A_t &= a_t + (a * M)_t = a_t + \left[a * \left(\sum_{k=1}^{\infty} F_k\right)\right]_t \\
&= a_t + \left[a * \left(F + \sum_{k=2}^{\infty} F_k\right)\right]_t \\
&= a_t + (a * F)_t + \left[a * \left(\sum_{k=2}^{\infty} (F_k)\right)\right]_t \\
&= a_t + (a * F)_t + \left[a * \left(\sum_{k=2}^{\infty} (F_{k-1} * F)\right)\right]_t \\
&= a_t + \left(\left(a + \left[a * \sum_{k=1}^{\infty} F_k\right]\right) * F\right)_t \\
&= a_t + ((a + a * M) * F)_t \\
&= a_t + (A * F)_t
\end{aligned}$$

which is Eqn. (8). Moreover, since a_t is bounded and M_t is non-decreasing we have for any $0 < T < \infty$

$$\sup_{0 \le t \le T} A_t \le \sup_{0 \le t \le T} |a_t| \left[1 + \int_0^T dM_r\right] \le \sup_{0 \le t \le T} |a_t| [1 + M_T] < \infty.$$

Thus, A_t is bounded on bounded intervals. Thus, Eqn. (9) gives one solution in the requisite class.

Now let B be any solution of Eqn. (8) which is bounded on bounded intervals. We have

$$\begin{aligned}
B_t &= a_t + (B * F)_t \\
&= a_t + ((a + B * F) * F)_t \\
&= a_t + (a * F)_t + (B * F_2)_t \\
&= \ldots \\
&= a_t + \left(a * \left(\sum_{k=1}^{n-1} F_k\right)\right)_t + (B * F_n)_t
\end{aligned} \tag{10}$$

for every $n \ge 1$. Further

$$|(B * F_n)_t| = \left|\int_0^t B_{t-r} dF_n(r)\right| \le \left(\sup_{0 \le r \le t} |B_r|\right) F_n(t).$$

Using Theorem 2 and Eqn. (20) of Sec. 7.6, we get that $F_n(t) \to 0$ as $n \to \infty$ for every $t > 0$. Hence

$$\lim_{n \to \infty} |(B * F_n)_t| = 0 \text{ for all } t > 0. \tag{11}$$

Also

$$\lim_{n\to\infty}\left(a*\left(\sum_{k=1}^{n-1}F_k\right)\right)_t=\left(a*\left(\sum_{k=1}^{\infty}F_k\right)\right)_t=(a*M)_t. \tag{12}$$

Thus, combining Eqns. (10) – (12), we get

$$B_t=a_t+\lim_{n\to\infty}\left(a*\left(\sum_{k=1}^{n-1}F_k\right)\right)_t+\lim_{n\to\infty}(B*F_n)_t=a_t+(a*M)_t.$$

This completes the proof.

As an immediate application we have the following exercise.

E6) Prove Lemma 1 using the above theorem.

To state the Basic renewal theorem we need to introduce one definition.

Definition 2 (Arithmetic Distribution Function): A distribution function G is said to be arithmetic if there exists a $\lambda > 0$ such that the set of all discontinuity points of G is a subset of $\{0, \pm\lambda, \pm 2\lambda, \ldots\}$.

The largest such λ is called the **span** of G.

Now we will state the Basic Renewal theorem without proof. This generalizes Theorem 4 to the solutions of the more general renewal type equations.

