UNIT 4 BESSEL FUNCTIONS

Structure **Page No.**

4.1 INTRODUCTION

In Unit 3, we studied Legendre, Hermite and Laguerre polynomials and discussed a number of their special properties. In this unit we shall study about Bessel functions. Bessel functions were first discovered in 1732 by D. Bernoulli (1700-1782), who provided a series solution (representing a Bessel function) for the oscillatory displacement of a heavy hanging chain. In 1764, Euler developed a series similar to that of Bernoulli, in connection with his work on the vibration of a circular drumhead. J.Fourier (1768-1836) also used Bessel functions in his classical treatise on heat in 1822, but it was Bessel the German astronomer who derived the differential equation bearing his name and studied the general properties of its solutions (now called Bessel functions) in his 1824 memoir. These functions occur so frequently in practice that they are undoubtedly the most important functions beyond the elementary ones.

We shall start the discussion in Sec.4.2 with the Bessel's differential equation and obtain its power series solution in terms of Bessel functions of first and second kind. We shall also discuss various properties of Bessel functions here. In Vibrational phenomena, one of the fundamental problems is to describe the wave motion of a natural or mechanical system: such as a vibrating membrane, subject to certain initial and boundary conditions. In Sec.4.4 we shall discuss this problem of vibrating membrane and illustrate the role played by Bessel functions in getting its solution. We shall also discuss here the problem concerning the buckling of vertical columns under a compressive load.

Objectives

After studying this unit you should be able to

- identify Bessel's differential equation;
- obtain the series solutions of Bessel's differential equation;
- derive Rodrigue's formula, generating function, recurrence relations and orthogonal property of Bessel functions;
- use recurrence relations and other properties of Bessel functions occurring in equations governing physical phenomenon, for finding their solutions.

4.2 BESSEL'S EQUATION

The differential equation

$$x^2 y'' + xy' + (x^2 - \upsilon^2)\, y = 0 \qquad (1)$$

where υ is a non-negative real number is known as **Bessel's equation of order** υ and its solutions are known as **Bessel functions.** The equation was studied by Friedrich Wilhelm Bessel (1784 – 1846), director of the astronomical observatory at Konigsberg, in connection with his work on planetary motion. Apart from this the equation appears in a wide range of applications such as steady and unsteady diffusion in cylindrical regions and one-dimensional wave propagation and diffusion in variable cross-section media etc. We shall now obtain the series solution of this equation.

Dividing Eqn.(1) throughout by the leading coefficient x^2, we see that it can be put in the form

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

where $P(x) = \dfrac{1}{x}$ and $Q(x) = (x^2 - \upsilon^2)/x^2$. This shows that there is one singular point $x = 0$, and that it is **regular singular** point because $x\,P(x) = 1$, and $x^2 Q(x) = x^2 - \upsilon^2$, are analytic at $x = 0$. Therefore, a Frobenius series solution exists for Eqn.(1). Here we shall be discussing two cases

i) $\upsilon \neq$ integer and ii) $\upsilon =$ integer.

Case when υ is not an integer

Consider the series solution of Eqn.(1) of the form

$$y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \quad c_0 \neq 0 \tag{2}$$

Substituting for y, y' and y'' in Eqn.(1), we obtain

$$\sum_{m=0}^{\infty}(m+k)(m+k-1)c_m x^{m+k} + \sum_{m=0}^{\infty}(m+k)c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} - \upsilon^2\sum_{m=0}^{\infty}c_m x^{m+k} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k)(m+k-1) + (m+k) - \upsilon^2\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k+\upsilon)(m+k-\upsilon)\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0$$

or, $$\sum_{m=0}^{\infty}\left[(m+k)^2 - \upsilon^2\right]c_m x^{m+k} + \sum_{m=0}^{\infty}c_m x^{m+k+2} = 0 \tag{3}$$

Equating to zero the coefficient of smallest power of x (i.e. x^k) to zero, we get the **indicial equation** as

$$c_0(k^2 - \upsilon^2) = 0 \Rightarrow k = \upsilon, -\upsilon \; (\because c_0 \neq 0) \tag{4}$$

Equating the coefficient of x^{k+1} to zero in Eqn.(3), we obtain

$$[(k+1)^2 - \upsilon^2)c_1 = 0, \quad \text{or} \quad c_1 = 0, \quad \text{for} \quad k = \pm\,\upsilon$$

To obtain the **recurrence** relation, we equate to zero the coefficients of x^{k+m} in Eqn.(3). This yields

$$(m+k+\upsilon)\;(m+k-\upsilon)\;c_m = -\,c_{m-2}$$

i.e, $$c_m = \frac{-1}{(m+k+\upsilon)\;(m+k-\upsilon)}\;c_{m-2} \tag{5}$$

For $\mathbf{k = \upsilon}$, relation (5) simplifies to

$$c_m = \frac{-1}{(m-2\upsilon)\,m}c_{m-2} \tag{6}$$

Since $c_1 = 0$, putting $m = 3, 5, 7, \ldots$ in relation (6), we get

$$c_3 = c_5 = c_7 = \ldots = 0 \tag{7}$$

and for $m = 2, 4, 6, \ldots$, we obtain

$$c_2 = \frac{-c_0}{2^2(1+\upsilon)},\; c_4 = \frac{-c_2}{2^3(2+\upsilon)} = \frac{c_0}{2^4(2!)(1+\upsilon)(2+\upsilon)}$$

$$c_6 = \frac{-c_4}{2^2(3)(3+\upsilon)} = \frac{-c_0}{2^6(3!)(1+\upsilon)(2+\upsilon)(3+\upsilon)^3}\cdots$$

In general, we can write

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m}(m!)(1+\upsilon)(2+\upsilon)\cdots(m+\upsilon)}, \quad m = 1, 2, \ldots \tag{8}$$

Putting these values of $c's$ in Eqn.(2), we obtain one of the linearly independent solution of the Bessel's equation as

$$y_1(x) = c_0 x^{\upsilon}\left[1 - \frac{x^2}{2^2(1+\upsilon)} + \frac{x^4}{2^4\,(2!)\,(1+\upsilon)\,(2+\upsilon)} - \cdots \right.$$
$$\left. \cdots + \frac{(-1)^m x^{2m}}{2^m\,(m!)\,(1+\upsilon)\,(2+\upsilon)\ldots(m+\upsilon)} + \cdots \right] \tag{9}$$

If υ were an integer, then the product $(1+\upsilon)(2+\upsilon)\ldots(m+\upsilon)$ could be simplified into closed form as $(\upsilon+m)!$ But if υ is not an integer then we can simplify the right hand side of Eqn.(9) by making a suitable choice of a value for c_0.

