UNIT 8 RENEWAL PROCESSES-II

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8.1 INTRODUCTION

In this unit, we will study the renewal function \( M_t \) in more detail. We will begin the unit by finding the Laplace transform of the function in Sec. 8.2. This will be used to show that the renewal function completely characterises the renewal process. We will then study the elementary and the basic renewal theorems and see some of their applications. We will also define the residual time \( \gamma_t \) and study its asymptotic distribution. By the end of the unit, this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- define the Laplace transform of the renewal function and compute it in some special cases;
- use the renewal argument to evaluate \( E[S_{N_t+1}] \) in terms of the renewal function \( M_t \) and the mean interoccurrence time \( \mu \);
- identify renewal type equations and find the solutions to these equations;
- use the Basic renewal theorem in applications;
- find the asymptotic distributions of the residual time \( \gamma_t \), the current age \( \delta_t \) and the total age \( \beta_t \).

8.2 LAPLACE TRANSFORM OF THE RENEWAL FUNCTION

In Section 7.6, we defined the renewal function and showed that it satisfies the renewal Eqn. (22). It is clear from Theorem 1 in Sec. 7.5 (see also Eqn. (20) in Sec. 7.6) that \( M_t \) is completely determined by \( F \), the cumulative distribution function of the interoccurrence times.

In this section, we will study some more properties of the renewal function \( M_t \). We will start by deriving the Laplace transform of \( M_t \). For this purpose we define the Laplace transform of a function in the form needed for our discussion.

Definition 1 (Laplace Transform): Let \( h : [0, \infty) \rightarrow [0, \infty) \) be a non-decreasing function. The Laplace transform (or Laplace Stiltjes transform) of \( h \), denoted by \( \tilde{h} \), is a function of \( t \) defined by
\[ \hat{h}(t) = \int_0^\infty e^{-st}dh(s), \]

for values of t for which the improper integral on the right hand side converges. It is known that the Laplace transform of a function completely characterises the function. In other words the following is true: Let \( \hat{h}_1 \) and \( \hat{h}_2 \) be the Laplace transforms of \( h_1 \) and \( h_2 \) respectively. If \( \hat{h}_1 = \hat{h}_2 \) then \( h_1 = h_2 \).

**Example 1:** Let \( X \) be a non-negative random variable with distribution function \( H \). Then \( H \) is clearly a non-negative and non-decreasing function. Thus,
\[ \hat{H}(t) = \int_0^\infty e^{-st}dH(s) = E[e^{-tX}]. \]
Thus, the Laplace transform of the cdf \( H \) evaluated at \( t > 0 \) is just the moment generating function of the random variable \( X \) evaluated at \(( -t )\).

Thus, \( \hat{H}(t) \) exists if moment generating function \( H(s) \) exists.

**Example 2:** Let \( Z_1 \) and \( Z_2 \) be two independent random variables with cdf's \( H_1 \) and \( H_2 \) respectively. Let \( \hat{H}_1 \) and \( \hat{H}_2 \) be the corresponding Laplace transforms. We have seen that if \( H \) is the cdf of \( Z_1 + Z_2 \) then \( H = H_1 \ast H_2 \). Then using Example 1 above the properties of moment generating functions we get
\[ \hat{H}(t) = E\left[e^{-(tZ_1 + tZ_2)}\right] = E\left[e^{-tZ_1}\right] E\left[e^{-tZ_2}\right] = (\hat{H}_1)(t) (\hat{H}_2)(t). \]
Iterating the above we can write for \( H = H_1 \ast H_2 \ast \ldots \ast H_n \),
\[ \hat{H}(t) = \prod_{j=1}^n (\hat{H}_j)(t) \]
as the Laplace transform of sum of \( n \) independent random variables.

In particular if the \( H_i \)'s are all the same or in other words if the underlying random variables are identically distributed with distribution function \( H \), then \( H = [H_j]^n \) and hence
\[ \hat{H}(t) = [(\hat{H}_j)(t)]^n. \]
We now want to calculate the Laplace transform of the renewal function \( M_1 \). Note that since \( M_1 = E(N_1) \) and \( N_1 \) counts the number of occurrences of the recurring event, \( M_1 \) is non-negative non-decreasing function. Thus \( M_1 \) is defined and Definition 1 can be used to calculate it. Recall the definition of \( F_n \), the \( n \)-fold convolution of \( F \). Then by using Eqn. (20) of Sec.7.6 and Example 2 above, we get
\[ \hat{M}_1 = \int_0^\infty e^{-st}dM_1 = \int_0^\infty e^{-st} \left( \sum_{k=1}^\infty \tilde{F}_k(s) \right) = \sum_{k=1}^\infty \int_0^\infty e^{-st}d\tilde{F}_k(s) = \sum_{k=1}^\infty \tilde{F}_k(t) = \sum_{k=1}^\infty (\tilde{F}(t))^k \]
Thus, if we know $\tilde{F}(t)$, the transform of $F$, then Eqn. (1) can be used to calculate $\tilde{M}_t$. The converse is also true which leads us to the following theorem.

**Theorem 1:** The renewal function $\{M_t : t \geq 0\}$ completely characterizes the renewal process $\{N_t : t \geq 0\}$.