Theorem 5 (Basic Renewal Theorem): Let F be the distribution function of a positive random variable with mean $\mu < \infty$. Suppose that a_t is Riemann integrable and A_t is the solution of the renewal type equation

$$A_t = a_t + (A*F)_t$$

1. If F is not arithmetic, then

$$\lim_{t-\infty} A_t = \frac{1}{\mu}\int_0^{\infty} a_x dx.$$

2. If F is arithmetic with span λ, then for all $c > 0$,

$$\lim_{n-\infty} A_{c+n\lambda} = \frac{\lambda}{\mu}\sum_{n=0}^{\infty} a_{c+n\lambda}$$

We note the following Corollary which is also called the **Blackwell's renewal theorem.**

Corollary 1: Let $h > 0$ be fixed. If F is not arithmetic then

$$\lim_{t\to\infty}[M_{t+h}-M_t]=\frac{h}{\mu}.$$

Proof: Define

$$a_t=\begin{cases}1 & \text{if } 0\le t<h\\ 0 & \text{if } h\le t.\end{cases}$$

Substituting in Eqn. (8), we get for $t > h$

$$A_t = 0 + \int_{t-h}^{t} dM_r = M_t - M_{t-h}$$

Further $\int_0^{\infty} a_r dr = h$. Thus, the corollary follows from Theorem 5.

The Basic renewal theorem is useful in many applications. We will see one such application in the next section.

8.5 ASYMPTOTIC DISTRIBUTION OF THE RESIDUAL TIME

In this section, we will study the asymptotics of various quantities associated with a renewal process. We will use the renewal theorems that we have studied in the last two sections for this purpose.

Let $X_1, X_2, \ldots$ be the sequence of inter-arrival times which are assumed to be i.i.d. with common cdf F. Let S_n be the associated renewal sequence and let N_t be the corresponding renewal process.

Thus, S_{N_t} denotes the time of the last occurrence of the event before time t and $S_{N_{t+1}}$ denotes the time of the 'next' occurrence. We define a few more related quantities.

Definition 3: The residual time at time t (or excess life) is defined by

$$\gamma_t = S_{N_{t+1}} - t \tag{13}$$

and is the time left for the next occurrence as seen at time t.

The current life (or age) at time t is the random variable defined by

$$\delta_t = t - S_{N_t} \tag{14}$$

The total life β_t is defined as

$$\beta_t = \gamma_t + \beta_t = S_{N_{t+1}} - S_{N_t} \tag{15}$$

We will see these quantities in the Poisson process example.

Example 6: Suppose that the renewal process N_t is a Poisson process with parameter λ. Then the residual time γ_t is an exponential random variable with parameter λ as can be seen using the stationary increments property of N_t as follows.

$$P(\gamma_t > x) = P(N_{t+x} - N_t = 0) = P(N_x = 0) = e^{-\lambda x}$$

A similar argument can be used to find the distribution of δ_t

Now try the following exercise.

E7) Find the cdf of δ_t.

Moreover, using the independent increment property of N_t we also get that δ_t and γ_t are independent random variables.

While finding the distributions (or even the joint distribution) of δ_t and γ_t was quite easy in the case of the Poisson process, this is certainly not the case in general. As in the case of the renewal function, we look at the asymptotic properties of these random variables. We have the following lemma.

Recall that μ denotes the mean interoccurrence time, i.e. $\mu = E(X_1)$. We assume $\mu < \infty$. We also assume for the rest of the subsection that F is continuous. Then, clearly F in non-arithmetic.

Lemma 2: For every $z > 0$

$$\lim_{t\to\infty} P(\gamma_t > z) = \frac{1}{\mu}\int_z^{\infty}(1 - F(s))ds.$$

Proof: For every $z > 0$ define the function A_t^z by

$$A_t^z = P(\gamma_t > z).$$

Then as usual conditioning on the first renewal occurring at $s > 0$, we get

$$P(\gamma_t > z \mid X_1 = s) = \begin{cases} 1 & \text{if} \quad s > t + z \\ 0 & \text{if} \quad t + z \ge s > t \\ A_{t-s}^z & \text{if} \quad t \ge s. \end{cases}$$

Unconditioning the above, we get

$$\begin{aligned} A_t^z &= \int_0^{\infty} P(\gamma_t > z \mid X_1 = s)dF(s) \\ &= \int_0^t A_{t-s}^z dF(s) + \int_t^{t+z} 0 dF(s) + \int_{t+z}^{\infty} 1 dF(s) \\ &= 1 - F(t+z) + \int_0^t A_{t-s}^z dF(s). \end{aligned}$$