Let the value of c_0 be chosen as

$$c_0 = \frac{1}{2^{\upsilon}\Gamma(\upsilon+1)} \tag{10}$$

$\Gamma(\upsilon+1) = \upsilon\Gamma\upsilon$

where Γ is the Gamma function defined by

$$\Gamma(\upsilon) = \int_0^{\infty} e^{-t} t^{\upsilon-1}\, dt \tag{11}$$

Substituting the value of c_0 from Eqn.(10) in Eqn.(8), we obtain

$$c_{2m} = \frac{(-1)^m}{2^{2m+\upsilon}(m!)\,\Gamma(\upsilon+1)[(1+\upsilon)(2+\upsilon)\ldots(m+\upsilon)]} = \frac{(-1)^m}{2^{2m+\upsilon}(m!)\Gamma(m+\upsilon+1)} \tag{12}$$

and the solution $y(x)$ given by Eqn.(9) reduces to

$$y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\upsilon+1)} \left(\frac{x}{2}\right)^{2m+\upsilon} \tag{13}$$

This $y(x)$ defines the **Bessel function of the first kind of order** υ and is denoted by $J_{\upsilon}(x)$. Therefore we have one solution of Eqn.(1) as

$$J_{\upsilon}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)} \left(\frac{x}{2}\right)^{2m+\upsilon}$$
$$= \left(\frac{x}{2}\right)^{\upsilon} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)} \left(\frac{x}{2}\right)^{2m} \tag{14}$$

To obtain a second linearly independent solution, we turn to the other indicial root, $k = -\upsilon$. There is no need to repeat all the steps; all we need to do is to change υ to $-\upsilon$ everywhere on the right side of Eqn.(14). Denoting the result as $J_{-\upsilon}(x)$, we have the second solution of Eqn.(1), the **Bessel function of the first kind of order** $-\boldsymbol{\upsilon}$ as

$$J_{-\upsilon}(x) = \left(\frac{x}{2}\right)^{-\upsilon} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\upsilon+1)} \left(\frac{x}{2}\right)^{2m} \tag{15}$$

Using ratio test it can be shown that both the series (14) and (15) converge for all x. The leading terms of the series in (14) and (15) are constant times x^{υ} and $x^{-\upsilon}$, respectively so neither of the solutions $J_{\upsilon}(x)$ and $J_{-\upsilon}(x)$ is a scalar multiple of the other. Thus they are linearly independent, and we conclude that, when υ **is not an integer**

$$\mathbf{y(x) = A\,J_{\upsilon}(x) + B\,J_{-\upsilon}(x)} \tag{16}$$

is the **general solution** of Bessel's Eqn.(1).

Writing Eqns.(14) and (15) in the form

$$J_{\upsilon}(x) = x^{\upsilon}\left[\frac{1}{\Gamma(\upsilon+1)\,2^{\upsilon}} - \frac{1}{\Gamma(\upsilon+2)\,2^{\upsilon+2}}x^2 + \cdots\right] \tag{17}$$

$$J_{-\upsilon}(x) = x^{-\upsilon}\left[\frac{1}{\Gamma(1-\upsilon)\,2^{-\upsilon}} - \frac{1}{\Gamma(2-\upsilon)\,2^{-\upsilon+2}}\,x^2 + \cdots\right] \qquad (18)$$

you may **observe** that the power series within the square brackets tend to $1/\Gamma(\upsilon+1)\,2^{\upsilon}$ and $1/\Gamma(1-\upsilon)\,2^{-\upsilon}$, respectively, as $x \to 0$. We see that $J_{\upsilon}(x) \sim \left[1/\Gamma(\upsilon+1)\,2^{\upsilon}\right]x^{\upsilon}$ and $J_{-\upsilon}(x) \sim \left[1/\Gamma(1-\upsilon)\,2^{-\upsilon}\right]x^{-\upsilon}$ as $x \to 0$. In more concise form we can write $J_{\upsilon}(x) = O(x^{\upsilon})$ and $J_{-\upsilon}(x) = O(x^{-\upsilon})$ as $x \to 0$. Thus, $J_{\upsilon}(x)'s$ tend to zero and the $J_{-\upsilon}(x)'s$ tend to infinity as $x \to 0$. In Fig.1, we have plotted $J_{1/2}(x)$ and $J_{-1/2}(x)$, for $\upsilon = 1/2$.

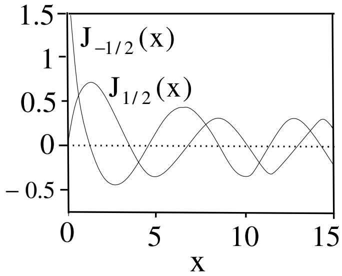

Fig.1

The cases where υ is half-integral i.e. $\upsilon = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots,$ are of special interest because these Bessel functions are actually elementary functions. In particular, the case $\upsilon = \frac{1}{2}$ leads to the interesting results where

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

We are leaving it for you to verify it yourself.

Let us consider the following examples.

Example 1: Find the solution of the differential equation

$$x^2 y'' + xy + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Solution: The given differential equation is the Bessel's equation with $\upsilon = \frac{1}{2}$. The general solution of the equation is

$$y(x) = A\,J_{1/2}(x) + B\,J_{-1/2}(x)$$

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Example 2: Transform the equation

$$x^2 y'' + xy' + 4(x^4 - \upsilon^2)y = 0$$

using the substitution $x^2 = z$ and hence find the general solution of the equation.

