UNIT 12 FINITE ELEMENT METHODS

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12.1 INTRODUCTION

In Unit 11, we have discussed the finite difference method for finding solutions of partial differential equations. In particular, we have solved the Laplace equation, Poisson equation and one-dimensional heat and wave equations. In this unit, we shall introduce another popular numerical method called finite element method for solving partial differential equations. Finite element methods are applicable to both initial and boundary value problems of ordinary and partial differential equations. However, we shall confine our discussion to the boundary value problems and in particular, discuss finite element methods for solving Laplace and Poisson equations in two dimensions. Finite element methods have some advantages over finite difference methods. One advantage of the finite element method over finite difference method is the relative ease with which the boundary conditions of the problem are handled. Irregularly shaped boundaries are handled in a natural manner by the finite element method in comparison to the finite difference method, where special formulas have to be developed for the boundaries.

In Sec. 12.2, we have started the discussion by defining triangular and rectangular finite elements usually used in the case of two-dimensional problems. In Sec. 12.3 Galerkin's finite element method which is a weighted residual method is discussed and its application to the Dirichlet boundary value problem for Poisson equation and Laplace equation is illustrated. Variational formulation of the Laplace and Poisson equations are also given in this section.

Objectives

After studying this unit you should be able to

- discretize a given two dimensional domain using triangular and rectangular finite elements;
- derive finite element Galerkin method for the solution of Laplace and Poisson equations with Dirichlet boundary conditions.

12.2 FINITE ELEMENTS

In finite element methods, we generate difference equations by using the variational principle or weighted residual methods. The closed domain \( R \), where the given partial differential equation holds, is divided into a finite number of non-overlapping subdomains \( R_1, R_2, \ldots, R_M \). These subdomains are called the finite elements. We use the straight line elements in the one dimensional case (see Fig. 1), that is in solving ordinary differential equations. In Fig. 1, the interval \([a, b]\) is subdivided into three straight line elements \( e_1, e_2, e_3 \). Generally, in two dimensions, we use the triangular or rectangular elements (see Figs. 2, 3). In Fig. 2, we have eight triangular elements numbered \( e_1, e_2, \ldots, e_8 \). In Fig. 3, we have four rectangular elements numbered \( e_1, e_2, e_3, e_4 \). The curved boundaries are handled in a natural manner.
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Fig.1: Straight line element  Fig.2: Triangular element  Fig.3: Rectangular element

In each of the elements $e$, we approximate the solution by a function say $w$, which is continuous and defined in terms of the nodal values belonging to that element. At the boundaries of the elements called the interfaces, the inter element conditions are to be satisfied. The general requirement at the interface is that the approximating function and its partial derivatives up to order one less than the highest order derivative occurring in the partial differential equation or its variational form must be continuous. The approximate solution $w$ is then substituted in the differential equation and the weighted residual method is applied. Alternatively, solution $w$ is substituted in the variational form of the partial differential equation. This gives rise to a system of linear or non-linear difference equations. The solution of this system gives the approximate solution of the partial differential equation at the nodal points in $R$. For simple networks, the difference equations derived by the finite difference and finite element methods are identical. As we have mentioned earlier, in this unit, we shall consider only the solution of Laplace and Poisson equations in two dimensions. The simple finite elements that can be used in this case are triangular and rectangular elements.

**Triangular and Rectangular Elements**

As you know, line segment elements (see Fig.1) are used to solve ordinary differential equations. For solving partial differential equations, the two basic elements that are used are triangular or rectangular elements. That is, the two dimensional domain $R$ is subdivided into triangular or rectangular elements.

Assemblage of triangles can always represent a two dimensional domain of any shape. Normally, we use equilateral triangles.

**Triangular element**

Consider the triangular element with corners at $(x_i, y_i), (x_j, y_j), (x_k, y_k)$, taken in the anti-clockwise direction. This type of element is called a three node triangle. We have three degrees of freedom (three nodal values at the corners), (see Fig.4). We write a linear approximation inside each element of the form

$$u^{(e)}(x,y) = a_1 + a_2 x + a_3 y$$  \hspace{1cm} (1)

At the nodes, we get

$$u_i = a_1 + a_2 x_i + a_3 y_i$$
$$u_j = a_1 + a_2 x_j + a_3 y_j$$
$$u_k = a_1 + a_2 x_k + a_3 y_k$$

For the solution of $a_1, a_2, a_3$, we have the system

$$
\begin{bmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
u_i \\
u_j \\
u_k
\end{bmatrix}
$$

Using Cramer's rule, the solution of this system is obtained as

$$a_1 = \frac{\Delta_1}{2\Delta}, \quad a_2 = \frac{\Delta_2}{2\Delta}, \quad a_3 = \frac{\Delta_3}{2\Delta}$$

where $\Delta$ is the area of the triangle and
\[
2 \Delta = 2 \text{(area of triangle)} = \begin{vmatrix}
1 & x_i & y_i \\
1 & x_j & y_j \\
1 & x_k & y_k
\end{vmatrix}
\]

\[
= (x_i - x_k)(y_j - y_k) - (x_j - x_k)(y_i - y_k)
\]

\[
\Delta_1 = \begin{vmatrix}
u_i & x_i & y_i \\
u_j & x_j & y_j \\
u_k & x_k & y_k
\end{vmatrix} = u_i(x_jy_k - x_ky_j) - u_j(x_iy_k - y_ix_k) + u_k(x_iy_j - y_ix_j)
\]

\[
\Delta_2 = \begin{vmatrix}
u_i & x_i & y_i \\
u_j & y_j & y_j \\
u_k & y_k & y_k
\end{vmatrix} = u_i(y_j - y_k) + u_j(y_k - y_i) + u_k(y_i - y_j)
\]

\[
\Delta_3 = \begin{vmatrix}
u_i & x_i & y_i \\
u_j & x_j & x_k \\
u_k & x_k & x_k
\end{vmatrix} = u_i(x_k - x_j) + u_j(x_i - x_k) + u_k(x_j - x_i)
\]

