UNIT 4 BRANCHING PROCESSES

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4.1 INTRODUCTION

Developments in this area of study in applied stochastic processes were initiated around 1874 through the research of Francis Galton and H. W. Watson in respect of the extinction of family surnames.

As is the practice in almost all societies, the family surname runs down the generations through the male offspring of the family. It is possible that all the offspring of a particular generation fail to reproduce a single child. Obviously then, in the subsequent generation there will be nobody to carry forward the family name, which then becomes extinct.

Since the number of children to be born to a couple cannot be exactly predicted because of inherent randomness, the obvious issue of interest would concern the probabilities of extinction of the family name at subsequent generations, or the probability of extinction. In a family tree, the branches are the descriptions of the generations hence the name of the process. We shall begin the unit by discussing the branching process and its mathematical formulations in terms of generating function, expectations and variance in Sec. 4.2. In Sec. 4.3, we will continue our discussion to the probability of extinction which is followed by some of the limiting properties of the probability of extinction.

Obviously, branching processes have considerable relevance in the context of studies in physics (neutrons), biology (bacteria), polymer chemistry (bonds), etc.

Objectives

After reading this unit, you should be able to:

- describe the branching process with examples;
- obtain the probability generating function, expectations and variance for a branching process;
- define the probabilities of extinction;
- express the various limiting properties for the probability of extinction.

4.2 MATHEMATICAL FORMULATIONS

Consider a population of X_0 elements, biological or otherwise (living beings or organisms like cells, neutrons, etc.), capable of reproducing offspring of the same kind. Suppose that each element, by the end of its life time, produces j offspring with probability p_j, $j = 0, 1, 2, \ldots,$ independently of any other offspring. Here, we are interested in the size of successive generations. We note that X_0 is the size of the 0-th generation, and these elements of this initial set are called **ancestors**. All the offspring born to X_0 elements of generation- 0 comprise the first generation. The elements generated by the elements of 0-generation are called the **direct descendents** of the ancestors. Let X_1 denote their total number, so that the size of the first generation is X_1 (which is, obviously, a random number).

Again, each element of the first generation (X_1 in number) would produce offspring according to the probability law $\{p_j\}$ independently of others, to form second generation and the process of reproduction continues in this manner.

Let X_n denotes the size of the n-th generation where $n = 0, 1, 2, 3, \ldots$.

Mathematically, in view of the above structure,

$$X_{n+1} = \xi_1 + \xi_2 + \cdots + \xi_{x_n},\ n = 0, 1, 2, \ldots$$

$$p_{ij} = P(X_{n+1} = j \mid X_n = i) = P(\xi_1 + \xi_2 + \cdots + \xi_i = j);\ i = 1, 2, \ldots, j = 0, 1, 2, \ldots \quad (1)$$

where $\xi_1, \xi_2, \ldots, \xi_i$, etc. are independent and identically distributed **(iid)** random variables with $P(\xi_k = r) = p_r$, where $r = 0, 1, 2, 3, \ldots$, and $k = 1, 2, \ldots, i$ and

$p_r \geq 0, \sum_{n=0}^{\infty} p_r = 1$.

The sequence $\{X_n\}$ constitutes a branching process which is a special case of a Markov chain with state space $(0, 1, 2, \ldots)$.

Now let us see the real life situations of the branching process defined above.

(i) Suppose, a series of plates are set up along the path of a current of electrons emitted by a mechanism. After hitting a reflector, an electron splits into a random number of new electrons. Suppose, X_0 is the number of electrons that hit the first plate and split into X_1 electrons which, in turn, after hitting the second plate split into X_2 electrons, and so on. The sequence $\{X_n, n \geq 0\}$ is a branching process.

(ii) Biological entities, such as human beings, and animals, that reproduce are examples of branching models.

(iii) Consider that a person with an infectious disease may transmit this disease to others. Now the number of infected persons in the n-th generation forms a branching process where all the infected persons of a generation are the offsprings of the immediately preceding generation.

Obviously, we can see that $\{X_n\}$ is a Markov chain with a transition probability matrix $((p_{ij}))$. This stochastic process is called a **discrete time branching process** or a **Galton-Watson process**. It is defined to be sub-critical, critical or super-critical as $\mu = \sum_{j=0}^{\infty} j p_j$ is '<', '=' or '>' 1. Notice that μ is the average number offspring of an individual element. For algebraic simplicity, let us take $X_0 = 1$. Then the probability generating function **(p.g.f.)** of the offspring distribution $\{p_j\}$ is given by

$$\phi(s) = E s^{X_1} = \sum_{j=0}^{\infty} s^j p_j,\quad 0 \leq s \leq 1 \quad (2)$$

We denote the p.g.f. of X_n by $\phi_n(s)$ so that $\phi_n(s) = E s^{X_n},\ 0 \leq s \leq 1$.
Notice that for $n \geq 1$,

$$\phi_{n+1}(s) = Es^{X_{n+1}}$$
$$= \sum_{j=0}^{\infty} s^j P(X_{n+1} = j)$$
$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P(X_{n+1} = j \mid X_n = i)\, P(X_n = i) s^j$$
$$= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} P(\xi_1 + \xi_2 + \cdots + \xi_i = j)\, P(X_n = i) s^j,$$