Solution: For $x^2 = z$, we have

$$y' = \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = 2x\frac{dy}{dz} = 2\sqrt{z}\,\frac{dy}{dz}$$

and $$y'' = \frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2} = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}$$

On substituting y' and y'' from above, the given equation reduces to

$$z\left(4z\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right) + 2z\frac{dy}{dz} + 4(z^2 - \upsilon^2)\,y = 0$$

or, $$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - \upsilon^2)\,y = 0$$

which is a Bessel's differential equation of order υ. Hence the general solution is given by

$$y(z) = A\,J_{\upsilon}(z) + B\,J_{-\upsilon}(z)$$

or, $$y(x) = A\,J_{\upsilon}(x^2) + B\,J_{-\upsilon}(x^2)$$

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You may now try the following exercises.

E1) Using Eqns.(14) and (15) and the relations $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

E2) Show that the given functions are solutions of the corresponding equations

a) $y'' + k^2 xy = 0$, $\quad y = \sqrt{x}\, J_{1/3}\left(\frac{2kx^{3/2}}{3}\right)$

b) $y'' + k^2 x^2 y = 0$, $\quad y = \sqrt{x}\, J_{1/4}\left(\frac{k x^2}{2}\right)$

c) $y'' + k^2 x^4 y = 0$, $\quad y = \sqrt{x}\, J_{1/6}\left(\frac{kx^3}{3}\right)$

E3) Using the substitutions given alongside, transform the following differential equations and hence, find their general solution.

a) $4x^2 y'' + 4x y' + (x - \upsilon^2) y = 0;$ $\quad (\sqrt{x} = z)$

b) $xy'' - y' + xy = 0;$ $\quad (y = xu)$

c) $xy'' + (1 + 2k) y' + xy = 0;$ $\quad (y = u / x^k)$

We now take up the case when υ is an integer.

Case when υ is an integer

Let $\upsilon = n$ be an integer. Since $\Gamma(\upsilon + 1) = \upsilon\Gamma(\upsilon)$, we have for $\upsilon = n$ $\Gamma(1) = 1,\ \Gamma(2) = \Gamma(1) = 1!,\ \Gamma(3) = 2\Gamma(2) = 2!, \ldots, \Gamma(n+1) = n\Gamma(n) = n!$ and the expression for c_{2m} given by Eqn.(12) can be written as

$$c_{2m} = \frac{(-1)^m}{2^{2m+n} (m!)(m+n)!}, \quad \text{and} \quad c_0 = \frac{1}{2^n\, n!}, \text{ from Eqn.(10)} \tag{19}$$

The Bessel function $J_n(x)$ is then defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \tag{20}$$

The series given by Eqn.(20) converges for all x. Functions $J_{-\upsilon}(x)$ for $\upsilon = n$, reduces to

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} = \sum_{m=n}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{2m-n} \tag{21}$$

where we have used the fact that $\Gamma(m - n + 1)$ is undefined when its argument is zero or a negative integer, namely, for $m = 0, 1, \ldots, n-1$. To start the series at zero once again, we make the change of index $m - n = k$ in Eqn.(21), which yields

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+n} = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \tag{22}$$

If $(-1)^n$ is factored out the series that remains on the right hand side of En.(22) is the same as that given in Eqn.(20), so that

$$J_{-n}(x) = (-1)^n J_n(x) \tag{23}$$

This shows that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent. Thus, whereas $J_\upsilon(x)$ and $J_{-\upsilon}(x)$ are linearly independent and give the general solution (16) of Eqn.(1) when υ is not an integer, we have only one linearly independent solutions so far for the case where $\upsilon =$ integer, namely, $y_1(x) = J_n(x)$ given by Eqn.(20). The second linearly independent solution is yet to be determined which we shall take up later in Sec.4.2.2.

The Bessel functions that arise most frequently in practice are $\mathbf{J_0(x)}$ and $\mathbf{J_1(x)}$, whose series representations are

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \quad (24)$$

and $$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!}\left(\frac{x}{2}\right)^{2m+1}$$

$$= \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots \quad (25)$$

At $x = 0$, we see that $J_0(0) = 1$ while $J_1(0) = 0$ and from Eqn.(20) it follows that $J_n(0) = 0,\ n = 2, 3, 4, \ldots$. For x near zero, the $m = 0$ term of Eqn.(20) yields the asymptotic formulas

$$J_0(x) \sim 1 \ \text{ as } \ x \to 0^+ \quad (26)$$

$$J_n(x) \sim \frac{(x/2)^n}{n!} \ \text{ as } \ x \to 0^+,\ n = 1, 2, 3, \ldots \quad (27)$$

The graphs of $J_0(x)$, $J_1(x)$ are shown in Fig.2. **Observe** that these functions exhibit an oscillatory behaviour somewhat like that of the sinusoidal functions, except that the amplitude (maximum departure from the x-axis) of the Bessel function diminishes as x increases and the (infinitely many) zeros of these functions are not evenly spaced. The location of these zeros is of great theoretical and practical importance, but we shall not be going into the details of the theory here. However, the first five zeros of $J_0(x)$ and $J_1(x)$ are given below:

$J_0(x)$: 2.405, 5.520, 8.654, 11.792, 14.931

$J_1(x)$: 3.832, 7.016, 10.173, 13.324, 16.471

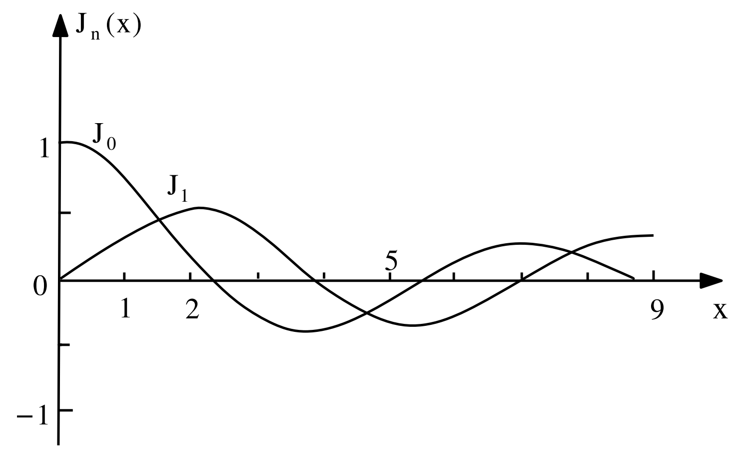

Fig.2

You may now try the following exercises.

