UNIT 8 SINGLESTEP METHODS FOR SOLVING INITIAL VALUE PROBLEMS

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8.1 INTRODUCTION

Many physical problems in science and engineering are modelled in the form of ordinary differential equations. The solution of the differential equation is the function that satisfies the differential equation and the given initial or boundary conditions. Such a solution is called the exact or the analytical solution. The methods for finding the exact solution are limited to certain special form of equations, mostly linear, homogeneous equations with constant coefficients or, to certain non-homogeneous equations with non-homogeneous terms restricted to some standard functions. However, numerical methods do not have any such limitations.

In this unit we shall discuss **singlestep** numerical methods for solving the **initial value problems** of first and higher orders. We shall start the unit with some preliminaries in Sec.8.2, which are required for further discussion in this unit. In Sec.8.3, we shall discuss Taylor series methods of first and higher order for solving single or system of initial value problems. Runge-Kutta methods of first and higher order will be discussed in Sec.8.4 both for single as well as system of equations. Finally, we shall discuss the stability of these singlestep methods in Sec.8.5.

Objectives

After studying this unit you should be able to

- explain singlestep and multistep methods for solving initial value problems;
- use Taylor series and Runge-Kutta methods for solving initial value problems;
- obtain the interval of absolute stability for a given singlestep method.

8.2 PRELIMINARIES

We start with giving some of the basic definitions in this section.
Consider the nth-order initial value problem (IVP)

$$y^{(n)}(x) = f(x, y, y', \ldots, y^{(n-1)})$$
$$y(x_0) = y_0, y'(x_0) = y'_0, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \qquad (1)$$

where x_0 is the initial point.

The nth order IVP (1) is equivalent to the following system of n first order initial value problems. We write $y = y_1$ and

$$\begin{aligned}
y' = y'_1 &= y_2 \quad &, \quad y_1(x_0) &= y_0 \\
y'_2 &= y_3 \quad &, \quad y_2(x_0) &= y'_0 \\
&\vdots & &\vdots \\
y'_{n-1} &= y_n \quad &, \quad y_{n-1}(x_0) &= y_0^{(n-2)} \\
y'_n &= f(x, y_1, y_2, \ldots, y_n) \quad &, \quad y_n(x_0) &= y_0^{(n-1)}
\end{aligned} \qquad (2)$$

In vector notation, this system can be written as

$$y'(x) = f(x, y), \quad y(x_0) = \alpha \tag{3}$$

where

$$y = [y_1', y_2', \ldots, y_n']^T,$$
$$f(x, y) = [y_2, y_3, \ldots, f(x, y_1, y_2, \ldots, y_n)]^T,$$
$$\alpha = [y_0, y_0', \ldots, y_0^{(n-1)}]^T = [y_1(x_0), y_2(x_0), \ldots, y_n(x_0)]^T.$$

Hence, it is sufficient to study numerical methods for solving the single first order IVP

$$y'(x) = f(x, y), \quad y(x_0) = y_0 \tag{4}$$

and then extend these methods for solving the system of first order IVP's (3) or, the nth order IVP(1).

We assume that the solution of the IVP(4) exists and is unique and also $f(x, y)$ has continuous partial derivatives with respect to x and y of as high order as required for all x in the interval $[x_0, b]$, in which the solution of the IVP(4) is to be obtained.

We divide the interval $[x_0, b]$ into N subintervals by the points

$$x_0 < x_1 < x_2 \ldots < x_N = b$$

The points $x_0, x_1, \ldots, x_N$ are called the **nodal point, grid points** or the **mesh-points**. We assume that the nodal points are equispaced with the spacing or the **step length**

$$h = \frac{x_N - x_0}{N}, \quad \text{where } x_N = b \tag{5}$$

We obtain

$$x_i = x_0 + ih \quad , \quad i = 1, 2, \ldots, N$$

In numerical methods, we determine y_i which is an approximate value of the exact solution y(x) at the nodal point x_i, that is

$$y_i \approx y(x_i) \tag{6}$$

The numerical methods for solving the initial value problems can be broadly classified into two categories

(i) singlestep methods, (ii) multistep methods.

In singlestep methods, the solution value y_{i+1} at the nodal point x_{i+1}, is obtained by using the solution value y_i at only the previous point x_i. Such a method is called an **explicit singlestep method** and is written as

$$y_{i+1} = y_i + h\phi(x_i, y_i, h), \quad i = 0, 1, \ldots \tag{7}$$

where h is the step size and ϕ is called the **increment function**. If the method also uses the solution value y_{i+1}, then the method is called an **implicit singlestep method.**

In multistep methods, the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using the previously calculated k-values y_{i-m} at the previous k nodal points x_{i-m}, $m = 0, 1, \ldots, (k-1)$.

If the error in the numerical method (truncation error)

$$TE = y(x_{i+1}) - y_{i+1}$$

can be written as

$$TE = C h^{p+1} y^{(p+1)}(\xi), \quad x_{i-m} < \xi < x_{i+1} \tag{8}$$

where C is a constant independent of h, then the method is said to be of **order** p. With this background we shall now discuss singlestep methods of various orders. In this unit we shall confine our discussion only to **explicit** singlestep methods.

8.3 TAYLOR SERIES METHODS

Consider the IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{9}$$

We assume that y(x) can be expanded in Taylor series about any point x_i, that is,

$$y(x) = y(x_i) + (x - x_i)\,y'(x_i) + \frac{1}{2!}(x - x_i)^2\,y''(x_i) + \cdots$$

$$+ \frac{1}{p!}(x - x_i)^p\,y^{(p)}(x_i) + \frac{1}{(p+1)!}(x - x_i)^{p+1}\,y^{(p+1)}(x_i) + \cdots \tag{10}$$

Substituting $x = x_{i+1}$ and $x_{i+1} - x_i = h$, we get

$$y(x_{i+1}) = y(x_i) + h\,y'(x_i) + \frac{h^2}{2!}y''(x_i) + \cdots$$

$$+ \frac{h^p}{p!}y^{(p)}(x_i) + \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x_i) + \cdots \tag{11}$$

which is an infinite series. If we retain terms upto pth powers of h, we obtain

$$y_{i+1} = y_i + h\,y'_i + \frac{h^2}{2!}y''_i + \cdots + \frac{h^p}{p!}y_i^{(p)} \tag{12}$$

where y_i is an approximation to $y(x_i)$.
The error term is given by

$$\text{error} = y(x_{i+1}) - y_{i+1}$$

$$= \frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x_i) + \frac{h^{p+2}}{(p+2)!}y^{(p+2)}(x_i) + \cdots$$

which is also an infinite series.
The principal error term, called the **truncation error** (TE) is obtained as

$$\text{TE} = \frac{h^{p+1}}{(p+1)}y^{(p+1)}(\xi) \quad , \quad x_i < \xi < x_{i+1}$$

Therefore, the method (12) is of order p.

The method (11) is called the Taylor's series method of order p. For different values of p, we obtain different order methods.

p = 1: first order Taylor series method

$$y_{i+1} = y_i + h\,y'_i$$

or,

$$y_{i+1} = y_i + h\,f(x_i, y_i)\,, \quad i = 0, 1, \cdots \tag{13}$$

$$\text{TE} = \frac{h^2}{2!}\,y''(\xi), \quad x_i < \xi < x_{i+1}$$

This method is also called the **Euler method**.

p = 2: second order Taylor series method

$$y_{i+1} = y_i + h\,y'_i + \frac{h^2}{2!}\,y''_i \tag{14}$$

$$\text{TE} = \frac{h^3}{3!}\,y'''(\xi)\,, \quad x_i < \xi < x_{i+1}$$

and so on.
The higher order derivatives $y_i^{(m)}, m = 1, 2, \ldots$ in Eqn.(12) can be obtained by differentiating the given differential equation repeatedly and evaluating them at the nodal point x_i.

System of first order IVP's

For the system of first order IVP's (3)

$$\mathbf{y}'(x) = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(x_0) = \boldsymbol{\alpha}$$

we write the method (12) as

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y}')_{x_i} + \frac{h^2}{2!}(\mathbf{y}'')_{x_i} + \cdots + \frac{h^p}{p!}(\mathbf{y}^{(p)})_{x_i} \tag{15}$$

where, $(\mathbf{y})_{x_i} = [y_{1i}, y_{2i}, \ldots, y_{ni}]^T$

$$(y^{(m)})_{x_i} = \left[y_{1i}^{(m)}, y_{2i}^{(m)}, \ldots, y_{ni}^{(m)}\right]^T, \quad m = 1, 2, \ldots, p$$

and $$y_{ji}^{(m)} = \frac{d^{m-1}}{dx^{m-1}} f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni}), \quad j = 1, 2, \ldots, n$$

Note that each term in Eqn.(15) is a column vector having the same number of elements as the number of first order differential equations.