[Where, naturally, $P(X_{n+1} = j \mid X_n = 0) = 0$ for all $j > 0$, as there is none in generation n to produce offspring].

$$= \sum_{i=0}^{\infty} P(X_n = i) \sum_{j=0}^{\infty} P(\xi_1 + \xi_2 + \cdots + \xi_i = j)\, s^j$$
$$= \sum_{i=0}^{\infty} P(X_n = i)\, \{\phi(s)\}^i = \phi_n(\phi(s)), \text{ for } n = 1, 2, \ldots \text{ [using Eqn.(2)]} \quad (3)$$

since, $\xi_1, \xi_2, \ldots, \xi_i$ being independent, the p.g.f. of their sum is the product of their p.g.f's which happen to be $\phi(s)$ for each ξ_r, $r = 1, 2, \ldots, i$. Here, it may be noted that $\phi_1(s) = \phi(s)$, is the generating functions of X_1.

By induction then,

$$\phi_{n+1}(s) = \phi_n(\phi(s))$$
$$= \phi_{n-1}(\phi(\phi(s)))$$
$$= \phi_{n-2}(\phi(\phi(\phi(s))))$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots$$
$$= \underbrace{\phi(\phi(\phi(\ldots(\phi(s))\ldots)))}_{n \text{ times}}$$
$$= \phi(\phi_n(s)), \text{ where } X_0 = 1 \quad (4)$$

If $X_0 \neq 1$, then consider that $X_0 = k$. In that case, each of the k members will reproduce its offspring independently. These k members individually constitute their own branching processes. Thus, in this case, the generating function of the sum will be $[\phi_n(s)]^k$. Let us now try to find the expectations and variance.

Let us first assume that $X_0 = 1$, then considering the usual relationship and Eqn. (4), we get:

$$E(X_1) = \frac{d}{ds}\phi(s)\big|_{s=1} = \phi'(1) = \mu,$$ which is the mean number of offsprings of an individual.

$$E(X_2) = \frac{d}{ds}\phi_2(s)\big|_{s=1} = \frac{d}{ds}\phi(\phi(s))\big|_{s=1}$$
$$= \phi'(\phi(s))\phi'(s)\big|_{s=1}$$
$$= \{\phi'(1)\} = \mu^2$$

Proceeding in this manner, for $n \geq 1$,

$$E(X_{n+1}) = \frac{d}{ds}\phi_{n+1}(s)\big|_{s=1}$$
$$= \frac{d}{ds}\phi_n(\phi(s))\big|_{s=1}$$

$$= \phi_n'(\phi(s))\phi'(s)|_{s=1}$$
$$= \phi_n'(1)\phi'(1) = E(X_n)E(X_1) = \mu E(X_n)$$

Using induction, we get $E(X_{n+1}) = \mu^{n+1}$, $n = 1, 2, \ldots$ (5)

Note that, when $X_0 = k \neq 1$, then $E(X_{n+1} | X_0 = k) = k\mu^{n+1}$ (6)

Similar computations will yield, for variance, $n \geq 1$,

$$\begin{aligned}
\text{Var}(X_{n+1}) &= E(X_{n+1}^2) - E(X_{n+1})^2 \\
&= E(X_{n+1}(X_{n+1} - 1)) + E(X_{n+1}) - (EX_{n+1})^2 \\
&= \frac{d^2}{ds^2}\phi_{n+1}(s)|_{s=1} + \mu^{n+1} - (\mu^{n+1})^2 \\
&= \frac{d}{ds}[\phi_n'(\phi(s))\phi'(s)|_{s=1}] \\
&= \phi_n''(\phi(s))(\phi'(s))^2 + \phi_n'(\phi(s))\phi''(s) \\
&= \begin{cases} \sigma^2\mu^n(\mu^n - 1)/(\mu^2 - \mu), & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}
\end{aligned} \quad (7)$$

where $\sigma^2 = \text{Var } X_1$.

Notice from Eqn. (5) that

$$E(X_{n+1}) \rightarrow \begin{cases} \infty; & \mu > 1 \\ 1; & \mu = 1 \\ 0; & \mu < 1 \end{cases}$$

In simple words, on the average, the generation size eventually **explodes** if $\mu > 1$, i.e., if the average number of offspring born to an individual is more than one, or, on average an individual is replaced by more than one offspring; similarly, the process dies out (becomes extinct) eventually if $\mu < 1$, i.e. if an individual is replaced, on the average, by less than one offspring. On the other hand, the average generation size remains fixed at 1, irrespective of the generation, if every individual is replaced by exactly one offspring, only on the average.

Here, it can be interpreted that the average of a branching process increases exponentially fast for super-critical, decreases exponentially fast for sub-critical and remains stable for critical.

Also, if $X_0 = k \neq 1$, then the variance is given below:

$$\begin{aligned}
\text{var}(X_{n+1}) &= k \ \text{var}(X_{n+1} | X_0 = 1) \\
&= k\begin{cases} \sigma^2\mu^n(\mu^n - 1)/(\mu^2 - \mu), & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}
\end{aligned}$$

Let us illustrate the concepts given above in the following examples.