E4) Show that $J_n(x)$ is an even function for n even and an odd function for n odd where n is an integer.

E5) Using the series expansion for $J_0(x)$ and $J_1(x)$ show that $J_0'(x) = -J_1(x)$.

We shall now discuss some of the properties of the Bessel functions.

4.2.1 Bessel Function of the First Kind

Bessel functions of first kind satisfy a number of properties that can be developed directly from their series definition. We shall now prove some of these properties.

Recurrence relations

Multiplying the Bessel function $J_\upsilon(x)$, given in Eqn.(14) by x^υ, we get

$$x^\upsilon J_\upsilon(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2\upsilon}}{m!\Gamma(m+\upsilon+1)2^{2m+\upsilon}}$$

Differentiating this equation with respect to x, and using the relation $\Gamma(m+\upsilon+1) = (m+\upsilon)\Gamma(m+\upsilon)$, we get

$$\frac{d}{dx}\left[x^\upsilon J_\upsilon(x)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2\upsilon)\, x^{2m+2\upsilon-1}}{m!2^{2m+\upsilon}\Gamma(m+\upsilon+1)}$$

$$= x^\upsilon \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon)} \left(\frac{x}{2}\right)^{2m+\upsilon-1}$$

$$= x^\upsilon J_{\upsilon-1}(x)$$

Therefore, $$\frac{d}{dx}[x^\upsilon J_\upsilon(x)] = x^\upsilon J_{\upsilon-1}(x) \qquad (28)$$

Setting $\upsilon = 0$, we get $J_0'(x) = J_{-1}(x) = -J_1(x)$, since $J_{-n}(x) = (-1)^n J_n(x)$.
Integrating Eqn.(28), we can write

$$\int x^\upsilon J_{\upsilon-1}(x)\,dx = x^\upsilon J_\upsilon(x) + c \qquad (29)$$

Similarly, if we multiply the Bessel function $J_\upsilon(x)$ by $x^{-\upsilon}$ and differentiate it with respect to x, we get

$$\frac{d}{dx}[x^{-\upsilon} J_\upsilon(x)] = \sum_{m=0}^{\infty} \frac{(-1)^m (2m)\, x^{2m-1}}{m!2^{2m+\upsilon}\Gamma(m+\upsilon+1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{(m-1)!2^{2m+\upsilon-1}\Gamma(m+\upsilon+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{k!2^{2k+\upsilon+1}\Gamma(k+\upsilon+2)} \qquad \text{[setting } m-1=k]$$

$$= x^{-\upsilon} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\Gamma(k+\upsilon+2)} \left(\frac{x}{2}\right)^{2k+\upsilon+1}$$

$$= -x^{-\upsilon} J_{\upsilon+1}(x)$$

Therefore, $$\frac{d}{dx}[x^{-\upsilon} J_\upsilon(x)] = -x^{-\upsilon} J_{\upsilon+1}(x) \qquad (30)$$

Integrating Eqn.(30), we obtain

$$\int x^{-\upsilon} J_{\upsilon+1}(x)\,dx = -x^{-\upsilon} J_\upsilon(x) + c \qquad (31)$$

If we carry out the differentiation on the left-hand sides of Eqns.(28) and (30) and divide the results by the factors $x^{\upsilon-1}$ and $x^{-\upsilon-1}$, respectively, we deduce that

$$x J_\upsilon'(x) + \upsilon J_\upsilon(x) = x\, J_{\upsilon-1}(x) \qquad (32)$$

and $$x J_\upsilon'(x) - \upsilon J_\upsilon(x) = -x J_{\upsilon+1}(x) \qquad (33)$$

When υ is an integer n, we write

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x) \qquad (34)$$

and $$x J_n'(x) = n J_n(x) - x\, J_{n+1}(x) \qquad (35)$$

Finally, the sum and difference of relations (32) and (33) yields, respectively

$$2J'_\upsilon(x) = J_{\upsilon-1}(x) - J_{\upsilon+1}(x) \tag{36}$$

and $$2\upsilon J_\upsilon(x) = x\, J_{\upsilon-1}(x) + x\, J_{\upsilon+1}(x) \tag{37}$$

Let us look at the following examples.

Example 3: Show that $J_4(x) = \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1 - \left(\frac{24}{x^2} - 1\right) J_0$.

Solution: Using recurrence relation (37), we obtain

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$\therefore J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$$

$$= \frac{6}{x}\left\{\left(\frac{8}{x^2} - 1\right) J_1 - \frac{4}{x} J_0\right\} - \frac{2}{x} J_1 + J_0$$

$$= \frac{8}{x}\left(\frac{6}{x^2} - 1\right) J_1 - \left(\frac{24}{x^2} - 1\right) J_0$$

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Example 4: Show that

a) $[x\, J_n(x)\, J_{n+1}(x)]' = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

b) $J_1''(x) = \frac{1}{x} J_2(x) - J_1(x)$

c) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{1}{x^2}(3 - x^2)\cos x + \frac{3}{x}\sin x\right]$

