UNIT 9 MULTISTEP AND PREDICTOR – CORRECTOR METHODS FOR SOLVING INITIAL VALUE PROBLEMS

Structure **Page No.**

9.1	Introduction Objectives	29
9.2	Multistep Methods Interpolation Approach Method of Undetermined Parameters	29
9.3	Predictor-Corrector Methods	43
9.4	Stability of Multistep Methods	46
9.5	Summary	51
9.6	Solutions/Answers	52
	Practical Assignments	55

9.1 INTRODUCTION

In Unit 8 you have studied singlestep (Taylor series and Runge-Kutta) methods for solving the initial value problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{1}$$

These methods require only the immediate previous value y_i at the nodal point x_i to calculate y_{i+1} at the nodal point x_{i+1}. The order of the highest derivative used in Taylor series methods and the number of function evaluations per step in the Runge-Kutta method increases with the order of the method. Thus very high order singlestep methods are not used for solving the initial value problems.

There is another class of methods for solving IVP's which require the k solution values y_{i-m} at previous k nodal points x_{i-m}, $m = 0, 1, \ldots, (k-1)$ to obtain the value of y_{i+1} at the nodal point x_{i+1}. Such methods are called **multistep** or **k-step** methods. These methods are classified as **explicit** or **implicit** according to whether they **do not use** or **use** y_{i+1} in addition to the previously calculated k values. These methods required only one extra function evaluation per step irrespective of the order of the method. In this unit we shall discuss both explicit and implicit multistep methods.

In Sec.9.2 we have used two approaches to discuss the derivation and implementation of various explicit and implicit multistep methods. The explicit and implicit methods are combined to define a new class of methods called the **predictor-corrector** methods which we shall discuss in Sec.9.3. Finally, in Sec.9.4 we have discussed the stability of the multistep methods.

Objectives

After reading this unit you should be able to

- explain the multistep methods;
- explain the predictor-corrector methods;
- derive the explicit and implicit multistep methods;
- obtain the solution of initial value problems using multistep and predictor-corrector methods;
- obtain the interval of absolute stability of multistep methods.

9.2 MULTISTEP METHODS

A numerical method for solving the IVP(1) viz.,

$$y' = f(x, y), \; y(x_0) = y_0$$

is called a multistep or a k-step method if it uses the values of $y(x)$ and $f(x, y)$ at the k nodal points x_{i-m}, $m = 0, 1, \ldots, (k-1)$ to determine the solution value y_{i+1} at the nodal point x_{i+1}. Such a method is called an **explicit** or a **predictor method.** If the method also uses the values of $y(x)$ and $f(x, y)$ at the nodal point x_{i+1}, then the method is called an **implicit** or a **corrector method.** The value y_i is the initial value and the values $y_{i-1}, y_{i-2}, \ldots, y_{i-k+1}$ are called the starting values. The starting values are obtained using some singlestep method before using the multistep method. Generally, Taylor series methods and Runge-Kutta methods are used for starting a multistep method. Thus multistep methods are not self-starting methods. A predictor method is used to predict a value of y_{i+1} and a corrector method is used to improve upon this value. In this section we shall use two approaches for the derivation of multistep methods.

9.2.1 Interpolation Approach

Integrating $y' = f(x, y)$ in the interval $[x_i, x_{i+1}]$, we obtain

$$\int_{x_i}^{x_{i+1}} y'\, dx = \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

or

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y)\, dx \tag{2}$$

To derive multistep methods, we replace $f(x, y)$ in the integral in Eqn.(2) by some suitable interpolating polynomial. We shall now discuss some of these methods.

Adams-Bashforth methods

These are explicit methods. Through the k data values $(x_i, f_i), (x_{i-1}, f_{i-1}), \ldots, (x_{i-k+1}, f_{i-k+1})$, we fit the Newton's backward difference interpolating polynomial of degree $k-1$ as

$$\begin{aligned} P_{k-1}(x) = f_i &+ \frac{(x - x_i)}{h}\nabla f_i + \frac{(x - x_i)(x - x_{i-1})}{2!\, h^2}\nabla^2 f_i + \ldots \\ &+ \frac{(x - x_i)(x - x_{i-1})\ldots(x - x_{i-k+2})}{(k-1)!\, h^{k-1}}\nabla^{k-1} f_i. \end{aligned} \tag{3}$$

The error of interpolation is given by

$$TE = \frac{(x - x_i)(x - x_{i-1})\ldots(x - x_{i-k+1})}{k!} f^{(k)}(\xi), \tag{4}$$

where ξ lies in some interval containing the points $x_i, x_{i-1}, \ldots, x_{i-k+1}$ and x. Replacing $f(x,y)$ by $P_{k-1}(x)$ in Eqn.(2), we get

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} \left[f_i + \frac{(x - x_i)}{h}\nabla f_i + \frac{(x - x_i)(x - x_{i-1})}{2!h^2}\nabla^2 f_i + \cdots \right] dx.$$

Set $x - x_i = hs$. Then $x - x_m = x - x_i + x_i - x_m = (s + i - m)h$. Hence, we get

$$y_{i+1} = y_i + \int_0^1 \left[f_i + s\nabla f + \frac{1}{2} s(s+1)\nabla^2 f_i + \cdots \right] ds$$

$$= y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \cdots \right]. \tag{5}$$

Error term can be obtained by integrating Eqn.(4) in the interval $[x_i, x_{i+1}]$ or, directly by using Taylor series. For different values of k, we obtain different methods.

k = 1: we get a singlestep method $y_{i+1} = y_i + h\, f_i$ (6)

which is the **Euler method.** The error term is given by

$$TE = y(x_{i+1}) - y_{i+1} = y(x_{i+1}) - [y_i + hf_i]$$

$$= y(x_{i+1}) - y_i - hy'_i$$

where $f_i = y'_i$.

Expanding each term in Taylor series about the point x_i, we obtain

$$TE = \frac{h^2}{2} y''(\xi), \quad x_i < \xi < x_{i+1}.$$

This method is of order 1.

k = 2:
$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\nabla f_i\right]$$

$$= y_i + h\left[f_i + \frac{1}{2}(f_i - f_{i-1})\right] = y_i + \frac{h}{2}(3f_i - f_{i-1}) \qquad (7)$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [y_i + h(3y'_i - y'_{i-1})]$$
$$= \frac{5}{12}h^3 y'''(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

This method is of order 2.

k = 3:
$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i\right]$$

$$= y_i + h\left[f_i + \frac{1}{2}(f_i - f_{i-1}) + \frac{5}{12}(f_i - 2f_{i-1} + f_{i-2})\right]$$

$$= y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}] \qquad (8)$$

The error term is given by

$$TE = y(x_i + h) - \left[y_i + \frac{h}{12}(23y'_i - 16y'_{i-1} + 5y'_{i-2})\right]$$

$$= \frac{3}{8}h^4 y^{(4)}(\xi), \quad x_{i-2} < \xi < x_{i+1}$$

This method is of order 3.

In general a k-step method of the form (5) gives a method of order k. We require one function evaluation per step for the application of the multistep method.

Example 1: Find the approximate value of $y(1.0)$ using the multistep method (Adams-Bashforth second order method)

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

with h = 0.2 for the initial value problem

$$y' = -2xy^2, \; y(0) = 1$$

Calculate the starting value using the Runge-Kutta second order method

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)$$
$$K_1 = hf(x_i, y_i)$$
$$K_2 = h\, f(x_i + h, y_i + K_1)$$

with the same step size h.

Solution : For the given multistep method

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

or,

$$y_{i+2} = y_{i+1} + \frac{h}{2}(3f_{i+1} - f_i) \qquad (9)$$

we get for $i = 0$

$$y_2 = y_1 + \frac{h}{2}(3f_1 - f_0)$$

Now y_0 is the given initial value and y_1 is the starting value. We calculate the starting value using the given second order Runge-Kutta method. We obtain from

$$f(x\ y) = -2x\ y^2,\ x_0 = 0,\ y_0 = 1,\ h = 0.2$$

$$K_1 = hf(x_0, y_0) = h\ f(0, 1) = 0$$

$$K_2 = hf(x_0 + h, y_0 + K_1) = h\ f(0.2, 1) = -0.08$$

$$y_1 \quad y(0.2) = y_0 + \frac{1}{2}(K_1 + K_2) = 0.96$$

After the starting value has been determined, we use the multistep method (9). We obtain for

i = 0: $x_0 = 0, y_0 = 1,\ f_0 = -2x_0\ y_0^2 = 0;$

$$x_1 = 0.2, y_1 = 0.96,\ f_1 = -2x_1\ y_1^2 = -0.36864$$

$$y_2 \approx y(0.4) = y_1 + \frac{h}{2}(3f_1 - f_0) = 0.849408.$$

i = 1: $x_2 = 0.4, y_2 = 0.849408,\ f_2 = -2x_2\ y_2^2 = -0.577195$

$$y_3 \approx y(0.6) = y_2 + \frac{h}{2}(3f_2 - f_1) = 0.713114.$$

i = 2: $x_3 = 0.6, y_3 = 0.713114,\ f_3 = -2x_3\ y_3^2 = -0.610237$

$$y_4 \approx y(0.8) = y_3 + \frac{h}{2}(3f_3 - f_2) = 0.587762.$$

i = 3: $x_4 = 0.8, y_4 = 0.5887762,\ f_4 = -2x_4\ y_4^2 = -0.552743$

$$y_5 \approx y(1.0) = y_4 + \frac{h}{2}(3f_4 - f_3) = 0.482963.$$

***

Example 2: Solve the initial value problem

$$y' = x + y^2,\ y(0) = 1$$

in the interval $[0,1]$ with step length $h = 0.2$ using the multistep method

$$y_{i+1} = y_i + \frac{h}{3}(23f_i - 16f_{i-1} + 5f_{i-2})$$

(Adams-Bashforth third order method).

Compute the starting values using third order Taylor series method with the same step size h.