The **truncation error** associated with the method (15) is given by

$$TE = \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi), \quad x_i < \xi < x_{i+1}$$

and hence the method (15) is of order p.

Example 1: **Obtain the approximate value of $y(1.4)$ for the initial value problem**

$$y' = -2xy^2, \; y(1) = 1$$

using.

(a) Taylor series method of first order (Euler method)

(b) Taylor series method of second order with (i) $h = 0.1$, (ii) $h = 0.2$. Compare the results with the exact solution $y(x) = \frac{1}{x^2}$.

Solution:

(a) Taylor series method of **first order** is given by

$$y_{i+1} = y_i + h y_i', \; i = 0, 1, \ldots$$

(i) **For h = 0.1,** we obtain for

i = 0 : $x_0 = 1, y_0 = 1, y_0' = -2x_0 y_0^2 = -2$

$y_1 \approx y(1.1) = y_0 + h y_0' = 1 + 0.1(-2) = 0.8$

i = 1 : $x_1 = 1.1, y_1 = 0.8, y_1' = -2x_1 y_1^2 = -1.408$

$y_2 \approx y(1.2) = y_1 + h y_1' = 0.8 + (0.1)(-1.408) = 0.6592$

i = 2 : $x_2 = 1.2, y_2 = 0.6592, y_2' = -2x_2 y_2^2 = -1.042907$

$y_3 \approx y(1.3) = y_2 + h y_2' = 0.544909$

i = 3 : $x_3 = 1.3, y_3 = 0.544909, y_3' = -2x_3 y_3^2 = -0.772007$

$y_4 \approx y(1.4) = y_3 + h y_3' = 0.467708$

We also have from the exact solution

$y(1.4) = 1/(1.4)^2 = 0.510204$

Actual error at x = 1.4 = $y(1.4) - y_4 = 0.0425$.

(ii) **For h = 0.2,** we obtain

i = 0 : $x_0 = 1, y_0 = 1, y_0' = -2x_0 y_0^2 = -2$

$y_1 \approx y(1.2) = y_0 + h y_0' = 1 + 0.2(-2) = 0.6$

i = 1 : $x_1 = 1.2, y_1 = 0.6, y_1' = -2x_1 y_1^2 = -0.864$

$y_2 \approx y(1.4) = y_1 + h y_1' = 0.4272$

Actual error at x = 1.4 = $y(1.4) - y_2 = 0.0830$.

(b) Taylor series method of **second order** is given by

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'', \; i = 0, 1, \ldots$$

We have,

$y' = -2xy^2, \qquad y_i' = -2x_i y_i^2$

$y'' = -2y^2 - 4xyy', \qquad y_i'' = -2y_i^2 - 4x_i y_i y_i'$.

(i) **For h = 0.1**, we obtain

$i = 0$: $x_0 = 1, y_0 = 1, y_0' = -2, y_0'' = 6$

$$y_1 \approx y(1.1) = y_0 + h y_0' + \frac{h^2}{2} y_0'' = 0.83$$

$i = 1$: $x_1 = 1.1, y_1 = 0.83, y_1' = -1.51558, y_1'' = 4.157098$

$$y_2 \approx y(1.2) = y_1 + h y_1' + \frac{h^2}{2} y_1'' = 0.699227$$

$i = 2$: $x_2 = 1.2, y_2 = 0.699227, y_2' = -1.173404, y_2'' = 2.960447$

$$y_3 \approx y(1.3) = y_2 + h y_2' + \frac{h^2}{2} y_2'' = 0.596689$$

$i = 3$: $x_3 = 1.3, y_3 = 0.596689, y_3' = -0.925698, y_3'' = 2.160163$

$$y_4 \approx y(1.4) = y_3 + h y_3' + \frac{h^2}{2} y_3'' = 0.514920$$

Actual error at $x = 1.4 = y(1.4) - y_4 = -0.0047$.

(ii) **For h = 0.2**, we obtain

$i = 0$: $x_0 = 1, y_0 = 1, y_0' = -2, y_0'' = 6$

$$y_1 \approx y_0 + h y_0' + \frac{h^2}{2} y_0'' = 0.72$$

$i = 1$: $x_1 = 1.2, y_1 = 0.72, y_1' = -1.24416, y_1'' = 3.263017$

$$y_2 \approx y(1.4) = y_1 + h y_1' + \frac{h^2}{2} y_1'' = 0.536428$$

Actual error at $x = 1.4 = y(1.4) - y_4 = -0.0262$.

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Remark: You may notice that the actual error in magnitude decreases when (i) h is reduced or (ii) higher order method is used.

Example 2: Solve the initial value problem $y' = x^2 + y^2, y(0) = 1$ upto $x = 0.4$ using fourth order Taylor series method with $h = 0.2$.

Solution: The **fourth order** Taylor series method is given by

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i''' + \frac{h^4}{24} y_i^{iv}. \qquad i = 0, 1, \ldots$$

We have ,

$$y' = x^2 + y^2 \quad , \quad y_i' = x_i^2 + y_i^2$$

$$y'' = 2x + 2y\,y' \quad , \quad y_i'' = 2x_i + 2y_i\,y_i'$$

$$y''' = 2 + 2(y')^2 + 2y\,y'' \quad , \quad y_i''' = 2 + 2(y_i')^2 + 2y_i\,y_i''$$

$$y^{iv} = 6y'\,y'' + 2y\,y''' \quad , \quad y_i^{iv} = 6y_i'\,y_i'' + 2y_i\,y_i'''$$

For h = 0.2, we obtain

$i = 0$: $x_0 = 0, y_0 = 1, y_0' = 1, y_0'' = 2, y_0''' = 8, y_0^{iv} = 28$

$$y_1 \approx y(0.2) = y_0 + h y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{6} y_0''' + \frac{h^4}{24} y_0^{iv}$$

$$= 1 + 0.2(1) + \frac{0.04}{2}(2) + \frac{0.008}{6}(8) + \frac{0.0016}{24}(28)$$

$$= 1.252533$$

$i = 1$: $x_1 = 0.2, y_1 = 1.252533, y_1' = 1.608839, y_1'' = 4.430248,$

$y_1''' = 18.274789, y_1^{iv} = 88.544887$

$$y_2 \approx y(0.4) = y_1 + h\,y_1' + \frac{h^2}{2} y_1'' + \frac{h^3}{6} y_1''' + \frac{h^4}{24} y_1^{iv}$$

$$= 1.252533 + 0.2(1.608839) + \frac{0.04}{2}(4.430248)$$

$$+ \frac{0.008}{6}(18.274789) + \frac{0.0016}{24}(88.544887)$$

$$= 1.693175$$

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Example 3: Solve the system of equations

$$y_1' = -3y_1 + 2y_2 \quad , \quad y_1(0) = 0$$

$$y_2' = 3y_1 - 4y_2 \quad , \quad y_2(0) = 1/2$$

on the interval [0, 0.4] using Taylor series second order method with h = 0.2.