Example 1: Consider a branching chain originating from a single element, where each individual element either replaces itself with probability p_1, or fails to replace itself (i.e., dies out without producing an offspring) with probability $p_0 = 1 - p_1$, $0 < p_0 < 1$. Obviously, $\mu = 0(p_0) + 1(p_1) = p_1 < 1$.

$E(X_n) = \mu^n \rightarrow 0$ as $n \rightarrow \infty$.

Here, $\phi(s) = E(s^x) = s^0(p_0) + s(p_1) = p_0 + p_1 s$.

Example 2: Consider a branching chain originated by a single element, where each individual element either replaces itself by k offspring with probability p_k, or fails to replace itself (i.e., dies out without producing an offspring) with probability $p_0 = 1 - p_k$ where $0 < p_0 < 1$.

Obviously, $\mu = 0(p_0) + k(p_k) = k\, p_k$ and $E(X_n) = \mu^n = (kp_k)^n \to 0, 1$ or ∞ as $n \to \infty$, depending on whether $k\, p_k < 1$, or $= 1$, or > 1. Here, $\phi(s) = E(s^x) = s^0(p_0) + s^k(p_k) = p_0 + p_k s^k$.

Example 3: Consider a branching chain where each individual in a generation either generates 2 offspring with probability p_2, or replaces itself with probability p_1, or fails to replace itself with probability $p_0, p_0 + p_1 + p_2 = 1$ where $0 < p_i < 1, i = 0, 1, 2$. Obviously, $\mu = 0(p_0) + 1(p_1) + 2(p_2) = p_1 + 2p_2$; also, $\phi(s) = E(s^x) = s^0(p_0) + s(p_1) + s^2(p_2) = p_0 + p_1 s + p_2 s^2$. Now, the limiting behaviour of $E(X_n)$ will depend on the numerical values of p_0, p_1 and p_2.

Example 4: Consider the probability distribution of the number of offspring in a branching chain originated by a single individual, where each individual element generates three offspring, following the Poisson law, with an average $\lambda > 0$, which is given by

$$p_k = (\lambda^k / k!)e^{-\lambda},\ k = 0, 1, 2, \ldots$$

thus,
$$\begin{aligned}\phi(s) = E(s^x) &= s^0(p_0) + s(p_1) + s^2(p_2) + \cdots \\ &= \{s^0(\lambda^0) + s(\lambda) + s^2(\lambda^2/2) + \cdots\}\, e^{-\lambda} \\ &= e^{s\lambda}e^{-\lambda} \\ &= \exp\{-\lambda(1-s)\}.\end{aligned}$$

and $\mu = \lambda . E(x_n) = \lambda^n \to 0$, or 1, or ∞ according as $\lambda < 1$, or $= 1$, or > 1.

Now, try the exercises that follow.

E1) Give two real life situations of the branching process.

E2) In a branching process $\{X_n : n \geq 1\}$, show that $E(X_{n+1}^2 \mid X_n = k) = k\sigma^2 + k^2\mu^2$. Hence find variance.

E3) Let $\{X_n\}$ be a branching process, where the probability distribution of the numbers of offspring be geometric, with $p_n = P[\xi = n] = qp^n$, $n = 0, 1, 2, \ldots$ and $q = 1 - p$. Then, find the probability generating function of $\{X_n\}$.

So far, we have discussed some mathematical concepts concerning the branching process. In a branching process there may arise a case when no further branch is reproduced. This is known as extinction. This may happen at any generation.

In the following section, we shall study the probability of extinction of a branching process in detail.

4.3 PROBABILITY OF EXTINCTION

By extinction we mean the event that the process $\{X_n\}$ will consist of all zeros but except for the first finitely n. It may also be noted in this context that in view of the structure of the process, once $X_n = 0$ for some n, say $n = k$, then X_n is zero for all subsequent n, i.e. $n = k+1, k+2, \ldots$ as there will be nobody left in generation k to reproduce generation $k+1, k+2, \ldots$. Let us denote the event $\{X_n = 0\}$ by $A_n, n \geq 1$. Since $X_n = 0$ implies $X_{n+1} = 0$, $A_n \subset A_{n+1}$, $n \geq 1$, so that $q_n = P(A_n) \uparrow$ as $n \uparrow$.

Recall that we had taken $\{p_n\}$ as the offspring distribution of an individual,

$$p_n = P(\xi = n),\ n = 0, 1, 2, \ldots$$

If $p_0 = 1$, then trivially, $P(X_1 = 0) = P(A_1) = 1$ so that the system is trivially extinct. On the other hand, if $p_0 = 0$, then every individual element reproduces at least one offspring so that X_n, the size of the n-th generation, which essentially comprise of the offspring of the elements of the previous generation, can never be zero with positive probability, so that the process can never be extinct. Thus, in what we discuss below, we take $0 < p_0 < 1$.

$$\therefore\ P\ (\text{extinction of the process}) = P(X_n = 0 \text{ for some } n)$$

$$= P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= P\left(\lim_{n\to\infty} A_n\right),\ \text{since } A_n \uparrow$$

$$= \lim_{n\to\infty} P(A_n)$$

$$= \lim_{n\to\infty} P(X_n = 0)$$

$$= \lim_{n\to\infty} \phi_n(0) \text{ since } \phi_n(s) = Es^{X_n} = \sum_{j=0}^{\infty} s^j P(X_n = j) \quad (8)$$

Obviously, $\phi(s) = Es^{X_1}$ $0 \leq s \leq 1$ is non-decreasing, and whenever $p_0 > 0$,

$$q_1 = P(X_1 = 0) = \phi(0) = P_0 > 0$$

$$q_1 = P(X_2 = 0) = \phi_2(0) = \phi(\phi(0)) = \phi(P_0) = \phi(q_1)$$

And, $\phi(q_1) > \phi(0) = q_1$, since $q_1 = p_0 > 0$.