Substituting the values of \( a, a_2, a_3 \) in Eqn.(1) and simplifying we obtain the approximation as

\[
u^{(e)}(x, y) = \frac{1}{2\Delta}(\Delta_1 + \Delta_2 x + \Delta_3 y)
\]

where,

\[
N_i^{(e)}(x, y) = \frac{1}{2\Delta}[(x_jy_k - x_ky_j) + (y_j - y_k)x + (x_k - x_j)y],
\]

\[
N_j^{(e)}(x, y) = \frac{1}{2\Delta}[(x_ky_i - x_iy_k) + (y_k - y_i)x + (x_i - x_k)y],
\]

\[
N_k^{(e)}(x, y) = \frac{1}{2\Delta}[(x_iy_j - x_jy_i) + (y_i - y_j)x + (x_j - x_i)y].
\]

\( N_i^{(e)}, N_j^{(e)}, N_k^{(e)} \) are called the shape functions of the approximation. Setting

\( x = x_i, y = y_i \) in Eqn.(2), we get

\[
u^{(e)}(x_i, y_i) = u_i
\]

\[
= N_i^{(e)}(x_i, y_i)u_i + N_j^{(e)}(x_i, y_i)u_j + N_k^{(e)}(x_i, y_i)u_k
\]

Hence,

\[
N_i^{(e)}(x_i, y_i) = 1,
\]

\[
N_j^{(e)}(x_i, y_i) = 0,
\]

\[
N_k^{(e)}(x_i, y_i) = 0.
\]

Similarly, substituting \( x = x_j, y = y_j \) and \( x = x_k, y = y_k \) in Eqn.(2), we find that the shape functions satisfy

\[
N_i^{(e)}(x_j, y_j) = 1 = N_j^{(e)}(x_j, y_j) = N_k^{(e)}(x_j, y_j)
\]

and \( N_m^{(e)}(x_r, y_r) = 0 \), for \( m \neq r, r = i, j, k \).

We can verify these results directly from Eqn.(3) also.

In vector notation, we can write Eqn.(2) as

\[
u^{(e)}(x, y) = [N_i^{(e)} N_j^{(e)} N_k^{(e)}] \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = [N^{(e)}] \begin{bmatrix} u^{(e)} \end{bmatrix}
\]

where \([N^{(e)}] = [N_i^{(e)} N_j^{(e)} N_k^{(e)}]\) and \([u^{(e)}] = [u_i u_j u_k]^T\).

If the domain contains \( K \) elements, the representation of the solution variable over the whole domain is given by
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\[ u(x, y) = \sum_{n=1}^{K} u^{(n)}(x, y) = \sum_{n=1}^{K} [N^{(n)}(x, y)] \{u^{(n)}\} \]  

(5)

On the interfaces, that is, the sides of the triangles, \( u(x, y) \) has no discontinuities. The elements that we are considering here are called conforming elements. Two adjoining elements are conforming if the value of \( u \) is determined uniquely on their common edge.

**Rectangular element**

Let the domain \( R \) be divided into rectangular elements. Assume that the sides of the element are parallel to \( x \) and \( y \) axes respectively. Since there are four vertices for a rectangle a piecewise linear approximation of the type (1), (which has only three parameters) is not possible. Hence, we choose the piecewise polynomial in the form

\[ u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy \]  

(6)

Let \( i, j, k \) and \( m \) denote the corners \( P(x_i, y_i), Q(x_j, y_j), R(x_j, y_m) \) and \( S(x_i, y_m) \), (see Fig. 5). We call these the nodes of the rectangle. At the nodes, (setting \( (x, y) = (x_i, y_i), (x, y) = (x_j, y_j), (x, y) = (x_j, y_m), (x, y) = (x_i, y_m) \), in Eqn.(6)), we get

\[ u_i = a_1 + a_2 x_i + a_3 y_i + a_4 x_i y_i \]  

(7)

\[ u_j = a_1 + a_2 x_j + a_3 y_j + a_4 x_j y_j \]  

(8)

\[ u_k = a_1 + a_2 x_j + a_3 y_m + a_4 x_j y_m \]  

(9)

\[ u_m = a_1 + a_2 x_i + a_3 y_m + a_4 x_i y_m \]  

(10)

Subtracting Eqn.(10) from Eqn.(7), we get

\[ (y_j - y_m) a_3 + y_i (y_i - y_m) a_4 = u_i - u_m \]

Subtracting Eqn.(9) from Eqn.(8), we get

\[ a_3 (y_j - y_m) + x_j (y_i - y_m) a_4 = u_j - u_k \]

Subtracting Eqn.(12) from Eqn.(11), we obtain

\[ a_4 (x_i - x_j) = \frac{u_i - u_m - u_j + u_k}{y_i - y_m} \]

or,

\[ a_4 = \frac{u_i - u_m - u_j + u_k}{(x_i - x_j)(y_i - y_m)} \]

Hence, \( a_3 = \frac{u_i - u_m - a_4 x_i}{y_i - y_m} \).

Subtracting Eqn.(8) from Eqn.(7), we obtain

\[ a_2 (x_i - x_j) + a_4 y_i (x_i - x_j) = u_i - u_j \]

Hence, \( a_2 = \frac{u_i - u_j - a_4 y_i}{x_i - x_j} \).

From Eqn.(7), we get

\[ a_i = u_i - a_2 x_i - a_3 y_i - a_4 x_i y_i \]

Substituting these values of \( a_1, a_2, a_3 \) and \( a_4 \) in Eqn.(6) and simplifying, we get

\[ u(x, y) = a_1 + a_2 x + a_3 y + a_4 xy \]

\[ = N_i^{(e)} u_i + N_j^{(e)} u_j + N_k^{(e)} u_k + N_m^{(e)} u_m \]  

(13)

where,

\[ N_i^{(e)} = \frac{(x - x_j)(y - y_m)}{(x_i - x_j)(y_i - y_m)}, \quad N_j^{(e)} = \frac{(x - x_j)(y - y_m)}{(x_j - x_i)(y_i - y_m)} \]
Note that the nodes are \(P(x_i, y_i), Q(x_j, y_i), R(x_j, y_m)\) and \(S(x_i, y_m)\).