Solution: The **second order** Taylor series method is given by

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y}')_{x_i} + \frac{h^2}{2}(\mathbf{y}'')_{x_i}, \qquad i = 0, 1, \ldots$$

We have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix},$$

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}$$

For h = 0.2, we obtain

$$\mathbf{i = 0}: \quad x_0 = 0, \; (\mathbf{y})_{x_0} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix},$$

$$(\mathbf{y}')_{x_0} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} -3(0) + 2(1/2) \\ 3(0) - 4(1/2) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_0} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}_{x_0} = \begin{bmatrix} -3(1) + 2(-2) \\ 3(1) - 4(-2) \end{bmatrix} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}$$

$$(\mathbf{y})_{x_1} \approx \mathbf{y}(0.2) = (\mathbf{y})_{x_0} + h(\mathbf{y}')_{x_0} + \frac{h^2}{2}(\mathbf{y}'')_{x_0}$$

$$= \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + 0.2\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{(0.2)^2}{2}\begin{bmatrix} -7 \\ 11 \end{bmatrix} = \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix}$$

$$\mathbf{i = 1}: \quad x_1 = 0.2, (\mathbf{y})_{x_1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix}$$

$$(\mathbf{y}')_{x_1} = \begin{bmatrix} -3y_1 + 2y_2 \\ 3y_1 - 4y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} -3(0.06) + 2(0.32) \\ 3(0.06) - 4(0.32) \end{bmatrix} = \begin{bmatrix} 0.46 \\ -1.10 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_1} = \begin{bmatrix} -3y_1' + 2y_2' \\ 3y_1' - 4y_2' \end{bmatrix}_{x_1} = \begin{bmatrix} -3(0.46) + 2(-1.10) \\ 3(0.46) - 4(-1.10) \end{bmatrix} = \begin{bmatrix} -3.58 \\ 5.78 \end{bmatrix}$$

$$(\mathbf{y})_{x_2} = \mathbf{y}(0.4) = (\mathbf{y})_{x_1} + (\mathbf{y}')_{x_1} + \frac{h^2}{2}(\mathbf{y}'')_{x_1}$$

$$= \begin{bmatrix} 0.06 \\ 0.32 \end{bmatrix} + 0.2\begin{bmatrix} 0.46 \\ -1.10 \end{bmatrix} + \frac{0.04}{2}\begin{bmatrix} -3.58 \\ 5.78 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0804 \\ 0.2156 \end{bmatrix}$$

Hence, $y_1(0.4) \approx 0.0804$, $y_2(0.4) \approx 0.2156$.

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Example 4: Reduce the second order initial value problem

$$y'' = y' + 1$$
$$y(0) = 1,\ y'(0) = 1$$

to a system of first order initial value problems. Hence, find an approximate value of $y(0.4)$ and $y'(0.4)$ using Taylor series second order method with $h = 0.2$.

Solution: Set $y = y_1$, we get

$$y_1' = y_2 \quad , \quad y_1(0) = 1$$
$$y_2' = 1 + y_2 \ , \quad y_2(0) = 1$$

we have

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}$$

$$\mathbf{y}'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} y_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}$$

The **second order** Taylor series method is given by

$$(\mathbf{y})_{x_{i+1}} = (\mathbf{y})_{x_i} + h(\mathbf{y})_{x_i} + \frac{h^2}{2}(\mathbf{y}'')_{x_i}, \quad i = 0, 1, \ldots$$

For h = 0.2, we get

$\mathbf{i = 0:}$

$$x_0 = 0, (\mathbf{y})_{x_0} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, (\mathbf{y}')_{x_0} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(\mathbf{y}'')_{x_0} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}_{x_0} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$(\mathbf{y})_{x_1} \approx \mathbf{y}(0.2) = (\mathbf{y})_{x_0} + h(\mathbf{y}')_{x_0} + \frac{h^2}{2}(\mathbf{y}'')_{x_0}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{0.04}{2}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix}$$

$\mathbf{i = 1:}$

$$x_1 = 0.2, (\mathbf{y})_{x_1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix}$$

$$(\mathbf{y}')_{x_1} = \begin{bmatrix} y_2 \\ 1 + y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 1.44 \\ 2.44 \end{bmatrix}, (\mathbf{y}'')_{x_1} = \begin{bmatrix} 1 + y_2 \\ 1 + y_2 \end{bmatrix}_{x_1} = \begin{bmatrix} 2.44 \\ 2.44 \end{bmatrix}$$

$$(\mathbf{y})_{x_2} \approx \mathbf{y}(0.4) = (\mathbf{y})_{x_1} + h(\mathbf{y}')_{x_1} + \frac{h^2}{2}(\mathbf{y}'')_{x_1}$$

$$= \begin{bmatrix} 1.24 \\ 1.44 \end{bmatrix} + 0.2\begin{bmatrix} 1.44 \\ 2.44 \end{bmatrix} + \frac{0.04}{2}\begin{bmatrix} 2.44 \\ 2.44 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5768 \\ 1.9768 \end{bmatrix}$$

Hence, $y(0.4) \approx 1.5768$ and $y'(0.4) \approx 1.9768$.

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You may now try the following exercises.

E1) Obtain the approximate value $y(1.2)$ for the initial value problem

$$y' = -y^2, \quad y(1) = 1$$

using Euler method with $h = 0.1$.

E2) Solve the initial value problem

$$y' = 2x + 3y, \ y(0) = 1$$

in the interval [0, 0.4] using (a) Taylor series second order method, (b) Taylor series fourth order method. Take h = 0.2 in each case.

E3) Solve the initial value problem

$$y' = x + \sin y,\ y(1) = 1$$

in the interval [1, 1.2] using (a) Taylor series second order method, (b) Taylor series fourth order method. Take h = 0.1 in each case.

E4) Find the approximate value of y (0.4) and u (0.4) for the system of equations

$$y' = 2y + u \ ,\ y(0) = 1$$
$$u' = 3y + 4u \ ,\ u(0) = 1$$

using Taylor series second order method with h = 0.2

E5) Write the second order initial value problem

$$y'' = -2yy',\ y(0) = 1,\ y'(1) = 1$$

as a system of two first order initial value problems. Hence, obtain approximate value of $y(1.4)$ and $y'(1.4)$ using second order Taylor series method with $h = 0.2$.

Taylor series method of any arbitrary order for solving the initial value problem (4) can be easily obtained. However, from application point of view, Taylor series method has a serious disadvantage. The method requires evaluation of derivatives of higher order for the function $f(x, y)$ of two variables x and y. These higher order derivatives have to be obtained manually for each problem. Therefore, we need to develop methods, which do not require the evaluation of higher order derivatives and still compare with the Taylor series methods. One such class of methods is the Runge-Kutta methods which we shall discuss in the next section.

8.4 RUNGE-KUTTA METHODS

To illustrate the idea behind Runge-Kutta methods, we integrate $y' = f(x, y)$ in the interval $[x_i, x_{i+1}]$ and obtain

$$\int_{x_i}^{x_{i+1}} \left(\frac{dy}{dx}\right) dx = \int_{x_i}^{x_{i+1}} f(x, y)\, dx$$

or, $$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y)\, dx \tag{16}$$

We note that $f(x, y)$ is the slope of the solution curve and it changes continuously in $[x_i, x_{i+1}]$. Let the continuously varying slope $f(x, y)$ in this interval be approximated by the fixed slope at $x = x_i$, that is, $f(x, y) \approx f(x_i, y_i)$ in $[x_i, x_{i+1}]$. Then from Eqn.(16), we obtain

$$y_{i+1} = y_i + f(x_i, y_i) \int_{x_i}^{x_{i+1}} dx$$

or, $$y_{i+1} = y_i + h f(x_i, y_i) \tag{17}$$

which is the **Euler method** (13).

Now, approximate $f(x, y)$ in $[x_i, x_{i+1}]$ by the slope at the point $x = x_{i+1}$, that is, $f(x, y) \approx f(x_{i+1}, y_{i+1})$.

Then, we obtain from Eqn.(16)

$$y_{i+1} = y_i + f(x_{i+1}, y_{i+1}) \int_{x_i}^{x_{i+1}} dx$$

or, $$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) \tag{18}$$

which is the **backward Euler method**. This method is an implicit method. The non-linear equation in y_{i+1} for non-linear problems can be solved using the Newton-Raphson method.

Now, approximate $f(x,y)$ in $[x_i, x_{i+1}]$ by the mean of the slopes at $x = x_i$ and $x = x_{i+1}$, that is, $f(x,y) = \frac{1}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$. Then we obtain the implicit method.

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \tag{19}$$

which is called the **trapezoidal method**.

If we approximate y_{i+1} in the right side of Eqn.(19) by the Euler method (17), then we obtain the explicit method

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_i + h, y_i + h f_i)]$$

where $f_i = f(x_i, y_i)$. This method is called the **Heun's method** or **Euler Cauchy method**. We can re-write this method in the form

$$\begin{aligned} y_{i+1} &= y_i + \frac{1}{2}(K_1 + K_2) \\ \text{where, } K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + h, y_i + K_1) \end{aligned} \tag{20}$$

The philosophy behind the Runge-Kutta methods is to consider a weighted average of slopes or approximate slopes at a number of points in $[x_i, x_{i+1}]$. If we use ν slopes, then the method is written as

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2 + \cdots + w_\nu K_\nu, \tag{21}$$

where,

$$\begin{aligned} K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) \\ &\vdots \\ K_\nu &= h f\left(x_i + \alpha_\nu h, y_i + \sum_{j=1}^{\nu-1} \beta_{\nu j} K_j\right) \end{aligned}$$

w_i's, α_i's and β_{ij}'s are parameters to be determined such that the method (21) agrees with the Taylor series method of certain order.