Solution:

a) $[x J_n(x)\, J_{n+1}(x)]' = [\{x^{-n} J_n(x)\}\{x^{n+1} J_{n+1}^{(x)}\}]'$

$$= x^{-n} J_n(x)[x^{n+1} J_n(x)] + x^{n+1} J_{n+1}[-x^{-n} J_{n+1}(x)]$$

[using relations (28) and (30)]

$$= x\, J_n^2(x) - x\, J_{n+1}^2(x)$$

b) Using relation (36) with $\upsilon = 1$, we get

$$2J_1'(x) = J_0(x) - J_2(x)$$

Differentiating it we obtain

$$2J_1''(x) = J_0'(x) - J_2'(x)$$

Using relation (32) with $\upsilon = 2$, and the fact that $J_0'(x) = -J_1(x)$, we get

$$2J_1''(x) = -J_1(x) - \left[J_1(x) - \frac{2}{x} J_2(x)\right]$$

or, $$J_1''(x) = \frac{1}{x} J_2(x) - J_1(x).$$

c) Using relation (37) for $\upsilon = -3/2$ and $\upsilon = -1/2$, we obtain

$$-3J_{-3/2}(x) = x\,[J_{-5/2}(x) + J_{-1/2}(x)]$$

and $$-J_{-1/2}(x) = x\,[J_{-3/2}(x) + J_{1/2}(x)]$$

Eliminating $J_{-3/2}(x)$ from the above two equations we obtain

$$x^2 J_{-5/2}(x) = 3J_{-1/2}(x) - x^2 J_{-1/2}(x) + 3x\, J_{1/2}(x)$$

Using $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ and $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$, in the above equation we have,

$$J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{1}{x^2}(3-x^2)\cos x+\frac{3}{x}\sin x\right]$$

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Example 5: Prove the following

a) $\int J_0(x)\ J_1(x)\,dx=-\frac{1}{2}J_0^2(x)+c$

b) $\int x\,J_0^2(x)\,dx=\frac{x^2}{2}[J_0^2(x)+J_1^2(x)]+c$

c) $\int J_3(x)\,dx=-J_2(x)-\frac{2}{x}J_1(x)+c$

Solution: a) We know that

$$[J_0^2(x)]'=2J_0(x)J_0'(x)=-2J_0(x)J_1(x) \qquad [\because J_0'(x)=-J_1(x)]$$

$$\therefore \quad \int J_0(x)J_1(x)\,dx=\frac{-1}{2}J_0^2(x)+c$$

b) $\int x\,J_0^2(x)\,dx=\frac{x^2}{2}J_0^2(x)+\int x^2J_0(x)\ J_1(x)\,dx \qquad [\because J_0'(x)=-J_1(x)]$

$$=\frac{x^2}{2}J_0^2(x)+\int (x\,J_0(x))\ (x\ J_1(x))\,dx$$

$$=\frac{x^2}{2}J_0^2(x)+(x\,J_1(x))^2-\int (x\,J_1(x))(x\ J_1(x))'\,dx \quad \text{[using relation (29)]}$$

$$=\frac{x^2}{2}J_0^2(x)+(x\,J_1(x))^2-\frac{(x\,J_1(x))^2}{2}+c$$

$$=\frac{x^2}{2}[J_0^2(x)+J_1^2(x)]+c$$

c) $\int J_3(x)\,dx=\int x^2(x^{-2}J_3(x))\,dx$

$$=x^2(-x^{-2}J_2(x))+\int x^{-2}J_2(x)\,2x\,dx \quad \text{[using relation (31)]}$$

$$=-J_2(x)+2(-x^{-1}J_1(x))+c \qquad \text{[using relation (31)]}$$

$$=-J_2(x)-\frac{2}{x}\ J_1(x)+c$$

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Example 6: Evaluate $\int x^3J_3(x)\,dx$.

Solution: $\int x^3J_3(x)\,dx=\int x^5[x^{-2}J_3(x)]\,dx$

$$=x^5[-x^{-2}J_2(x)]-\int[-x^{-2}J_2(x)]5x^4dx \quad \text{[using relation (31)]}$$

$$=-x^3J_2(x)+5\int x^2J_2(x)\,dx$$

Now $\int x^2J_2(x)\,dx=\int x^3[x^{-1}J_2(x)]\,dx$

$$=x^3[-x^{-1}J_1(x)]-\int[-x^{-1}J_1(x)]3x^2\,dx$$

$$=-x^2J_1(x)+3\int x\,J_1(x)\,dx$$

$$\int x\,J_1(x)\,dx=-\int x\,J_0'(x)\,dx=-[x\,J_0(x)-\int J_0(x)\,dx]$$

$$=-x\,J_0(x)+\int J_0(x)\,dx$$

Then $\int x^3J_3(x)\,dx=-x^3J_2(x)-5x^2J_1(x)-15x\,J_0(x)+15\int J_0(x)\,dx$

The integral $\int J_0(x)\,dx$ cannot be obtained in closed form.

***

You may now try the following exercises.

E6) Express the Bessel functions $J_2(x)$ and $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

E7) Prove the following:

a) $J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$

b) $J_n''(x) = \frac{1}{4}\,[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$

c) $[J_\upsilon^2(x)]' = \frac{x}{2\upsilon}[J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x)]$

d) $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{x\sin x + \cos x}{x}\right)$

e) $J_{1/2}^2(x) + J_{-1/2}^2(x) = \frac{2}{\pi x}$.

E8) Show that

a) $4J_0'''(x) + 3\,J_0'(x) + J_3(x) = 0$

b) $J_0^2(x) + 2(J_1^2(x) + J_2^2(x) + J_3^2(x) + \cdots) = 1$

E9) Prove the following:

a) $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2J_n(x) + c$

b) $\int J_2(x)\,dx = -2J_1(x) + \int J_0(x)\,dx + c$

c) $\int J_5(x)\,dx = -J_4(x) - \frac{4}{3}J_3(x) - \frac{8}{x^2}J_2(x) + c$

d) $\int x^3 J_0(x)\,dx = x(x^2 - 4)J_1(x) + 2x^2 J_0(x) + c$

e) $\int x^5 J_0(x)\,dx = x^5 J_1(x) - 4x^4 J_2(x) + 8x^3 J_3(x) + c$

f) $\int_0^a x\,J_0(r,x)\,dx = \frac{a}{r}J_1(ar)$, $\quad a, r$ are constant.

g) $\int_0^1 (x - x^3)J_0(x)\,dx = 4J_1(1) - 2J_0(1)$

h) $\int J_1(x)\sin x\,dx = x\,J_1(x)\sin x + J_0(x)(x\cos x - \sin x) + c$.

We shall now show that the Bessel function of the first kind, $J_n(x)$ can also be obtained using a generating function.

Generating Function

We shall show that

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n\ ,\ \ t \neq 0. \tag{38}$$

The function on the left hand side is called the **generating function** of the Bessel function $J_n(x)$.