Again,

$$q_3 = \phi_3(0) = \phi(\phi_2(0)) = \phi(q_2)$$

$$> \phi(q_1), \text{since } q_2 > q_1 \text{ from above}$$

$$> \phi(0) = q_1;$$

By induction,

$$q_{n+1} = \phi(q_n)$$

So that, from Eqn. (8), the extinction probability

$$\Pi_0 = \lim_{n\to\infty} \phi_n(0)$$

$$= \lim_{n\to\infty} q_{n+1}$$

$$= \lim_{n\to\infty} \phi(q_n)$$

$$= \phi\left(\lim_{n\to\infty} q_n\right), \text{ by continuity of } \phi$$

$$= \phi(\Pi_0).$$

Thus, the extinction probability Π_0 is the solution of the equation

$$\Pi_0 = \phi(\Pi_0) \quad (9)$$

We may now summarize the above discussion as follows:

Theorem 1: Let $X_0, X_1, X_2, \ldots$ be a Galton-Watson process driven by the offspring distribution $\{p_0, p_1, \ldots\}$. Let $X_0 = 1$ and $\phi(s) = \sum_{j=0}^{\infty} s^j p_j$ be the probability generating function. Then, the extinction probability Π_0 of the process is zero, or one, depending on whether $p_0 = 0$ or 1; otherwise, Π_0 is a solution of the equation $s = \phi(s), 0 \le s \le 1$.

Notice that $s = 1$ is a root of the equation $s = \phi(s)$; but there can be other roots too! Then, which solution of $s = \phi(s), 0 \le s \le 1$ will yield Π_0? We will get further insight about Π_0 from the following theorem.

Theorem 2 (Fundamental Theorem): Suppose $p_0 > 0$. Then Π_0 is the smallest positive root of $s = \phi(s)$. If further $p_0 + p_1 < 1$, then $\Pi_0 = 1$ if and only if $\mu \le 1$.

Proof: We note that

$$s = \phi(s) = \sum_{j=0}^{\infty} s^j p_j \ge s^0 p_0 = q_1 .$$

Now, assume, $s \ge q_n$, then:

$$\begin{aligned} q_{n+1} &= P(X_{n+1} = 0) \\ &= \sum_{j=0}^{\infty} P(X_{n+1} = 0, X_1 = j) \\ &= \sum_{j=0}^{\infty} P(X_{n+1} = 0 \mid X_1 = j) P(X_1 = j) . \end{aligned} \tag{10}$$

Now, because of the structure of the process, under $X_1 = k$, the random vector $(X_2, X_3, \ldots, X_{n+1})$ is distributed as the sum of k independent vectors, exactly one originating from each of the k elements of generation 1. So that under $X_1 = k$, the distribution of $(X_2, X_3, \ldots, X_{n+1})$ is the same as the distribution of the sum of k i.i.d vectors each being distributed like $(X_1, X_2, \ldots, X_n)$. In particular

$P(X_{n+1} = 0 \mid X_1 = j)$

$= P$ (Size of the n-th generation of the j elements of generation-1 is zero)

$= P(\bigcap_{k=1}^{j}$ {Size of n-th generation of k^{th} element of the first generation is zero})

$= \prod_{k=1}^{j} P$ (Size of n-th generation of k^{th} element of the first generation is zero)

$= \prod_{k=1}^{j} P(X_n = 0)$

$= \{P(X_n = 0)\}^j = q_n^j .$

Thus, from Eqn.(10),

$$\begin{aligned} q_{n+1} &= \sum_{j=0}^{\infty} q_n^j p_j \\ &\le \sum_{j=0}^{\infty} s^j p_j = \phi(s) = s. \end{aligned}$$

Thus, by induction it follows that

$$s \geq q_n \,, n = 1, 2, 3, \ldots$$
$$\therefore\ s \geq \lim_{n\to\infty} q_n = \lim_{n\to\infty} P(A_n) = \Pi_0$$

Thus, Π_0 is the smallest of the roots of $s = \phi(s)$. This proves the first part of the theorem. Now, let $p_0 + p_1 < 1$, i.e., the number of offspring of an element can be more than one with positive probability. Then,

$$\begin{aligned}
\phi''(s) &= \frac{d^2}{ds^2}\phi(s) \\
&= \frac{d^2}{ds^2}\sum_{j=0}^{\infty} s^j p_j \\
&= \sum_{j=0}^{\infty} j(j-1)\, s^{j-2} p_j \\
&> 0, \quad s \in (0, 1).
\end{aligned}$$

Therefore, ϕ is strictly convex in $(0, 1)$; also, $\phi'(s) = \sum_{j=0}^{\infty} js^{j-1}\ p_j$

> 0, since $p_0 < 1$,

so that $\phi(s)$ is strictly increasing in $s \in (0, 1)$. We now consider two cases.