At the node \(P(x_i, y_i)\), we have
\[
N_{x_i}^{(e)}(x_i, y_i) = 1, \quad N_{y_i}^{(e)}(x_i, y_i) = 0, \quad N_{x_i y_i}^{(e)}(x_i, y_i) = 0, \quad N_{y_i}^{(e)}(x_i, y_i) = 0.
\]
At \(Q(x_j, y_i)\), we have
\[
N_{x_j}^{(e)}(x_j, y_i) = 0, \quad N_{y_i}^{(e)}(x_j, y_i) = 1, \quad N_{x_j y_i}^{(e)}(x_j, y_i) = 0, \quad N_{y_i}^{(e)}(x_j, y_i) = 0.
\]
At \(R(x_j, y_m)\), we get
\[
N_{x_j}^{(e)}(x_j, y_m) = 0, \quad N_{y_m}^{(e)}(x_j, y_m) = 0, \quad N_{x_j y_m}^{(e)}(x_j, y_m) = 1, \quad N_{y_m}^{(e)}(x_j, y_m) = 0.
\]
At \(S(x_i, y_m)\), we get
\[
N_{x_i}^{(e)}(x_i, y_m) = 0, \quad N_{y_m}^{(e)}(x_i, y_m) = 0, \quad N_{x_i y_m}^{(e)}(x_i, y_m) = 0, \quad N_{y_m}^{(e)}(x_i, y_m) = 1.
\]

Therefore, the shape functions have the value 1 at the node where it is defined and have the value 0 at all the other nodes.

In general, when the sides of the rectangle are not parallel to the axes, the shape functions satisfy the condition
\[
N_{x_i}^{(e)}(x_i, y_s) = 1, \quad \text{for } r = s
\]
\[
= 0, \quad \text{for } r \neq s.
\]

Note that the shape functions derived in Eqn.(14) are the Lagrange bivariate interpolations. It is important that the interpolation polynomial \(u(x, y)\) is not confused with the shape or interpolation functions \(N^{(e)}\). The interpolation polynomial \(u(x, y)\) applies to an element, whereas \(N_{x_i}^{(e)}(x, y)\) is the shape functions for \(u_i\) in an element.

Let \(R\) be a two-dimensional domain. Consider a typical node \(i\), called an apex node. It is the node which is common to all the neighboring elements. The nodes marked \(\bullet\) in Fig.2 and 3 are apex nodes. In Fig.2, the apex \(i\) is common to six triangular elements. In Fig.3, the apex \(i\) is common to four rectangular elements. The piecewise approximating function \(u(x, y)\) over the whole domain \(R\) can be written as
\[
u(x, y) = \sum_{i=1}^{M} N_i(x, y) u_i
\]
where \(M\) is the number of nodes contained in \(R\), \(N_i\) are the interpolating functions and \(u_i\) are the values at the nodes. The shape functions \(N_i(x, y)\) are defined by
\[
N_i(x, y) = N_{x_i}^{(e)}(x, y), \text{ given by Eqns.(3) or (14) if } (x, y) \text{ is in } e \text{ and has } i \text{ as an apex}
\]
\[
= 0, \text{ otherwise.}
\]

Since we have defined some elements that can be used in the two-dimensional domain, we are ready to introduce finite element methods for the solution of Laplace and Poisson equations. Finite element method can be applied either to extremize the variational form of the partial differential equation or directly through the use of a weighted residual method. Galerkin's method uses the weighted residual approach.

In the next section, we shall discuss the Galerkin method and derive the finite element Galerkin method. Here we shall be giving the method for two-dimensional boundary value problem although the method can be generalized to any dimension.

### 12.3 GALERKIN METHOD

Consider the boundary value problem
\[
L(u) = f(x, y); \quad x, y \in R
\]
\[
B_i[u] = g_i(x, y); \quad x, y \in \Gamma
\]
where $L$ is the second order linear differential operator, $\Gamma$ is the boundary of $\mathcal{R}$ and $\mathcal{B}_i$ are the boundary conditions. We assume the solution of the partial differential equation in the form

$$u(x, y) \approx w(x, y, a_1, a_2, \ldots, a_m) = \phi_0(x, y) + \sum_{i=1}^{m} a_i \phi_i(x, y)$$  \hspace{1cm} (18)

which depends on the parameters $a_1, a_2, \ldots, a_m$ and $\phi_i(x, y)$ are basis functions satisfying the boundary conditions and taken from a complete set of functions (can be taken as polynomials, orthogonal functions etc).

Normally, $\phi_0(x, y)$ satisfies the inhomogeneous boundary conditions and $\phi_i(x, y), i = 1, 2, \ldots, m$ satisfy the homogeneous boundary conditions

$$\mathcal{B}_i[\phi_0] = g_i, \quad \mathcal{B}_i[\phi_j] = 0, \quad j = 1, 2, \ldots, m.$$  \hspace{1cm} (18)

The error in satisfying the differential equation is given by

$$E(x, y, a) = L[w(x, y, a)] - f(x, y)$$  \hspace{1cm} (19)

where $a = [a_1, a_2, \ldots, a_m]^T$.

In the Galerkin method, the error is made orthogonal over the whole domain, to the functions $\phi_i(x, y), i = 1, 2, \ldots, m$. In other words,

$$\int_{\mathcal{R}} E(x, y, a) \phi_i(x, y) \, dx \, dy = 0, \quad j = 1, 2, \ldots, m$$  \hspace{1cm} (20)

This gives a $m \times m$ system of equations for the solution of $a_1, a_2, \ldots, a_m$.