**Proof:** Clearly by Eqn. (1) if the renewal function is completely known then so is $\{\tilde{M}_t : t \geq 0\}$. Further it follows from Eqn. (1) that

$$\tilde{F}(t) = \frac{\tilde{M}_t}{1 + \tilde{M}_t}.$$  \hspace{1cm} (2)

Thus, $\{\tilde{F}(t) : t \geq 0\}$ is completely known. Since Laplace transform of a function completely determines the function, we get that $\{F(t) : t \geq 0\}$ is completely determine. Theorem of Sec.7.5 now implies that the distribution of $\{N_t : t \geq 0\}$ is completely determined. This completes the proof.

We recall the renewal Eqn. (22) of Sec. 7.6, which is

$$M_t = F(t) + \int_0^t M_{t-s} dF(s).$$

We can rewrite this using the convolution notation as

$$M_t = F(t) + (M * F)_t.$$  \hspace{1cm} (3)

**Remark:** Note that if the distribution function $F$ has a density $f$ then

$$\int_0^t M_{t-s} dF(s) = \int_0^t M_{t-s} f(s) ds = (M * f)_t,$$

where the convolution appearing in the last term above is as in Definition 4 of Sec. 7.5. However we will also continue to use the convolution notation for distribution functions as in Eqn. (3).

If we take the Laplace transform on both sides of Eqn. (3), we get

$$\tilde{M}_t = \tilde{F}(t) + \tilde{M}_t \tilde{F}(t).$$  \hspace{1cm} (4)

Note that this is same as Eqn. (2).

We will now calculate the Laplace transform of $M_t$ in some special cases.

**Example 3:** Let the interoccurrence times be exponentially distributed with rate $\lambda > 0$. Then we know that $\{N_t : t \geq 0\}$ is a Poisson process with rate $\lambda$ and hence $M_t = \lambda t$. Thus,

$$\tilde{M}_t = \int_0^\infty e^{-\lambda s} ds = \int_0^\infty e^{-s} \lambda ds = \frac{\lambda}{t}.$$ 

Note that one can also use Eqn. (1) to compute the above. We leave this as an exercise.

**Example 4:** Let $N_t$ be the binomial process introduced in Example 3 and Eqn. (3) of Unit 7. Then $N_t = N_{t[t]}$ is a Binomial random variable with parameters $[t]$ and $(1-p)$. Here $[t]$ denotes the largest integer smaller than or equal to $t$. Thus, the
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The Laplace transform of \( M_t \) is the moment generating function (m.g.f.) (evaluated at \((-t)\)) of a binomial random variable and, thus, we get
\[
\tilde{M}_t = [1 + (1 - p) (e^{-t} - 1)]^{[t]} = [p + (1 - p) e^{-t}]^{[t]}
\]

**Example 5:** Let \( N_t \) be the negative binomial process introduced in Example 2 and E5 of Unit 7. Then \( N_t \) is negative binomial random variable with parameters \([t]\) and \(p\). As in Example 4 we can find the m.g.f. of the random variable in order to find the Laplace transform of \( M_t \). However, in this case the interoccurrence distribution is very simple with the corresponding random variable taking only two values 0 and 1 with probabilities \(1 - p\) and \(p\) respectively. Thus, \( \tilde{F}(t) \) is just the m.g.f. of this simple random variable and equals \((1 - p) + pe^{-t}\). Thus, using Eqn. (1) we get
\[
\tilde{M}_t = \frac{\tilde{F}(t)}{1 - \tilde{F}(t)} = \begin{cases} 
(1 - p) + pe^{-t} & \text{if } t > 0 \\
p - pe^{-t} & \text{if } t = 0
\end{cases}
\]

Let us now try to solve the following exercises.

**E1)** For Example 3 compute \( \tilde{F}(t) \) and use this to calculate \( \tilde{M}_t \).

**E2)** Let \( \{X_n : n \geq 1\} \) be an i.i.d. sequence of interoccurrence times with common probability mass function given by
\[
P(X_n = 0) = 0.2, \quad P(X_n = 1) = 0.3, \quad P(X_n = 2) = 0.5.
\]
Let \( N_t, t \geq 0 \) be the corresponding renewal process. Find the Laplace transform \( \tilde{M}_t \) of the renewal function \( M_t \).

**E3)** Let \( \{X_n : n \geq 1\} \) be an i.i.d. sequence of interoccurrence times with common probability density function given by
\[
f(x) = \begin{cases} 
e^{-(x-1)} & \text{if } x > 1 \\
0 & \text{otherwise}
\end{cases}
\]
Let \( N_t, t \geq 0 \) be the corresponding renewal process. Find the Laplace transform \( \tilde{M}_t \) of the renewal function \( M_t \).

Though we have calculated the renewal function and its Laplace transform in a few cases, it is not always easy to compute these for all the renewal processes. Thus, we study the asymptotic behaviour of \( M_t \) as \( t \to \infty \) which we will do in the next section.

### 8.3 ELEMENTARY RENEWAL THEOREM

At the end of the last subsection we remarked that it is not always easy to compute the renewal function \( M_t \) for all finite \( t \). However, in this subsection, we will see that by studying the asymptotic properties of the function we can approximately calculate the function for large values of \( t \). We begin with a couple of exercises which will be used in the proof of Theorem 2 below.

