UNIT 7 RENEWAL PROCESSES-I

Structure

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7.1 INTRODUCTION

Many situations arise in day-to-day life, whether at home or in the office, where a certain event recurs again and again. Further, the time interval between two consecutive occurrences, though not fixed, follows the same probabilistic pattern. Study of such processes helps in a better understanding of these events. Such processes are called renewal processes. We will start the study of these processes in Sec. 7.2 of this unit. In Sec. 7.3 and 7.4, we shall cite few examples to understand the renewal processes. In Sec. 7.5, we shall discuss the distribution of S_n, the time of the n-th occurrence of the event, and the distribution of renewal process N_t, the number of occurrences till time t. In the last section of this unit, we shall discuss the renewal equation. By the end of the unit this is what you should be able to do.

Objectives

After studying this unit, you should be able to:

- identify a renewal process;
- find the law of the process (S_n or N_t) in a few simple cases;
- use the renewal argument to find certain quantities of interest;
- find the renewal function;
- derive the renewal equation.

7.2 DEFINITION

Let us start with a question. If a bulb lasts for 12 hours, then how many bulbs are required for one week continuous lighting? The answer of this seems very simple with various assumptions like all the bulbs are alike, all the bulbs are having same life time, also one bulb is used at anytime, i.e., no simultaneous use of bulbs, etc. In this situation, the life times of bulbs are random variables with common cumulative distribution function (cdf) $F(x)$. This can now be considered as a stochastic model, in which all the random variables are mutually independent and non-negative. In other words, this is the study of properties of a sequence $\{x_n\}$ of iid non-negative random variables, which is a representation of the life times of the bulbs that are being renewed (or replaced). The life time distribution of the renewed bulb is same as that of the previous bulb. The sequence $\{x_n\}$ is said to constitute a renewal sequence which the assumption that the failure of the bulbs is complete and sudden and are replaced instantaneously.

Consider an event whose time of occurrence is a random variable X with cumulative distribution function (cdf) F. i.e. $F(x) = P(X \le x)$. Since $X \ge 0$, F will satisfy

$F(0) = 0$.

Suppose this event occurs repeatedly and such that the occurrences are independent of one another. Such a process can be modelled via an i.i.d. sequence of random variables $X_1, X_2, \ldots$ with common cdf F, where X_n can be thought of as the time between the $(n-1)$-th and the n-th of the event.

Let the partial sums of the sequence be S_n, which is given by

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i,\ n = 1, 2, 3, \ldots \tag{1}$$

Here S_n denotes the total time elapsed before the n-th event occurs, this is, time instant of n-th occurrence of the event and is known as renewal epochs.

X_i is the interval between occurrences of two successive renewals, which is called the renewal period of the process.

The number of occurrences of the event in any given time interval can be counted. Let N_t be the number of renewals upto time t and $\{N_t : t \ge 0\}$ denote this counting process, which can be defined as follows.

$$N_0 = 0,\ N_t = \inf\{k : S_{k+1} > t\} \tag{2}$$

This process $\{N_t : t \ge 0\}$ is called a renewal process. Note that for any $t > 0$, exactly k occurrences take place upto time t if and only if $S_k \le t < S_{k+1}$, i.e.

$$\{N_t = k\} = \{S_k \le t < S_{k+1}\} \tag{3}$$

$\{N_t : t \ge 0\}$, $\{X_n\}$ and $\{S_n\}$ can be used to define a renewal process and also can be determined from Eqn. (1) and Eqn. (2).

Note that there is an equivalent way of defining N_t which is given in the following exercise.

E1) Let $\{\hat{N}_t : t \ge 0\}$ be defined by

$\hat{N}_0 = 0,\ \hat{N}_t = \sup\{k : S_k \le t\}$.

Show that $\hat{N}$ coincides with the renewal process N.

Let us now define renewal process in formal definition.

Definition 1 (Renewal Process): The renewal process $\{N_t : t \ge 0\}$ is a non-negative integer valued counting process that counts the number of occurrences of an event during the time interval (0,t], where the times between occurrences of two successive events are non-negative, independent, identically distributed random variables.