Let us now consider the application of the method to the solution of the boundary value problem defined by Eqns.(16), (17). Let $k$ be the number of nodes in an element. We write the approximation in an element $e$ as

$$u(x, y) = \sum_{i=1}^{k} N_i u_i = N^{(e)} u^{(e)}$$  \hspace{1cm} (21)

where $N^{(e)} = [N_1, N_2, \ldots, N_k]$ and $u^{(e)} = [u_1, u_2, \ldots, u_k]^T$. The residual is

$$E = L(u) - f$$

Here, we choose the weight function as $N_i$. Therefore, we require

$$\int_{\mathcal{R}} [L(u) - f] N_i \, dx \, dy = 0, \quad i = 1, 2, \ldots, k$$  \hspace{1cm} (22)

Since the region is divided into $M$ elements, similar equation holds for each element

$$\int_{\mathcal{R}} [L(u) - f] N_i \, dx \, dy = 0, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, M$$  \hspace{1cm} (23)

where $N_i$ are now the shape functions in the element and $u_i$ are the values of the variable at the nodes in the element. The elemental equations are assembled (the contribution of all the common elements at an apex $i$) and the fixed boundary conditions are applied.

Let us consider the application of this method to the Dirichlet boundary value problem for the Poisson equation defined by

$$L u + G(x, y) = u_{xx} + u_{yy} + G(x, y) = 0 \quad \text{in} \quad \mathcal{R}$$  \hspace{1cm} (24)

$$u(x, y) = 0 \quad \text{on} \quad \Gamma$$  \hspace{1cm} (25)

Let $\mathcal{R}$ contain $M$ elements, each element with $k$ nodes. We write the approximation in the whole domain as (see Eqn.(15))

$$u(x, y) = \sum_{i=1}^{M} N_i(x, y) u_i$$  \hspace{1cm} (26)

Substituting $u(x, y)$ in the Eqn.(24) and using Eqn.(22), we obtain
Let $R$ be the domain as given in Fig. 6.

$R$ is bounded by the curves CDA and ABC.
Equation of curve CDA: $x = h_1(y)$.
Equation of curve ABC: $x = h_2(y)$.
Therefore, $R$ is defined as

$$
R: \quad p_1 \leq y \leq p_2, \quad h_1(y) \leq x \leq h_2(y)
$$

(28)

We can also say that $R$ is bounded by the curves DAB and BCD.
Equation of curve DAB: $y = g_1(x)$.
Equation of curve BCD: $y = g_2(x)$.
Therefore, $R$ is also defined as

$$
R: \quad q_1 \leq x \leq q_2, \quad g_1(x) \leq y \leq g_2(x)
$$

(29)

Integrating the first term of Eqn.(27) and using the definition of $R$ as given by Eqn.(28), we get (see. Fig. 6)

$$
\sum_{j=1}^{M} \left[ \int_{R} (N_i)_{xx} (N_j)_{x} \, dx \, dy \right] u_i + \int_{R} G(x, y) N_j \, dx \, dy = 0, \quad j = 1, 2, \ldots, M
$$

(27)

The first integral is integration along the boundary $\Gamma$ (line integral).
Similarly, using the definition of $R$ as given by Eqn.(29), the second term of Eqn.(27) gives

$$
\sum_{j=1}^{M} \left[ \int_{R} (N_i)_{yy} (N_j)_{y} \, dx \, dy \right] u_i = \int_{\Gamma} G(x, y) N_j \, d\Gamma - \sum_{j=1}^{M} \int_{R} (N_i)_{yy} (N_j)_{y} \, dx \, dy
$$

(31)

The contributions of the first term on the right hand side of Eqns(28) and (31) are zero for all the elements inside. Only the elements which have a part of the boundary as its sides, have contributions from the natural boundary conditions. When no natural boundary conditions are given, the contributions of the first terms are taken as zero. For the boundary value problems that we are studying, we take the contribution of these terms as zero. Thus, from Eqns.(27), (30) and (31), we obtain

$$
\sum_{j=1}^{M} \left[ \int_{R} (N_i)_{xx} (N_j)_{x} + (N_i)_{yy} (N_j)_{y} \right] \, dx \, dy \, u_i = \int_{R} G(x, y) N_j \, dx \, dy, \quad j = 1, 2, \ldots, M.
$$

(32)

For the Laplace equation $\nabla^2 u = 0$, we have $G(x, y) = 0$ on the right hand side of Eqn.(32). The integrals are evaluated for each element. The element equations are then assembled. The equation at each node $i$ (apex $i$) is written from the contribution of all the elements common to it. The coefficient matrix of the final system is a band matrix.

We shall now illustrate the method through examples.

**Example 1:** Find the solution of the boundary value problem

$$
\nabla^2 u = x^2 + y^2, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1
$$

$$
u = \frac{1}{12} (x^4 + y^4) \quad \text{on the boundary}
$$

using the Galerkin method with (i) triangular elements, (ii) rectangular elements and one internal node ($h = 1/2$).
Solution: (i) Triangular elements: The mesh is given in Fig. 7. At the apex 1, six triangular elements $e_1, e_2, e_3, e_4, e_5$, and $e_6$ contribute.

The boundary values are

$$u_2 = u(0,0) = 0, \quad u_3 = u\left(\frac{1}{2}, 0\right) = \frac{1}{192} = u_s,$$

$$u_4 = u\left(1, \frac{1}{2}\right) = \frac{17}{192} = u_g, \quad u_5 = u(1,1) = \frac{1}{6}.$$

Comparing the given equation with Eqn.(24) we have in this case

$$G(x,y) = -(x^2 + y^2).$$

We use Eqn.(32), for obtaining the difference equation at the node $i$. We have

$$\sum_{m=1}^{M} \left[ \iint \left(N_m, (N_i)_{x} + (N_m)_{y} \right) dx \, dy \right] u_m + \iint (x^2 + y^2) N_i dx \, dy = 0 \quad (33)$$

Let us now obtain the contribution of each element. The apex node is always denoted by $i$ and the remaining nodes are numbered as $j, k$ in the anti-clockwise direction.