Case I: $\mu \leq 1$

Then, ϕ being convex, $\phi'(s) < \phi'(1) = \mu \leq 1 = \frac{d}{ds} s$, so that

$$\frac{d}{ds}(\phi(s) - s) < 0.$$

This implies that $\psi(s) = \phi(s) - s$ is strictly decreasing in $s \in (0, 1)$. Thus, for $0 < s < 1$

$$\psi(s) > \psi(1)$$

i.e. $\quad \phi(s) - s > \phi(1) - 1 = 0$

i.e. $\quad \phi(s) > s, \ 0 < s < 1$

$\Rightarrow \quad \phi(s) = s$ does not have a solution in $(0, 1)$.

So that the only solution of $\phi(s) = s$ is $s = 1$, since $s = 0$ is not a solution as $\phi(0) = p_0 > 0$. Thus, we have been able to show that if $\mu \leq 1$, then $\Pi_0 = 1$.

Case II: $\mu > 1$

When $\phi'(1) = \sum_{j=0}^{\infty} j\, s^{j-1} p_j \big|_{s=1} = \mu > 1$, by the continuity of ϕ', $\exists\, s_0$, such that $\phi'(s) > 1$ for $s_0 < s < 1$, then:

expanding $\phi(s)$ about s_0, then at $s \neq 1$ we get

$$1 = \phi(1) = \phi(s_0) + (1 - s_0)\phi'(s^*), \ s_0 < s^* < 1$$

$$\therefore \ \frac{1 - \phi(s_0)}{1 - s_0} = \phi'(s^*) > 1$$

i.e. $\phi(s_0) - s_0 < 0$.

Let us now try to visualize the graph of $\phi(s)$. We know the following already:

(i) $\quad \phi(s) - s$ is continuous,

(ii) $\quad \phi(0) = p_0 > 0, \ \phi(1) = 1, \ \phi(s_0) < s_0$

(iii) $\quad \phi(s)$ is convex

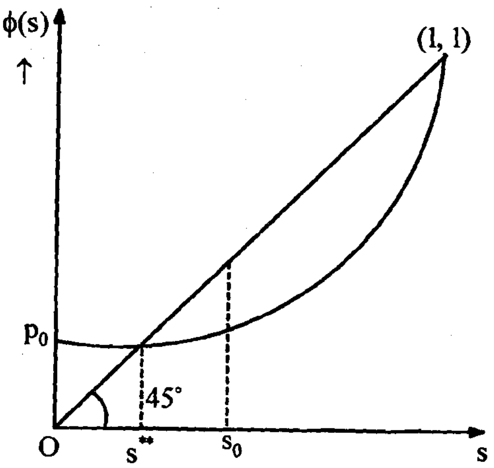

Fig.1

From (ii), then, $\exists\, s^{**} < s_0 \ni \phi(s^{**}) = s^{**}$. We will be through if we now show that $\phi(s)$ does not have any other root, obviously, smaller than s^{**}. If false, suppose $s^{***} < s^{**}$ is such a root so that $\phi(s^{***}) = s^{***}$, and $\psi(s) = \phi(s) - s = 0$ has three roots, viz, $s = s^{***}, s^{**}, 1$. Then, $\psi'(s) = \phi'(s) - 1 = 0$ has at least two roots in $(0, 1)$ and $\psi''(s) = \phi''(s) = 0$ has at least one root in $(0, 1)$, but $\phi''(s) = \sum_{j=0}^{\infty} j\,(j-1) s^{j-2} p_j > 0$ since $p_3 + p_4 + \cdots = 1 - (p_0 + p_1) > 0$ as $p_0 + p_1 < 1$. Thus, we confront a contradiction. Hence $\phi(s)$ has only one root in $(0, 1)$. This completes the proof of the theorem.

Let us now illustrate it.

Example 5: Refer to Example 1. Here, $\phi(s) = E(s^x) = p_0 + p_1 s$, so that $s = 1$ is the only solution of the equation $\phi(s) = s$, as $p_0 + p_1 = 1$. As such, the probability Π_0 that the process will be extinct is 1.

Example 6: Refer to Example 2. Here, $\phi(s) = E(s^x) = s^0(p_0) + s^k(p_k) = p_0 + p_k s^k$; now, the equation $\phi(s) = s$ would imply that we need to solve $s - p_k s^k - p_0 = 0$, i.e., $s(1 - p_k s^{(k-1)}) = p_0$ for s. For $k > 1$, from Theorem 2, as $p_0 < 1$ (since $p_1 = 0$), the probability Π_0 of extinction of the process is 1, if and only, if $\mu = kp_k \le 1$.

Example 7: Refer to Example 4. Here, $\phi(s) = \exp\{-\lambda(1 - s)\}$. Check that $p_0 + p_1 = e^{-\lambda} + \lambda e^{-\lambda} = (1 + \lambda)e^{-\lambda} \le 1$ so that from Theorem 2, the probability Π_0 of extinction of the process is 1 provided $\lambda \le 1$; otherwise, $\Pi_0 < 1$.