This is a renewal type equation as in Eqn. (8). Further the function $(1 - F(t+z))$ as a function of t is clearly bounded. Hence by Theorem 4 its unique solution is given by

$$A_t^z = 1 - F(t+z) + \int_0^t (1 - F(t+z-s))dM_s \tag{16}$$

But, Eqn. (16) may not be easy to compute. But note that since F is the cdf of a nonnegative random variable X_1, we have

$$\int_0^{\infty}(1 - F(s))ds = \int_0^{\infty} P(X_1 > s)ds = E(X_1) = \mu < \infty.$$

Then

$$\int_0^{\infty}(1 - F(t+z))dt = \int_z^{\infty}(1 - F(s))ds < \infty.$$

Thus, the function $(1 - F(t+z))$ is Riemann integrable and the Basic renewal theorem can be applied. Theorem 5 now implies

$$\lim_{t\to\infty} A_t^z = \frac{1}{\mu}\int_0^{\infty}(1 - F(t+z))dt = \frac{1}{\mu}\int_z^{\infty}(1 - F(s))ds.$$

This completes the proof.

An immediate consequence of the above Lemma is the following Corollary which gives the asymptotic joint distribution of γ_t and δ_t.

Corollary 2: For every $y > 0, z > 0$

$$\lim_{t\to\infty} P(\delta_t > y, \gamma_t > z) = \frac{1}{\mu}\int_{y+z}^{\infty}(1 - F(s))ds.$$

Proof: Note that for any $y > 0, z > 0$ and any $t > 0$

$$\{\gamma_t > z, \delta_t > y\} = \{\gamma_{t-y} > z + y\}.$$

Thus, using Lemma 2, we get

$$\lim_{t\to\infty} P(\gamma_t > z, \delta_t > y) = \lim_{t\to\infty} P(\gamma_{t-y} > z + y) = \frac{1}{\mu}\int_{y+z}^{\infty}(1 - F(s))ds.$$

We further get that asymptotically γ_t and δ_t behave similarly.

Corollary 3: For every $y > 0$,

$$\lim_{t\to\infty} P(\delta_t > y) = \frac{1}{\mu}\int_y^\infty (1 - F(s))ds.$$

Proof: The proof follows immediately from Corollary 2 since

$$\begin{aligned}\lim_{t\to\infty} P(\delta_t > y) &= \lim_{t\to\infty}\lim_{z\downarrow 0} P(\gamma_t > z, \delta_t > y)\\ &= \lim_{z\downarrow 0}\lim_{t\to\infty} P(\gamma_t > z, \delta_t > y)\\ &= \lim_{z\downarrow 0}\frac{1}{\mu}\int_{z+y}^\infty (1 - F(s))ds\\ &= \frac{1}{\mu}\int_y^\infty (1 - F(s))ds.\end{aligned}$$

Finally, we find the asymptotic distribution of the total lifetime β_t as a consequence of the Basic renewal theorem.

Lemma 3: For every $x > 0$

$$\lim_{t\to\infty} P(\beta_t > x) = \frac{1}{\mu}\int_x^\infty s dF(s).$$

Proof: Fix $x > 0$. Define $B_t^x = P(\beta_t > x)$. Once again employing the familiar renewal argument, we get

$$P(\beta_t > x \mid X_1 = s) = \begin{cases} 1 & \text{if} \quad s > \max(t, x) \\ B_{t-s}^x & \text{if} \quad t \geq s \\ 0 & \text{otherwise.}\end{cases}$$

Unconditioning the above, we get

$$\begin{aligned}B_t^x &= \int_0^\infty P(\beta_t > x \mid X_1 = s)dF(s)\\ &= \int_0^t B_{t-s}^x dF(s) + \int_{\max(t,x)}^\infty 1\, dF(s)\\ &= 1 - F(\max(t,x)) + \int_0^t B_{t-s}^x dF(s).\end{aligned}$$

The basic renewal theorem now gives

$$\begin{aligned}\lim_{t\to\infty} B_t^x &= \lim_{t\to\infty} P(\beta_t > x)\\ &= \frac{1}{\mu}\int_0^\infty [1 - F(\max(t,x))]dt\\ &= \frac{1}{\mu}\left[\int_0^x [1 - F(x)]dt + \int_x^\infty [1 - F(t)]dt\right]\\ &= \frac{1}{\mu}\int_x^\infty s dF(s).\end{aligned}$$

The last equality above follows using integration by parts.