Solution: From the given multistep method

$$y_{i+1} = y_i + \frac{h}{3}(23f_i - 16f_{i-1} + 5f_{i-2})$$

or,

$$y_{i+3} = y_{i+2} + \frac{h}{3}(23f_{i+2} - 16f_{i+1} + 5f_i) \quad (10)$$

we obtain for $i = 0$

$$y_3 = y_2 + \frac{h}{3}(23f_2 - 16f_1 + 5f_0)$$

Now y_0 is the given initial value, whereas y_1 and y_2 are the starting values. We compute the starting values using the third order Taylor series method

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2} y_j'' + \frac{h^3}{6} y_j''',\quad j = 0,1 \quad (11)$$

We have

$$y' = x + y^2,\quad y'' = 1 + 2y y',\quad y''' = 2y y'' + 2(y')^2$$

We obtain from Eqn.(11) for h = 0.2 and

j = 0: $x_0 = 0, y_0 = 1, y_0' = 1,\ y_0'' = 3, y_0''' = 8$

$$y_1 \approx y(0.2) = y_0 + h\,y_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0'''$$
$$= 1 + 0.2(1) + \frac{(0.2)^2}{2}(3) + \frac{(0.2)^3}{6}(8)$$
$$= 1.270667.$$

$j = 1$: $x_1 = 0.2, y_1 = 1.270667, y_1' = 1.814595,$
$y_1'' = 5.611492,\ y_1''' = 20.846185$ and

$$y_2 \approx y(0.4) = y_1 + h\,y_1' + \frac{h^2}{2}y_1'' + \frac{h^3}{6}y_1'''$$
$$= 1.270667 + 0.2(1.814595)$$
$$= \frac{(0.2)^2}{2}(5.611492) + \frac{(0.2)^3}{6}(20.846185)$$
$$= 1.773611.$$

After determining the starting values, we use the multistep method (10). We have $h = 0.2$ and

$$x_0 = 0,\ y_0 = 1,\ f_0 = x_0 + y_0^2 = 1,$$
$$x_1 = 0.2, y_1 = 1.270667,\ f_1 = x_1 + y_1^2 = 1.814595,$$
$$x_2 = 0.4, y_2 = 1.773611, f_2 = x_2 + y_2^2 = 3.545696$$

We obtain from Eqn.(10) for

$i = 0$: $$y_3 = y(0.6) = y_2 + \frac{h}{12}(23f_2 - 16f_1 + 5f_0)$$
$$= 1.773611 + \frac{0.2}{12}[23(3.545696) - 16(1.814595) + 5(1)]$$
$$= 2.732236.$$

$i = 1$: $x_3 = 0.6, y_3 = 2.732236, f_3 = x_3 + y_3^2 = 8.065112$
$$y_4 \approx y(0.8) = y_3 + \frac{h}{12}(23f_3 - 16f_2 + 5f_1)$$
$$= 2.732236 + \frac{0.2}{12}[23(8.065112) - 16(3.545696) + 5(1.814595)]$$
$$= 5.029560.$$

$i = 2$: $x_4 = 0.8, y_4 = 5.029560, f_4 = x_4 + y_4^2 = 26.096470$
$$y_5 \approx y(1.0) = y_4 + \frac{h}{12}[23f_4 - 16f_3 + 5f_2]$$
$$= 5.029560 + \frac{0.2}{12}[23(26.096470) - 16(8.065112) + 5(3.545696)]$$
$$= 13.177985.$$

Adams-Moulton methods

These are implicit methods. Through the $k + 1$ data values $(x_{i+1}, f_{i+1}), (x_i, f_i), \ldots, (x_{i-k+1}, f_{i-k+1})$, we fit the Newton's backward difference interpolating polynomial of degree k as

$$P_k(x) = f_{i+1} + \frac{(x - x_{i+1})}{h}\nabla f_{i+1} + \frac{(x - x_{i+1})\,(x - x_i)}{2!h^2}\nabla^2 f_{i+1} + \cdots$$
$$+ \frac{(x - x_{i+1})(x - x_i)\ldots(x - x_{i-k+2})}{k!h^k}\nabla^k f_{i+1}. \qquad (12)$$

Note that we are using the current data point (x_{i+1}, f_{i+1}) also.
The error of interpolation is given by

$$TE = \frac{1}{(k+1)!}(x - x_{i+1})(x - x_i)\dots(x - x_{i-k+1})f^{(k+1)}(\xi) \qquad (13)$$

where ξ lies in some interval containing the points $x_{i+1}, x_i, \dots, x_{i-k+1}$ and x. Replacing $f(x, y)$ by $P_k(x)$ in Eqn.(2), we get

$$y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} \left[f_{i+1} + \frac{(x - x_{i+1})}{h}\nabla f_{i+1} + \frac{(x - x_{i+1})(x - x_i)}{2!h^2}\nabla^2 f_{i+1} + \cdots \right] dx$$

Set $x - x_i = hs$. Then $x - x_m = (s + i - m)\,h$. Hence, we obtain

$$y_{i+1} = y_i + h\int_0^1 \left[f_{i+1} + (s-1)\nabla f_{i+1} + \frac{(s-1)s}{2!}\nabla^2 f_{i+1} + \cdots \right] ds$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} - \cdots \right] \qquad (14)$$

The error term can be obtained by integrating (13) in the interval $[x_i, x_{i+1}]$ or, directly by using the Taylor series. For different values of k, we obtain different methods.

k = 0 : We get a singlestep method. Retaining one term inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h f_{i+1} \qquad (15)$$

which is the **backward Euler method**. The error term is given by

$$TE = y'(x_{i+1}) = y_{i+1} = y(x_i + h) - [y_i + h y'_{i+1}]$$

where $f_{i+1} = y'_{i+1}$. Expanding each term in Taylor series about x_i, we get

$$TE = -\frac{h^2}{2} y''(\xi), \qquad x_i < \xi < x_{i+1}$$

This method is of order 1.

k = 1 : We again get a singlestep method. Retaining two terms inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} \right]$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) \right] = y_i + \frac{h}{2}[f_i + f_{i+1}] \qquad (16)$$

which is called the **trapezoidal method.**

The error term is given by

$$TE = (x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - \left[y_i + \frac{h}{2}(y'_i + y'_{i+1}) \right]$$

$$= -\frac{h^3}{12} y'''(\xi), \qquad x_i < \xi < x_{i+1}$$

This method is of order 2.

k = 2 : Retaining three terms inside the brackets in Eqn.(14), we get the method

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\nabla f_{i+1} - \frac{1}{12}\nabla^2 f_{i+1} \right]$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) - \frac{1}{12}(f_{i+1} - 2f_i + f_{i-1}) \right]$$

$$= y_i + \frac{h}{12}[5 f_{i+1} + 8f_i - f_{i-1}] \qquad (17)$$

The error term is obtained as

$$TE = y(x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - y_i - \frac{h}{12}[5y'_{i+1} + 8y'_i - y'_{i-1}]$$

$$= \frac{h^4}{24} y^{(iv)}(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

This method is of order 3.

In general, the k-step method of the form (14) is of order $(k+1)$.

Since the methods are implicit, we obtain a non-linear equation in y_{i+1} for non-linear initial value problem. The solution can be obtained by using Newton-Raphson method or, any other iterative method. Generally we take the initial approximation $y_{i+1} = y_i$. For a linear initial value problem, we can obtain y_{i+1} directly without iteration.

Milne-Simpson methods

These are implicit methods. If we integrate $y' = f(x, y)$ in the interval $[x_{i-1}, x_{i+1}]$, we get

$$y(x_{i+1}) = y(x_{i-1}) + \int_{x_{i-1}}^{x_{i+1}} f(x, y)\, dx \tag{18}$$

Now we replace $f(x, y)$ in Eqn.(18) by the interpolating polynomial $P_k(x)$ given by Eqn.(12) based on the data values $(x_{i+1}, f_{i+1}), (x_i, f_i), \ldots, (x_{i-k+1}, f_{i-k+1})$. Using the substitution $x - x_i = hs$, we obtain as in the case of Adams-Moulton methods

$$y_{i+1} = y_{i-1} + h \int_{-1}^{1} \left[f_{i+1} + (s-1)\nabla f_{i+1} + \frac{(s-1)s}{2!} \nabla^2 f_{i+1} + \cdots \right] ds$$

$$= y_{i-1} + h\left[2f_{i+1} - 2\nabla f_{i+1} + \frac{1}{3}\nabla^2 f_{i+1} + O\nabla^3 f_{i+1} + \cdots \right] \tag{19}$$

By choosing different values of k, we get different methods.

$\mathbf{k = 0}$: Retaining one term inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + 2h f_{i+1} \tag{20}$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [\, y_{i-1} + 2h\, y'_{i+1}\,]$$
$$= -2h^2\, y''(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

This method is of order 1.

$\mathbf{k = 1}$: Retaining two terms inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + h[\, 2f_{i+1} - 2\nabla f_{i+1}\,]$$
$$= y_{i-1} + h[\, 2f_{i+1} - 2(f_{i+1} - f_i)\,]$$
$$= y_{i-1} + 2h f_i \tag{21}$$

This is an explicit method. The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - [\, y_{i-1} + 2h\, y'_i\,]$$
$$= \frac{h^3}{3} y'''(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

This method is of order 2.

$\mathbf{k = 2}$: Retaining three terms inside the brackets in Eqn.(19), we get the method

$$y_{i+1} = y_{i-1} + h\left[2f_{i+1} - 2\nabla f_{i+1} + \frac{1}{3}\nabla^2 f_{i+1} \right]$$

$$= y_{i-1} + h\left[2f_{i+1} - 2(f_{i+1} - f_i) + \frac{1}{3}(f_{i+1} - 2f_i + f_{i-1}) \right]$$

$$= y_{i-1} + \frac{h}{3}[\, f_{i+1} + 4f_i + f_{i-1}\,] \tag{22}$$

The error term is given by

$$TE = y(x_{i+1}) - y_{i+1}$$

$$= y(x_i + h) - \left[y_{i-1} + \frac{h}{3}(y'_{i+1} + 4y'_i + y'_{i-1}) \right]$$

$$= -\frac{h^5}{90} y^{(5)}(\xi), \quad x_{i-1} < \xi < x_{i+1}$$

Thus the method is of order 4. The method (22) is called the **Milne-Simpson method of order 4**.