The shape functions $N_i, N_j$ and $N_k$ are given as fellows (see Eqn.(3))

$$N_i = \frac{1}{2\Delta} \left[ (x_j y_k - x_i y_j) + (y_j y_k) x + (x_k - x_j) y \right],$$

$$N_j = \frac{1}{2\Delta} \left[ (x_k y_i - x_j y_k) + (y_k y_i) x + (x_i - x_k) y \right],$$

$$N_k = \frac{1}{2\Delta} \left[ (x_i y_j - x_k y_i) + (y_i y_j) x + (x_j - x_i) y \right],$$

where $\Delta = \text{area of the triangle}.$

Now, we evaluate the integrals in Eqn.(33) for each element.

Element $e_i$: $i = \left(\frac{1}{2}, \frac{1}{2}\right), \quad j = (0,0), \quad k = \left(\frac{1}{2}, 0\right), \quad \Delta = \frac{1}{8}$. (see Fig. 8)

$$N_i = 4 \left[ \frac{1}{2} y \right] = 2y, \quad N_j = 4 \left[ \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) x \right] = 1 - 2x$$

$$N_k = 4 \left[ \frac{1}{2} x - \frac{1}{2} y \right] = 2(x - y).$$

We have $(N_i)_x = 0, (N_i)_y = 2, (N_j)_x = -2, (N_j)_y = 0, (N_k)_x = 2, (N_k)_y = -2$.

Substituting the above values in Eqn.(33), we obtain

$$\iint_{e_i} f dx \, dy \, u_i + \iint_{e_i} (0) dx \, dy \, u_2 + \iint_{e_i} (x^2 + y^2)(2y) dx \, dy \, u_j + \iint_{e_i} (x^2 + y^2)(2x) dx \, dy \, u_k$$

Now, $\iint f(x,y) dx \, dy = \int_{x=0}^{1} \int_{y=0}^{1} f(x,y) dy \, dx$

Carrying out the integrations, we get

$$4 \left(\frac{1}{8}\right) u_i + 4 \left(\frac{1}{8}\right) u_3 + 2 \left(\frac{3}{4}\right) \left(\frac{1}{160}\right) = \frac{1}{2} u_1 - \frac{1}{2} \left(\frac{1}{192}\right) + \frac{3}{320} u_2 + \frac{1}{2} u_i + \frac{13}{1920}.$$
Element $e_2: i = \left(\frac{1}{2}, \frac{1}{2}\right), j = \left(\frac{1}{2}, 0\right), k = \left(1, \frac{1}{2}\right), \Delta = \frac{1}{8}$ (see Fig.9)

$N_i = \frac{1}{16} \left[\frac{1}{2} \left(\frac{1}{2}\right) + \left(-\frac{1}{2}\right) x + \left(1 - \frac{1}{2}\right) y\right] = 1 - 2x + 2y,$

$N_j = \frac{1}{16} \left[\frac{1}{2} \left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right) y\right] = 1 - 2y,$

$N_k = \frac{1}{16} \left[\frac{1}{2} \left(-\frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) x\right] = -1 + 2x.$

We have, $(N_i)_x = -2, (N_i)_y = 2, (N_j)_x = 0, (N_j)_y = -2, (N_k)_x = 2, (N_k)_y = 0.$

Substituting these values in Eqn.(33), we obtain

\[
\begin{align*}
\left[\int e_2 (-2 + 2) dx dy\right] u_1 + \left[\int e_2 (-2) dx dy\right] u_3 \\
+ \left[\int e_2 (2 - 2) dx dy\right] u_6 + \left[\int e_2 (x^2 + y^2) (1 - 2x + 2y) dx dy\right]
\end{align*}
\]

Now, \[
\int e_2 f(x, y) dx dy = \int \int e_2 f(x, y) dy dx
\]

Carrying out the integrations, we get

\[
8 \left(\frac{1}{8}\right) u_1 - 4 \left(\frac{1}{8}\right) \left(\frac{17}{192}\right) - 4 \left(\frac{1}{8}\right) \left(\frac{17}{192}\right) + \frac{11}{480} = u_1 = -\frac{23}{960}.
\]

Element $e_3: i = \left(\frac{1}{2}, \frac{1}{2}\right), j = \left(1, \frac{1}{2}\right), k = (1, 1), \Delta = \frac{1}{4}$ (see Fig.10)

$N_i = \frac{1}{16} \left[\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right) x\right] = 2(1 - x),$

$N_j = \frac{1}{16} \left[\left(-\frac{1}{2}\right) x + \left(\frac{1}{2}\right) y\right] = 2(x - y),$

$N_k = \frac{1}{16} \left[\frac{1}{2} \left(\frac{1}{2}\right) - \frac{1}{2} + \left(1 - \frac{1}{2}\right) y\right] = -1 + 2y.