We shall only discuss the cases $\nu = 2$ and $\nu = 4$ which provide the methods of orders 2 and 4 respectively.

Second order Runge-Kutta method (two stage method)

Consider the following two-stage method

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2 \tag{22}$$

where

$$\begin{aligned} K_1 &= h f(x_i, y_i) \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) \end{aligned}$$

Expanding K_1 and K_2 in Taylor series about the point x_i, we get

$$\begin{aligned} K_1 &= h f(x_i, y_i) = h f_i \\ K_2 &= h f(x_i + \alpha_2 h, y_i + \beta_{21} h f_i) \\ &= h\left[f_i + \left(\alpha_2 h \frac{\partial f}{\partial x} + \beta_{21} h f_i \frac{\partial f}{\partial y}\right) \right. \\ &\quad \left. + \frac{1}{2}\left(\alpha_2^2 h^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2 \beta_{21} h^2 f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 h^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots \right] \end{aligned}$$

$$= h f_i + h^2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right)$$

$$+ \frac{h^3}{2}\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

where the partial derivatives are evaluated at the point (x_i, y_i). Substituting the expressions for K_1 and K_2 in Eqn.(20), we get

$$y_{i+1} = y_i + w_1 h f_i + w_2\left[h f_i + h^2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right)\right.$$

$$\left. + \frac{h^3}{2}\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_i \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots\right] \qquad (23)$$

$$= y_i + (w_1 + w_2) h f_i + h^2 w_2\left(\alpha_2 \frac{\partial f}{\partial x} + \beta_{21} f_i \frac{\partial f}{\partial y}\right)$$

$$+ \frac{1}{2} h^3 w_2\left(\alpha_2^2 \frac{\partial^2 f}{\partial x^2} + 2\alpha_2\beta_{21} f_1 \frac{\partial^2 f}{\partial x \partial y} + \beta_{21}^2 f_i^2 \frac{\partial^2 f}{\partial y^2}\right) + \cdots$$

From the given differential equation, we get

$$y' = f(x, y)$$

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}$$

$$y''' = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dx}\right) + f\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 f}{\partial x \partial y}\right) + f\left(\frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}\right)$$

$$= \left(\frac{\partial^2 f}{\partial x^2} + 2f \frac{\partial^2 f}{\partial x \partial y} + f^2 \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}\right)$$

Now, Taylor series method for solving the given differential equation is given by

$$y(x_{i+1}) = y(x_i) + h\, y'(x_i) + \frac{h^2}{2!} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \cdots$$

$$= y_i + h f_i + \frac{h^2}{2}\left(\frac{\partial f}{\partial x} + f_i \frac{\partial f}{\partial y}\right)$$

$$+ \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial x^2} + 2f_i \frac{\partial^2 f}{\partial x \partial y} + f_i^2 \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f_i \frac{\partial f}{\partial y}\right)\right] + \cdots \qquad (24)$$

where partial derivatives are evaluated at the point (x_i, y_i).

Comparing the coefficients of h and h^2 in Eqns.(23) and (24) we obtain

$$w_1 + w_2 = 1$$

$$w_2 \alpha_2 = \frac{1}{2}$$

$$w_2 \beta_{21} = \frac{1}{2} \qquad (25)$$

It is not possible to compare the coefficient of h^3, as there are three terms in Eqn.(23) and five terms in Eqn.(24).

The solution of the system of Eqn.(25) is obtained as

$$\beta_{21} = \alpha_2, w_2 = \frac{1}{2\alpha_2}, w_1 = 1 - \frac{1}{2\alpha_2}$$

and $\alpha_2 \neq 0$ is arbitrary. The method (22) becomes

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha_2}\right)K_1 + \frac{1}{2\alpha_2}K_2 \qquad (26)$$

where

$$K_1 = h f(x_i, y_i)$$
$$K_2 = h f(x_i + \alpha_2 h, y_i + \alpha_2 K_1)$$

The truncation error associated with the method is given by

$$TE = y(x_{i+1}) - y_{i+1}$$
$$= h^3\left[\left(\frac{1}{6} - \frac{1}{2}w_2\alpha_2^2\right)\left(\frac{\partial^2 f}{\partial x^2} + 2f_i\frac{\partial^2 f}{\partial x \partial y} + f_i^2\frac{\partial^2 f}{\partial y^2}\right) + \frac{1}{6}\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + f_i\frac{\partial f}{\partial y}\right)\right] + O(h^4)$$

Therefore, the two stage Runge-Kutta method (26) is a second order method for all values of α_2. For different values of α_2 we obtain different second order methods.

For $\alpha_2 = 1$, we get, $\beta_{21} = 1, w_1 = w_2 = \frac{1}{2}$ and the method is

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2) \qquad (27)$$
$$K_1 = h f(x_i, y_i)$$
$$K_2 = h f(x_i + h, y_i + K_1)$$

which is the **Euler-Cauchy method** or the **Heun method** (18)

For $\alpha_2 = \frac{1}{2}$, we get $\beta_{21} = \frac{1}{2}, w_1 = 0, w_2 = 1$ and the method is

$$y_{i+1} = y_i + K_2 \qquad (28)$$
$$K_1 = h f(x_i, y_i)$$
$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right)$$

which is also called the **modified Euler-Cauchy method.**

Fourth order Runge-Kutta method (Four-stage method)

Without derivation, we list one of the four stage Runge-Kutta methods

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad (29)$$
$$K_1 = h f(x_i, y_i)$$
$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right)$$
$$K_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\right)$$
$$K_4 = h f(x_i + h, y_i + K_3)$$

which is called the **classical Runge-Kutta method** and is of fourth order.

Remarks

1. Both Taylor series method and Runge-Kutta methods are singlestep methods.
2. We require two function evaluations per step for a second order Runge-Kutta method and four function evaluations per step for a fourth order Runge-Kutta method.

System of first order IVP's

For the system of n first order IVP's (3)

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \mathbf{y}(x_0) = \boldsymbol{\alpha}$$

we write the Runge-Kutta methods as follows:

Second order method (27)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \tag{30}$$

where

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}$$

and

$$K_{j1} = h f_j(x_i, y_{1i}, y_{2i}, \ldots, y_{ni})$$
$$K_{j2} = h f_j(x_i + h, y_{1i} + K_{11}, y_{2i} + K_{21}, \ldots, y_{ni} + K_{n1})$$
$$j = 1, 2, \ldots, n.$$

Fourth order method (29)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + 2\mathbf{K}_4) \tag{31}$$

where, $\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix}, \mathbf{K}_4 = \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}$

and

$$K_{j1} = h f_j(x_1, y_{1i}, y_{2i}, \ldots, y_{ni})$$
$$K_{j2} = h f_j\left(x_i + \frac{h}{2}, y_{1i} + \frac{1}{2}K_{11}, y_{2i} + \frac{1}{2}K_{21}, \ldots, y_{ni} + \frac{1}{2}K_{n1}\right)$$
$$K_{j3} = h f_j\left(x_i + \frac{h}{2}, y_{1i} + \frac{1}{2}K_{12}, y_{2i} + \frac{1}{2}K_{22}, \ldots, y_{ni} + \frac{1}{2}K_{n2}\right)$$
$$K_{j4} = h f_j(x_i + h, y_{1i} + K_{13}, y_{2i} + K_{23}, \ldots, y_{ni} + K_{n3})$$
$$j = 1, 2, \ldots, n.$$

We shall now illustrate the methods discussed above through examples.

Example 5: Solve the initial value problem

$$y' = x^2 + y^2, y(1) = 2$$

on the interval [1, 1.4] using the second order Runge-Kutta method (27) with h = 0.2.

Solution: We are given that $f(x, y) = x^2 + y^2$ and $h = 0.2$.