For-$\mathbf{k} = \mathbf{3}$, we obtain the same method as (22). Hence, for $k = 2$ and $k = 3$, we obtain the same method.

Example 3: Find the approximate value of $y(0.5)$ for the initial value problem

$$y' = x + y, \; y(0) = 1,$$

using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

(Milne-Simpson fourth order method) with $h = 0.1$

Compute the starting value using classical Runge-Kutta fourth order method with the same step length h.

Solution: The given multistep method is an implicit method. Since the given initial value problem is linear we do not require any iteration. We can compute y_{i+1} directly. From the given multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

or,

$$y_{i+2} = y_i + \frac{h}{3}(f_{i+2} + 4f_{i+1} + f_i) \tag{23}$$

we obtain for $i = 0$

$$y_2 = y_1 + \frac{h}{3}(f_2 + 4f_1 + f_0)$$

Now y_0 is the given initial value, whereas y_1 is the starting value. We compute y_1 using classical fourth order Runge-Kutta method.

We have $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$. We obtain

$$K_1 = h\, f(x_0, y_0) = h\, f(0, 1) = 0.1$$

$$K_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = h\, f(0.05, 1.05) = 0.11$$

$$K_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = h\, f(0.05, 1.055) = 0.1105$$

$$K_4 = h\, f(x_0 + h, y_0 + K_3) = h\, f(0.1, 1.1105) = 0.12105$$

$$y_1 \approx y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.110342$$

After determining the starting value, we use the multistep method (23). From Eqn.(23), we obtain

$$y_{i+2} = y_i + \frac{h}{3}[(x_{i+2} + y_{i+2}) + 4(x_{i+1} + y_{i+1}) + (x_i + y_i)]$$

or,

$$\left(1 - \frac{h}{3}\right) y_{i+2} = y_i + \frac{h}{3}[4y_{i+1} + y_i + x_{i+2} + 4x_{i+1} + x_i]$$

Using $x_{i+2} = x_i + 2h$, $x_{i+1} = x_i + h$, we get

$$\left(1 - \frac{h}{3}\right) y_{i+2} = y_i + \frac{h}{3}[4y_{i+1} + y_i + 6x_i + 6h]$$

$$= \frac{4h}{3} y_{i+1} + \left(1 + \frac{h}{3}\right) y_i + \frac{h}{3}(6x_i + 6h) \tag{24}$$

Using $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $y_1 = 1.110342$, $h = 0.1$, we obtain from Eqn.(24) for

i = 0: $\frac{29}{30}y_2 = \frac{4}{30}y_1 + \frac{31}{30}y_0 + \frac{1}{30}\left(6x_0 + \frac{6}{10}\right)$

$= 1.201379$

or, $y_2 \approx y(0.2) = 1.242806, \quad x_2 = 0.2$

i = 1: $\frac{29}{30}y_3 = \frac{4}{30}y_2 + \frac{31}{30}y_1 + \frac{1}{30}\left(6x_1 + \frac{6}{10}\right)$

$= 1.353061$

or, $y_3 \approx y(0.3) = 1.399718, \quad x_3 = 0.3$

i = 2: $\frac{29}{30}y_4 = \frac{4}{30}y_3 + \frac{31}{30}y_2 + \frac{1}{30}\left(6x_2 + \frac{6}{10}\right)$

$= 1.530862$

or, $y_4 \approx y(0.4) = 1.583650, \quad x_4 = 0.4$

i = 3: $\frac{29}{30}y_5 = \frac{4}{30}y_4 + \frac{31}{30}y_3 + \frac{1}{30}\left(6x_3 + \frac{6}{10}\right)$

$= 1.737529$

or, $y_5 \approx y(0.5) = 1.797443$

***

Example 4: Find an approximate value of $y(1.2)$ for the initial value problem

$$y' = x^2 + y^2, \; y(1) = 2$$

using the Adams-Moulton third order method

$$y_{i+1} = y_i + \frac{h}{12}[\, 5f_{i+1} + 8f_i - f_{i-1}]$$

with $h = 0.1$. Calculate the starting value using third order Taylor series method with $h = 0.1$.

Solution: The Adams-Moulton third order method can be written as

$$y_{i+2} = y_{i+1} + \frac{h}{12}[\, 5f_{i+2} + 8f_{i+1} - f_i], \qquad i = 0, 1, \ldots \tag{25}$$

For $i = 0$, we get

$$y_2 = y_i + \frac{h}{12}[\, 5f_2 + 8f_1 - f_0]$$

Now, y_0 is the initial value and y_1 is the starting value. We calculate y_1 using Taylor series third order method

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' \tag{26}$$

with $h = 0.1$. We have

$$y_0 = 2,\; y_0' = x_0^2 + y_0^2 = 5 \;\; y_0'' = 2x_0 + 2y_0\, y_0' = 22,$$

$$y_0''' = 2 + 2y_0\, y_0'' + 2(y_0')^2 = 140$$

Therefore, we obtain from Eqn.(26)

$$y(1.1) \approx y_1 = 2 + (0.1)\,(2) + \frac{(0.1)^2}{2}\,(22) + \frac{(0.1)^3}{6}(140)$$

$$= 2.633333$$

After determining the starting value, we use the multistep method (25). From Eqn.(25), we obtain for $i = 0$.

$$y(1.2) \approx y_2 = y_1 + \frac{h}{12}\left[\, 5(x_2^2 + y_2^2) + 8(x_1^2 + y_1^2) - (x_0^2 + y_0^2)\right]$$

Using $y_0 = 2$, $y_1 = 2.633333$, $x_0 = 1$, $x_1 = 1.1$, $x_2 = 1 + 2h = 1.2$ and simplifying, we get

$y_2 = 0.041667\ y_2^2 + 3.194629$, which is a nonlinear equation in y_2.

We use Newton-Raphson method to determine y_2.

Let $F(y_2) = 0.041667 y_2^2 - y_2 + 3.194629$.

We get, $F'(y_2) = 0.083334 y_2 - 1$

The Newton-Raphson method is given by

$$y_2^{(s+1)} = y_2^{(s)} - \frac{F(y_2^{(s)})}{F'(y_2^{(s)})},\ s = 0, 1, \ldots$$

Starting with $y_2^{(0)} = y_1 = 2.633333$, we get

$$y_2^{(1)} = y_2^{(0)} - \frac{F(y_2^{(0)})}{F'(y_2^{(0)})} = 2.633333 - \frac{0.850233}{(-0.780554)} = 3.722602$$

$$y_2^{(2)} = y_2^{(1)} - \frac{F(y_2^{(1)})}{F'(y_2^{(1)})} = 3.722602 - \frac{0.049439}{(-0.689781)} = 3.794275$$

$$y_2^{(3)} = y_2^{(2)} - \frac{F(y_2^{(2)})}{F'(y_2^{(2)})} = 3.794275 - \frac{0.000214}{(-0.683808)} = 3.794588$$

Since $F(y_2^{(3)}) = 0.0000001$, we take $y(1.2) \approx y_2 = 3.794588$

***

Note that we have to use iteration at every step.

And now some exercises for you.

E1) Find the truncation error and the order of the method

$$y_{i+3} = y_i + \frac{3h}{8}[\ y'_i + 3y'_{i+1} + 3y'_{i+2} + y'_{i+3}]$$

for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$.

E2) Find an approximate value of $y(0.6)$ for the initial value problem

$$y' = x - y^2,\ y(0) = 1$$

using the multistep method

$$y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

with the step length $h = 0.2$. Compute the starting value using Taylor series second order method with the same steplength h.

E3) Find an approximate value of $y(1.5)$ for the initial value problem

$$y' = x^2 + y^2,\ y(1) = 0$$

using the multistep method

$$y_{i+1} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

with $h = 0.1$. Calculate the starting values using third order Taylor series method with the same steplength h.

E4) Find an approximate value of $y(2)$ for the initial value problem

$$y' = y + y^2,\ y(1) = 1$$

using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}[\ 7f_i - 2f_{i-1} + f_{i-2}]$$

with $h = 0.2$. Compute the starting values using the Heun's method with the same steplength h.

E5) Solve the initial value problem

$$y' = 2x + 3y,\ y(1) = 2$$

on the interval [1, 1.4] using the multistep method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(y'_{i+1} + 4y'_i + y'_{i-1})$$

with $h = 0.1$. Calculate the starting value using classical fourth order Runge-Kutta method, with the same steplength h.

Let us now discuss the second approach for the derivation of multistep methods.

9.2.2 Method of Undetermined Parameters

The general multistep or k-step method for the solution of the initial value problem (1) can be written as

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + h\left[b_0 y'_{i+1} + b_1 y'_i + \cdots + b_k y'_{i-k+1}\right]$$

or,
$$y_{i+1} = \sum_{m=1}^{k} a_m y_{i-m+1} + h\sum_{m=0}^{k} b_m y'_{i-m+1} \tag{27}$$

Changing i to $i+k-1$ in Eqn.(24), we can also write the method as

$$y_{i+k} = \sum_{m=1}^{k} a_m y_{i+k-m} + h\sum_{m=0}^{k} b_m y'_{i+k-m} \tag{28}$$

In method (27), y_i is the initial value and $y_{i-1}, \ldots, y_{i-k+1}$ are the starting values.

If $b_0 = 0$, the method (27) defines an explicit method and if $b_0 \neq 0$, the method (27) defines an implicit method.