We have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{xt/2}e^{-x/2t} = \left\{\sum_{r=0}^{\infty}\frac{(xt/2)^r}{r!}\right\}\left\{\sum_{m=0}^{\infty}\frac{(-x/2t)^m}{m!}\right\}$$

$$= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{r+m} t^{r-m}}{r!m!} \tag{39}$$

Our goal is to obtain a single series in powers of t. Thus we make the change of index $n = r - m$, and because both r and m have an infinite range of values, it follows that n varies from $-\infty$ to ∞. Then the sum on the right hand side of Eqn.(39) becomes

$$\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m} t^n}{(n+m)!m!} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!} \right\} t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, t^n.$$

which proves formula (38).

In many situations the integral representation of $J_n(x)$ is more convenient to use than its series representation. We shall now see how the generating function relation can be used to derive the integral representation of Bessel function of the first kind.

Integral Representation

We start with the generating function relation

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \quad t \neq 0.$$

and set $t = e^{i\theta}$. Then

$$e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} = \sum_{n=-\infty}^{\infty} J_n(x)[\cos n\theta + i\sin n\theta]$$

$$\therefore \cos(x\sin\theta) + i\sin(x\sin\theta) = \{J_0(x) + [J_{-1}(x) + J_1(x)]\cos\theta + [J_{-2}(x) + J_2(x)]\cos 2\theta + \cdots\}$$
$$+ i\{[J_1(x) - J_{-1}(x)]\sin\theta + [J_2(x) - J_{-2}(x)]\sin 2\theta + \cdots\}$$

$$= \{J_0(x) + 2J_2(x)\cos 2\theta + \cdots\} + i\{2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + \cdots\} \tag{40}$$

where in Eqn.(40), we have used the fact that $J_{-n}(x) = (-1)^n J_n(x)$.

Equating real and imaginary parts of Eqn.(40), we obtain

$$\cos(x\sin\theta) = J_0(x) + 2\{J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\} \tag{41}$$

and $$\sin(x\sin\theta) = 2\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\} \tag{42}$$

These series are usually called the **Jacobi series.** Multiplying both sides of Eqn.(41) by $\cos n\theta$ and integrating over the interval $[0, \pi]$, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta = \begin{cases} J_n(x), & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases} \tag{43}$$

where we have used the result $\int_0^{\pi} \cos m\theta \cos n\theta\, d\theta = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$.

Similarly, multiplying both sides of Eqn.(42) by $\sin n\theta$, integrating over the interval $[0, \pi]$ and using $\int_0^{\pi} \sin m\theta \sin n\theta\, d\theta = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta\, d\theta = \begin{cases} J_n(x), & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases} \tag{44}$$

Adding Eqns.(43) and (44), we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos n\theta \cos(x\sin\theta) + \sin n\theta \sin(x\sin\theta)\; d\theta = J_n(x)$$

or, $$\boldsymbol{\frac{1}{\pi}\int_0^{\pi} \cos(n\,\theta - x\sin\theta)\; d\theta = J_n(x)} \qquad (45)$$

for all **integral values of** n.

Eqn.(45) gives the **Bessel's integral formula.**

Let us consider the following examples.

Example 7: Show that

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)\; J_k(y) \qquad (46)$$

Solution: Using formula (38), we can write

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\, e^{\frac{y}{2}\left(t-\frac{1}{t}\right)} = e^{\left[\frac{x+y}{2}\left(t-\frac{1}{t}\right)\right]} = \sum_{n=-\infty}^{\infty} J_n(x+y)\; t^n$$

Now the product of two exponentials on the left is also

$$\left[\sum_{j=-\infty}^{\infty} J_j(x)\,t^j\right]\left[\sum_{k=-\infty}^{\infty} J_k(y)\; t^k\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} J_{n-k}(x)\; J_k(y)\right] t^n$$

Equating the coefficients of t^n in the above two expressions, we obtain

$$J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x)\; J_k(y).$$

***

Example 8: Using the generating function for $J_n(x)$, prove that $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$, for integer values of n.

Solution: We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n .$$

Differentiating both sides with respect to t, we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\,\frac{x}{2}\left(1+\frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} n\; J_n(x)\; t^{n-1}$$

or, $$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} J_n(x)\; t^n = \sum_{n=-\infty}^{\infty} n\; J_n(x)\; t^{n-1}$$

i.e., $$\sum_{n=-\infty}^{\infty} \frac{x}{2}\left(1+\frac{1}{t^2}\right) J_n(x)\,t^n = \sum_{n=-\infty}^{\infty} n\; J_n(x)\; t^{n-1}$$

The above equation can be written as

$$\sum_{n=-\infty}^{\infty} \frac{x}{2}\, J_n(x)\; t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2}\, J_n(x)\; t^{n-2} = \sum_{n=-\infty}^{\infty} n\; J_n(x)\; t^{n-1}$$

or, $$\sum_{n=-\infty}^{\infty} \frac{x}{2}\, J_n(x)\; t^n + \sum_{n=-\infty}^{\infty} \frac{x}{2}\, J_{n+2}(x)\; t^n = \sum_{n=-\infty}^{\infty} (n+1)\; J_{n+1}(x)\; t^n$$

i.e. $$\sum_{n=-\infty}^{\infty}\left[\frac{x}{2}\, J_n(x) + \frac{x}{2}\, J_{n+2}(x)\right] t^n = \sum_{n=-\infty}^{\infty} (n+1)\; J_{n+1}(x)\; t^n$$

Equating the coefficients of t^n on both the sides, we obtain

$$\frac{x}{2} J_n(x) + \frac{x}{2} J_{n+2}(x) = (n+1) J_n(x)$$

Replacing n by $(n-1)$ in the above equation, the required result is obtained.