Definition 2 (Renewal Sequence): Let $\{N_t : t \ge 0\}$ be a renewal process and let S_n, $n = 1, 2, \ldots$ be the time of occurrence of the n-th event so that Eqn.(2) holds and let $S_0 \equiv 0$. Then the sequence $\{S_n\}$ is called a renewal sequence.

There are lots of processes in real life which can be modelled as renewal processes. The variety of situations are replacement, queueing, maintenance, simulation, manpower studies, demography, reliability, inventory control, etc. In the next section, we start off with the most basic example, viz., that of a Poisson process.

7.3 AN EXAMPLE: POISSON PROCESS

The Poisson process has already been introduced in Unit 5 as an example of a pure birth process. It is also a renewal process as we will see below. First we will give an alternative definition of the Poisson process.

Definition 3: A process N_t is a Poisson process with parameter λ if

1. $N_0 \equiv 0$.
2. For $t > 0$, N_t has a Poisson distribution with parameter λt.
3. For $s, t > 0$,

$$N_{t+s} - N_s \stackrel{d}{=} N_t$$

and is independent of $\{N_u : 0 \le u \le s\}$.

Here $V \stackrel{d}{=} W$ means random variables V and W have the same probability distribution.

The last property is called the **stationary independent increments** property of the Poisson Process.

Example 1 (Poisson Process): Consider the counting process when the interoccurrence times are i.i.d. random variables which are exponentially distributed with parameter $\lambda > 0$. Let $X_1, X_2, \ldots$ denote this sequence. Then the probability density function (pdf) of X_1 (and hence of all X_i) is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} \ ; & x > 0 \\ 0 \ ; & \text{otherwise} \end{cases}$$

and its cdf is given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x} \ ; & x \ge 0 \\ 0 \ ; & \text{otherwise.} \end{cases}$$

Further, it is well known that for every $n \ge 1$, S_n (defined by (Eqn.(1)) is a gamma random variable with parameters n and λ. Thus the cdf of S_n is given by

$$F_n(x) = \begin{cases} 1 - \sum_{j=0}^{n-1} \dfrac{e^{-\lambda x} (\lambda x)^j}{j!}, & x \ge 0 \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

In this case we can find the distribution of the renewal process $\{N_t : t \ge 0\}$ as follows. Fix $t > 0$, then

$$P(N_t = 0) = P(S_1 > t) = P(X_1 > t) = e^{-\lambda t} \qquad (5)$$

Further for any positive integer k, using Eqn.(3) and Eqn.(4), we get

$$\begin{aligned} P(N_t = k) &= P(S_k \le t < S_{k+1}) \\ &= P(S_{k+1} > t) - P(S_k > t) \\ &= \sum_{j=0}^{k} \frac{e^{-\lambda t} (\lambda t)^j}{j!} - \sum_{j=0}^{k-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!} \\ &= \frac{e^{-\lambda t} (\lambda t)^k}{k!}. \end{aligned} \qquad (6)$$

Eqns (5) and (6) now imply that for every $t > 0$, N_t is a Poisson random variable with parameter (λt). To show that N_t is a Poison process with parameter λ we will show that N has stationary independent increments. See Definition 3. For this, we need the following memory less property of the exponential distribution: If $Y \sim \exp(\alpha)$, then

$$P(Y > s+t \mid Y > s) = P(Y > t) = e^{-\alpha t} \tag{7}$$

Now fix $s, t > 0$ and consider $N_{s+t} - N_s$ which counts the number of occurrences of the event in the time interval $(s, s, +t]$. By definition of S and N (see Definitions 1 and 2), the time of occurrence of the first event after s is S_{N_s+1} and S_{N_s} is the time of occurrence of the most recent or last event before time s. Then,

$$\begin{aligned} P(N_{s+t} - N_s = 0 \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\ &= P(S_{N_s+1} - s > t \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\ &= P(S_k + X_{k+1} - s > t \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\ &= P(X_{k+1} > s+t-x \mid X_{k+1} > s-x) \\ &= e^{-\lambda t}. \end{aligned} \tag{8}$$