In method (27) a'_ms and b'_ms are unknowns which are to be determined such that the method is of a particular order. We write the error term as

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= y(x_i + h) - \sum_{m=1}^{k} a_m y(x_{i-m+1}) - h\sum_{m=0}^{k} b_m y'(x_{i-m+1}) \tag{29}$$

Expanding each term in (29) in Taylor series about the point x_i and collecting the terms of various powers of h, we can write

$$TE = C_0 y(x_i) + C_1 h\, y'(x_i) + \cdots + C_p h^p y^{(p)}(x_i) + C_{p+1} h^{p+1} y^{(p+1)}(x_i) + \cdots \tag{30}$$

where C'_ms are independent of h and depend on a'_ms and b'_ms. The multistep method (27) is said to be of order p, if

$$C_0 = C_1 = \cdots = C_p = 0 \text{ and } C_{p+1} \neq 0$$

The truncation error in this case is given by

$$TE = C_{p+1} h^{p+1} y^{(p+1)}(x_i) + O(h^{p+2})$$

Solving the system of equations obtained from

$$C_0 = C_1 = \cdots = C_p = 0$$

we determine the constants a'_ms and b'_ms and hence the method. The first non-zero term in Eqn.(30) gives the error term and the order of the method.

A method is said to be **consistent** if $p \geq 1$.

Note that we have (2k + 1) arbitrary unknowns in the method (27) and we need $2k+1$ equations to determine these unknowns. Hence, a method of the form (27) can be of maximum order 2k. However, the stability conditions restrict the order of the method to k for an explicit method. For an implicit method, the order is restricted to k+1, when k is odd; and k+2, when k is even. We shall discuss the stability of these methods in Sec.9.4.

Adams-Bashforth method

The form of a few important class of methods is

$$y_{i+1} = y_i + h\sum_{m=1}^{k} b_m y'_{i-m+1}$$

This is an explicit method and is of order k.

Nystrom method

$$y_{i+1} = y_{i-1} + h\sum_{m=1}^{k} b_m y'_{i-m+1}$$

This is an explicit method and is of order k.

Adams-Moulton method

$$y_{i+1} = y_i + h\sum_{m=0}^{k} b_m y'_{i-m+1}$$

This is an implicit method and is of order k+1.

Milne-Simpson method

$$y_{i+1} = y_{i-1} + h\sum_{m=0}^{k} b_m y'_{i-m+1}$$

This is an implicit method and is of order $k+1$, when k is odd and of order $k+2$, when k is even.

We illustrate the derivation of multistep methods by taking some examples.

Example 5: Obtain the constants a_1, b_1 and b_2 in the explicit multistep method

$$y_{i+1} = a_1 y_i + h[\, b_1 y'_i + b_2 y'_{i-1}]$$

Determine the truncation error and the order of the method.

Solution: We write

$$\begin{aligned} TE &= y(x_{i+1}) - y_{i+1} \\ &= y(x_{i+1}) - [\, a_1 y(x_i) + h\{b_1 y'(x_i) + b_2 y'(x_{i-1})\}] \\ &= y(x_i + h) - a_1 y(x_i) - h[\, b_1 y'(x_i) + b_2 y'(x_i - h)] \end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various power of h, we get

$$\begin{aligned} TE &= (1 - a_1) y(x_i) + \{1 - (b_1 + b_2)\} h\, y'(x_i) \\ &\quad + \left(\frac{1}{2} + b_2\right) h^2 y''(x_i) + \left(\frac{1}{6} - \frac{1}{2} b_2\right) h^3 y'''(x_i) + \cdots \end{aligned} \tag{31}$$

We have three unknowns and we need three equations to determine these unknowns. Setting the constant term and the coefficients of h, h^2 in Eqn.(31) to zero, we obtain the system of equations

$$\begin{aligned} 1 - a_1 &= 0, &\quad \text{or,} \quad a_1 &= 1 \\ 1 - (b_1 + b_2) &= 0, &\quad \text{or,} \quad b_1 + b_2 &= 1 \\ \frac{1}{2} + b_2 &= 0, &\quad \text{or,} \quad b_2 &= -\frac{1}{2} \end{aligned}$$

whose solution is $a_1 = 1$, $b_1 = \frac{3}{2}$, $b_2 = -\frac{1}{2}$. Thus we obtain the method as

$$y_{i+1} = y_i + \frac{h}{2}[\, 3y'_i - y'_{i-1}] \tag{32}$$

and from Eqn.(31), we obtain

$$\begin{aligned} TE &= \left(\frac{1}{6} - \frac{1}{2} b_2\right) h^3 y'''(x_i) + O(h^4) \\ &= \frac{5}{12} h^3 y'''(x_i) + O(h^4) \end{aligned}$$

Therefore, the method is of order 2.

The method (32) is the Adams-Bashforth method of order 2.

***

Example 6: Derive the constants in the explicit method

$$y_{i+1} = a_1 y_i + h\{b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}\}$$

Determine also the truncation error and the order of the method.

Solution: We write

$$\begin{aligned} TE &= y(x_{i+1}) - y_{i+1} \\ &= y(x_{i+1}) - [a_1 y(x_i) + h\{b_1 y'(x_i) + b_2 y'(x_{i-1}) + b_3 y'(x_{i-2})\}] \\ &= y(x_i + h) - a_1 y(x_i) - h[b_1 y'(x_i) + b_2 y'(x_i - h) + b_3 y'(x_i - 2h)] \end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various powers of h, we get

$$\begin{aligned} TE &= (1 - a_1) y(x_i) + \{1 - (b_1 + b_2 + b_3)\} h y'(x_i) \\ &+ \left\{\frac{1}{2} + (b_2 + 2b_3)\right\} h^2 y''(x_i) + \left\{\frac{1}{6} - \frac{1}{2}(b_2 + 4b_3)\right\} h^3 y'''(x_i) \\ &+ \left\{\frac{1}{24} + \frac{1}{6}(b_2 + 8b_3)\right\} h^4 y^{iv}(x_i) + \cdots \end{aligned} \quad (33)$$

We have four unknowns and we need four equations to determine these unknowns. Equating the constant terms and the coefficients of h, h^2 and h^3 in Eqn.(33) to zero, we obtain the system of equations.

$$\begin{aligned} &1 - a_1 = 0 && \text{or,} \quad a_1 = 1 \\ &1 - (b_1 + b_2 + b_3) = 0 && \text{or,} \quad b_1 + b_2 + b_3 = 1 \\ &\frac{1}{2} + (b_2 + 2b_3) = 0 && \text{or,} \quad b_2 + 2b_3 = -\frac{1}{2} \\ &\frac{1}{6} - \frac{1}{2}(b_2 + 4b_3) = 0 && \text{or,} \quad b_2 + 4b_3 = \frac{1}{3} \end{aligned}$$

Solving this system of equations, we obtain

$$a_1 = 1, \; b_1 = \frac{23}{12}, \; b_2 = -\frac{16}{12}, \; b_3 = \frac{5}{12}$$

and the method is given by

$$y_{i+1} = y_i + \frac{h}{12}[23 y'_i - 16 y'_{i-1} + 5 y'_{i-2}] \quad (34)$$

which is Adams-Bashforth third order method. From Eqn.(33), we get

$$\begin{aligned} TE &= \left[\frac{1}{24} + \frac{1}{6}(b_2 + 8b_3)\right] h^4 y^{iv}(x_i) + O(h^5) \\ &= \frac{3}{8} h^4 y^{iv}(x_i) + O(h^5). \end{aligned}$$

***

Example 7: Derive the method

$$y_{i+1} = a_1 y_i + a_2 y_{i-2} + h(b_0 y'_{i+1} + b_1 y'_i + b_2 y'_{i-1})$$

Determine the truncation error and order of the method.

Solution: We write

$$\begin{aligned} TE &= y(x_{i+1}) - y_{i+1} \\ &= y(x_{i+1}) - [a_1 y(x_i) + a_2 y(x_{i-2}) + h\{b_0 y'(x_{i+1}) + b_1 y'(x_i) + b_2 y'(x_{i-1})\}] \\ &= y(x_i + h) - [a_1 y(x_i) + a_2 y(x_i - 2h) + h\{b_0 y'(x_i + h) \\ &\qquad + b_1 y'(x_i) + b_2 y'(x_i - h)\}] \end{aligned}$$

Expanding each term in Taylor series about the point x_i and collecting the coefficients of various power of h, we get

$$TE = (1 - a_1 - a_2)\, y(x_i) + [1 + 2a_2 - (b_0 + b_1 + b_2)]\, h\, y'(x_i)$$
$$+ \left[\frac{1}{2} - 2a_2 - (b_0 - b_2)\right] h^2 y''(x_i) + \left[\frac{1}{6} + \frac{4}{3}a_2 - \frac{1}{2}(b_0 + b_2)\right] h^3 y'''(x_i)$$
$$+ \left[\frac{1}{24} - \frac{2}{3}a_2 - \frac{1}{6}(b_0 - b_2)\right] h^4 y^{iv}(x_i) + \left[\frac{1}{120} + \frac{4}{15}a_2 - \frac{1}{24}(b_0 + b_2)\right] h^5 y^{v}(x_i)$$
$$+ \cdots \qquad (35)$$

We have five unknowns, and we need five equations to determine these unknowns. Setting the constant term and the coefficients of h, h^2, h^3 and h^4 in Eqn.(35) to zero, we obtain the system of equations

$$1 - a_1 - a_2 = 0, \quad \text{or} \quad a_1 + a_2 = 1$$
$$1 + 2a_2 - (b_0 + b_1 + b_2) = 0, \quad \text{or} \quad b_0 + b_1 + b_2 - 2a_2 = 1$$
$$\frac{1}{2} - 2a_2 - (b_0 - b_2) = 0, \quad \text{or} \quad b_0 - b_2 + 2a_2 = \frac{1}{2}$$
$$\frac{1}{6} + \frac{4}{3}a_2 - \frac{1}{2}(b_0 + b_2) = 0, \quad \text{or} \quad b_0 + b_2 - \frac{8}{3}a_2 = \frac{1}{3}$$
$$\frac{1}{24} - \frac{2}{3}a_2 - \frac{1}{6}(b_0 - b_2) = 0, \quad \text{or} \quad b_0 - b_2 + 4a_2 = \frac{1}{4}$$

Solving these equations, we obtain

$$a_1 = \frac{9}{8}, \quad a_2 = -\frac{1}{8}, \quad b_0 = \frac{3}{8}, \quad b_1 = \frac{6}{8}, \quad b_2 = -\frac{3}{8}$$

and the method is given by

$$y_{i+1} = \frac{1}{8}(9y_i - y_{i-2}) + \frac{h}{8}[3y'_{i+1} + 6y'_i - 3y'_{i-1}] \qquad (36)$$

We obtain from (35)

$$TE = \left[\frac{1}{120} + \frac{4}{15}a_2 - \frac{1}{24}(b_0 + b_2)\right] h^5 y^{v}(x_i) + O(h^6)$$
$$= -\frac{1}{40} h^5 y^{v}(x_i) + O(h^6)$$

Hence, the method (36) is implicit and is of order 4.