**E4)** Show that for every fixed \( t > 0 \), \( F_n(t) \) is monotonically decreasing in \( n \). Use this to show that for all nonnegative integers \( n, r, k \) with \( 0 \leq k \leq r - 1 \)
\[
F_{n+k}(t) \leq [F_r(t)]^n F_k(t),
\]
where we define \( F_0(t) = 1 \).

E5) Let \( F(0) < 1 \). Use E4) to show that for every \( t > 0 \) there exists an \( r \) such that \( F_r(t) < 1 \).

Now we can prove the following result. We assume that \( F(0) < 1 \).

**Theorem 2:** \( M_t < \infty \) for all \( t \geq 0 \).

**Proof:** Fix \( t > 0 \). By E5) there exists a \( r \) such that \( F_r(t) < 1 \). Then for this \( r \), using Eqn. (20), we get

\[
M_t = \sum_{j=1}^{\infty} F_j(t)
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{r-1} F_{n+k}(t) - F_0(t)
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{r-1} [F_k(t)]^n F_k(t) - F_0(t)
\]

\[
= \frac{1}{1 - F_0(t)} \sum_{k=0}^{r-1} F_k(t) - F_0(t)
\]

\[
< \infty , \text{ where we define } F_0(t) = 1
\]

This proves the theorem.

We will now prove an auxiliary lemma.

**Lemma 1:** Assume that \( E(X_i) < \infty \). Then

\[
E(S_{N+1}) = E(X_i) (M_t + 1).
\]

**Proof:** Note that \( S_{N+1} > t \) denotes the time of first occurrence of the event after time \( t \). We will prove the lemma using the renewal argument. Let \( A_t = E[S_{N+1}] \). The conditioning on \( X_i \), the time of occurrence of the first event, we get

\[
E[S_{N+1} | X_i = x] = \begin{cases} x & \text{if } x > t \\ x + A_{t-x} & \text{if } x \leq t \end{cases}
\]

Then

\[
A_t = E[S_{N+1}]
\]

\[
= \int_0^\infty E[S_{N+1} | X_i = x] dF(x)
\]

\[
= \int_0^t [x + A_{t-x}] dF(x) + \int_t^\infty x dF(x)
\]

\[
= \int_0^t xdF(x) + \int_0^t A_{t-x} dF(x)
\]

\[
= E(X_i) + \int_0^t A_{t-x} dF(x)
\]

\[
= E(X_i) + (A \ast F)_t.
\]

Now taking Laplace transforms on both sides and noting that the Laplace transform of a constant function is itself we get

\[
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\]
\[ \bar{A}_t = \bar{E}(X_t) + \bar{A}_t\bar{F}(t). \]

Thus,
\[
\bar{A}_t = \frac{E(X_t)}{1 - \bar{F}(t)} = E(X_t) \left[ 1 + \frac{\bar{F}(t)}{1 - \bar{F}(t)} \right] = E(X_t) \left[ 1 + M_t \right].
\]

Since Laplace transforms of non-negative functions characterise the functions and since the right hand side in the above equation is the Laplace transform of the function \( g(t) = E(X_t) \left[ 1 + M_t \right] \), we get
\[ A_t = E(X_t) \left[ 1 + M_t \right], \quad t \geq 0 \]
and this proves the result.

Alternatively, this lemma can be proved by the following argument
\[ S_{N_t + 1} = \sum_{i=1}^{N_t+1} X_i, \]
where \( X_i \) being i.i.d.
\[
E(S_{N_t + 1}) = E \left( E(S_{N_t + 1} / N_t) \right) = E(N_t + 1) E(X_t) = E(X_t) \left( M_t + 1 \right).
\]

Now we will state and prove our first asymptotic result, the Elementary Renewal Theorem. We restate all the assumptions in the statement of the theorem.

**Theorem 3:** Let \( X_1, X_2, \ldots \) be i.i.d. non-negative random variables with common distribution function \( F \). Let \( E(X_t) = \mu < \infty \). Let \( S_n \) be the corresponding renewal sequence and let \( N_t \) be the associated renewal process. Then for the renewal function \( M_t = E(N_t) \), the following holds.

\[
\lim_{t \to \infty} \frac{M_t}{t} = \frac{1}{\mu}.
\]

**Proof:** Since by definition, \( S_{N_t + 1} \) is the time instant of the first occurrence of the event after time \( t \) we have \( S_{N_t + 1} > t \). Now using Lemma 1, we get
\[ t < E[S_{N_t + 1}] = \mu[1 + M_t]. \]

Dividing both side by \( \mu t \) and rearranging terms, we get
\[
\frac{M_t}{t} > \frac{1}{\mu t}.
\]

Taking \( \lim \inf \) as \( t \to \infty \) on both sides, we get
\[
\lim \inf_{t \to \infty} \frac{M_t}{t} \geq \frac{1}{\mu}. \quad (5)
\]

Now fix \( c > 0 \). Let \( X_i^c \) be the truncated random variables defined by
\[
X_i^c = \begin{cases} 
X_i & \text{if } X_i \leq c \\
\ c & \text{if } X_i > c
\end{cases}
\]