The last equality above follows from the memoryless property of X_{k+1}. Similarly, for $i \ge 1$, we can write

$$\begin{aligned} & P(N_{s+t} - N_s = i \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\ &= P(S_k + X_{k+1} + \cdots + X_{k+i+1} - s > t, \\ &\quad S_k + X_{k+1} + \cdots + X_{k+i} - s \le t \mid N_s = k, S_{N_s} = x, N_u : 0 \le u \le s) \\ &= P(X_{k+1} + \cdots + X_{k+i+1} > s+t-x \\ &\quad X_{k+1} + \cdots + X_{k+i} \le s+t-x \mid X_{k+1} > s-x) \\ &= P(X_{k+1} + \cdots + X_{k+i+1} > s+t-x \mid X_{k+1} > s-x) \\ &\quad - P(X_{k+1} + \cdots + X_{k+i} \le s+t-x \mid X_{k+1} > s-x) \\ &= \sum_{j=0}^{i} \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \sum_{j=0}^{i-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \frac{e^{-\lambda t}(\lambda t)^i}{i!}, \end{aligned} \tag{9}$$

where once again the last equality follows from the memoryless property of X_{k+1} and the fact that the X_j's are i.i.d. exponential (λ). Thus, the increments $N_{s+t} - N_s$ are independent of $\{N_u : 0 \le u \le s\}$ and have the same distribution as N_t for all $s, t > 0$. Thus, the renewal process N_t satisfies all the properties of definition 3, and hence is a Poisson Process with parameter λ.

As mentioned earlier, Poisson process is the most basic example of a renewal process. Many of the results that we will study in later units in this block can be first viewed in terms of the Poisson process. In the next section, we will look at more examples of renewal processes.

7.4 SOME MORE EXAMPLES

In the last section, we focused on a specific example of a renewal process when the time between successive events are i.i.d. exponential random variables. The Poisson process is a very important example and we studied it in some detail. In this section, we will discuss some more examples. It is left to the reader to check that the processes mentioned in these examples are indeed renewal processes.

Clearly if the interoccurrence time distribution, i.e. the distribution of X_i, changes we will get a different renewal process. Note that it is possible that X_i takes value 0. In fact, $P(X_i = 0) = F(0)$ can be strictly greater than 0. Note, further, that $X_n = 0$ means that the $S_n = S_{n-1}$ so that the $(n-1)$th and n-th events occur simultaneously. We will give a couple of examples where X_i is a discrete random variable.

Example 2 (Negative Binomial Process): Let X_n be i.i.d. Bernoulli $(p), 0 < p < 1$, i.e.

$$P(X_n = 0) = 1 - p, P(x_n = 1) = p.$$

then the corresponding renewal process $\{N_t, t = 0, 1, 2, \ldots\}$ is called a negative binomial process.

Example 3 (Binomial Process): Let X_n be i.i.d. each having geometric distribution

$$P(X_n = i) = (1-p)p^{i-1}, \qquad i = 1,2,3,\ldots$$

where $0 < p < 1$. Then the corresponding renewal process $\{N_t, t = 0, 1, 2, \ldots\}$ is called the binomial process.

Now we will see a few real life situations where such processes arise. We will study some of these processes in detail later.

Example 4 (Recurrent Events of Markov Chains): Let $\{Y_n : n = 0,1,2,\ldots\}$ be a Markov chain with state space $\Gamma = \{0, 1, 2, \ldots\}$ and such that $Y_0 = 0$. Let S_n be the number of transitions or the time for the n-th visit of the chain Y_n to 0. Here S_0 can be thought of as the time of the zero-th visit to zero which is 0. Thus

$$S_n = \min\{k > S_{n-1} : Y_k = 0\}.$$

Then it follows from the Markov property that the sequence $\{X_n\}$ defined by

$$X_n = S_n - S_{n-1}, \; n = 1, 2, \ldots$$

is an i.i.d. sequence. Hence S_n is a renewal sequence and the associated counting process $\{N_t, t = 0, 1, 2, \ldots\}$ is a renewal process.