We obtain from Eqn.(27) for

$i = 0$: $x_0 = 1, y_0 = 2$

$$K_1 = h f(x_0, y_0) = h f(1, 2) = 0.2[(1)^2 + (2)^2] = 1$$
$$K_2 = h f(x_0 + h, y_0 + K_1) = h f(1.2, 3) = 0.2[(1.2)^2 + 3^2] = 2.088$$
$$y_1 \approx y(1.2) = y_0 + \frac{1}{2}(K_1 + K_2) = 3.544$$

$i = 1$: $x_1 = 1.2, y_1 = 3.544$

$$K_1 = h f(x_1, y_1) = h f(1.2, 3.544)$$
$$= 0.2[(1.2)^2 + (3.544)^2] = 2.799987$$
$$K_2 = h f(x_1 + h, y_1 + K_1) = h f(1.4, 6.343987)$$
$$= 0.2[(1.4)^2 + (6.343987)^2] = 8.441235$$
$$y_2 \approx y(1.4) = y_1 + \frac{1}{2}(K_1 + K_2) = 9.164611.$$

***

Example 6: Find an approximate value of y(0.8) for the initial value problem.

$$y' = \sqrt{x+y}, \quad y(0.4) = 0.41$$

using the second order Runge-Kutta method (26) with h = 0.2.

Solution: We are given that $f(x,y) = \sqrt{x+y}$ and h = 0.2.
We obtain from Eqn.(26), for

i = 0: $x_0 = 0.4, \; y_0 = 0.41$

$$K_1 = hf(x_0, y_0) = hf(0.4, 0.41) = 0.2\sqrt{0.4+0.41} = 0.18$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}K_1\right) = hf(0.5, 0.5) = 0.2\sqrt{0.5+0.5} = 0.2$$

$$y_1 \approx y(0.6) = y_0 + K_2 = 0.41 + 0.2 = 0.61$$

i = 1: $x_1 = 0.6, \; y_1 = 0.61$

$$K_1 = hf(x_1, y_1) = hf(0.6, 0.61) = 0.2\sqrt{0.6+0.61} = 0.22$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_1\right) = hf(0.7, 0.72)$$

$$= 0.2\sqrt{0.7+0.72} \qquad = 0.2383$$

$$y_2 \approx y(0.8) = y_1 + K_2 = 0.61 + 0.2383 = 0.8483$$

***

Example 7: Find an approximate value of y(0.2) for the initial value problem

$$y' = 1 + y^2, \quad y(0) = 1$$

using the classical Runge-Kutta fourth order method (29) with h = 0.1.

Solution: We are given that $f(x,y) = 1 + y^2$ and $h = 0.1$.
We obtain from Eqn.(29) for

i = 0: $x_0 = 0, y_0 = 1$

$$K_1 = hf(x_0, y_0) = hf(0,1) = 0.1[1 + (1)^2] = 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = hf(0.05, 1.1) = 0.1[1 + (1.1)^2] = 0.221$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = hf(0.05, 1.1105)$$

$$= 0.1[1 + (1.1105)^2] = 0.223321$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = hf(0.1, 1.223321)$$

$$= 0.1[1 + (1.223321)^2] = 0.249651$$

$$y_1 \approx y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.223049$$

i = 1: $x_1 = 0.1, \; y_1 = 1.223049$

$$K_1 = hf(x_1, y_1) = hf(0.1, 1.223049) = 0.1[1 + (1.223049)^2] = 0.249585$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = hf(0.15, 1.347841)$$

$$= 0.1[1 + (1.347841)^2] = 0.281668$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}K_2\right) = hf(0.15, 1.363883)$$

$$= 0.1[1 + (1.363883)^2] = 0.286018$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = hf(0.2, 1.509067)$$
$$= 0.1[1 + (1.509067)^2] = 0.327728$$
$$y_2 \approx y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1.508497$$

***

Example 8: Solve the system of IVP's

$$y' = 2y + z, \; y(0) = 1$$
$$z' = 3y + 4z, \; z(0) = 1$$

on the interval [0, 0.4] using second order Runge-Kutta method (30) with $h = 0.2$.

Solution: We are given that $f_1(x, y, z) = 2y + z$, $f_2(x, y, z) = 3y + 4z$ and $h = 0.2$. We obtain from Eqn.(30), for

i = 0: $x_0 = 0, y_0 = 1, z_0 = 1$

$$K_{11} = hf_1(x_0, y_0, z_0) = hf_1(0, 1, 1) = 0.6$$
$$K_{21} = hf_2(x_0, y_0, z_0) = hf_2(0, 1, 1) = 1.4$$
$$K_{12} = hf_1(x_0 + h, y_0 + K_{11}, z_0 + K_{21}) = hf_1(0.2, 1.6, 2.4) = 1.12$$
$$K_{22} = hf_2(x_0 + h, y_0 + K_{11}, z_0 + K_{21}) = hf_2(0.2, 1.6, 2.4) = 2.88$$
$$y_1 \approx y(0.2) = y_0 + \frac{1}{2}[K_{11} + K_{12}] = 1 + \frac{1}{2}[0.6 + 1.12] = 1.86$$
$$z_1 \approx z(0.2) = z_0 + \frac{1}{2}[K_{21} + K_{22}] = 1 + \frac{1}{2}[1.4 + 2.88] = 3.14$$

i = 1: $x_1 = 0.2, y_1 = 1.86, z_1 = 3.14$

$$K_{11} = hf_1(x_1, y_1, z_1) = hf_1(0.2, 1.86, 3.14) = 1.372$$
$$K_{21} = hf_2(x_1, y_1, z_1) = hf_2(0.2, 1.86, 3.14) = 3.628$$
$$K_{12} = hf_1(x_1 + h, y_1 + K_{11}, z_1 + K_{21}) = hf_1(0.4, 3.232, 6.768) = 2.6464$$
$$K_{22} = hf_2(x_1 + h, y_1 + K_{11}, z_1 + K_{21}) = hf(0.4, 3.232, 6.768) = 7.3536$$
$$y_2 \approx y(0.4) = y_1 + \frac{1}{2}[K_{11} + K_{12}] = 1.86 + \frac{1}{2}[1.372 + 2.6464] = 3.8692$$
$$z_2 \approx z(0.4) = z_1 + \frac{1}{2}[K_{21} + K_{22}] = 3.14 + \frac{1}{2}[3.628 + 7.3536] = 8.6308$$

* * *

And now some exercises for you.

E6) Solve the IVP

$$y' = x + y^2, \; y(0) = 1$$

on the interval [0, 0.4] using Runge-Kutta second order method (27) with $h = 0.2$.

E7) Find an approximate value of y(1.2) for the initial value problem

$$y' = \frac{y - x}{y + x}, \qquad y(1) = 2$$

using Runge-Kutta second order method (28) with $h = 0.1$.

E8) Find an approximate value of y(2.4) for the initial value problem

$$y' = x(y - x), \; y(2) = 3$$

using classical fourth order Runge-Kutta method with $h = 0.2$.

E9) Solve the initial value problem

$$y' = x + \cos y, \; y(1) = 1$$

in the interval [1, 1.2] using classical fourth order Runge-Kutta method with $h = 0.1$.

E10) Solve the system of IVP's

$$y' = xz + 1, \; y(0) = 0$$
$$z' = xy \;, \quad z(0) = 1$$

on the interval [0, 0.4] using second order Runge-Kutta method (30) with $h = 0.2$.

E11) Write the second order IVP $y'' = \frac{3}{2}y^2, y(0) = 1, y'(0) = 1$, as a system of two first order IVP's. Obtain approximate value of $y(0.4)$ and $y'(0.4)$ by solving this system using Rung-Kutta fourth order method (31) with h = 0.2.

We shall now find the interval of stability for various methods discussed above.

8.5 STATBILITY OF SINGLE STEP METHODS

The numerical solution y_i is an approximation of $y(x_i)$ and contains errors (both truncation error and round off error). The error

$$\epsilon_i = y(x_i) - y_i$$

is called the **numerical error** at the nodal point x_i. If the error remains bounded as $i \to \infty$ (or $h \to 0$), then the numerical method is said to be **stable**.

We discuss the stability of the single step methods by applying the method to the **test equation.**

$$y' = \lambda y, \quad y(x_0) = y_0 \tag{32}$$

where λ is a positive or a negative constant. The exact solution of Eqn.(30) is obtained as

$$y(x) = y(x_0)e^{\lambda(x-x_0)}$$

Substituting $x = x_i$ and $x = x_{i+1}$, we get

$$y(x_i) = y(x_0)e^{\lambda(x_i - x_0)} = y(x_0)e^{\lambda ih}$$
$$y(x_{i+1}) = y(x_0)e^{\lambda(x_{i+1} - x_0)} = y(x_0)e^{\lambda(i+1)h}$$

Dividing the above two equations, we get

$$\frac{y(x_{i+1})}{y(x_i)} = \frac{y(x_0)e^{\lambda(i+1)h}}{y(x_0)e^{\lambda ih}} = e^{\lambda h}$$

Hence, we obtain

$$y(x_{i+1}) = e^{\lambda h}\, y(x_i) \tag{33}$$

and h is the step size.