Then note that the random variables \( X_1^c, X_2^c, \ldots \) form another i.i.d. sequence with \( E(X_1^c) = \mu^c \) (say). Let the associated renewal sequence, renewal process and renewal function be \( S_n^c, N_t^c \) and \( M_t^c \) respectively. Since \( S_n^c \leq t \) and \( X_i^c \leq c \) for all \( i \), we get
\[ S_{N_{i+1}}^c = S_{N_{i}}^c + X_{N_{i+1}}^c \leq t + c. \]

Note that since \( X_i^c \leq X_i \) we get that \( N_i^c \geq N_i \) which implies \( M_i^c \geq M_i \). Then once again using Lemma 1 we have

\[ t + c \geq E[S_{N_{i+1}}^c] = \mu^c[1 + M_i^c] \geq \mu^c[1 + M_i]. \]

Thus, dividing by \( \mu^c \)

\[ \frac{M_i}{t} \leq \frac{1}{\mu^c} + \frac{c}{\mu^c} - \frac{1}{t}. \]

Taking limsup as \( t \to \infty \) on both sides, we get for all \( c > 0 \)

\[ \limsup_{t \to \infty} \frac{M_i}{t} \leq \frac{1}{\mu^c}. \quad (6) \]

Finally since \( \mu^c \) is the mean of the non-negative random variable \( X_i^c \) and hence can be got by

\[ \mu^c = \int_{0}^{c} P(X_i^c > x) \, dx = \int_{0}^{c} (1 - F(x)) \, dx, \]

Therefore

\[ \lim_{c \to \infty} \int_{0}^{c} (1 - F(x)) \, dx = \mu. \]

Thus, taking limit as \( c \to \infty \) in Eqn. (6) we get

\[ \limsup_{t \to \infty} \frac{M_i}{t} \leq \frac{1}{\mu}. \quad (7) \]

Now Eqn. (5) and Eqn. (7) together imply

\[ \lim_{t \to \infty} \frac{M_i}{t} = \frac{1}{\mu} \]

and this completes the proof.

If follows directly from the elementary renewal theorem that the average number of renewals up to time \( t \), namely \( M_i \), increases to \( \infty \) as \( t \to \infty \). It also gives the rate at which \( M_i \) increases.

In the next section, we will state a more general version of the above theorem called the Basic Renewal Theorem.

### 8.4 BASIC RENEWAL THEOREM

In the previous section, we studied the Elementary renewal theorem. The same type of arguments give a more general theorem which we will study in this section. We recall the definition of the convolution \( g_{i} * g_{2} \) of two functions \( g_{i} \) and \( g_{2} \) given in Unit 7.

Let \( a_{i} \) be a bounded function. As usual \( F(t) \) denote the cdf of the interoccurrence time and \( M_{i} \) the renewal function. We look at the following renewal-type equation (compare with the renewal equation, Eqn. (3)).

\[ A_{i} = a_{i} + (A * F), \quad (8) \]

We are interested in finding solution \( A_{i} \) of Eqn. (8). The following theorem identifies such solution.
**Theorem 4:** Suppose $a_t$ is a bounded function. There exists one and only one function $A_t$, which is bounded on all finite intervals and which satisfies Eqn. (8). The unique solution is given by

$$A_t = a_t + \int_0^t a_{t-s} dM_s = a_t + (a * M)_t$$

where $M_t$ is the renewal function.

**Proof:** Let $A_t$ be defined by Eqn. (9). We will show that this is a solution of Eqn. (8). Using Eqn. (20) of Sec. 7.6 and Eqn. (9), we get

$$\begin{align*}
A_t &= a_t + (a * M)_t = a_t + \left[ a \left( \sum_{k=1}^{\infty} F_k \right) \right]_t \\
&= a_t + \left[ a \left( F + \sum_{k=2}^{\infty} F_k \right) \right]_t \\
&= a_t + (a \cdot F)_t + \left[ a \left( \sum_{k=2}^{\infty} (F_k) \right) \right]_t \\
&= a_t + (a \cdot F)_t + \left[ a \left( \sum_{k=2}^{\infty} (F_{k-1} \cdot F) \right) \right]_t \\
&= a_t + \left( \left( a + \left( a \cdot \sum_{k=1}^{\infty} F_k \right) \right) \cdot F \right)_t \\
&= a_t + (a + a \cdot M) \cdot F)_t \\
&= a_t + (A \cdot F)_t
\end{align*}$$

which is Eqn. (8). Moreover, since $a_t$ is bounded and $M_t$ is non-decreasing we have for any $0 < T < \infty$

$$\sup_{0 \leq s \leq T} A_s \leq \sup_{0 \leq s \leq T} |a_s| \left[ 1 + \int_0^T dM_s \right] \leq \sup_{0 \leq s \leq T} |a_s| |1 + M_T| < \infty.$$ 

Thus, $A_t$ is bounded on bounded intervals. Thus, Eqn. (9) gives one solution in the requisite class.