Example 5 (A Queueing Example): Consider a bank counter where a clerk attends to the customers who are standing in a queue one after another. Assume that customers arrive at the counter at random times in an independent fashion so that the inter-arrival times are i.i.d. exponential. Also assume that the successive service times for the customers also an i.i.d sequence of random variables. Then there are several different renewal processes associated with this system. We give a few examples below.

1. The process N_t^1 which denotes the total number of arrivals upto time t is a Poisson renewal process.

2. The process N_t^2 which counts the number of busy periods completed upto (and including) time t is also a renewal process. More precisely, let $\{Q_t : t \geq 0\}$ be the queue-length process which just denotes the number of persons in the queue at time t. This obviously equals the number of arrivals minus the number of services completed by the bank clerk upto time t. Assume $Q_0 = 0$ and let $S_0 = 0$. Define

$$S_s = \inf\{t > S_{n-1} : Q_t = 0\}, \; X_n = S_n - S_{n-1}.$$

Here S_n denotes the time when the n-th busy period gets over, X_n is the time duration of the n-th busy period. S_n forms a renewal sequence and the associated counting process N_t^2 is a renewal process.

3. The process N_t^3 which counts the start of successive busy periods is also a renewal process, if $Q_0 = 1$ and the service has just started.

Example 6 (Counter Process): Consider a system where electrical impulses or signals are arriving and are being recorded on a recording device (counter). If the successive electrical impulses are assumed to be arriving in an i.i.d. fashion, then the process N_t^1 which counts the number of signals arriving in the time interval $(0, t]$ is clearly a renewal process.

Further, due to mechanical restrictions, suppose that as soon as a signal is recorded the counter gets locked for a small but fixed amount of time and any signal arriving in that time period will not be recorded. Then the process N_t^2 which counts the number of recorded signals in the time interval $(0, t]$ is also a renewal process.

Example 7 (Replacement Models): Let $X_1, X_2, \ldots$ denote the lifetimes of a particular type of components used in a machine. We will assume that the lifetimes are i.i.d. Suppose that whenever a component fails, it is replaced immediately by the next one. Then clearly the process which counts the number of components that fail upto (and including) time t is a renewal process.

Now, many times for preventive maintenance it is economical to replace the component at some fixed age, say T, even if it has not failed till then. Of course, if the component fails earlier, it is replaced on failure. Then each of the following processes is a renewal process.

1. number of failures in the interval $(0, t]$.
2. number of planned preventive replacements in the interval $(0, t]$.
3. number of replacements in the interval $(0, t]$.

Example 8 (Upper Records of a Renewal Process): Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with $E(\xi_1) \geq 0$, so that the random variables take non-negative values with positive probability. Let

$$U_0 = 0,\ U_n = \sum_{i=1}^{n} \xi_i, \qquad n \geq 1,$$

and let S_n be the upper record times for the sequence U_n defined by

$$S_0 = 0, S_n = \inf\{k : U_k > U_{S_{k-1}}, k = 1, 2, \ldots\}$$

Then S_n is called the n-th upper record time of the sequence $\{U_n\}$. This new sequence is also a renewal sequence and the associated counting process is a renewal process.

Now, try the following exercises.

E2) Let N_t be the renewal process in Example 2. Then show that N_n has a negative binomial distribution for all non-negative integers n.

E3) For the renewal process N_t of Example 3, show that N_n has a binomial distribution for all non-negative integers n.

So far, we have discussed the renewal process with examples. Now in this section, we shall focus on the distribution of the renewal times and the renewal process.

7.5 DISTRIBUTION OF THE RENEWAL TIMES AND THE RENEWAL PROCESS

By now you have seen that renewal processes arise in a variety of situations. We will now study some of its properties. We will need the following definition.

Definition 4 (Convolution of Functions): Let g_1 and g_2 be any two non-negative functions. The convolution of g_1 and g_2 is a function denoted by $g_1 * g_2$ and defined by

$$g_1 * g_2(x) = \int_0^x g_1(x-y)g_2(y)dy.$$

Note that $g_1 * g_2 = g_2 * g_1$.