Now try the following exercises.

E8) Refer to E9) of Unit 7, where a system is fitted with two identical components – only one of which is used at a time. The system fails when both the components fail. Assuming the life time distribution of the component is exponential with parameter λ, find the Laplace transform of the interoccurrence time distribution.

E9) Refer to E8). Find the renewal function.

E10) In the above exercise, find the long term average number of failures.

Now before ending this unit, let us go over its main points.

8.6 SUMMARY

In this unit, we studied the following:

1. We defined the Laplace transform of a non-negative function and computed the Laplace transform of the renewal function M_t, which is denoted by $\tilde{M}_t$.

2. We saw the relationship between $\tilde{M}_t$ and $\tilde{F}_t$. With this we were able to conclude that the renewal function uniquely characterizes the distribution of a renewal process.

3. We computed $\tilde{M}_t$ in some special cases.

4. We proved the relation $E\left[S_{N_{t+1}}\right] = E(X_1)(M_t + 1)$.

5. We proved the elementary renewal theorem.

6. We looked at renewal type equations and found the unique solutions to such equations. We also stated the Basic renewal theorem.

7. We applied the basis renewal theorem to find asymptotic distributions of the residual time γ_t, the current age δ_t and the total age β_t.

8.7 SOLUTIONS/ANSWERS

E1) Here $F(t) = 1 - e^{-\lambda t}, t \geq 0$. Hence

$$\tilde{F} = \int_0^\infty e^{-ts} dF(s) = \int_0^\infty e^{-ts}(\lambda e^{\lambda s}) ds = \frac{\lambda}{\lambda + t}$$

Thus,

$$\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{\lambda}{t}.$$

E2) As in Example 5, we will first find $\tilde{F}_t$. Note

$$\tilde{F}_t = E\left[e^{-tX_1}\right] = 0.2 + 0.3e^{-t} + 0.5e^{-2t}$$

Thus,

$$\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{0.2 + 0.3e^{-t} + 0.5e^{-2t}}{0.8 - 0.3e^{-t} - 0.5e^{-2t}}.$$

E3) Here

$$\tilde{F}_t = E\left[e^{-tX_1}\right] = \int_1^\infty e^{-tx} e^{-(x-1)} dx$$

$$= \left.\frac{e^{-x(t+1)+1}}{t+1}\right|_{x=1}^{x=\infty}$$

$$= \frac{e^{-t}}{t+1}.$$

Thus, $\tilde{M}_t = \frac{\tilde{F}_{t'}}{1-\tilde{F}_t} = \frac{e^{-t}}{t+1-e^{-t}}.$

E4) By definition for $1 \le m \le n$

$$F_n(t) = [F(t)]^{*n} = \left[F(t)^{*(n-m)} * [F(t)]^{*m}\right]$$

$$= F_{n-m}(t) * F_m(t) = \int_0^t F_{n-m}(t-s)dF_m(s)$$

$$\le F_{n-m}(t)F_m(t) \le F_m(t)$$

The last inequalities follows since F_{n-m} is a cumulative distribution function and hence non-decreasing. This proves the first part.

Using the second last inequality above, we get

$$F_{nr+k}(t) \le F_{(n-1)r+k}(t)F_r(t).$$

Now iterating this inequality n times, we get

$$F_{nr+k}(t) \le F_k(t)[F_r(t)]^n.$$

where we define $F_0(t) = 1$.

E5) Since $F(0) < 1$, we get using right continuity of F that there exists a $t_0 > 0$ such that $F(t_0) < 1$. Hence for all $0 \le t \le t_0, F(t) < 1$.