Example 8: Let us consider the following offspring distribution:

$$p_j = b\, p^{j-1}, \quad j = 1, 2, \ldots$$

$$p_0 = 1 - \sum_{j=1}^{\infty} p_j = 1 - b/(1-p) = \frac{1-p-b}{1-p}$$

where $b, p > 0,\ p + b < 1$

During the 1920 census in the USA, this model was found to give a good fit with $b = 0.2126$ and $p = 0.5893$ in the context of American males. Now,

$$\phi(s)=\sum_{j=0}^{\infty} p_j\, s^j$$

$$=p_0+bs\sum_{j=1}^{\infty} p^{j-1}s^{j-1}$$

$$=p_0+\frac{bs}{1-ps}=\frac{1-p-b}{1-p}+\frac{bs}{1-ps}$$

$$\phi'(s)=\frac{b}{1-ps}+\frac{bs}{(1-ps)^2}.p$$

$$\therefore\quad \mu=\phi'(s)\big|_{s=1}=\frac{b}{1-p}+\frac{bp}{(1-p)^2}$$

$$=\frac{b-bp+bp}{(1-p)^2}=\frac{b}{(1-p)^2}.$$

Also, $\phi(s)=s$ would mean $1-\frac{b}{1-p}+\frac{bs}{1-ps}=s$

i.e. $\quad b.\frac{-1+ps+s-ps}{(1-p)\,(1-ps)}=s-1$

i.e. $\quad \frac{b(s-1)}{(1-p)\,(1-ps)}=(s-1)$

i.e. $\quad \frac{b}{(1-p)\,(1-ps)}=1 \;\text{ if }\; s\neq 1$

i.e. $\quad 1-ps=\frac{b}{1-p}$

i.e. $\quad s_0=\frac{1-p-b}{p(1-p)}$

Thus, 1 and s_0 are the two roots of $\phi(s)=s$. Note that

$$\mu \gtreqless 1$$

if $\quad b\gtreqless(1-p)^2$

which would mean

$$s_0=\frac{1-p-b}{p(1-p)}$$

$$\lesseqgtr\frac{1-p-(1-p)^2}{p(1-p)}$$

$$=1.$$

In summary,

$$\mu\gtreqless 1\Leftrightarrow s_0\lesseqgtr 1.$$

Thus, the extinction probability, according to Theorem 2, is $\Pi_0=1$ if $\mu\le 1$ and $\Pi_0=\frac{1-p-b}{p(1-p)}$ if $\mu>1$.

In this context, we may also be interested in determining the probabilities of extinction at the n-th generation. Here, algebra is interesting.

For any two points, μ and v,

$$\phi(s)-\phi(u)=b\left\{\frac{s}{1-ps}-\frac{u}{1-pu}\right\}=\frac{b(s-u)}{(1-ps)(1-pu)}$$

$$\therefore \quad \frac{\phi(s)-\phi(u)}{\phi(s)-\phi(v)} = \frac{s-u}{s-v} \cdot \frac{1-pv}{1-pu}.$$

Take $u = s_0$ and $v = 1$, so that for $\mu \neq 1$ (i.e. $s_0 \neq 1$)

$$\frac{1-p}{1-ps_0} = \left\{\frac{\phi(s)-s_0}{s-s_0}\right\}\left\{\frac{\phi(s)-1}{s-1}\right\}^{-1}$$

Taking the limit as $s \to 1$ (check for yourself)

$$\frac{1-p}{1-ps_0} = \frac{1}{\mu}$$

As such then, for all $s \neq 1$

$$\frac{\phi(s)-s_0}{\phi(s)-1} = \frac{1}{\mu} \cdot \frac{s-s_0}{s-1}.$$

Then, from Eqn. (3)

$$\frac{\phi_2(s)-s_0}{\phi_2(s)-1} = \frac{\phi(\phi(s))-s_0}{\phi(\phi(s))-1}$$

$$= \frac{1}{\mu} \cdot \frac{\phi(s)-s_0}{\phi(s)-1} = \frac{1}{\mu^2} \cdot \frac{s-s_0}{s-1}.$$

Iterating this way, for all $s \neq 1$

$$\frac{\phi_n(s)-s_0}{\phi_n(s)-1} = \frac{1}{\mu^n} \cdot \frac{s-s_0}{s-1}$$

which, when solved explicitly, yields, for $s \neq 1$

$$\phi_n(s) = 1-\mu^n\left(\frac{1-s_0}{\mu^n-s_0}\right) + \frac{\mu^n\left(\frac{1-s_0}{\mu^n-s_0}\right)^2 s}{1-\left(\frac{\mu^n-1}{\mu^n-s_0}\right)s}, \text{ if } \mu \neq 1. \qquad (11)$$

If $\mu = 1$, then $s_0 = 1$ so that

$$\phi(s) = \frac{p-(2p-1)s}{1-ps}$$

as $b = (1-p)^2$.

Using Eqn. (3), it can be shown that for $n \geq 1$,

$$\phi_n(s) = \frac{np-\{(n+1)p-1\}s}{1+(n-1)p-nps} \qquad (12)$$

Thus, from Eqn. (11) and (12), the probability that the process will be extinct at generation n is

$$P(x_n = 0) = \phi_n(0)$$

$$= \begin{cases} 1-\mu^n\left(\dfrac{1-s_0}{\mu^n-s_0}\right), & \text{if } \mu \neq 1 \\ \dfrac{np}{1+(n-p)p}, & \text{if } \mu = 1 \end{cases}.$$

Now, try these exercises.