Now consider the renewal sequence S_n defined by Definition (1). Assume that the times duration of occurrence between two successive events, X_i's, are i.i.d. with common cumulative distribution function F. Assume, $F(0) = 0$.

Independence of X_i's and Eqn.(1) imply that the joint distribution of $S_1, S_2, \ldots, S_n$, for any n is completely determined by the cdf F.

Now let F_n denote the cdf of S_n. Note that $F_1 \equiv F$, since $S_1 = X_1$. Now using the fact that S_n is a sum of two independent random variables X_n and S_{n-1} we can get the cdf F_n by convoluting F_{n-1} and F as follows.

$$\begin{aligned} F_n(x) &= P(S_n \le x) \\ &= \int_0^x P(S_{n-1} \le x-y)\ dF_1(y) \\ &= \int_0^x F_{n-1}(x-y)\ dF(y) \\ &= F_{n-1} * F(x). \end{aligned} \tag{10}$$

Iterating the above we get that F_n is a n-fold convolution of F with itself. We will denote this by

$$F_n(x) = F^{*n}(x) \tag{11}$$

Let N_t be the renewal process associated with the sequence S_n. Then the probability law of the renewal process is also completely characterized by the cdf F. In fact, for

$$0 < t_1 < t_2 < \cdots < t_n \text{ and } 0 \le k_1 \le k_2 \le \cdots \le k_n$$

$$\begin{aligned} &P(N_{t_1} = k_1, N_{t_2} = k_2, \ldots, N_{t_n} = k_n) \\ &= P(S_{k_1} \le t_1 < S_{k_1+1}, S_{k_2} \le t_2 < S_{k_2+1}, \ldots, S_{k_n} \le t_n < S_{k_n+1}) \end{aligned} \tag{12}$$

The assertion now follows since the RHS is completely determined by F. We will restate it in the form of a theorem.

Theorem 1: The distribution of the processes $\{S_n : n = 1, 2, \ldots\}$ and $\{N_t, t \ge 0\}$ is completely determined by the F, where F is the cdf of the interoccurrence times. Further, Eqn.(3) implies

$$P(N_t \ge k) = P(S_k \le t) = F_k(t), \qquad t \ge 0,\ k = 0, 1, 2, \ldots,\ F_0(t) = 1 \tag{13}$$

Thus,

$$P(N_t = k) = P(N_t \ge k) - P(N_t \ge k+1) = F_k(t) - F_{k+1}(t) \tag{14}$$

We have already used this last assertion in Eqn.(6) while studying the Poisson renewal process.

We have already commented that as soon as an event occurs the process starts afresh or is renewed. This notion can be made more precise as follows. Note that the random variable N_t denotes the number of occurrences of the event in the time interval $(0,t]$. Similarly, N_{t+X_1} will denote the number of occurrences of the event in the time interval $(0,t+X_1]$. This last number is at least 1 since the first occurrence has taken place at time X_1. If we discount for that occurrence or in other words look at $N_{t+X_1}-1$, then we will get the number of occurrences of the event in the time interval $(X_1,\, t+X_1]$. As the process is a renewal process, therefore the following theorem looks logical.

Theorem 2: The processes $\{N_t,\, t\ge 0\}$ and $\{N_{t+X_1}-1, t\ge 0\}$ have the same probability law.

Proof: We have to show that the finite dimensional distributions of the two processes are the same. First note that since X_i's are i.i.d., $S_n - X_1 = \sum_{i=2}^{n} X_i$ has the same law as that of $S_{n-1} = \sum_{i=1}^{n-1} X_i$. Similarly

$$(S_1,S_2,\ldots,S_n) \overset{d}{=} (S_2-X_1,S_3-X_1,\ldots,S_{n+1}-X_1). \tag{15}$$