Now let $B$ be any solution of Eqn. (8) which is bounded on bounded intervals. We have

$$\begin{align*}
B_t &= a_t + (B \cdot F)_t \\
&= a_t + ((a + B \cdot F) \cdot F)_t \\
&= a_t + (a \cdot F)_t + (B \cdot F)_t \\
&= \ldots \\
&= a_t + \left( \left( a + \left( a \cdot \sum_{k=1}^{\infty} F_k \right) \right) \cdot F \right)_t \\
&= a_t + (A \cdot F)_t
\end{align*}$$

which is Eqn. (8). Moreover, since $a_t$ is bounded and $M_t$ is non-decreasing we have for any $0 < T < \infty$

$$\sup_{0 \leq s \leq T} B_s \leq \sup_{0 \leq s \leq T} |a_s| \left[ 1 + \int_0^T dM_s \right] \leq \sup_{0 \leq s \leq T} |a_s| |1 + M_T| < \infty.$$ 

Thus, $B_t$ is bounded on bounded intervals. Thus, Eqn. (9) gives one solution in the requisite class.

Now let $B$ be any solution of Eqn. (8) which is bounded on bounded intervals. We have

$$\begin{align*}
B_t &= a_t + (B \cdot F)_t \\
&= a_t + ((a + B \cdot F) \cdot F)_t \\
&= a_t + (a \cdot F)_t + (B \cdot F)_t \\
&= \ldots \\
&= a_t + \left( \left( a + \left( a \cdot \sum_{k=1}^{\infty} F_k \right) \right) \cdot F \right)_t \\
&= a_t + (A \cdot F)_t
\end{align*}$$

for every $n \geq 1$. Further

$$|B \cdot F_n| = \left| \int_0^t B_{t-s} dF_n(s) \right| \leq \left( \sup_{0 \leq s \leq t} |B_s| \right) F_n(t).$$

Using Theorem 2 and Eqn. (20) of Sec. 7.6, we get that $F_n(t) \to 0$ as $n \to \infty$ for every $t > 0$. Hence

$$\lim_{n \to \infty} |B \cdot F_n| = 0 \text{ for all } t > 0.$$

(11)
Also
\[
\lim_{n \to \infty} \left( a \ast \left( \sum_{k=1}^{n} F_k \right) \right) = \left( a \ast \left( \sum_{k=1}^{n} F_k \right) \right) = (a \ast M)_t. \tag{12}
\]
Thus, combining Eqns. (10) – (12), we get
\[
B_t = a_t + \lim_{n \to \infty} \left( a \ast \left( \sum_{k=1}^{n-1} F_k \right) \right) + \lim_{n \to \infty} (B \ast F_n)_t = a_t + (a \ast M)_t.
\]
This completes the proof.

As an immediate application we have the following exercise.

E6) Prove Lemma 1 using the above theorem.

To state the Basic renewal theorem we need to introduce one definition.

**Definition 2 (Arithmetic Distribution Function):** A distribution function \( G \) is said to be arithmetic if there exists a \( \lambda > 0 \) such that the set of all discontinuity points of \( G \) is a subset of \( \{0, \pm \lambda, \pm 2\lambda, \ldots\} \).

The largest such \( \lambda \) is called the span of \( G \).

Now we will state the Basic Renewal theorem without proof. This generalizes Theorem 4 to the solutions of the more general renewal type equations.

**Theorem 5 (Basic Renewal Theorem):** Let \( F \) be the distribution function of a positive random variable with mean \( \mu < \infty \). Suppose that \( a_t \) is Riemann integrable and \( A_t \) is the solution of the renewal type equation
\[
A_t = a_t + (A \ast F)_t.
\]

1. If \( F \) is not arithmetic, then
\[
\lim_{t \to \infty} A_t = \frac{1}{\mu} \int_0^\infty a_x dx.
\]
2. If \( F \) is arithmetic with span \( \lambda \), then for all \( c > 0 \),
\[
\lim_{n \to \infty} A_{t+n\lambda} = \frac{\lambda}{\mu} \sum_{n=0}^{\infty} a_{c+n\lambda}
\]

We note the following Corollary which is also called the Blackwell's renewal theorem.

**Corollary 1:** Let \( h > 0 \) be fixed. If \( F \) is not arithmetic then
\[
\lim_{t \to \infty} [M_{t+h} - M_t] = \frac{h}{\mu}.
\]

**Proof:** Define
\[
a_t = \begin{cases} 1 & \text{if } 0 \leq t < h \\ 0 & \text{if } h \leq t. \end{cases}
\]
Substituting in Eqn. (8), we get for \( t > h \)
\[
A_t = 0 + \int_{t-h}^t dM_r = M_t - M_{t-h}
\]
Further \( \int_0^\infty a_x dx = h \). Thus, the corollary follows from Theorem 5.
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The Basic renewal theorem is useful in many applications. We will see one such application in the next section.

8.5 ASYMPTOTIC DISTRIBUTION OF THE RESIDUAL TIME

In this section, we will study the asymptotics of various quantities associated with a renewal process. We will use the renewal theorems that we have studied in the last two sections for this purpose.

Let $X_1, X_2, \ldots$ be the sequence of inter-arrival times which are assumed to be i.i.d. with common cdf $F$. Let $S_n$ be the associated renewal sequence and let $N_t$ be the corresponding renewal process.