E4) Suppose in a branching process, the offspring distribution is as follows:

$p_k = ({}^N C_k) p^k q^{(N-k)}$, $q = 1-p$, $0 < p < 1$

$k = 0, 1, 2, \ldots$. Discuss the probability of extinction of this branching process.

E5) Following the notations introduced in this unit, for a branching process with $X_0 = 1$, define for $n = 1, 2, 3, \ldots$

$Y_n = 1 + X_1 + X_2 + \cdots + X_n$;

Thus, Y_n is the total number of individuals born in the family (progeny) up to and including the n-th generation. Let $\psi_n(s)$ denotes the probability generating function of Y_n. Show that

$\psi_{n+1}(s) = s\phi(\psi_n(s))$.

E6) Suppose in a branching process, the offspring distribution is as follows:

$p_k = pq^k, q = 1-p, 0 < p < 1$,

$k = 0, 1, 2, \ldots$. Discuss the probability of extinction in this branching process.

E7) Consider a branching chain where each individual element either replaces itself by 2 offspring with probability p, or fails to replace itself (i.e. dies out without producing an offspring) with probability $q = 1-p$, where $0 < p < 1$. We define $\psi_n(s)$ as in E5), and $\psi(s) = \lim_{n\to\infty} \psi_n(s)$. Show that if $p < 1/2$, then

$$\psi(s) = \frac{1-\sqrt{1-4pqs^2}}{2ps}.$$

What would be the expression for the limiting probability generating function $\psi(s)$ had the offspring distribution been geometric, as in E6).

We now end this unit by giving a summary of what we have covered in it.

4.4 SUMMARY

In this unit, we have discussed the following points.

1. The branching process is a Markov chain with a transition probability matrix $((P_{ij}))$.
2. The probability generation function of the offspring distribution $\{p_j\}$ is given by $\phi(s) = \sum_{j=0}^{\infty} s^j p_j;\ 0 \le s \le 1$.
3. The expectations and variance for the branching process are proved.
4. The extinction probability is given by $\Pi_0 = \phi(\Pi_0)$
5. The fundamental theorem for the probability of extinction is stated and proved.

4.5 SOLUTIONS/ANSWERS

E1) Any two real life situations constituting the branching process.

E3) The generating function

$$\phi(s) = \sum_j p_j s^j = \sum_{j=0}^{\infty} qp^j s^j = \frac{q}{1-ps}.$$

E4) The generating function

$$\phi(s) = \sum_j p_j s^j = \sum_{j=0}^{N} {}^N C_j p^j q^{N-j} s^j = (ps+q)^N$$

Now, the extinction probability has to be a solution of $s = (ps+q)^N$. Then, argue in terms of Theorems 1 and 2.

E5) Clearly, by definition,

$$\Psi_{n+1}(s) = \sum_{j=0}^{\infty} s^j P(Y_{n+1} = j)$$

$$= \sum_{j=1}^{\infty} s^j P(Y_n + X_{n+1} = j)$$

$$= \sum_{j=1}^{\infty} s^j \sum_{k=1}^{j} P(Y_n + X_{n+1} = j \mid Y_n = k) P(Y_n = k)$$

$$= \sum_{j=1}^{\infty} s^j \sum_{k=1}^{j} P(X_{n+1} = j-k \mid Y_n = k) P(Y_n = k)$$

$$= \sum_{k=1}^{\infty} s^k \sum_{j=k}^{\infty} P(\xi_{n1} + \xi_{n2} + \cdots + \xi_{nk} = j-k \mid Y_n = k) P(Y_n = k) s^{j-k}$$

where ξ_{nr} is the number of offspring born to the r-th member of generation n. The above line then equals to

$$= \sum_{k=1}^{\infty} s^k \sum_{j=k}^{\infty j} P(\xi_{n1} + \xi_{n2} + \cdots + \xi_{nk} = j-k) P(Y_n = k) s^{j-k}$$

$$= \sum_{k=1}^{\infty} s^k P(Y_n = k) \sum_{j=k}^{\infty} P(\xi_{n1} + \xi_{n2} + \cdots + \xi_{nk} = j-k) s^{j-k}$$

$$= \sum_{k=1}^{\infty} s^k P(Y_n = k) \sum_{r=0}^{\infty} P(\xi_{n1} + \xi_{n2} + \cdots + \xi_{nk} = r) s^r$$

$$= \sum_{k=0}^{\infty} s^k P(Y_n = k) \{\phi(s)\}^k \text{, since } \xi\text{'s are IID}$$

$$= \sum_{k=0}^{\infty} \{\phi(s)s\}^k P(Y_n = k) = \psi_n(s\phi(s)).$$

Alternately, let us look at

$$Y_{n+1} = 1 + X_1 + X_2 + \cdots + X_{n+1};$$

Thus, the total number of individuals born in the family till generation $n+1$, starting from generation 0 can be looked upon as the sum total of individuals born in n generations started by X_1 individuals of generation 1. Alternately, if $Y_1 = k$, i.e., $X_1 = k-1$, we should be able to write