Now fix $0<t_1<t_2<\ldots<t_n$ and integers $0\le k_1\le k_2\le\ldots\le k_n$

$$\begin{aligned}
&P(N_{t_1+X_1}-1=k_1, N_{t_2+X_1}-1=k_2,\ldots,N_{t_n+X_1}-1=k_n)\\
&=P(N_{t_1+X_1}=k_1+1, N_{t_2+X_1}=k_2+1,\ldots,N_{t_n+X_1}=k_n+1)\\
&=P(S_{k_1+1}\le t_1+X_1<S_{k_1+2},\ldots,S_{k_n+1}\le t_n+X_1<S_{k_n+2})\\
&=P(S_{k_1+1}-X_1\le t_1<S_{k_1+2}-X_1,\ldots,S_{k_n+1}-X_1\le t_n<S_{k_n+2}-X_1)\\
&=P(S_{k_1}\le t_1<S_{k_1+1}, S_{k_2}\le t_2<S_{k_2+1},\ldots,S_{k_n}\le t_n<S_{k_n+1})\\
&=P(N_{t_1}=k_1, N_{t_2}=k_2,\ldots,N_{t_n}=k_n).
\end{aligned} \tag{16}$$

where the second last equality follows from Eqn. (15) above. This completes the proof of the theorem.

Then above theorem is also helpful in deriving integral equalities concerning some quantities which are naturally associated with a renewal process. We illustrate the point with an example below.

Example 9: Let $p_n(t)=P(N_t=n)$. We will use the above theorem to show that

$$p_n(t)=\int_0^t p_{n-1}(t-s)dF(s) \text{ for all } n\ge 1. \tag{17}$$

Fix $n>0$. We will condition on the time of first renewal which is same as the time of the first occurrence of the concerned event. Note that if $S_1>t$ iff $N_t=0$. If, however, $S_1<t$ then $\{N_t=n\}$ if and only if there are exactly $n-1$ occurrences of the event in the time interval $(S_1,t]$. But, Theorem 2 implies that the conditional probability of the above event given $S_1<t$ is exactly same as that of the event where exactly $n-1$ events occur in the time interval $(0,t-S_1]$. Summarising we can write

$$P(N_t=n\,|\,S_1=s)=\begin{cases}0 & \text{if } s>t\\ p_{n-1}(t-s) & \text{if } s\le t.\end{cases}$$

Thus, unconditioning on S_1 we get

$$p_n(t) = \int_0^\infty P(N_t = n \mid S_1 = s)dF(s)$$
$$= \int_0^t p_{n-1}(t-s)dF(s).$$

The exercise given below is the discrete time version of the example above and can be solved similarly. The trick is to condition on the time of first occurrence and use Theorem 2.

Note that in discrete time version $F(0)$ may be positive.

Try an exercise.

E4) In Examples 9 above also assume that X_i's are discrete random variables with

$$P(X_1 = i) = q_i$$

Then show that

$$p_n(j) = \sum_{l=0}^{j} p_{n-1}(j-l)q_l. \qquad (18)$$

So far we have discussed renewal process with various examples and distributions S_n and N_t. In this section, we shall learn about the renewal equation.

7.6 RENEWAL EQUATION

We will continue the study of renewal processes. An important ingredient is the renewal equation which is an equation satisfied by the renewal function . Before we study renewal equation, let us first define the renewal function formally in the definition given below.

Definition 5 (Renewal Function): The renewal function M_t is defined as the expected number of renewals upto time t, i.e.

$$M_t = E(N_t) \qquad (19)$$

The renewal function plays a very important role in renewal theory. We will see that it completely characterizes the renewal process. Note that the Eqn.(19) along with Eqn.(14) gives

$$M_t = \sum_{k=0}^{\infty} kP(N_t = k)$$
$$= \sum_{k=0}^{\infty} k(F_k(t) - F_{k+1}(t))$$
$$= \sum_{k=0}^{\infty} F_k(t). \qquad (20)$$

Eqn.(20) can be used to compute the renewal function M_t. We can also use Theorem 2 for this purpose as follows.

First, let us look at an example.

Example 10: For the Poisson process with rate λ, the renewal function is

$$M_t = E(N_t) = \lambda t.$$

Now, try to find the solution of the exercises given below.