Thus, $S_{N_t}$ denotes the time of the last occurrence of the event before time $t$ and $S_{N_t+1}$ denotes the time of the ‘next’ occurrence. We define a few more related quantities.

Definition 3: The residual time at time $t$ (or excess life) is defined by

$$Y_t = S_{N_t+1} - t$$

and is the time left for the next occurrence as seen at time $t$.

The current life (or age) at time $t$ is the random variable defined by

$$\delta_t = t - S_{N_t}$$

The total life $\beta_t$ is defined as

$$\beta_t = Y_t + \delta_t = S_{N_t+1} - S_{N_t}$$

We will see these quantities in the Poisson process example.

Example 6: Suppose that the renewal process $N_t$ is a Poisson process with parameter $\lambda$. Then the residual time $Y_t$ is an exponential random variable with parameter $\lambda$, as can be seen using the stationary increments property of $N_t$ as follows.

$$P(Y_t > x) = P(N_{t+x} - N_t = 0) = P(N_x = 0) = e^{-\lambda x}$$

A similar argument can be used to find the distribution of $\delta_t$.

Now try the following exercise.

E7) Find the cdf of $\delta_t$.

Moreover, using the independent increment property of $N_t$, we also get that $\delta_t$ and $Y_t$ are independent random variables.

While finding the distributions (or even the joint distribution) of $\delta_t$ and $Y_t$ was quite easy in the case of the Poisson process, this is certainly not the case in general. As in the case of the renewal function, we look at the asymptotic properties of these random variables. We have the following lemma.

Recall that $\mu$ denotes the mean interoccurrence time, i.e. $\mu = E(X_1)$. We assume $\mu < \infty$. We also assume for the rest of the subsection that $F$ is continuous. Then, clearly $F$ in non-arithmetic.
Lemma 2: For every \( z > 0 \)

\[
\lim_{t \to \infty} P(\gamma_t > z) = \frac{1}{\mu} \int_z^{\infty} (1 - F(s)) \, ds.
\]

Proof: For every \( z > 0 \) define the function \( A^z_t \) by

\[
A^z_t = P(\gamma_t > z).
\]

Then as usual conditioning on the first renewal occurring at \( s > 0 \), we get

\[
P(\gamma_t > z \mid X_t = s) = \begin{cases} 
1 & \text{if } s > t + z \\
0 & \text{if } t + z \geq s > t \\
A^z_{t-s} & \text{if } t \geq s.
\end{cases}
\]

Unconditioning the above, we get

\[
A^z_t = \int_0^t P(\gamma_t > z \mid X_t = s) \, dF(s) = \int_0^t A^z_{t-s} \, dF(s) + \int_t^{t+z} 0 \, dF(s) + \int_{t+z}^{\infty} 1 \, dF(s) \\
= 1 - F(t + z) + \int_0^t A^z_{t-s} \, dF(s).
\]

This is a renewal type equation as in Eqn. (8). Further the function \((1 - F(t + z))\) as a function of \( t \) is clearly bounded. Hence by Theorem 4 its unique solution is given by

\[
A^z_t = 1 - F(t + z) + \int_0^t (1 - F(t + z - s)) \, dM_s. \tag{16}
\]

But, Eqn. (16) may not be easy to compute. But note that since \( F \) is the cdf of a nonnegative random variable \( X_t \), we have

\[
\int_0^\infty (1 - F(s)) \, ds = \int_\infty^0 P(X_t > s) \, ds = E(X_t) = \mu < \infty.
\]

Then

\[
\int_0^\infty (1 - F(t + z)) \, dt = \int_0^\infty (1 - F(s)) \, ds < \infty.
\]

Thus, the function \((1 - F(t + z))\) is Riemann integrable and the Basic renewal theorem can be applied. Theorem 5 now implies

\[
\lim_{t \to \infty} A^z_t = \frac{1}{\mu} \int_0^\infty (1 - F(t + z)) \, dt = \frac{1}{\mu} \int_0^\infty (1 - F(s)) \, ds.
\]

This completes the proof.

An immediate consequence of the above Lemma is the following Corollary which gives the asymptotic joint distribution of \( \gamma_t \) and \( \delta_t \).

Corollary 2: For every \( y > 0, z > 0 \)

\[
\lim_{t \to \infty} P(\delta_t > y, \gamma_t > z) = \frac{1}{\mu y + z} \int_0^\infty (1 - F(s)) \, ds.
\]

Proof: Note that for any \( y > 0, z > 0 \) and any \( t > 0 \)

\[
\{ \gamma_t > z, \delta_t > y \} = \{ \gamma_{t-t} > z + y \}.
\]

Thus, using Lemma 2, we get

\[
\lim_{t \to \infty} P(\gamma_t > z; \delta_t > y) = \lim_{t \to \infty} P(\gamma_{t-t} > z + y) = \frac{1}{\mu} \int_0^\infty (1 - F(s)) \, ds.
\]

We further get that asymptotically \( \gamma_t \) and \( \delta_t \) behave similarly.
Corollary 3: For every $y > 0$,
\[
\lim_{t \to \infty} P(\delta_t > y) = \frac{1}{\mu} \int_y^\infty (1 - F(s)) \, ds.
\]