***

And now some exercises for you.

E6) Derive the method

$$y_{i+1} = a_1 y_{i-1} + h[b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}]$$

for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

E7) Derive the method

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + h[b_1 y'_i + b_2 y'_{i-1} + b_3 y'_{i-2}]$$

for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

E8) Derive the method

$$y_{i+1} = a_1 y_i + h[b_0 y'_{i+1} + b_1 y'_i + b_2 y'_{i-1}]$$

for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and order of the method.

E9) Derive the method

$$y_{i+1} = a_1 y_i + a_2 y'_{i-1} + h b_0 y'_{i+1}$$

for solving the initial value problem $y' = f(x, y),\ y(x_0) = y_0$. Find the truncation error and the order of the method.

So far we have developed a number of explicit and implicit multistep methods to solve an initial value problem $y' = f(x, y), \quad y(x_0) = y_0$. Based on the stability properties of the method which we shall be discussing in Sec.9.4, it is seen that all the explicit methods converge only if the step size h is sufficiently small. Hence, if the solution of the initial value problem is required in a large interval, then we may require a large number of integration steps, which besides increasing the round-off error, is not economical. On the other hand, most implicit methods have strong stability properties, that is one can choose sufficiently large value of h for integration. However, we need to solve a non-linear equation by iteration at each step, which is also expensive. Hence, we combine the explicit and implicit methods to define a new class of methods called the **predictor-corrector methods** which we shall discuss now.

9.3 PREDICTOR-CORRECTOR METHODS

The explicit methods are called the **predictors** which predict or give an initial approximation to $y(x_{i+1})$. This initial approximation is denoted by $y_{i+1}^{(p)}$. The implicit methods are called the **correctors** which use $y_{i+1}^{(p)}$ to obtain an improved value of y_{i+1}. This corrected value is denoted by $y_{i+1}^{(c)}$. The corrector method is used till convergence is obtained. We define predictor-corrector methods (P-C sets) as follows:

P : Predict an approximate value of $y(x_{i+1})$. This approximate value is denoted by $y_{i+1}^{(p)}$ or $y_{i+1}^{(0)}$.

C : Correct this value using a corrector method to obtain $y_{i+1}^{(c)}$. We take $y(x_{i+1}) = y_{i+1}^{(c)}$.

When the predictor and corrector methods are of the same order, we need only one or two corrector iterations per step. However, if predictor is of lower order, we may require more number of corrector iterations for convergence. The predictor-corrector methods (P-C methods) are used as sets. Some examples of P-C sets are

1. P: $y_{i+1}^{(0)} = y_i + h f(x_i, y_i)$ (Euler method)

   C: $y_{i+1}^{(s+1)} = y_i + h f(x_{i+1}, y_{i+1}^{(s)})$ (backward Euler method)

   $s = 0, 1, \ldots$

The order of both the methods is one.

2. P: $y_{i+1}^{(0)} = y_i + h f(x_i, y_i)$ (Euler method)

   C: $y_{i+1}^{(s+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$ (Trapezoidal method)

   $s = 0, 1, \ldots$

The predictor is of first order and the corrector is of second order.

3. P: $y_{i+1}^{(0)} = y_{i-3} + \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2})$

   C: $y_{i+1}^{(s+1)} = y_{i-1} + \frac{h}{3}\left[f_{i-1} + 4f_i + f(x_{i+1}, y_{i+1}^{(s)})\right]$

   $s = 0, 1, \ldots$

Both the methods are of fourth order. This P-C set is also called **Milne's P-C set.** The starting values are obtained using some singlestep explicit method like Taylor series or Runge-Kutta method. The order of the singlestep method may be same or less than that of the corrector method.

Example 8: Obtain the approximate value of $y(0.3)$ for the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 1$$

using the method:

P = predictor: $y_{i+1}^{(p)} = y_i + h f_i$

C = corrector: $y_{i+1}^{(c)} = y_i + \frac{h}{2}[f_i + f(x_{i+1}, y_{i+1})]$ (37)

with the step length h = 0.1. Perform two corrector iterations per step.

Solution: Since both the predictor and corrector methods are singlestep methods, we do not need any starting value. We write the methods (37) as

$$P: \quad y_{i+1}^{(0)} = y_i + h f(x_i, y_i) = y_i + h\left[x_i^2 + y_i^2\right]$$

$$C: \quad y_{i+1}^{(s+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$$

$$= y_i + \frac{h}{2}\left[x_i^2 + y_i^2 + x_{i+1}^2 + \left\{y_{i+1}^{(s)}\right\}^2\right], \quad s = 0, 1,$$

$i = 0, 1, 2.$

since $f(x, y) = x^2 + y^2$.

Using the given predictor method for h = 0.1, we get

i = 0: $x_0 = 0, \quad y_0 = 1,$

$$P: \ y_1^{(0)} = y_0 + h(x_0^2 + y_0^2) = 1.1$$

$$C: \ y_i^{(s+1)} = y_0 + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(s)})\right]$$

$$s = 0: y_1^{(1)} = y_0 + \frac{h}{2}\left[x_0^2 + y_0^2 + x_1^2 + (y_1^{(0)})^2\right] = 1.111$$

$$s = 1: y_1^{(2)} = y_0 + \frac{h}{2}\left[x_0^2 + y_0^2 + x_1^2 + (y_1^{(1)})^2\right] = 1.112216$$

Therefore, after two corrector iterations, we get

$$y_1 \approx y(0.1) = 1.112216$$

i = 1: $x_1 = 0.1, \quad y_1 = 1.112216$

$$P: \ y_2^{(0)} = y_1 + h(x_1^2 + y_1^2) = 1.236918$$

$$C: \ y_2^{(s+1)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(s)})^2\right]$$

$$s = 0: y_2^{(1)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(0)})^2\right] = 1.253066$$

$$s = 1: y_2^{(2)} = y_1 + \frac{h}{2}\left[x_1^2 + y_1^2 + x_2^2 + (y_2^{(1)})^2\right] = 1.255076$$

Therefore, after two corrector iterations, we obtain .

$$y_2 \approx y(0.2) = 1.255076$$

i = 2: $x_2 = 0.2, \quad y_2 = 1.255076$

$$P: \ y_3^{(0)} = y_2 + h(x_2^2 + y_2^2) = 1.416598$$

$$C: \ y_3^{(s+1)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(s)})^2\right]$$

$$s = 0: y_3^{(1)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(0)})^2\right] = 1.440674$$

$$s = 1: y_3^{(2)} = y_2 + \frac{h}{2}\left[x_2^2 + y_2^2 + x_3^2 + (y_3^{(1)})^2\right] = 1.444114$$

Therefore, after two corrector iterations, we obtain

$$y_3 \approx y(0.3) = 1.444114$$

***

Example 9: Solve the initial value problem

$$y' = x^2 + y^3, \quad y(1) = 0$$

on the interval [1, 1.6] using the predictor-corrector method

$$P:\ y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$C:\ y_{i+1} = y_i + \frac{h}{12}(5f_{i+1} + 8f_i - f_{i-1}) \qquad (38)$$

with the step length h = 0.2. Perform three corrector iterations per step. Compute the starting value using Taylor series second order method with the same step length h.

Solution: We have $f(x,y) = x^2 + y^3$. We write the method (38) as

$$P:\quad y_{i+1}^{(0)} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$\text{or,}\quad y_{i+2}^{(0)} = y_{i+1} + \frac{h}{2}(3f_{i+1} - f_i) \qquad (39)$$

$$C:\quad y_{i+1}^{(s+1)} = y_i + \frac{h}{12}(5f(x_{i+1}, y_{i+1}^{(s)}) + 8f_i - f_{i-1})$$

$$\text{or,}\quad y_{i+2}^{(s+1)} = y_{i+1} + \frac{h}{12}\left[5f(x_{i+2}, y_{i+2}^{(s)}) + 8f_{i+1} - f_i\right] \qquad (40)$$

where $s = 0, 1, 2$ and $i = 0, 1, \ldots$

For $i = 0$, we get from Eqns.(39) and (40)

$$P:\quad y_2^{(0)} = y_1 + \frac{h}{2}(3f_1 - f_0)$$

$$C:\quad y_2^{(s+1)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(s)}) + 8f_1 - f_0\right],\ s = 0, 1, \ldots$$

Now y_0 is the given initial value, whereas y_1 is the starting value. We compute the starting value using the Taylor series second order method

$$y_1 = y_0 + h y_0' + \frac{h^2}{2} y_0''$$

with h = 0.2. We have

$$y' = x^2 + y^3, \qquad y'' = 2x + 3y^2 y'$$

Therefore, for $x_0 = 1,\ y_0 = 0$, we obtain $y_0' = 1, \quad y_0'' = 2$

and $y_1 \approx y(1.2) = 0 + 0.2(1) + \frac{(0.2)^2}{2}(2) = 0.24$

After determining the starting value, we use the predictor-corrector method given by Eqns.(39) and (40). We obtain for

i = 0: $x_0 = 1,\ y_0 = 0,\ x_1 = 1.2,\ y_1 = 0.24$

$f_0 = x_0^2 + y_0^3 = 1,\ f_1 = x_1^2 + y_1^3 = 1.453824$

P: $y_2^{(0)} = y_1 + \frac{h}{2}[\,3f_1 - f_0] = 0.576147$

Now $x_2 = 1.4,\ f(x_2, y_2^{(0)}) = x_2^2 + (y_2^{(0)})^3 = 2.151250$

C: $s = 0: y_2^{(1)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(0)}) + 8f_1 - f_0)\right] = 0.596447$

and $f(x_2, y_2^{(1)}) = x_2^2 + (y_2^{(1)})^3 = 2.172185$

$s = 1: y_2^{(2)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(1)}) + 8f_1 - f_0\right] = 0.598192$

and $f(x_2, y_2^{(2)}) = x_2^2 + (y_2^{(2)})^3 = 2.174053$

$s = 2: y_2^{(3)} = y_1 + \frac{h}{12}\left[5f(x_2, y_2^{(2)}) + 8f_1 - f_0\right] = 0.598348$