E5) Find the renewal function for the Negative Binomial process of Example 2.

E6) Find the renewal function for the Binomial Process of Example 3.

In continuation to renewal function, let us discuss now renewal argument.

As we did for solving Example 9 and (E4) we will condition on the first renewal time S_1. If $s > t$ then $N_t = 0$ with probability one and hence $M_t = 0$. However, if $S_1 \le t$, then by Theorem 2 the expected number of renewals in the time interval $(S_1, t]$ given $S_1 = s$ coincides with M_{t-s}, the expected number of renewals in the time interval $(0, t-s]$. Since one renewal has already occurred at time s, we get that, conditioned on $S_1 \le t$ the expectation of N_t is summarising, we get

$$E(N_t \mid S_1 = s) = \begin{cases} 0 & \text{if } s > t \\ 1 + M_{t-s} & \text{if } s \le t. \end{cases} \tag{21}$$

Unconditioning on S_1 or in other words taking expectation with respect to S_1 we get

$$\begin{aligned} M_t &= \int_0^\infty E(N_t \mid S_1 = s)dF(s) \\ &= \int_0^t [1 + M_{t-s}]dF(s) \\ &= F(t) + \int_0^t M_{t-s}dF(s). \end{aligned} \tag{22}$$

The above method where we conditioned on the first renewal time S_1 and then integrated with respect to S_1 is called the **renewal argument.**

Definition 6 (Renewal Equation): The above Eqn. (22) is called the renewal equation.

The renewal equation plays a very important role in the study of renewal processes.

Now try the following exercises.

E7) Consider Example 6, the counter process. Suppose that the electrical impulses are arriving at time which are i.i.d. exponential (λ) random variables. Assume that the counter gets locked for a fixed time $\varepsilon > 0$ on the arrival of every signal (whether it is recorded or not). Also assume that to begin with (at time 0) the counter is similarly locked. Let N_t^1 and N_t^2 be as defined in the example and let $Y_1^i, Y_2^i, \ldots, i = 1, 2$ denote the corresponding inter-arrival times. Find the distribution of Y_j^2. Find the corresponding renewal functions M_t^1 and M_t^2.

E8) Consider a system with two components which are arranged in series. The system fails if either of the two components fail. On failure, a component is replaced instantaneously. Suppose each component works independently and has life and exponential life time distribution with parameter λ. Let N_t denotes the number of failures for the system in the time interval $[0, t)$. Find the distribution of N_t and also the renewal function.

E9) A particular component in a system is very critical. Hence there is a built-in redundancy – there are two identical components in the system. However, only one of them is used at a time. Whenever the first one fails the system is automatically switched on to the second (redundant) component. The system fails whenever the second component also fails. At that time both the components are replaced instantaneously.

Assume that the life time distribution of the component is exponential with parameter λ. Express this system in a "renewal process" framework. Find the inter-occurrence time distribution.

Now before ending this unit let us go over its main points.

7.7 SUMMARY

In this unit, you studied the following:

1. There are many situations that we come across where an event recurs again and again and such that the time between two consecutive occurrences are i.i.d. The number of occurrences of the event upto time t can be modeled as a process $N_t, t \geq 0$ which is called the **renewal process.** The sequence of the time of occurrences of the event S_n is called the **renewal sequence.**

2. **Poisson process** is the basic example of a renewal process. We have seen several other examples of renewal process.

3. We have studied the properties of the distribution of S_n and N_t. We now know that the distributions are completely determined by the distribution function F of the interoccurrence time. We have also learnt how to condition on the first occurrence time to deduce properties of the renewal process. Such a method is called the **renewal argument.**

4. The expected value of numbers of renewals, M_t is called the **renewal function.** We have derived the **renewal equation** satisfied by M_t.

7.8 SOLUTIONS/ANSWERS

E1) Note that $\{\hat{N}_t = k\} = \{S_k \leq t < S_{k+1}\} = \{N_t = k\}$.