***

Example 9: Show that

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots$$

and $$\sin(x\cos\theta) = 2\,[\,J_1\cos\theta - J_3\cos 3\theta + \cdots]$$

Solution: From Eqn.(38), we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + J_1(x)\,t + J_2(x)\,t^2 + \cdots + J_n(x)\,t^n + \cdots$$
$$-J_1(x)\,t^{-1} + J_2(x)\,t^{-2} - \cdots + (-1)^n J_n(x)\,t^{-n} + \cdots$$
$$= J_0(x) + \left(t - \frac{1}{t}\right) J_1(x) + \left(t^2 + \frac{1}{t^2}\right) J_2(x) + \cdots \tag{47}$$

Now if we put $t = i\,e^{i\theta}$ in Eqn.(47), then

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{i x\cos\theta} = \cos(x\cos\theta) + i\sin(x\cos\theta) \tag{48}$$

The right hand side of Eqn.(47) becomes

$$J_0 + 2i\,J_1\cos\theta - 2J_2\cos 2\theta - 2i\,J_3\cos 3\theta + \cdots$$
$$= [J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots] + 2i[J_1\cos\theta - J_3\cos 3\theta + \cdots] \tag{49}$$

Comparing Eqns.(48) and (49), we obtain

$$\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \cdots \tag{50}$$

and $$\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + \cdots] \tag{51}$$

***

Example 10: Show that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\,d\theta$

Solution: Using the result obtained in Example 9 and integrating Eqn.(50) over the interval $[0,\pi]$, we obtain

$$\int_0^{\pi} \cos(x\cos\theta)\,d\theta = J_0(x)\int_0^{\pi} d\theta = \pi J_0(x)\,.$$

since all other terms vanish.

***

It is often necessary in mathematical physics to expand a given function in terms of Bessel functions, where the particular type of expansion depends on the problem at hand. The simplest and most useful expansions of this kind are series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n J_\upsilon(\lambda_n x) = a_1 J_\upsilon(\lambda_1 x) + a_2 J_\upsilon(\lambda_2 x) + \cdots \tag{52}$$

where $f(x)$ is defined on the interval $0 \le x \le 1$, and the λ_n are the positive zeros of some fixed Bessel function $J_\upsilon(x)$ with $\upsilon \ge 0$. We have chosen the interval $0 \le x \le 1$ only for the sake of simplicity. By a simple change of variable a function defined on an interval of the form $0 \le x \le R$, can be considered. Such expansions play a major role in physical problems. We shall illustrate the use of expansion (52) in solving the two-dimensional wave equation for a vibrating circular membrane in Sec.4.3. Determination of the coefficients in Eqn.(52) depend on the orthogonality property of the Bessel functions $J_\upsilon(\lambda_n x)$ which we shall discuss now.

Orthogonality of Bessel Functions

In λ_n and λ_m are the positive zeros of Bessel functions $J_\upsilon(x)$ with $\upsilon \ge 0$ then we shall show that

$$\int_0^1 x\, J_\upsilon(\lambda_m x)\, J_\upsilon(\lambda_n x)\, dx = \begin{cases} 0 & , \quad \text{if } m \neq n \\ \frac{1}{2} J^2_{\upsilon+1}(\lambda_n), & \text{if } m = n \end{cases} \tag{53}$$

Formula (53) shows that the functions $J_\upsilon(\lambda_n x)$ are orthogonal with respect to the weight functions x on the interval $0 \le x \le 1$.

To establish (53), we begin with the fact that $y = J_\upsilon(x)$ is a solution of

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\upsilon^2}{x^2}\right) y = 0$$

If a and b are distinct positive constants, it follows that the function $u(x) = J_\upsilon(ax)$ and $v(x) = J_\upsilon(bx)$ satisfy the equations

$$u'' + \frac{1}{x} u' + \left(a^2 - \frac{\upsilon^2}{x^2}\right) u = 0 \tag{54}$$

and
$$v'' + \frac{1}{x} v' + \left(b^2 - \frac{\upsilon^2}{x^2}\right) v = 0 \tag{55}$$

Multiply Eqns.(54) and (55) by v and u, respectively, and subtract the results, to obtain

$$\frac{d}{dx}(u'v - v'u) + \frac{1}{x}(u'v - v'u) = (b^2 - a^2)uv$$

After multiplication with x, the above equation becomes

$$\frac{d}{dx}[x\,(u'v - v'u)] = (b^2 - a^2)\; xuv \tag{56}$$

Integrating Eqn.(56) from $x = 0$ to $x = 1$, we get

$$(b^2 - a^2) \int_0^1 x\; u\; v\, dx = [x\,(u'v - v'u)]_0^1 \tag{57}$$

In Eqn.(57), the expression in brackets on the right hand side clearly vanishes at $x = 0$. Now since we have $u(1) = J_\upsilon(a)$ and $v(1) = J_\upsilon(b)$, it follows that if a and b are distinct positive zeros λ_m and λ_n of $J_\upsilon(x)$, then integral on the left is zero. That is, we have

$$\int_0^1 x\; J_\upsilon(\lambda_m x)\; J_\upsilon(\lambda_n x) = 0 \tag{58}$$

which is the first part of (53). Now evaluation of integral (58) has to be done when $m = n$.

Multiplying Eqn.(54) by $2x^2 u'$, it becomes

$$2x^2 u'\, u'' + 2x\; u'^2 + 2a^2 x^2\, u\, u' - 2\upsilon^2\, u\, u' = 0$$

or,
$$\frac{d}{dx}(x^2 u'^2) + \frac{d}{dx}(a^2 x^2 u^2) - 2a^2 x\; u^2 - \frac{d}{dx}(\upsilon^2 u^2) = 0$$

Integrating the above equation from $x = 0$ to $x = 1$, we obtain

$$2\, a^2 \int_0^1 x\, u^2\; dx = [x^2\; u'^2 + (a^2 x^2 - \upsilon^2)\; u^2]_0^1 \tag{59}$$

When $x = 0$, the expression in brackets vanishes, and since $u'(1) = a\, J'_\upsilon(a)$, Eqn.(59) yields

$$\int_0^1 x\; J_\upsilon^2(ax)\; dx = \frac{1}{2}\; J'^2_\upsilon(a) + \frac{1}{2}\left(1 - \frac{\upsilon^2}{a^2}\right) J_\upsilon^2(a)$$

Putting $a = \lambda_n$, we obtain

$$\int_0^1 x\ J_\upsilon^2(\lambda_n x)\,dx = \frac{1}{2}\ J_\upsilon'^2(\lambda_n) = \frac{1}{2}\ J_{\upsilon+1}^2(\lambda_n) \tag{60}$$

where we have used the relation $J'_\upsilon(x) - \frac{\upsilon}{x} J_\upsilon(x) = J_{\upsilon+1}(x)$ in Eqn.(60).