We have $(N_i)_x = -2, (N_i)_y = 0, (N_j)_x = 2, (N_j)_y = -2, (N_k)_x = 0, (N_k)_y = 2.$

Substituting these values in Eqn.(33), we obtain

\[
\begin{align*}
\left[\int e_3 dx dy\right] u_1 + \left[\int e_3 - 4 dx dy\right] u_6 + \left(0\right) u_6 + 2 \left[\int e_3 (x^2 + y^2) (1 - x) dx dy\right]
\end{align*}
\]

Now, \[
\int e_3 f(x, y) dx dy = \int \int e_3 f(x, y) dy dx
\]

Carrying out the integrations, we get

\[
4 \left(\frac{1}{8}\right) u_1 - 4 \left(\frac{1}{8}\right) u_6 + \left(\frac{39}{960}\right) = \frac{1}{2} u_1 - \frac{1}{2} \left(\frac{17}{192}\right) + \frac{39}{960} = \frac{1}{2} u_1 - \frac{7}{1920}.
\]
Element $e_4$: $i = \left(\frac{1}{2}, \frac{1}{2}\right)$, $j = (1,1)$, $k = \left(\frac{1}{2}, 1\right)$, $\Delta = \frac{1}{8}$. (see Fig.11)

We obtain

$$N_i = 2(1 - y), \quad N_j = -1 + 2x, \quad N_k = 2(-x + y)$$

and

$$x = 0, \quad (N_i)_y = -2, \quad (N_j)_x = 2, \quad (N_j)_y = 0, \quad (N_k)_x = -2, \quad (N_k)_y = 2.$$

Substituting the above values in Eqn.(33), we obtain

$$\int \int_{e_4} 4dx \, dy \, u_i + (0)u_9 \int \int_{e_4} -4dx \, dy \, u_8 + \int \int_{e_4} (x^2 + y^2) 2(1 - y) \, dx \, dy$$

Now,

$$\int \int_{e_4} f(x, y) dx \, dy = \int_{x=0}^{1/2} \int_{y=1/2}^{1} f(x, y) dx \, dy.$$

Carrying out the integrations, we get

$$(4) \left(\frac{1}{8}\right) u_1 - 4 \left(\frac{1}{8}\right) \left(\frac{17}{192}\right) + 39 \left(\frac{39}{960}\right) = \frac{u_1}{2} = \frac{7}{1920}.$$

Element $e_5$: $i = \left(\frac{1}{2}, \frac{1}{2}\right)$, $j = \left(\frac{1}{2}, 1\right)$, $k = \left(0, \frac{1}{2}\right)$, $\Delta = \frac{1}{8}$. (see Fig.12)

We obtain

$$N_i = 1 + 2x - 2y, \quad N_j = -1 + 2y, \quad N_k = 1 - 2x.$$ 

and

$$x = 2, \quad (N_i)_y = -2, \quad (N_j)_x = 0, \quad (N_j)_y = 2, \quad (N_k)_x = -2, \quad (N_k)_y = 0.$$

Substituting these values in Eqn.(33), we obtain

$$\int \int_{e_5} 4dx \, dy \, u_i + \left(\int \int_{e_5} -4dx \, dy \, u_9 + \int \int_{e_5} -4dx \, dy \, u_9\right) + \int \int_{e_5} (x^2 + y^2)(1 + 2x - 2y) \, dx \, dy$$

Now,

$$\int \int_{e_5} f(x, y) dx \, dy = \int_{x=0}^{1/2} \int_{y=1/2}^{1/2} f(x, y) dx \, dy.$$

Carrying out the integrations, we get

$$(8) \left(\frac{1}{8}\right) u_1 - 4 \left(\frac{1}{8}\right) \left(\frac{1}{192}\right) - 4 \left(\frac{1}{8}\right) \left(\frac{1}{192}\right) + 11 \left(\frac{11}{480}\right) = u_1 = \frac{23}{960}.$$

Element $e_6$: $i = \left(\frac{1}{2}, \frac{1}{2}\right)$, $j = \left(0, \frac{1}{2}\right)$, $k = (0,0)$, $\Delta = \frac{1}{8}$. (see Fig.13)

We obtain

$$N_i = 2x, \quad N_j = -2x + 2y, \quad N_k = 1 - 2y.$$ 

And

$$x = 0, \quad (N_i)_y = 0, \quad (N_j)_x = -2, \quad (N_j)_y = 2, \quad (N_k)_x = 0, \quad (N_k)_y = -2.$$

Substituting these values in Eqn.(33), we obtain

$$\int \int_{e_6} 4dx \, dy \, u_i + \left(\int \int_{e_6} -4dx \, dy \, u_8 + (0)u_2 \int \int_{e_6} (x^2 + y^2) \, dx \, dy + \int \int_{e_6} (x^2 + y^2) \, dx \, dy.$$
Now, \( \iint_{e_b} f(x,y) \, dx \, dy = \int_0^{1/2} \left( \int_{y=0}^{1/2} f(x,y) \, dy \right) \, dx \)

Carrying out the integrations, we get

\[
(4) \left( \frac{1}{8} \right) u_i - 4 \left( \frac{1}{8} \right) \left( \frac{1}{192} \right) + \frac{9}{960} = \frac{1}{2} u_i + \frac{13}{1920}.
\]

Adding all the contributions, we obtain the difference equation at the node 1 as

\[
\left( \frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{1}{2} \right) u_i + \left( \frac{13}{1920} - \frac{23}{960} - \frac{7}{1920} - \frac{23}{960} + \frac{13}{1920} \right) = 0
\]

or, \( 4u_i - \frac{1}{24} = 0 \).

Hence, \( u_i = 1/96 \).

It may be noted that for the given problem, \( u(x,y) = (x^4 + y^4)/12 \) satisfies both the differential equation and the boundary condition. Hence, \( u = (x^4 + y^4)/12 \) is the exact solution. Note that we have obtained (coincidentally) the finite element solution which is the same as the exact solution \( u \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{96} \).

Now we solve Example 1 using rectangular elements.