Now recall that F_2 is the cdf of $X_1 + X_2$ where X_1, X_2 are i.i.d. with common distribution function F. Thus, we have

$$P(X_1 + X_2 > 2t_0) \ge P(X_1 > t_0, X_2 > t_0) = P(X_1 > t_0)P(X_2 > t_0).$$

This is same as $1 - F_2(2t_0) \ge [1 - F(t_0)]^2 > 0$.

Hence for $0 \le t \le 2t_0$

$$F_2(t) \le F_2(2t_0) < 1.$$

By induction, we get

$$F_{2^j}(t) \le F_{2^j}(2^j t_0) < 1 \text{ for all } 0 \le t \le 2^j t_0.$$

Now using the monotone property of the sequence F_n proved in E4), we get the result.

E6) In the proof of the Lemma 1 we have shown that $A_t = E\left[S_{N_{t+1}}\right]$ satisfies the renewal type equation

$$A_t = E(X_1) + (A * F)_t.$$

This is same as Eqn.(8) with $a_t \equiv E(X_1)$. By Theorem 4 the unique solution is given by Eqn.(9) which in this case becomes

$$A_t = E(X_1) + \int_0^t E(X_1)dM_\tau = E(X_1)[1 + M_t].$$

E7) By Eqn. (14) it is evident that $\delta_t \le t$. Thus, $P(\delta_t \le x) = 1$ for all $x \ge t$. Now let, $x < t$. Then

$$P(\delta_t > x) = P(N_t - N_{t-x} = 0) = P(N_x = 0) = e^{-\lambda x}$$

E8) Let F denotes the cdf of an exponential random variable with parameter λ and let us denote the distribution of $X^1 + X^2$ by G, i.e., G is the cdf of a gamma random variable with parameters 2 and λ. Then $G = F * F$. Let the corresponding renewal functions be denoted by M_t^F and M_t^G. We know that $M_t^F = \lambda t$. (See Example 10 of Unit 7).

Further, by Example 2 (of Unit 8) the corresponding Laplace transforms are given by $\tilde{G} = \tilde{F}^2$. Now

$$\tilde{F}(t) = \int_0^\infty e^{-ts} dF(s) = \int_0^\infty \lambda e^{-ts} e - \lambda s ds = \frac{\lambda}{t+\lambda}.$$

Thus,

$$\tilde{G}(t) = \left(\frac{\lambda}{t+\lambda}\right)^2.$$

E9) As we have seen earlier

$$\tilde{G}(t) = \left(\frac{\lambda}{t+\lambda}\right)^2.$$

Now Eqn. (1) (of this unit) implies

$$\tilde{M}^G(t) = \frac{\tilde{G}(t)}{1-\tilde{G}(t)} = \frac{\lambda}{t} \cdot \frac{\lambda}{t+2\lambda}.$$

We have already noted (see Example 3) that $\tilde{M}^F(t) = \lambda / t$. Thus, we get that $\tilde{M}^G(t)$ is a product of two Laplace transform $\tilde{M}^F(t)$ and $\tilde{H}(t)$ where

$$\tilde{H}(y) = \frac{2\lambda}{t+2\lambda}.$$

Note $\tilde{M}^G(t) = \dfrac{\tilde{M}^F(t)\tilde{H}(t)}{2}$. Thus, we conclude using Example 2 that $M_t^G = H_t * M_t^F$ where $H_t = 1 - e^{-2\lambda t}$. Hence

$$M_t^G = \frac{1}{2}\int_0^\infty (1 - e^{-2\lambda(t-s)})\lambda ds = \frac{1}{2}\left[\lambda t = \frac{1}{2}(1 - e^{-2\lambda t})\right].$$

E10) The required quantity is given by the elementary renewal theorem to be

$$\lim_{t\to\infty} \frac{M_t^G}{t} = \frac{\lambda}{2}.$$

This is expected since the long term average number of failures will be half the number when there is no redundant component built into the system.

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