Therefore, after three corrector iterations

$y_2 \approx y(1.4) = 0.598348$

i = 1: $x_1 = 1.2,\ y_1 = 0.24,\ x_2 = 1.4,\ y_2 = 0.598348$

$f_1 = 1.453824,\ f_2 = x_2^2 + y_2^3 = 2.174220$

$$P:\ y_3^{(0)} = y_2 + \frac{h}{2}[3f_2 - f_1] = 1.105232$$

Now $x_3 = 1.6,\ f(x_3, y_3^{(0)}) = x_3^2 + (y_3^{(0)})^3 = 3.910083$

$$C:\ s = 0: y_3^{(1)} = y_2 + \frac{h}{12}[5f(x_3, y_3^{(0)}) + 8f_2 - f_1] = 1.189854$$

and $f(x_3, y_3^{(1)}) = x_3^2 + (y_3^{(1)})^3 = 4.244539$

$$s = 1: y_3^{(2)} = y_2 + \frac{h}{12}[5f(x_3, y_3^{(1)}) + 8f_2 - f_1] = 1.217725$$

and $f(x_3, y_3^{(2)}) = x_3^2 + (y_3^{(2)})^3 = 4.365709$

$$s = 2: y_3^{(3)} = y_2 + \frac{h}{12}[5f(x_3, y_3^{(2)}) + 8f_2 - f_1] = 1.227823$$

Hence, after three corrector iterations

$y_3 \approx y(1.6) = 1.227823$

***

E10) Solve the initial value problem

$$y' = y + \sin y,\quad y(1) = 1$$

in the interval $[1, 1.6]$ using the predictor-corrector method

$$P:\ y_{i+1} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

$$C:\ y_{i+1} = y_i + \frac{h}{2}(f_{i+1} + f_i)$$

with $h = 0.2$. Calculate the starting value using second order Taylor series method with the same steplength. Perform two corrector iterations per step.

E11) Obtain the approximate value of $y(1.0)$ for the initial value problem

$$y' = x^2 + y^2,\quad y(0) = 1$$

using the predictor-corrector method

$$P:\ y_{i+1} = y_{i-1} + \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2})$$

$$C:\ y_{i+1} = y_{i-1} + \frac{h}{3}(f_{i+1} + 4f_i + f_{i-1})$$

with $h = 0.2$. Calculate the starting values using the Euler method with the same steplength. Perform two corrector iterations per step.

Let us now discuss the stability of multistep methods.

9.4 STABILITY OF MULTISTEP METHODS

If we apply the multistep method

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + h[b_0 y'_{i+1} + b_1 y'_i + \cdots + b_k y'_{i-k+1}] \quad (41)$$

to the test equation

$$y' = \lambda y,\quad y(x_0) = y_0$$

we get

$$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + \lambda h[b_0 y_{i+1} + b_1 y_i + \cdots + b_k y_{i-k+1}]$$

or,
$$y_{i+1} - a_1 y_i - a_2 y_{i-1} - \cdots - a_k y_{i-k+1} - \lambda h\left[b_0 y_{i+1} + b_1 y_i + \cdots + b_k y_{i-k+1}\right] = 0 \quad (42)$$

The exact solution of Eqn.(42) is given by

$$y(x) = y(x_0)\, e^{(x-x_0)\lambda}$$

Therefore, we get

$$y(x_i) = y(x_0)\, e^{(x_i - x_0)\lambda} = y(x_0)\, e^{\lambda hi}$$

and $y(x_{i+1}) = y(x_0)\, e^{(x_{i+1}-x_0)\lambda} = y(x_0)\, e^{\lambda h(i+1)}$

Hence, we obtain

$$\frac{y(x_{i+1})}{y(x_i)} = e^{\lambda h} \quad \text{or,} \quad y(x_{i+1}) = y(x_i)\, e^{\lambda h}$$

We find that the exact solution of Eqn.(41) grows $(\lambda > 0)$ or, decays $(\lambda < 0)$ by the factor $e^{\lambda h}$.

Substituting $y_i = y(x_0)\, e^{\lambda hi}$ in Eqn.(42), we get

$$\left\{e^{\lambda h(i+1)} - a_1 e^{\lambda hi} - \cdots - a_k e^{\lambda h(i-k+1)}\right\} y(x_0) - \lambda h\left\{ b_0 e^{\lambda h(i+1)} + b_1 e^{\lambda hi} + \cdots + b_k e^{\lambda h(i-k+1)}\right\} y(x_0) = 0$$

Dividing by $e^{\lambda h(i-k+1)}$, we get

$$[(e^{\lambda hk} - a_1 e^{\lambda h(k-1)} - \cdots - a_k) - \lambda h(b_0 e^{\lambda hk} + b_1 e^{\lambda h(k-1)} + \cdots + b_k)] = 0 \quad (43)$$

Substituting $e^{\lambda h} = \xi$ in Eqn.(43), we get

$$(\xi^k - a_1 \xi^{k-1} - \cdots - a_k) - \lambda h(b_0 \xi^k + b_1 \xi^{k-1} + \cdots + b_k) = 0$$

or,

$$\rho(\xi) - \lambda h\, \sigma(\xi) = 0 \quad (44)$$

where,

$$\rho(\xi) = \xi^k - a_1 \xi^k - \cdots - a_k$$

$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^k + \cdots + b_k$$

We note that $\sigma(\xi)$ is a polynomial of degree k for an implicit method $(b_0 \neq 0)$ and a polynomial of degree $k-1$ for an explicit method $(b_0 = 0)$.

The polynomial equation $\rho(\xi) - \lambda h\, \sigma(\xi) = 0$ is called the **characteristic equation** associated with the multistep method (41).

The polynomial $\rho(\xi) = 0$ is called the **reduced characteristic equation**.

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the roots of $\rho(\xi) = 0$ and $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ be the roots of $\rho(\xi) - \lambda h\, \sigma(\xi) = 0$.

We note that for a consistent method, $\xi = 1$ is always a root of $\rho(\xi) = 0$.

The multistep method (41) is said to be

stable : if $|\xi_i| < 1,\ i \neq 1$ and $\xi_1 = 1$

unstable: if $|\xi_i| > 1$ for some i or, if there is a multiple root of $\rho(\xi) = 0$ of modulus unity.

conditionally stable (weakly stable): if ξ_i's are simple and if more than one of these roots have modulus unity.

For a stable method, we further define

absolutely stable : if $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i

relatively stable : if $\lambda > 0$ and $|\xi_{ih}| < e^{\lambda h}$ for all i

interval of absolute stability : The set of values of λh for which the method is absolutely stable, that is the interval $(\lambda h, 0)$, $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i

Similarly, we define the interval of relative stability.

We generally use the **Routh-Hurwitz criterion** to obtain the interval of absolute stability. To use this criterion, we substitute

$$\xi = \frac{1+z}{1-z} \quad (45)$$

in the characteristic equation $\rho(\xi) - \lambda h\, \sigma(\xi) = 0$ and obtain the **transformed equation.**

$$P(z) = \nu_0 z^k + \nu_1 z^{k-1} + \cdots + \nu_{k-1} z + \nu_k = 0 \qquad (46)$$

where ν_i' s depend on λh.

Let

$$D = \begin{vmatrix} \nu_1 & \nu_3 & \nu_5 & \cdots & \nu_{2k-1} \\ \nu_0 & \nu_2 & \nu_4 & \cdots & \nu_{2k-2} \\ 0 & \nu_1 & \nu_3 & \cdots & \nu_{2k-3} \\ 0 & \nu_0 & \nu_2 & \cdots & \nu_{2k-4} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \nu_k \end{vmatrix}$$

then the method is absolutely stable if

(i) $\nu_i > 0$ for all i and

(ii) leading principal minors of D are positive.

We obtain the condition of absolute stability as

$\mathbf{k=1:}\quad \nu_0 > 0,\quad \nu_1 > 0$

$\mathbf{k=2:}\quad \nu_0 > 0,\quad \nu_1 > 0,\quad \nu_2 > 0$

$\mathbf{k=3:}\quad \nu_0 > 0,\quad \nu_1 > 0,\quad \nu_2 > 0$ and $\nu_1 \nu_2 - \nu_0 \nu_3 > 0$

Note that if any of ν_i' s is zero and other ν_i' s are positive, then it indicates that a root lies on the unit circle $|\xi| = 1$. If any of ν_i' s is negative, then there is at least one root for which $|\xi_i| > 1$.

In both the cases, the method is unstable.

Without giving proof we state the following result.

Theorem 1 The order of a stable explicit k-step method of the form (24) cannot exceed k and the order of a stable implicit k-step method of the form (24) cannot exceed $k+1$ if k is odd and $k+2$ if k is even.

We now take up a few examples

Example 10: Find the interval of absolute stability of the Adams-Bashforth methods

(i) $y_{i+1} = y_i + \frac{h}{2}(3y_i' - y_{i-1}')$

(ii) $y_{i+1} = y_i + \frac{h}{12}(23\, y_i' - 16\, y_{i-1}' + 5 y_{i-2}')$

Solution: We apply the method to the test equation $y' = \lambda y$.