(ii) Rectangular elements: The mesh is given in Fig.14. At the apex 1, four rectangular elements contribute. The boundary values are same as earlier, and in addition we have \( u_4 = u(1,0) = \frac{1}{12} = u_7 \). Let us now obtain the contribution of each element. The apex node is always denoted by \( i \) and the remaining nodes \( j, k, m \) are numbered in the anti-clockwise direction. The shape functions \( N_i, N_j, N_k \) and \( N_m \) are given as follows (see Eqn.(14)).

\[
N_i = \frac{(x-x_i) \, (y-y_m)}{(x_i-x_j) \, (y_j-y_m)} , \quad N_j = \frac{(x-x_j) \, (y-y_m)}{(x_i-x_j) \, (y_j-y_m)} , \\
N_k = \frac{(x-x_i) \, (y-y_j)}{(x_j-x_i) \, (y_j-y_i)} , \quad N_m = \frac{(x-x_i) \, (y-y_j)}{(x_j-x_i) \, (y_j-y_i)}
\]

Element \( e_1: i = \left( \frac{1}{2}, \frac{1}{2} \right), j = \left( 0, \frac{1}{2} \right), k = (0,0), m = \left( \frac{1}{2}, 0 \right) \) (see Fig.15)

\[
N_i = \frac{(x-0) \, (y-0)}{(1/2) \, (1/2)} = 4xy, N_j = \frac{x \, (1/2)}{(-1/2) \, (1/2)} = -2(2x-1)y \\
N_k = \frac{x \, (1/2)}{(-1/2) \, (-1/2)} = (2x-1)(2y-1) \\
N_m = \frac{x \, (1/2)}{(1/2) \, (-1/2)} = -2x(2y-1)
\]

substituting these values in Eqn.(33), we obtain...
Numerical Solutions of PDEs

\[ \int_{e_1} \left[ \int_0^1 \left( \frac{16}{24} u_1 - \frac{1}{6} \left( \frac{1}{192} \right) \right) - 0 - \frac{1}{6} \left( \frac{1}{192} \right) + \frac{1}{64} = \frac{2}{3} u_1 + \frac{1}{72} \right] \mathrm{d}x \mathrm{d}y \]

Carrying out the integrations, we get

Element \( e_2 \): \( i = \left( \frac{1}{2}, \frac{1}{2} \right), \ j = \left( \frac{1}{2}, 0 \right), \ k = (1, 0), \ m = \left( \frac{1}{2}, 1 \right) \) (see Fig. 16).

\[ N_1 = -4(x-1)y, \ N_2 = 2(x-1) (2y-1) \]
\[ N_3 = -(2x-1) (2y-1), \ N_4 = 2(2x-1)y. \]

Eqn.(33), yields

\[ \int_{e_2} \left[ \int_0^{1/2} \left( \frac{16}{24} u_1 - \frac{1}{6} \left( \frac{1}{192} \right) \right) - 0 - \frac{1}{6} \left( \frac{1}{192} \right) + \frac{1}{64} = \frac{2}{3} u_1 + \frac{1}{72} \right] \mathrm{d}x \mathrm{d}y \]

Carrying out the integrations, we get

Element \( e_3 \): \( i = \left( \frac{1}{2}, \frac{1}{2} \right), \ j = \left( 1, \frac{1}{2} \right), \ k = (1, 1), \ m = \left( \frac{1}{2}, 1 \right) \) (see Fig. 17)

\[ N_1 = -4(x-1) (y-1), \ N_2 = -2(x-1) (y-1) \]
\[ N_3 = (2x-1) (2y-1), \ N_4 = -2(x-1) (2y-1). \]

Eqn.(33), yields

\[ \int_{e_3} \left[ \int_0^{1/2} \left( \frac{16}{24} u_1 - \frac{1}{6} \left( \frac{1}{192} \right) \right) - 0 - \frac{1}{6} \left( \frac{1}{192} \right) + \frac{1}{64} = \frac{2}{3} u_1 + \frac{1}{72} \right] \mathrm{d}x \mathrm{d}y \]
Carrying out the integrations, we get
\[ \frac{2}{3} u_1 - \frac{1}{6} \left( \frac{17}{192} \right) - \frac{1}{3} \left( \frac{1}{6} \right) - \frac{1}{6} \left( \frac{17}{192} \right) + \frac{11}{192} = \frac{2}{3} u_1 - \frac{1}{36}. \]

Element 1: \( i = \left( \frac{1}{2}, \frac{1}{2} \right), j = \left( \frac{1}{2}, 1 \right), k = (0, 1), m = \left( 0, \frac{1}{2} \right) \) (see Fig.18).

We obtain
\[ N_i = -4(y - 1)(y - 1) , N_j = 2(2y - 1) \]
\[ N_k = -(2x - 1)(2y - 1), N_m = 2(2x - 1)(y - 1). \]

Eqn.(33) gives
\[ \int_{e_1} (16(y - 1)^2 + 16x^2) \, dx \, dy \, u_1 + \int_{e_1} (2(2y - 1)(-4)(y - 1) + 4x(-4x)) \, dx \, dy \, u_5 \]
\[ + \int_{e_1} (-2(2y - 1)(-4)(y - 1) - 2(2x - 1)(-4x)) \, dx \, dy \, u_7 \]
\[ + \int_{e_1} (4(y - 1)(-4)(y - 1) + 2(2x - 1)(-4x)) \, dx \, dy \, u_9 \]
\[ + \int_{e_1} (x^2 + y^2)(-4)x(y - 1) \, dx \, dy \]

Carrying out the integrations, we get
\[ \frac{2}{3} u_1 - \frac{1}{6} \left( \frac{17}{192} \right) - \frac{1}{3} \left( \frac{1}{12} \right) - \frac{1}{6} \left( \frac{17}{192} \right) + \frac{11}{192} = \frac{2}{3} u_1 - \frac{1}{144}. \]

Adding all the contributions, we obtain the difference equation at the node 1 as
\[ \left( \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \right) u_1 + \frac{1}{72} - \frac{1}{144} - \frac{1}{36} - \frac{1}{144} = 0 \]
or,
\[ \frac{8}{3} u_1 - \frac{1}{36} = 0 \]

We obtain \( u_1 = \frac{1}{96} \) which is the same as the exact solution.

**Remark:** In Example 1, coincidentally, the finite element solution obtained in both the cases are the same as the exact solution. However, in practice, large number of elements (triangular or rectangular) are required in order to obtain accurate solutions which in turn, increase the number of computations and level of difficulty for obtaining the elemental equations manually. Normally, we solve the problem with say, \( M \) elements. We compute the problem again by increasing the number of elements. Convergence of the solution values at the nodes can be studied to stop the computations.