The exact solution of Eqn.(32) grows (when $\lambda > 0$) or decays (when $\lambda < 0$) by the **growth-factor** $e^{\lambda h}$.

When we apply a numerical method to solve Eqn.(32), we obtain a relation of the form

$$y_{i+1} = E(\lambda h)\, y_i \tag{34}$$

where $E(\lambda h)$ is some approximation to $e^{\lambda h}$. The numerical solution grows or decays by the growth factor $E(\lambda h)$. The growth factor $E(\lambda h)$ should behave similar to $e^{\lambda h}$ for meaningful results.

Substituting $y_i = y(x_i) + \epsilon_i$ and $y_{i+1} = y(x_{i+1}) + \epsilon_{i+1}$, we get

$$y(x_{i+1}) + \epsilon_{i+1} = E(\lambda h)[y(x_i) + \epsilon_i]$$

Since $y(x_{i+1}) = y(x_i)\, e^{\lambda h}$, we get

$$y(x_i)\, e^{\lambda h} + \epsilon_{i+1} = E(\lambda h)\,[y(x_i) + \epsilon_i]$$

or,

$$\epsilon_{i+1} = \left[E(\lambda h) - e^{\lambda h}\right] y(x_i) + E(\lambda h)\epsilon_i \qquad (35)$$

Thus the error ϵ_{i+1} at the nodal point x_{i+1} consists of two parts.

The first part $E(\lambda h) - e^{\lambda h}$ is the **local truncation error** and can be made as small as we like by increasing the order of the numerical method or by reducing the step size h. We find that this error $\to 0$ as $h \to 0$.

The second part $E(\lambda h)\epsilon_i$ is called the **propagation error**. We consider the following two cases

Case 1: $\lambda < 0$

In this case the exact solution decreases. For meaningful numerical results, the growth factor $E(\lambda h)$ of the numerical solution should decay at least as fast as $e^{\lambda h}$, the growth factor of the exact solution. Since $e^{\lambda h} < 1$ for $\lambda < 0$, we obtain the condition

$$|E(\lambda h)| < 1, \ \lambda < 0 \qquad (36)$$

and the method in this case is called **absolutely stable**. In this case $y_i \to 0$ as $i \to \infty$. The set of values of λh which satisfy Eqn.(36) give the **interval of absolute stability** $(\lambda h, 0)$ for the given method. If the interval of absolute stability is $]-\infty, 0[$, then the method is called **A-stable**.

Case 2: $\lambda > 0$

In this case the exact solution increases. For meaningful numerical results, the growth factor $E(\lambda h)$ of the numerical solution should not increase faster than the growth factor $e^{\lambda h}$ of the exact solution. Therefore, we obtain the condition

$$E(\lambda h) < e^{\lambda h}, \ \lambda > 0 \qquad (37)$$

and the method is called **relatively stable.**

The set of the values of λh which satisfy Eqn.(37) give the **interval of relative stability** $(0, \lambda h)$.

Remarks

1) If $\lambda < 0$, we consider absolute stability and if $\lambda > 0$, we consider relative stability
2) The test Eqn.(32) is the linearised form of $y' = f(x, y)$.
3) For the general initial value problem $y' = f(x, y), y(x_0) = y_0$ we generally take $\lambda = (\partial f / \partial y)_{x_0}$.

Now, we obtain the intervals of absolute stability for the Taylor series and Runge-Kutta methods discussed earlier.

Taylor series method

For the IVP

$$y' = \lambda y, \ y(x_0) = y_0 \qquad (38)$$

we have

$$y'' = \lambda y' = \lambda^2 y$$
$$y''' = \lambda y'' = \lambda^3 y$$
$$\vdots$$
$$y^{(p)} = \lambda y^{(p-1)} = \lambda^p y$$

Applying the p th order Taylor series method

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \cdots + \frac{h^p}{p!}y_i^{(p)}$$

to the test equation (36), we get

$$y_{i+1} = y_i + \lambda h y_i + \frac{\lambda^2 h^2}{2!} y_i + \cdots + \frac{\lambda^p h^p}{p!} y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^p h^p}{p!}\right] y_i$$

$$= E(\lambda h)\ y_i$$

where

$$E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^p h^p}{p!} \qquad (39)$$

For $\lambda > 0$, $E(\lambda h) < e^{\lambda h}$ for all λh and therefore, the Taylor series method is always relatively stable.

For $\lambda < 0$, we obtain the intervals of absolute stability for different values of p.

p = 1: $\quad | E(\lambda h) | < | 1 + \lambda h | < 1$

We get,

$$-1 < 1 + \lambda h < 1 \text{ or, } -2 < \lambda h < 0$$

Hence, the interval of absolute stability is $]-2,\ 0[$

p = 2: $\quad | E(\lambda h) | = \left| 1 + \lambda h + \frac{\lambda^2 h^2}{2} \right| < 1$

We get,

$$-1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < 1 \text{ or, } -1 < \frac{1}{2}\left[(1+\lambda h)^2 + 1\right] < 1$$

or, $\quad -2 < (1+\lambda h)^2 + 1 < 2$

The left inequality is always satisfied. From the right inequality, we get

$$(1+\lambda h)^2 + 1 < 2 \text{ or, } (1+\lambda h)^2 < 1$$

or, $\quad |1+\lambda h| < 1$ or, $-1 < 1 + \lambda h < 1$

or, $\quad -2 < 1 + \lambda h < 0$

Hence, the interval of absolute stability is $]-2,\ 0[$

p = 3: $\quad |E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right| < 1$

It can be verified that $\lambda h \in (-2.5,\ 0)$

p = 4: $\quad |E(\lambda h)| = \left|1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right| < 1$

It can be verified that $\lambda h \in]-2.78,\ 0[$

RUNGE-KUTTA METHODS

Second order method

Applying the method (22) to the test equation

$$y' = \lambda y,\ y(x_0) = y_0$$

We get,

$$K_1 = h f(x_i, y_i) = \lambda h\ y_i$$

$$K_2 = h f(x_i + \alpha_2 h, y_i + \beta_{21} K_1) = \lambda h\ (y_i + \beta_{21} K_1)$$

$$= \lambda h\ (y_i + \beta_{21} \lambda^2 h^2\ y_i)$$

$$= (\lambda h + \beta_{21} \lambda^2 h^2)\ y_i$$

and

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

$$= y_i + w_1 \lambda h\, y_i + w_2 \left[\lambda h + \beta_{21} \lambda^2 h^2\right] y_i$$

$$= \left[1 + (w_1 + w_2)\lambda h + w_2 \beta_{21} \lambda^2 h^2\right] y_i$$

Using the conditions $w_1 + w_2 = 1$ and $w_2 \beta_{21} = \frac{1}{2}$ (see Eqn.(25)), we get

$$y_{i+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_i = E(\lambda h)\, y_i$$

where $E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2}$

For $\lambda > 0$, $E(\lambda h) < e^{\lambda h}$ for all λh. Therefore, this method is relatively stable for all λh.

For $\lambda < 0$, $|E(\lambda h)| < 1$ gives $\lambda h \in]-2, 0[$. Hence, the interval of absolute stability for second order Runge-Kutta method is $]-2, 0[$.

Remark: The interval of absolute stability of all second order Runge-Kutta methods is $]-2, 0[$.

Fourth order method

Applying the method (29) to the test equation

$$y' = \lambda y, \; y(x_0) = y_0$$

we get,

$$K_1 = h f(x_i, y_i) = \lambda h\, y_i$$

$$K_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right) = \lambda h\left[y_i + \frac{1}{2}K_1\right]$$

$$= \lambda h\left[y_i + \frac{1}{2}\lambda h\, y_i\right] = \left[\lambda h + \frac{\lambda^2 h^2}{2}\right] y_i$$

$$K_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\right) = \lambda h\left[y_i + \frac{1}{2}K_2\right]$$

$$= \lambda h\left[y_i + \frac{1}{2}\left(\lambda h + \frac{\lambda^2 h^2}{2}\right) y_i\right] = \left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right) y_i$$

$$K_4 = h f(x_i + h, y_i + K_3) = \lambda h (y_i + K_3)$$

$$= \lambda h\left[y_i + \left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right) y_i\right]$$

$$= \left(\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{2} + \frac{\lambda^3 h^3}{4}\right) y_i$$

and

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \left[1 + \frac{1}{6}(\lambda h) + \frac{2}{6}\left(\lambda h + \frac{\lambda^2 h^2}{2}\right) + \frac{2}{6}\left(\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right)\right.$$

$$\left. + \frac{1}{6}\left(\lambda h + \lambda^2 h^2 + \frac{\lambda^3 h^3}{2} + \frac{\lambda^3 h^3}{4}\right)\right] y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right] y_i$$

$$= E(\lambda h)\, y_i$$

where $E(\lambda h) = 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}$

Since $E(\lambda h) < e^{\lambda h}$, $\lambda > 0$, this method is relatively stable for all λh.