(i) we get the characteristic equation $(k = 2)$ as

$$\xi^2 - \xi - \frac{\lambda h}{2}(3\xi - 1) = 0$$

or $$\xi^2 - (1 + \frac{3\lambda h}{2})\,\xi + \frac{\lambda h}{2} = 0$$

Substituting $\xi = \frac{1+z}{1-z}$, we get

$$\left(\frac{1+z}{1-z}\right)^2 - \left(1 + \frac{3\lambda h}{2}\right)\left(\frac{1+z}{1-z}\right) + \frac{\lambda h}{2} = 0$$

or, $$(1+z)^2 - \left(1 + \frac{3\lambda h}{2}\right)(1 - z^2) + \frac{\lambda h}{2}(1-z)^2 = 0$$

or, $$(2 + 2\lambda\, h)\, z^2 + (2 - \lambda\, h)\, z + [-\lambda\, h] = 0$$

or, $$\nu_0\, z^2 + \nu_1\, z + \nu_2 = 0.$$

where, $\nu_0 = 2 + 2\lambda h$, $\nu_1 = 2 - \lambda h$, $\nu_2 = -\lambda h$.

We require that

$$\nu_0 > 0,\ \nu_1 > 0\ \ \nu_2 > 0 \text{ when } \lambda h < 0$$

We find that ν_0 and ν_1 are always positive.

Now $\nu_2 > 0 \Rightarrow 2(1+\lambda h) > 0$ or, $\lambda h > -1$

Hence, interval of absolute stability is $]-1, 0[$

(ii) we obtain the characteristic equation $(k = 3)$ as

$$(\xi^3 - \xi^2) - \frac{\lambda h}{12}(23\,\xi^2 - 16\,\xi + 5) = 0$$

or, $$\xi^3 - \left(1 + \frac{23H}{12}\right)\xi^2 + \frac{16H}{12}\xi - \frac{5H}{12} = 0$$

where, $H = \lambda h$.

Substituting $\xi = \dfrac{1+z}{1-z}$, we get

$$(1+z)^3 - \left(1 + \frac{23}{12}H\right)(1+z)^2\ (1-z) + \frac{16H}{12}(1+z)\ (1-z)^2 - \frac{5H}{12}\ (1-z)^3 = 0$$

or, $$(1+3z+3z^2+z^3) - \left(1 + \frac{23}{12}H\right)(1+z-z^2-z^3) + \frac{16H}{12}(1-z-z^2+z^3) - \frac{5H}{12}(1-3z+3z^2-z^3) = 0$$

or, $$\left(2 + \frac{44H}{12}\right)z^3 + \left(4 - \frac{2H}{3}\right)z^2 + (2-2\,H)\,z - H = 0$$

Comparing with $\nu_0 z^3 + \nu_1 z^2 + \nu_2 z + \nu_3 = 0$, we get

$\nu_0 = \frac{1}{3}(6+11H),\ \nu_1 = \frac{2}{3}(6-H),\ \nu_2 = 2(1-H),\ \nu_3 = -H$. Since $\lambda h < 0$, we have $H < 0$. We find that ν_1, ν_2, ν_3 are always positive. From $\nu_0 > 0$, we get the condition $H > -6/11$. We also find that $\nu_1\nu_2 - \nu_0\nu_3 = 5H^2 - (22/3)H + 8$, which is positive for all $H < 0$. Hence, the interval of absolute stability is $]-6/11, 0[$.

***

Example 11: Find the interval of absolute stability of the methods.

(i) $y'_{i+1} = y_i + \frac{h}{2}[\,y'_{i+1} + y'_i]$

(ii) $y'_{i+1} = y_i + \frac{h}{12}[\,5y'_{i+1} + 8y'_i - y'_{i-1}]$

Solution:

(i) We have $k = 1$. The characteristic equation is obtain as

$$(\xi - 1) - \frac{\lambda h}{2}(\xi + 1) = 0$$

Solving for ξ, we get

$$\xi = \frac{1 + \lambda h/2}{1 - \lambda h/2}$$

For $\lambda h < 0$, $|\xi| < 1$ for all λh. Hence, the method is absolutely stable for all h. The interval of absolute stability is $]-\infty, 0[$ and therefore the method is A-stable.

(ii) We have $k = 2$. The characteristic equation is given by

$$\xi^2 - \xi = \frac{\lambda h}{12}(5\,\xi^2 + 8\,\xi - 1)$$

or, $\left(1-\frac{5H}{12}\right)\xi^2 - \left(1+\frac{2H}{3}\right)\xi + \frac{H}{12} = 0$

where $H = \lambda h$.

Substituting $\xi = \frac{1+z}{1-z}$, we get

$$\left(1-\frac{5H}{12}\right)(1+z)^2 - \left(1+\frac{2H}{3}\right)(1+z)(1-z) + \frac{H}{12}(1-z)^2 = 0$$

or, $\left(1-\frac{5H}{12}\right)(1+2z+z^2) - \left(1+\frac{2H}{3}\right)(1-z^2) + \frac{H}{12}(1-2z+z^2) = 0$

or, $\left(2+\frac{H}{3}\right)z^2 + (2-H)\,z - H = 0$

Comparing with $v_0 z^2 + v_1 z + v_2 = 0$, we get

$$v_0 = 2 + \frac{H}{3},\quad v_1 = 2 - H,\quad v_2 = -H$$

Since $\lambda < 0$, we find that $H < 0$. Now v_1 an v_2 are both positive for $H < 0$. $v_0 > 0$ if $H > -6$. Hence, the interval of absolute stability is $]-6, 0[$.

***

Example 12: Show that the Milne-Simpson method

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n+1} + 4y'_n + y'_{n-1})$$

is not absolutely stable for any h.

Solution: Here $k = 2$. The characteristic equations is obtained as

$$\xi^2 - 1 - \frac{\lambda h}{3}(\xi^2 + 4\xi + 1) = 0$$

or, $\left(1-\frac{H}{3}\right)\xi^2 - \frac{4H}{3}\xi - \left(1+\frac{H}{3}\right) = 0,\quad H = \lambda h$

Note that the roots of $\rho(\xi) = \xi^2 - 1 = 0$ are ± 1. Since two roots of $\rho(\xi) = 0$ are simple and of modulus 1, the method is conditionally stable.

Alternatively

Substitute $\xi = \frac{1+z}{1-z}$ in the characteristic equation and obtain

$$\left(1-\frac{H}{3}\right)(1+z)^2 - \frac{4H}{3}(1+z)(1-z) - \left(1+\frac{H}{3}\right)(1-z)^2 = 0$$

$$\left(1-\frac{H}{3}\right)(1+2z+z^2) - \frac{4H}{3}(1-z^2) - \left(1+\frac{H}{3}\right)(1-2z+z^2) = 0$$

or, $\frac{2H}{3}z^2 + 4z - \frac{2H}{3} = 0$

Comparing with $v_0 z^2 + v_1 z + v_2 = 0$, we get

$$v_0 = \frac{2H}{3},\quad v_1 = 4,\quad v_2 = -\frac{2H}{3}.$$

Since $\lambda < 0,\ H < 0$. For $H < 0$, v_1 and v_2 are always positive. However, v_0 is not positive for any $H < 0$. Hence, the method is not absolutely stable for any $H < 0$.

***

And now some exercises for you.

E12) Determine the interval of absolute stability for the method

$$y_{i+1} = y_{i-1} + \frac{h}{3}(7y'_i - 2y'_{i-1} + y'_{i-2})$$

when applied to the test equation $y' = \lambda y,\ \lambda < 0$.

E13) Show that the method

$$y_{i+1} = 8y_i + 9y_{i-1} + \frac{h}{3}(17y'_i + 14y'_{i-1} - y'_{i-2})$$

is not stable when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

E14) Determine the interval of absolute stability for the method

$$y_{i+1} = \frac{9}{8}y_i - \frac{1}{8}y_{i-2} + \frac{3h}{8}(y'_{i+1} + 2y'_i - y'_{i-1})$$

when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

E15) Show that the method

$$y_{i+1} = \frac{4}{3}y_i - \frac{1}{3}y_{i-1} + \frac{2h}{3}y'_{i+1}$$

is A-stable when applied to the test equation $y' = \lambda y$, $\lambda < 0$.

We now end the unit by giving a summary of what we have covered in it.

9.5 SUMMARY

In this unit, we have learnt the following points:

1. In **multistep** methods for solving the IVP(1)

   $$y' = f(x, y), \quad y(x_0) = y_0$$

   the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using previously, calculated k values y_{i-m} at the nodal points $x_{i-m}, m = 0, 1, \ldots, k-1$. Such a method is called an **explicit multistep method.** If the method also uses the solution value y_{i+1}, then the method is called **implicit multistep method.**
2. The multistep methods are not self starting. y_i is called the initial value and $y_{i-1}, y_{i-2}, \ldots, y_{i-k+1}$ are called the starting values. The starting values are to be calculated using some singlestep method before the multistep method can be used.
3. A general multistep method or a **k-step multistep method** can be written in the form

   $$y_{i+1} = a_1 y_i + a_2 y_{i-1} + \cdots + a_k y_{i-k+1} + h(b_0 y'_{i+1} + b_i y'_i + \cdots + b_k y'_{i-k+1})$$

   The method is explicit if $b_0 = 0$ and implicit if $b_0 \neq 0$. The unknowns a_i's and b_i's are obtained using (i) interpolation methods (ii) method of undetermined parameters.

   We have the following form of a few important class of methods

   (a) **Adams-Bashforth method:** $a_1 = 1, a_2 = a_3 = \cdots a_k = 0, b_0 = 0$. These methods are explicit methods and are of order k.

   (b) **Nystrom method:** $a_2 = 1, a_1 = a_3 = \cdots = a_k = 0, b_0 = 0$. These methods are explicit methods and are of order k.

   (c) **Adams-moulton methods:** $a_1 = 1, a_2 = a_3 = \cdots = a_k = 0, b_0 \neq 0$. These methods are implicit methods and are of order $k+1$.

   (d) **Milne-Simpson methods:** $a_2 = 1, a_1 = a_3 = \cdots = a_k = 0, b_0 \neq 0$. These methods are implicit methods and are of order $k+1$ if k is odd and of order $k+2$ if k is even.
4. An explicit method is called a **predictor** and an implicit method is called a **corrector.** We use the explicit method to predict the value $y^{(p)}_{i+1}$ for y_{i+1} and use the corrector, which uses $y^{(p)}_{i+1}$ iteratively to obtain y_{i+1}, the improved value of y_{i+1}. After the required number of corrector iterations, we take $y_{i+1} = y^{(c)}_{i+1}$.