E2) By definition

$$\{K : S_k \leq t < S_{k+1}).$$

Since each X_i takes only two possible values 0 or 1, and since S_k is the time of occurrence of the k-th event, we get $\{S_k \leq n < S_{k+1}\}$ occurs if and only if **(i)** $X_k = 1$ and **(ii)** out of the first $(k-1)X_i$'s exactly $n-1$ are 1's and the rest 0. Hence

$$P(N_n = k) = \binom{k-1}{n-1} p^n (1-p)^{k-n}, k = n, n+1, n+2, \ldots$$

Thus, N_n has a negative binomial distribution with parameters n and p.

E3) Note that X_i has a geometric distribution with parameter $(1-p)$. Thus, it follows that S_k has a negative binomial distribution with parameters k and $(1-p)$. Once again we need to look at the event $\{S_k \leq n < S_{k+1}\}$. This is same as the event in which the first n Bernoulli trials result in exactly k successes and $n-k$ failures while the $(k+1)$ st success occurs at the j-th trail for some $j > n$, the intermediate $j-n-1$ trials having resulted in failures. Thus,

$$P(N_n = k) = P(S_k \le n < S_{k+1})$$
$$= \binom{n}{k}(1-p)^k p^{n-k} \sum_{j=n+1}^{\infty} (1-p)p^{j-n-1}$$
$$= \binom{n}{k}(1-p)^k p^{n-k}.$$

Thus, N_n has a binomial distribution with parameters n and $(1-p)$.

E4) Note that when $n = o$

$$p_0(j) = P(N_j = 0) = P(X_1 > j) = 1 - \sum_{l=0}^{j} q_l, \quad j = 0,1,2,\ldots$$

Using Theorem 2 we get for $n \ge 1$ and $j \ge 0$

$$P(N_j = n \mid s_1 = l) = \begin{cases} 0 & \text{if } l = j+1, j+2, \ldots \\ p_{n-1}(j-l) & \text{if } l = 0,1,2,\ldots,j. \end{cases}$$

Unconditioning we get

$$p_n(j) = \sum_{l=0}^{j} p_{n-1}(j-l) q_l .$$

E5) Since the distribution of X_i is discrete, the events occur only at integral times. Hence $N_t = N_{[t]}$ where $[t]$ denotes the largest integer smaller than or equal to t. Since $N_{[t]}$ has a negative binomial distribution with parameters $[t]$ and p we get

$$M_t = \frac{[t]}{p}.$$

E6) Once again $N_t = N_{[t]}$ and further the distribution is binomial with parameters $[t]$ and $1-p$. Thus,

$$M_t = (1-p)[t], t \ge 0.$$

E7) Note Y_j^2 denotes the time between the $(j-1)$-th and j-th recorded signals. The random variables Y_j^2 and i.i.d. Further, and arriving signal gets recorded if and only if it has arrived after a gap of time more than ε, i.e., if and only if $Y_j^2 > \varepsilon$. Thus

$$P(Y_1^2 \le x) = P(X \le x \mid X > \varepsilon) = 1 - e^{-\lambda(x-\varepsilon)}, \ x \ge \varepsilon$$

where X is a exponential random variable with parameter λ and denotes the time of arrival of the impulse.

E8) The failure time distribution is the distribution of the minimum of two i.i.d. exponential (λ) random variables. We know that this is also exponential (2λ). Thus, the renewal process N_t is a Poisson process with parameter 2λ and hence the renewal function is given by $M_t = E(N_t) = 2\lambda t$.

E9) Let X^1 denotes the failure time of the first component and X^2 that of the second component. The system fails at time $X^1 + X^2$. Let $(X_j^1, X_j^2): j = 1, 2, \ldots$ denote the successive failure times of the two components. The renewal event is the failure of the system. Then the associated renewal sequence is given by

$$S_n = \sum_{j=1}^{n} \left(X_j^1 + X_j^2 \right).$$

Let N_t denotes the associated renewal process.

Now, we want to find the distribution of $X_1^1 + X_1^2$. Since X_1^1, X_1^2 are i.i.d. exponential (λ) the inter-occurrence time distribution is gamma with parameters 2 and λ and its cdf is given by Eqn. (4) for $n = 2$.

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