$$Y_{n+1} = (1 + X_{21} + \cdots + X_{n+1,1}) + (1 + X_{22} + \cdots + X_{n+1,2}) + (1 + X_{23} + \cdots + X_{n+1,3}) + \cdots + (1 + X_{2\,k-1} + \cdots + X_{n+1,\,k-1})$$

where X_{ij} is the number of family members in (original) generation i of the branch started by the j-th individual (out of $k-1$ individuals) of the first generation. We also remember that the same stochastic conditions apply (reproduction distributions are the same for all individuals who behave independently of each other). Now,

$$\Psi_{n+1}(s) = \sum_{j=1}^{\infty} s^j P(Y_{n+1} = j)$$

$$= \sum_{j=1}^{\infty} s^j \sum_{k=1}^{j} P(Y_{n+1} = j \mid Y_1 = k)\, P(Y_1 = k)$$

$$= \sum_{j=1}^{\infty} s^j \sum_{k=1}^{j} P(Y_{n+1} = j \mid Y_1 = k) P(Y_1 = k)$$

$$= \sum_{j=1}^{\infty} s^j \sum_{k=1}^{j} P(1 + X_{21} + \cdots + X_{n+1,1}) + (1 + X_{22} + \cdots + X_{n+1,2})$$
$$+ (1 + X_{23} + \cdots + X_{n+1,3}) +$$
$$\cdots + (1 + x_{2k-1} + \cdots + X_{n+1,k-1} = j) P(Y_1 = k)$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} s^j P((1 + X_{21} + \cdots + X_{n+1,1}) + (1 + X_{22} + \cdots + X_{n+1,2})$$
$$+ (1 + X_{23} + \cdots + X_{n+1,3}) +$$
$$\cdots + (1 + X_{2k-1} + \cdots + X_{n+1,k-1}) = j) P(Y_1 = k)$$

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} s^j P(Z_1 + Z_2 + \cdots + Z_{k-1} = j)\ P(Y_1 = j)$$

where, for $r = 1, 2, \ldots, k-1$, $Z_r = 1 + X_{2r} + \cdots + X_{n+1,r}$: here, Z_r is the number of family members born to the r-th member of the generation 1 till the $(n+1)$th generation counting from the first ever individual, and in n number of generations starting from the the r-th individual of the first generation. Notice that interpretation-wise, Z_r has a similar interpretation as that of Y_n, because Z_r is the number of family members caused in n generations by the r-th individual of the first generation. Thus, since the Z's are IID, being ascribed by different individuals of generation - 1, then the above is equal to (recalling that the pgf of sum of $(k-1)$ IID random variables is the $(k-1)$-th power of the pgf of any one of them)

$$= \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} s^j \{P(Z_1 + Z_2 + \cdots + Z_{k-1} = j)\} P(Y_1 = k)$$

$$= \sum_{k=1}^{\infty} \{\psi_n(s)\}^{k-1} P(X_1 = k-1)$$

$$= \sum_{k=0}^{\infty} \{\psi_n(s)\}^{k} P(X_1 = k)$$

$$\phi(s\psi_n(s)).$$

E6) $p_k = pq^k \quad q = 1-p \quad 0 < p < 1 \quad p = 0,1,2\ldots$

probability of extinction

$$\phi(s) = \sum_{j=0}^{\infty} p_j s^j$$

$$= \sum_{j=0}^{\infty} pq^j s^j$$

$$= \frac{p}{1-qs}$$

$$\phi'(s) = \frac{-p(-q)}{(1-qs)^2} = \frac{pq}{(1-qs)^2}$$

$$\mu = \phi'(s)|_{s=1} = \frac{pq}{(1-q)^2}$$

Also, $\phi(s) = s \Rightarrow \frac{p}{1-qs} = s$

i.e., $s_0 = p/q$.

Therefore 1 and s_0 are the two roots of $\phi(s) = s$.

It may also be noted that

$$\mu \lesseqgtr 1 \text{ if } q \lesseqgtr p$$

i.e., $\mu \lesseqgtr 1$ if $s_0 \lesseqgtr p$.

Thus, the extinction probability is $\Pi_0 = 1$ if $\mu \leq 1$, and $\Pi_0 = \frac{p}{1-p}$ if $\mu > 1$.

E7) Here, the offspring distribution is as follows:

$$p_0 = q, p_1 = 0, p_{12} = p, p_r = 0, r \geq 3$$

so that

$$\varphi(s) = q + ps^2.$$

Now, from E5), $\Psi_{n+1}(s) = s\varphi(\psi_n(s))$, and taking limit as $n \to \infty$ on both sides, we get

$$\Psi(s) = s\varphi(\psi(s))$$

and hence,

$$\Psi(s) = s\{q + p\psi^2(s))$$

i.e., $ps\Psi^2(s) - \psi(s)sq = 0$

so that $\Psi(s) = \frac{1-\sqrt{1-4pqs}^{\,2}}{2ps}$.

Whenever the offspring distribution is geometric, the pgf will be

$$\varphi(s) = \frac{p}{1-sq}$$

so that, arguing as before,

$$\psi(s) = \frac{ps}{1-q\Psi(s)}$$

and as such,

$$q\Psi^2(s) - \psi(s) + ps = 0.$$

Hence, $\Psi(s) = \frac{1-\sqrt{1-4pqs}}{2q}$.

---x---