5. If we apply the multistep method (44) to the test equation $y' = \lambda y$, we obtain the **characteristic equation**

$$\rho(\xi) - \lambda h \sigma(\xi) = 0$$

where

$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - \cdots - a_k$$

$$\sigma(\xi) = b_0 \xi^k + b_1 \xi^{k-1} + \cdots + b_k$$

Let $\xi_1, \xi_2, \ldots, \xi_k$ be the roots of the **reduced characteristic equation** $\rho(\xi) = 0$ and $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ be the roots of the characteristic equation. Then, the method is

(i) **stable:** if $|\xi_i| < 1$, $i \neq 1$ and $\xi_1 = 1$.

(ii) **unstable:** if $|\xi_i| > 1$ for some i, or if there is a multiple root of $\rho(\xi) = 0$ of modulus one.

(iii) **conditionally stable (weakly stable):** if ξ_i's are simple and if more than one of these roots have modulus one.

The method is called

absolutely stable: if $\lambda < 0$ and $|\xi_{ih}| < 1$ for all i. The set of values of λh which satisfy this condition give the **interval of absolute stability.** If this interval is $]-\infty, 0[$, the method is called **A-stable.**

relatively stable: if $\lambda > 0$ and $|\xi_{ih}| < e^{\lambda h}$ for all i. The set of values of λh which satisfy this condition give the **interval of relative stability.**

The order of a stable k-step explicit method cannot exceed k and the order of a stable k-step implicit method cannot exceed $k+1$ if k is odd and $k+2$ if k is even.

9.6 SOLUTIONS/ANSWERS

E1) $TE = -\frac{3}{80} h^5 y^{v}(x_i) + O(h^6)$; order: 4

E2) $x_0 = 0,\ y_0 = 1,\ y'_0 = -1,\ y''_0 = 3$; starting value: $y_1 \approx y(0.2) = 0.86$;

We obtain from the multistep method

$f_0 = y'_0 = -1,\ f_1 = y'_1 = -0.5396,\ y_2 \approx y(0.4) = 0.79812;$

$f_2 = y'_2 = -0.236996,\ y_3 \approx y(0.6) = 0.780981.$

E3) $x_0 = 1,\ y_0 = 0,\ y'_0 = 1,\ y''_0 = 2,\ y'''_0 = 4$; Starting value:

$y_1 \approx y(1.1) = 0.110667$

$x_1 = 1.1,\ y_1 = 0.110667,\ y'_1 = 1.222247,\ y''_1 = 2.861296,$

$y'''_1 = 5.621078$; starting value $y_2 \approx y(1.2) = 0.248135$.

From the multistep method we get

$f_0 = 1,\ f_1 = 1.222247,\ f_2 = 1.501571;\ y_3 \approx y(1.3) = 0.414637;$

$f_3 = 1.861924;\ y_4 \approx y(1.4) = 0.622223;$

$f_4 = 2.347162;\ y_5 \approx y(1.5) = 0.886405.$

E4) Starting values:

$x_0 = 1,\ y_0 = 1,\ K_1 = 0.4,\ K_2 = 0.672,\ y_1 \approx y(1.2) = 1.536;$

$x_1 = 1.2,\ y_1 = 1.536,\ K_1 = 0.779059,\ K_2 = 1.534912,$

$y_2 \approx (1.4) = 2.692985.$

From the multistep method, we get

$f_0 = 2,\ f_1 = 3.895296,\ f_2 = 9.945153,\ y_3 \approx y(1.6) = 5.791032;$

$f_3 = 39.327084,\ y \approx y(1.8) = 19.979290;$

$f_4 = 419.151320,\ y_5 \approx y(2.0) = 196.814380.$

E5) Starting value:

$x_0 = 1,\ y_0 = 2,\ K_1 = 0.8,\ K_2 = 0.93,\ K_3 = 0.9495,$
$K_4 = 1.10485,\ y_1 \approx y(1.1) = 2.943975.$

From the multistep method, we get

$$(1-h)y_{i+2} = y_i + \frac{h}{3}(2x_{i+2} + 4f_{i+1} + f_i),\ i = 0,1,2$$

We obtain

$x_0 = 1,\ y_0 = 2,\ f_0 = 8,\ x_1 = 1.1,$
$y_1 = 2.943975,\ f_1 = 11.031925,\ x_2 = 1.2$
$y_2 \approx y(1.2) = 4.241767;\ f_2 = 15.125301,\ x_3 = 1.3,$
$y_3 \approx y(1.3) = 6.016755;\ f_3 = 20.650264,\ x_4 = 1.4,$
$y_4 \approx y(1.4) = 8.436273.$

E6) $a_1 = 1,\ b_1 = \frac{7}{3},\ b_2 = -\frac{2}{3},\ b_3 = \frac{1}{3}.$

$TE = \frac{1}{3}h^4 y^{iv}(x_i) + O(h^5);$ order : 3.

E7) $a_1 = -8,\ a_2 = 9,\ b_1 = \frac{17}{3},\ b_2 = \frac{14}{3},\ b_3 = -\frac{1}{3}.$

$TE = \frac{1}{9}h^5 y^{v}(x_i) + O(h^6);$ order : 4.

E8) $a_1 = 1,\ b_0 = \frac{5}{12},\ b_1 = \frac{8}{12},\ b_2 = -\frac{1}{12}.$

$TE = -\frac{h^4}{24} y^{iv}(x_i) + O(h^5);$ order : 3

E9) $a_1 = \frac{4}{3},\ a_2 = -\frac{1}{3},\ b_0 = \frac{2}{3}.$

$TE = -\frac{1}{9}h^3 y'''(x_i) + O(h^4);$ order : 2

E10) Starting value:

$x_0 = 1,\ y_0 = 1,\ y'_0 = 1.841471,\ y''_0 = 2.381773,$
$y_1 \approx y(1.2) = 1.415930.$

From the given P-C set, we get

$f_0 = 1.841471,\ x_1 = 1.2,\ f_1 = 2.403962,\ x_2 = 1.4,$
$y_2^{(0)} = 1.952972;\ f(x_2, y_2^{(0)}) = 2.880828$
$y_2^{(1)} = 1.944409;\ f(x_2, y_2^{(1)}) = 2.875424$
$y_2^{(2)} \approx y(1.4) = y_2 = 1.943869,\ f_2 = 2.875081;\ x_3 = 1.6$
$y_3^{(0)} = 2.565997;\ f(x_3, y_3^{(0)}) = 3.110332,$
$y_3^{(1)} = 2.542410;\ f(x_3, y_3^{(1)}) = 3.106377,$
$y_3^{(2)} \approx y(1.6) = y_3 = 2.542015.$

E11) Starting values

$y_1 \approx y(0.2) = 1.2,\quad y_2 \approx y(0.4) = 1.496,$
$y_3 \approx y(0.6) = 1.9756.$

Using the P-C set, we get

$y_4^{(0)} = 3.4235,\quad y_4^{(1)} = 3.6167,\quad y_4^{(2)} = 3.7074$

Therefore, $y_4 \approx y(0.8) = 3.7074.$

$y_5^{(0)} = 9.0140, \quad y_5^{(1)} = 11.5792, \quad y_5^{(2)} = 15.1009$

Therefore, $y_5 \approx y(1.0) = 15.1009$

E12) Characteristic equation is obtained as

$$\xi^3 - \frac{7H}{3}\xi^2 - \left(1 - \frac{2H}{3}\right)\xi - \frac{H}{3} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we obtain

$$v_0 z^3 + v_1 z^2 + v_2 z + v_3 = 0$$

where

$$v_0 = \frac{10H}{3}, \; v_1 = 4 + \frac{2H}{3}, \; v_2 = 4 - 2H, \; v_3 = -2H$$

Since $v_0 < 0$ for all $H < 0$, method is not stable for any H.

E13) The roots of the reduced characteristic equation

$$\rho(\xi) = \xi^2 + 8\xi - 9 = 0$$

are $\xi_1 = 1, \xi_2 = -9$. Since $|\xi_2| > |$, the method is unstable.

E14) Characteristic equation is obtained as

$$\left(1 - \frac{3H}{8}\right)\xi^3 - \left(\frac{9}{8} + \frac{3H}{4}\right)\xi^2 + \frac{3H}{8}\xi + \frac{1}{8} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we get

$$v_0 z^3 + v_1 z^2 + v_2 z + v_3 = 0$$

where

$$v_0 = \frac{1}{4}(8+3H), \; v_1 = \frac{1}{4}(18-3H), \; v_2 = \frac{3}{4}(2-3H), \; v_3 = -\frac{3H}{4}$$

and $D = v_1 v_2 - v_0 v_3 = \frac{3}{4}[9 - 13H + 3H^2]$. Interval of absolute stability is $\left(-\frac{8}{3}, 0\right)$.

E15) The characteristic equation is obtained as

$$\left(1 - \frac{2H}{3}\right)\xi^2 - \frac{4}{3}\xi + \frac{1}{3} = 0, \quad H = \lambda h < 0.$$

Substituting $\xi = (1+z)/(1-z)$, we get

$$v_0 z^2 + v_1 z + v_2 = 0$$

where

$$v_0 = \frac{4}{3} - \frac{H}{3}, \; v_1 = \frac{2}{3} - \frac{2H}{3}, \; v_2 = -\frac{H}{3}$$

since v_0, v_1, v_2 are all positive for all $H < 0$, the method is A-stable.

—x—

PRACTICAL ASSIGNMENTS

Session 2:

1. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Euler method as predictor and Backward Euler method as corrector. Use the corrector three times. Input the step length h and the number of steps of integration.

2. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Euler method as predictor and Trapezoidal method as corrector. Use the corrector three times. Input the step length h and the number of steps of integration.

3. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Milne's predictor-corrector method. Required starting values are to be computed using the Euler method.

4. Write a program to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, using Milne's predictor-corrector method. Required starting values are to be computed using the Taylor series method of second order. Test your program on Example 9.