Proof: The proof follows immediately from Corollary 2 since
\[
\lim_{t \to \infty} P(\delta_t > y) = \lim_{z \to \infty} \lim_{t \to \infty} P(y, z, \delta_t > y)
\]
\[
= \lim_{z \to \infty} \lim_{t \to \infty} P(y, z, \delta_t > y)
\]
\[
= \lim_{z \to \infty} \frac{1}{\mu} \int_{2z+y}^\infty (1 - F(s)) \, ds
\]
\[
= \frac{1}{\mu} \int_y^\infty (1 - F(s)) \, ds.
\]

Finally, we find the asymptotic distribution of the total lifetime $\beta$, as a consequence of the Basic renewal theorem.

Lemma 3: For every $x > 0$
\[
\lim_{t \to \infty} P(\beta_t > x) = \frac{1}{\mu} \int_x^\infty s \, dF(s)
\]

Proof: Fix $x > 0$. Define $B^x_t = P(\beta_t > x)$. Once again employing the familiar renewal argument, we get
\[
P(\beta_t > x | X_1 = s) = \begin{cases} 
1 & \text{if } s > \max(t, x) \\
B_{t-s}^x & \text{if } t \geq s \\
0 & \text{otherwise.}
\end{cases}
\]

Unconditioning the above, we get
\[
B^x_t = \int_0^\infty P(\beta_t > x | X_1 = s) \, dF(s)
\]
\[
= \int_0^t B_{t-s}^x \, dF(s) + \int_{\max(t, x)}^\infty 1 \, dF(s)
\]
\[
= 1 - F(\max(t, x)) + \int_0^t B_{t-s}^x \, dF(s).
\]
The basic renewal theorem now gives
\[
\lim_{t \to \infty} B^x_t = \lim_{t \to \infty} P(\beta_t > x)
\]
\[
= \frac{1}{\mu} \int_0^\infty [1 - F(\max(t, x))] \, dt
\]
\[
= \frac{1}{\mu} \left[ \int_0^x [1 - F(x)] \, dt + \int_x^\infty [1 - F(t)] \, dt \right]
\]
\[
= \frac{1}{\mu} \int_x^\infty s \, dF(s).
\]
The last equality above follows using integration by parts.

Now try the following exercises.

E8) Refer to E9) of Unit 7, where a system is fitted with two identical components — only one of which is used at a time. The system fails when both the components fail. Assuming the life time distribution of the component is exponential with parameter $\lambda$, find the Laplace transform of the interoccurrence time distribution.
E9) Refer to E8). Find the renewal function.

E10) In the above exercise, find the long term average number of failures.

Now before ending this unit, let us go over its main points.

### 8.6 SUMMARY

In this unit, we studied the following:

1. We defined the Laplace transform of a non-negative function and computed the Laplace transform of the renewal function \( M_t \), which is denoted by \( \tilde{M}_t \).

2. We saw the relationship between \( \tilde{M}_t \) and \( \tilde{F}_t \). With this we were able to conclude that the renewal function uniquely characterizes the distribution of a renewal process.

3. We computed \( \tilde{M}_t \) in some special cases.

4. We proved the relation \( \mathbb{E}[S_{N_{1+}}] = \mathbb{E}(X_1)(M_t + 1) \).

5. We proved the elementary renewal theorem.

6. We looked at renewal type equations and found the unique solutions to such equations. We also stated the Basic renewal theorem.

7. We applied the basis renewal theorem to find asymptotic distributions of the residual time \( \gamma_t \), the current age \( \delta_t \) and the total age \( \beta_t \).

### 8.7 SOLUTIONS/ANSWERS

E1) Here \( F(t) = 1 - e^{-\lambda t}, t \geq 0 \). Hence

\[
\tilde{F} = \int_0^\infty e^{-st}dF(s) = \int_0^\infty e^{-st}(\lambda e^{\lambda s})ds = \frac{\lambda}{\lambda + 1}
\]

Thus,

\[
\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{\lambda}{t}
\]

E2) As in Example 5, we will first find \( \tilde{F}_t \). Note

\[
\tilde{F}_t = E\left[e^{-\lambda X_t}\right] = 0.2 + 0.3e^{-t} + 0.5e^{-2t}
\]

Thus,

\[
\tilde{M}_t = \frac{\tilde{F}_t}{1 - \tilde{F}_t} = \frac{0.2 + 0.3e^{-t} + 0.5e^{-2t}}{0.8 - 0.3e^{-t} - 0.5e^{-2t}}
\]

E3) Here

\[
\tilde{F}_t = E\left[e^{-\lambda X_t}\right] = \int_0^\infty e^{-\lambda x}e^{-(x-t)}dx
\]

\[
= e^{-(t+1)x}\bigg|_{x = \infty} - e^{-t}\bigg|_{x = 1}
\]
Thus, \( \hat{M}_t = \frac{\hat{F}_t}{1 - \hat{F}_t} = \frac{e^{-t}}{t + 1 - e^{-t}}. \)

**E4)** By definition for \( 1 \leq m \leq n \)
\[
F_n(t) = \left[ F(t) \right]^n = \left[ F(t)^{n - m} \star F(t)^m \right]
\]
\[
= F_{n-m}(t) \star F_m(t) = \int_0^t F_{n-m}(t - s)dF_m(s)
\]
\[
\leq F_{n-m}(t)F_m(t) \leq F_m(t)
\]
The last inequalities follows since \( F_{n-m} \) is a cumulative distribution function and hence non-decreasing. This proves the first part.