For $\lambda < 0$, $|E(\lambda h)| < 1$ gives the interval of stability as $]-2.78, 0[$.

Example 9: Find the interval of absolute stability for the second order Runge-Kutta method (27).

Solution: Applying the method (27) to the test equation

$$y' = \lambda y,\ y(x_0) = y_0, \quad \lambda < 0$$

we get

$$K_1 = hf(x_i, y_i) = \lambda h\ y_i$$

$$K_2 = hf(x_i + h,\ y_i + K_1) = \lambda h\ (y_i + K_1)$$

$$= \lambda h (y_i + \lambda h\ y_i)$$

$$= (\lambda h + \lambda^2 h^2)\ y_i$$

and

$$y_{i+1} = y_i + \frac{1}{2}[K_1 + K_2]$$

$$= y_i + \frac{1}{2}\lambda h\ y_i + \frac{1}{2}(\lambda h + \lambda^2 h^2)\ y_i$$

$$= \left[1 + \lambda h + \frac{\lambda^2 h^2}{2}\right] y_i$$

$$= E(\lambda h)\ y_i$$

where $E(\lambda h) = 1 + \lambda h + \dfrac{\lambda^2 h^2}{2}$

The condition $|E(\lambda h)| < 1$ gives the interval of absolute stability as $]-2, 0[$.

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Example 10: The second order Runge-Kutta method is applied to the IVP

$$y' = -100y,\ y(0) = 1$$

Find the values of h so that the method produces stable results.

Solution: Comparing the given problem with the test equation $y' = \lambda y$, we get $\lambda = -100$.

Since for stability : $|\lambda h| < 2$, we get $100h < 2$ or, $h < \dfrac{1}{50}$.

Hence, for $0 < h < \dfrac{1}{50}$, the method will produce stable results.

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And now some exercises for you.

E12) Find the interval of absolute stability of the two stage Runge-Kutta method given by

$$y_{i+1} = y_i + \frac{1}{4}(K_1 + 3K_2)$$

where $K_1 = hf(x_i, y_i)$

$$K_2 = hf\left(x_i + \frac{2h}{3},\ y_i + \frac{2}{3}K_1\right).$$

E13) Fourth order classical Runge-Kutta method is used to solve the initial value problem

$$y' = -200y, \quad y(0) = 1$$

Determine the value of h so that the method produces stable results.

We now end this unit by giving a summary of what we have covered in it.

8.6 SUMMARY

In this unit we have discussed the following points

1. In **singlestep** methods for solving IVP(4) viz.,
$$y'(x)=f(x,y), \quad y(x_0)=y_0, \quad x\in[x_0,b]$$
the solution value y_{i+1} at the nodal point x_{i+1} is obtained by using the solution value y_i at only the previous point x_i. Such a method is called an **explicit singlestep** method and is written as
$$y_{i+1}=y_i+h\phi(x_i,y_i,h), \quad i=0,1,\ldots$$
where h is the **step size** and ϕ is called the **increment function.** If the method also uses the solution value y_{i+1}, then the method is called an **implicit singlestep** method.
2. Taylor series method of order p for the solution of the IVP(4) is given by
$$y_{i+1}=y_i+h\,y_i'+\frac{h^2}{2!}y_i''+\cdots+\frac{h^p}{p!}y_i^{(p)}$$
where, $x_i=x_0+ih$, $i=0,1,2\ldots,N$, $x_N=b$ and y_i is an approximation to $y(x_i)$. The error of approximation is given by
$$TE=\frac{h^{p+1}}{(p+1)!}y^{(p+1)}(\xi), \quad x_i<\xi<x_{i+1}$$
3. Euler's method is the Taylor series method of order one.
4. Taylor series method of order p for the system of first order IVP's(3) is given by
$$(\mathbf{y})_{x_{i+1}}=(\mathbf{y})_{x_i}+h(\mathbf{y}')_{x_i}+\frac{h^2}{2!}(\mathbf{y}'')_{x_i}+\cdots+\frac{h^p}{p!}(\mathbf{y}^{(p)})_{x_i}$$
where $\quad (\mathbf{y})_{x_i}=[y_{1i},y_{2i},\ldots,y_{ni}]^T$
$$(\mathbf{y}^{(m)})_{x_i}=[y_{1i}^{(m)},y_{2i}^{(m)},\ldots,y_{ni}^{(m)}]^T, \quad m=1,2,\ldots,p$$
and $\quad y_{ji}^{(m)}=\frac{d^{m-1}}{d_{x_{m-1}}}f_j(x_i,y_{1i},y_{2i},\ldots,y_{ni}), \quad j=1,2,\ldots,n.$
5. A v-stage Runge-Kutta method for the solution of IVP(4)
$$y'=f(x,y),\ y'(x_0)=y_0,\ x\in[x_0,b]$$
where the mesh points are $x_i=x_0+ih$, $i=0,1,\ldots,n$, $x_n=b=x_0+nh$, is obtained by writing
$$y_{i+1}=y_i+h\,(\text{weighted sum of the slopes})$$
$$=y_i+\sum_{i=1}^{v}w_iK_i$$
where v slopes are used. These slopes are defined by
$$K_i=f\left[x_i+\alpha_i h,\ y_i+\sum_{j=1}^{i-1}\beta_{ij}K_j\right], \quad i=1,2,\ldots,v, \quad \alpha_i\neq 0$$
The unknowns α_i,β_{ij} and w_i are then obtained by expanding K_i's and y_{i+1} in Taylor series about the point (x_i,y_i) and comparing the coefficients of different powers of h.
6. Unlike Taylor series methods, Runge-Kutta methods do not need calculation of higher order derivatives of $f(x,y)$ but need only the evaluation of $f(x,y)$ at the off-step points.
7. The numerical solution y_i which is an approximation of $y(x_i)$ contains errors. The error at the nodal point x_i is given by

$$\epsilon_i = y(x_i) - y_i$$

and is called **numerical error**. If this error remains bounded as $i \to \infty$ (or $h \to 0$), then the numerical method is said to be **stable**.

8. When a numerical method is applied to the following linearised form of IVP(4), (test equation)

$$y' = \lambda y, \; y(x_0) = y_0, \; x \in [x_0, b]$$

with nodal points $x_i = x_0 + ih$, $i = 0, 1, \ldots, N$ and $x_n = b = x_0 + Nh$, a relation of the form

$$y_{i+1} = E(\lambda h) y_i$$

is obtained where $E(\lambda h)$ is some approximation to $e^{\lambda h}$. The error ϵ_{i+1} at the nodal point x_{i+1} consists of two parts and is given by

$$\epsilon_{i+1} = [E(\lambda h) - e^{\lambda h}] y(x_i) + E(\lambda h) \epsilon_i$$

The first part $E(\lambda h) - e^{\lambda h}$ is the **local truncation error** and the second part $E(\lambda h) \epsilon_i$ is called the **propagation error** and the following results hold

i) numerical method is called **absolutely stable** if

$$|E(\lambda h)| < 1, \;\; \lambda < 0$$

In this case $y_i \to 0$ as $i \to \infty$. The set of values of λh which satisfy the above condition give the **interval of absolute stability** $]\lambda h, 0[$ for the given method. If this interval is $]-\infty, 0[$, then the method is called **A-stable.**

ii) numerical method is called **relatively stable** if

$$E(\lambda h) < e^{\lambda h}, \; \lambda > 0$$

and the set of values of λh which satisfy the above condition give the **interval of relative stability.**