As we have mentioned earlier finite element methods can be applied to extremise the variational form of partial differential equations. We now give here the variational form of Laplace and Poisson equation.

**Variational Forms**

First, consider a functional in one independent variable \( x \) in the form
\[ I(u) = \int_{x_1}^{x_2} F(x, u, u_x, u_{xx}) \, dx \]  

The problem is to find \( u(x) \), called an **extremal**, so as to extremise the functional in Eqn.(34). From the theory of variations, we can show that \( u(x) \) that extremises (34)
is also the solution of the Euler-Lagrange equation

\[
\frac{\partial F}{\partial u} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_x} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial u_{xx}} \right) = 0
\]  

(35)

Therefore, the solution of the partial differential Eqn.(35) extremises the functional (34) and the function extremising (34) is the solution of the partial differential Eqn.(35). Geometric considerations can be used to find whether it gives a minimum of Eqn.(34) or not. Now, consider a functional in two independent variable \( x, y \) in the form

\[ I(u) = \iint_F F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \, dA \]  

(36)

The corresponding Euler-Lagrange equation can be written as

\[
\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial u_y} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial u_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial F}{\partial u_{yy}} \right) + \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial F}{\partial u_{xy}} \right) = 0
\]  

(37)

In deriving the variational form and hence the Euler-Lagrange form, we require that the following conditions are satisfied.

\[
\left[ \frac{\partial F}{\partial u_x} - \frac{d}{dx} \left( \frac{\partial F}{\partial u_{xx}} \right) \right]_{x_1}^{x_2} = 0 \quad \text{and} \quad \left[ \frac{\partial F}{\partial u_{xx}} \right]_{x_1}^{x_2} = 0.
\]

These conditions are called the natural boundary conditions of the problem.

**Example 2:** Consider a functional as

\[ I(u) = \iint_R \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 - 2u f(x, y) \right] \, dA \]  

(38)

Then, using Eqn.(37), the Euler-Lagrange equation is given by

\[
-2f - \frac{\partial}{\partial x} \left[ 2 \frac{\partial u}{\partial x} \right] - \frac{\partial}{\partial y} \left[ 2 \frac{\partial u}{\partial y} \right] = 0 \quad \text{or} \quad u_{xx} + u_{yy} + f(x, y) = 0
\]  

(39)

which is the Poisson equation.

If we set \( f = 0 \), then we get

\[ I(u) = \iint_R \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \, dA \]  

(40)

and its Euler-Lagrange equation is the Laplace equation \( u_{xx} + u_{yy} = 0 \).

Eqns.(40) and (38) are respectively called the variational formulations of the Laplace equation \( u_{xx} + u_{yy} = 0 \) and the Poisson equation \( u_{xx} + u_{yy} + f(x, y) = 0 \).

***

Note that the variational form contains partial derivatives that are one order lower than the order of the partial differential equation. This is an advantage of the variational formulation. However, in practice, it is not easy to write the variational form of a given partial differential equation. In the case of a boundary value problem as defined by Eqn.(16) if the operator \( L \) is self-adjoint, then application of the classical variational principle and the Galerkin’s method lead to the same matrix system. Thus, for the Laplace and Poisson equations, both these methods yield the same result. However, if the differential equation considered is not self-adjoint, then the difference equations obtained by the two methods are different. In this unit, since we are discussing the solution of only the Laplace and Poisson equations, application of Galerkin’s method is sufficient for our purpose.

You may now try the following exercises.

E1) Find the solution of the boundary value problem

\[
\nabla^2 u = 4, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1
\]

\[ u = x^2 + y^2, \quad \text{on the boundary} \]
using the Galerkin method with (i) triangular elements, (ii) rectangular elements and one internal node \( (h = 1/2) \).

E2) Find the solution of the boundary value problem

\[
\nabla^2 u = x^2 + 2y^2, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1
\]

\[
u = \frac{1}{12} (x^4 + 2y^4), \text{ on the boundary}
\]

using the Galerkin method with (i) triangular elements (ii) rectangular elements and one internal node \( (h = 1/2) \).

E3) Find the solution of the boundary value problem

\[
\nabla^2 u = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1
\]

\[
u = \frac{1}{6} (x^3 + y^3), \text{ on the boundary}
\]

using the Galerkin's method with rectangular elements and one internal node \( (h = 1/2) \).

E4) Find the solution of the boundary value problem

\[
\nabla^2 u = 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1
\]

\[
u = x + y, \text{ on the boundary}
\]

using the Galerkin method with triangular elements and one internal node \( (h = 1/2) \).

We now end this unit by giving a summary of what we have covered in it.

\textbf{12.4 SUMMARY}

In this Unit, we have learnt the following points.

1. In finite element methods of solving a partial differential equation in a domain \( \mathbb{R} \), we generate difference equations by dividing \( \mathbb{R} \) into a finite number of non-overlapping sub-domains called finite elements.

2. Straight line elements are used in the case of one-dimensional problems whereas, triangular or rectangular elements are used in two dimensions.

3. The required solution is approximated by piecewise continuous polynomials defined in terms of nodal values at the vertices. At each nodal point, difference equation is obtained. The solution of these equations gives the numerical solution of the differential equation.

4. In finite element methods, difference equations can be generated by using the variational principle or weighted residual methods.

5. Finite element Galerkin method is a weighted residual method and does not require the variational form of the given differential equation.

6. If the given pde is self-adjoint then application of the variational principle and the Galerkin's method lead to the same matrix system.

\textbf{12.5 SOLUTIONS/ANSWERS}

E1) \[
u_1 = u \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2}.
\]

E2) \[
u_2 = u \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{64}.
\]
E3) \( u_1 = u \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{24} \).

E4) \( u_1 = u \left( \frac{1}{2}, \frac{1}{2} \right) = 1 \).