Using the second last inequality above, we get
\[
F_{m+k}(t) \leq F_{(n-l)+k}(t)F_l(t).
\]
Now iterating this inequality \( n \) times, we get
\[
F_{m+k}(t) \leq F_t(t)\left[ F(t) \right]^n.
\]
where we define \( F_0(t) = 1. \)

**E5)** Since \( F(0) < 1 \), we get using right continuity of \( F \) that there exists a \( t_0 > 0 \) such that \( F(t_0) < 1 \). Hence for all \( 0 \leq t \leq t_0, F(t) < 1 \).

Now recall that \( F_2 \) is the cdf of \( X_1 + X_2 \) where \( X_1, X_2 \) are i.i.d. with common distribution function \( F. \) Thus, we have
\[
P(X_1 + X_2 > 2t_0) \geq P(X_1 > t_0, X_2 > t_0) = P(X_1 > t_0)P(X_2 > t_0).
\]
This is same as \( 1 - F_2(2t_0) \geq [1 - F(t_0)]^2 > 0. \)
Hence for \( 0 \leq t \leq 2t_0 \)
\[
F_2(t) \leq F_2(2t_0) < 1.
\]
By induction, we get
\[
F_2(t) \leq F_2(2^lt_0) < 1 \text{ for all } 0 \leq t \leq 2^lt_0.
\]
Now using the monotone property of the sequence \( F_n \) proved in E4), we get the result.

**E6)** In the proof of the Lemma 1 we have shown that \( A_t = E\left[ S_{N_t} \right] \) satisfies the renewal type equation
\[
A_t = E(X_t) + (A \star F)_t.
\]
This is same as Eqn.(8) with \( a_t = E(X_t) \). By Theorem 4 the unique solution is given by Eqn.(9) which in this case becomes
\[
A_t = E(X_t) + \int_0^t E(X_t)dM_t = E(X_t)[1 + M_t].
\]

**E7)** By Eqn. (14) it is evident that \( \delta_t \leq t. \) Thus, \( P(\delta_t \leq x) = 1 \) for all \( x \geq t. \) Now let, \( x < t. \) Then
\[
P(\delta_t > x) = P(N_t - N_{t-x} = 0) = P(N_x = 0) = e^{-\lambda x}
\]

**E8)** Let \( F \) denotes the cdf of an exponential random variable with parameter \( \lambda \) and let us denote the distribution of \( X^1 + X^2 \) by \( G \), i.e., \( G \) is the cdf of a gamma random variable with parameters 2 and \( \lambda. \) Then \( G = F \star F \). Let the
corresponding renewal functions be denoted by \( M_i^F \) and \( M_i^G \). We know that \( M_i^F = \lambda t \). (See Example 10 of Unit 7).

Further, by Example 2 (of Unit 8) the corresponding Laplace transforms are given by \( \tilde{G} = \tilde{F}^2 \). Now

\[
\tilde{F}(t) = \int_0^\infty e^{-st}dF(s) = \int_0^\infty \lambda e^{-st}e^{-\lambda s}ds = \frac{\lambda}{t + \lambda}.
\]

Thus,

\[
\tilde{G}(t) = \left( \frac{\lambda}{t + \lambda} \right)^2.
\]

E9) As we have seen earlier

\[
\tilde{G}(t) = \left( \frac{\lambda}{t + \lambda} \right)^2.
\]

Now Eqn. (1) (of this unit) implies

\[
\tilde{M}^G(t) = \frac{\tilde{G}(t)}{1 - \tilde{G}(t)} = \frac{\lambda}{t + 2\lambda}.
\]

We have already noted (see Example 3) that \( \tilde{M}^F(t) = \lambda/t \). Thus, we get that \( \tilde{M}^G(t) \) is a product of two Laplace transform \( \tilde{M}^F(t) \) and \( \tilde{H}(t) \) where

\[
\tilde{H}(y) = \frac{2\lambda y}{t + 2\lambda}.
\]

Note \( \tilde{M}^G(t) = \frac{\tilde{M}^F(t)\tilde{H}(t)}{2} \). Thus, we conclude using Example 2 that

\[
M_i^G = H_i \ast M_i^F \text{ where } H_i = 1 - e^{-2\lambda t}.
\]

Hence

\[
M_i^G = \frac{1}{2} \int_0^\infty (1 - e^{-2\lambda(t-s)}) \lambda ds = \frac{\lambda t}{2} \left( 1 - e^{-2\lambda t} \right).
\]

E10) The required quantity is given by the elementary renewal theorem to be

\[
\lim_{t \to \infty} \frac{M_i^G}{t} = \frac{\lambda}{2}.
\]

This is expected since the long term average number of failures will be half the number when there is no redundant component built into the system.

---x---