8.7 SOLUTIONS/ANSWERS

E1) $x_0 = 1, \; y_0 = 1, \; y_0' = -1$

$y_1 \approx y(1.1) = y_0 + h y_0' = 1 + (0.1)(-1) = 0.9$

$x_1 = 1.1, \; y_1 = 0.9, \; y_1' = -0.81$

$y_2 \approx y(1.2) = y_1 + h y_1' = 0.9 + (0.1)(-0.81) = 0.819.$

E2) a) $x_0 = 0, y_0 = 1, y_0' = 3, y_0'' = 11; \; y_1 \approx y(0.2) = 1.82.$

$x_1 = 0.2, y_1 = 1.82, y_1' = 5.86, y_1'' = 19.58, \;\; y_2 \approx y(0.4) = 3.3836.$

b) $x_0 = 0, \; y_0 = 1, \; y_0' = 3, \; y_0'' = 11, \;\; y_0''' = 33, y_0^{iv} = 99,$

$y_1 \approx y(0.2) = 1.8706.$

$x_1 = 0.2, \; y_1 = 1.8706, \; y_1' = 6.0118, \; y_1'' = 20.0354, \; y_1''' = 60.1062,$

$y_1^{iv} = 180.3186, \; y_2 \approx y(0.4) = 3.5658.$

E3) a) $x_0 = 1, y_0 = 1, y_0' = 1.841471, y_0'' = 1.540302, \; y_1 \approx y(1.1) = 1.191849.$

$x_1 = 1. \; y_1 = 1.191849, \; y_1' = 2.029055, \; y_1'' = 1.369943,$

$y_2 \approx y(1.2) = 1.401604$

b) $x_0 = 1, \; y_0 = 1, \; y_0' = 1.841471, \; y_0'' = 1.540302, \; y_0''' = -.841471,$

$y_0^{iv} = -0.540302, \; y_1 \approx y(1.1) = 1.191706.$

$x_1 = 1.1, \; y_1 = 1.191706, \; y_1' = 2.029002, \; y_1'' = 1.370076,$

$y_1''' = -0.929002, \; y_1^{iv} = -0.370076, \; y_2 \approx y(1.2) = 1.401300.$

E4) $x_0 = 0,\ \mathbf{y}_0 = [1,\ 1]^T,\ \mathbf{y}'_0 = [3,\ 7]^T,\ \mathbf{y}''_0 = [13,\ 37]^T$

$\mathbf{y}_1 \approx [y(0.2),\ u(0.2)]^T = [y_1,\ u_1]^T = [1.86,\ 3.14]^T$

$x_1 = 0.2,\ \mathbf{y}_1 = [1.86,\ 3.14]^T,\ \mathbf{y}'_1 = [6.86,\ 18.14]^T,$

$\mathbf{y}''_1 = [31.86,\ 93.14]^T$

$\mathbf{y}_2 \approx [y(0.4),\ u(0.4)]^T = [y_2,\ u_2]^T = [3.8692,\ 8.6308]^T$

E5) $y'_1 = y_2,\ y_1(1) = 1;\ y'_2 = -2y_1y_2,\ y_2(1) = 1$

$x_0 = 1,\ \mathbf{y}_0 = [1,\ 1]^T,\ \mathbf{y}'_0 = [1,\ -2]^T,\ \mathbf{y}''_0 = [-2,\ 2]^T$

$\mathbf{y}_1 \approx [y_1(1.2),\ y_2(1.2)]^T = [y_{11},\ y_{21}]^T = [1.16,\ 0.64]^T$

$x_1 = 1.2,\ \mathbf{y}_1 = [1.16,\ 0.64]^T,\ \mathbf{y}'_1 = [0.64,\ -1.4848]^T,$

$\mathbf{y}''_1 = [-1.4848,\ 2.6255]^T,$

$\mathbf{y}_2 \approx [y_1(1.4),\ y_2(1.4)]^T = [y_{12},\ y_{22}]^T = [1.2583,\ 0.3956]^T.$

E6) $x_0 = 0,\ y_0 = 1,\ K_1 = 0.2,\ K_2 = 0.328,\ y_1 \approx y(0.2) = 1.264.$

$x_1 = 0.2,\ y_1 = 1.264,\ K_1 = 0.359539,\ K_2 = 0.607176$

$y_2 \approx y(0.4) = 1.747338.$

E7) $x_0 = 1,\ y_0 = 2,\ K_1 = 0.033333,\ K_2 = 0.031522,$

$y_1 \approx y(1.1) = 2.031522.$

$x_1 = 1.1,\ y_1 = 2.031522,\ K_1 = 0.029747,\ K_2 = 0.028044,$

$y_2 \approx y(1.2) = 2.059566.$

E8) $x_0 = 2,\ y_0 = 3,\ K_1 = 0.4,\ K_2 = 0.462,\ K_3 = 0.47502,$

$K_4 = 0.5{,}16009,\ y_1 \approx y(2.2) = 3.472508.$

$x_1 = 2.2,\ y_1 = 3.472508,\ K_1 = 0.559904,\ K_2 = 0.668132,$

$K_3 = 0.693024,\ K_4 = 0.847455,\ y_2 \approx y(2.4) = 4.160787.$

E9) $x_0 = 1,\ y_0 = 1,\ K_1 = 0.154030,\ K_2 = 0.152396,\ K_3 = 0.148639,$

$K_4 = 0.150623,\ y_1 \approx y(1.1) = 1.152397.$

$x_1 = 1.1,\ y_1 = 1.152397,\ K_1 = 0.150630,\ K_2 = 0.148639,$

$K_3 = 0.148733,\ K_4 = 0.146641,\ y_2 \approx y(1.2) = 1.301066.$

E10) $x_0 = 0,\ y_o = 0,\ z_0 = 1,$

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} = \begin{bmatrix} 0.24 \\ 0.008 \end{bmatrix}, \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \approx \begin{bmatrix} y(0.2) \\ z(0.2) \end{bmatrix} = \begin{bmatrix} 0.22 \\ 1.004 \end{bmatrix}$$

$x_1 = 0.2,\ y_1 = 0.22,\ z_1 = 1.004$

$$\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix} = \begin{bmatrix} 0.24016 \\ 0.0088 \end{bmatrix},\ \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \end{bmatrix} = \begin{bmatrix} 0.281024 \\ 0.036813 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} \approx \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.480592 \\ 1.026807 \end{bmatrix}.$$

E11) $y'_1 = y_2,\ y_1(0) = 1;\ y'_2 = \frac{3}{2}y_1^2,\ y_2(0) = 1.$

$x_0 = 0,\ y_{10} = 1,\ y_{20} = 1$

$K_1 = [0.2,\ 0.3]^T,\ K_2 = [0.23,\ 0.363]^T,$

$K_3 = [0.2363,\ 0.374838]^T,\ K_4 = [0.274968,\ 0.458531]^T.,$

$[y_{11}, y_{21}]^T \approx [y_1(0.2), y_2(0.2)]^T = [1.234595, 1.372368]^T$.

$x_1 = 0.2,\ y_{11} = 1.234595,\ y_{21} = 1.372368,$

$K_1 = [0.274474, 0.457267]^T,\ K_2 = [0.320200, 0.564577]^T,$

$K_3 = [0.330931, 0.583552]^T,\ K_4 = [0.391184, 0.735261]^T,$

$[y_{12}, y_{22}]^T \approx [y_1(0.4), y_2(0.4)]^T = [1.562582, 1.953832]^T$.

E12) Apply the method to the test equation
$y' = \lambda y,\ \lambda < 0,$ we get

$$K_1 = \lambda h y_i,\ K_2 = \lambda h\left(1 + \frac{2\lambda h}{3}\right) y_i$$

$$y_{i+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) y_i = E(\lambda h) y_i$$

From $|E(\lambda h)| < 1$, we get $-2 < \lambda h < 0$.

E13) Comparing the given problem with the test equation $y' = \lambda y$, we get $\lambda = -200$.
For stability, we require $|\lambda h| < 2.78$. Therefore, $|-200h| < 2.78$ or,
$h < 0.0139$.

—x—

PRACTICAL ASSIGNMENTS

Session 1:

1. Write a program using Euler's method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercise E1).

2. Write a program using Taylor series method of second order to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. User shall give the expression for y''. Input the step length h and the number of steps of integration. Test your program on exercises E2) and E3).

3. Write a program using Runge-Kutta second order method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercises E6) and E7).

4. Write a program using fourth order classical Runge-Kutta method to solve the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$. Input the step length h and the number of steps of integration. Test your program on exercises E8) and E9).