Thus formula (53) is completely established.

As an illustration of this formula let us see how it is used in determining the coefficients of the series (52).

If an expansion of the form (52) is assumed to be possible, then multiplying Eqn.(52) throughout by $x\,J_\upsilon(\lambda_m x)$, integrating term by term from 0 to 1, and using formula (53), we obtain

$$\int_0^1 x\,f(x)\ J_\upsilon(\lambda_m x)\,dx = \frac{a_m}{2}\ J_{\upsilon+1}^2(\lambda_m)$$

and on replacing m by n we obtain the following formula for a_n:

$$a_n = \frac{2}{J_{\upsilon+1}^2(\lambda_n)} \int_0^1 x\ f(x)\ J_\upsilon(\lambda_n x)\ dx \tag{61}$$

The series (52), with its coefficients calculated by (61), is called the **Bessel series** or sometimes the **Fourier-Bessel series** of the function $f(x)$.

As an illustration consider the following example.

Example 11: Compute the Bessel series of the function $f(x) = 1$, for the interval $0 \le x \le 1$, in terms of the functions $J_\upsilon(\lambda_n x)$, where λ_n are the positive zeros of $J_0(x)$.

Solution: In this case Eqn.(61), yields

$$a_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 x\ J_0(\lambda_n x)\ dx$$

Now using the recurrence relation (29) for $\upsilon = 1$, we obtain

$$\int_0^1 x\ J_0(\lambda_n x)dx = \left[\frac{1}{\lambda_n} x\ J_1(\lambda_n x)\right]_0^1 = \frac{J_1(\lambda_n)}{\lambda_n}$$

$$\therefore \quad a_n = \frac{2}{\lambda_n J_1(\lambda_n)}$$

It follows then

$$1 = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_1(\lambda_n)}\ J_0(\lambda_n x), \quad (0 \le x \le 1)$$

is the desired Bessel series.

***

You may now try the following exercises.

E10) Show that the change of variables $y = u\,x^a$, $t = x^c$ and $z = bt$ reduces the differential equation

$$x^2 y'' + (1-2a)\,xy' + [b^2c^2x^{2c} + (a^2 - n^2c^2)]\,y = 0$$

to the Bessel's equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2 - n^2)\ u = 0.$$

E11) Prove the identity

$$1 = J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + \cdots$$

E12) If $f(x)$ is defined by

$$f(x)=\begin{cases}1, & 0\le x\le \frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ 0, & \frac{1}{2}<x\le 1\end{cases}$$

show that

$$f(x)=\sum_{n=1}^{\infty}\frac{J_1(\lambda_n/2)}{\lambda_n J_1^2(\lambda_n)}J_0(\lambda_n x)$$

where the λ_n are the positive zeros of $J_0(x)$.

E13) If $f(x)=x^{\upsilon}$ for the interval $0\le x<1$, show that its Bessel series in the functions $J_\upsilon(\lambda_n x)$, where the λ_n are the positive zeros of $J_\upsilon(x)$, is

$$x^{\upsilon}=\sum_{n=1}^{\infty}\frac{2}{\lambda_n J_{\upsilon+1}(\lambda_n)}J_\upsilon(\lambda_n x).$$

We have shown in Sec.4.2 that when υ is not an integer then $J_\upsilon(x)$ and $J_{-\upsilon}(x)$ are linearly independent solutions of Eqn.(1) and the complete solution is given by

$$y(x)=A J_\upsilon(x)+B J_{-\upsilon}(x).$$

However, when $\upsilon=n$ is an integer, $J_n(x)$ and $J_{-n}(x)$ are linearly dependent since $J_{-n}(x)=(-1)^n J_n(x)$. We therefore have only one independent solution. In order to find the general solution, we now obtain the second linearly independent solution of the Bessel's equation.

4.2.2 Bessel Function of the Second Kind

For obtaining the second solution of the Bessel's equation for the case when $\upsilon=n$ is an integer, we once again make use of the power series method discussed in Unit 2. Let us illustrate the method for the case $n=0$, for Bessel's equations of order zero

$$x y''+y'+xy=0 \qquad (62)$$

If we substitute the series solution $y(x)=\sum_{m=0}^{\infty}c_m x^{m+k}$ in Eqn.(62), we have the case of repeated indicial roots $k=0$. If you recall Case 3, discussed in Sec.2.4.2, Unit 2, then you know that two linearly independent solutions are given by $[y(x)]_{k=0}$ and $[\partial y/\partial k]_{k=0}$. Accordingly, if $y_1(x)=[y(x)]_{k=0}$, then second solution is of the form

$$\begin{aligned}y_2(x)&=y_1(x)\ln x+[c_1 x+c_2 x^2+\cdots],\ x>0\\ &=J_0(x)\ln x+[c_1 x+c_2 x^2+\cdots]\end{aligned}$$

since $y_1(x)=J_0(x)$.

Further, we have

$$y_2'(x)=J_0'\ln x+\frac{1}{x}J_0+[c_1+2c_2x+3c_3x^2+\cdots]$$

$$y_2''(x)=J_0''\ln x+\frac{2}{x}J_0'-\frac{1}{x^2}J_0+[2c_2+6c_3x+\cdots]$$

Substituting the above values of $y_2(x)$, $y_2'(x)$ and $y_2''(x)$ in Eqn.(62), we obtain

$$\begin{aligned}&[xJ_0''+J_0'+xJ_0]\ln x+2J_0'+[c_1+4c_2x+(c_1+9c_3)x^2+\cdots\\ &\qquad +[c_{m-1}+(m+1)^2c_{m+1}]x^m+\cdots]=0 \qquad (63)\end{aligned}$$

The first term of Eqn(63) vanishes as $J_0(x)$ is a solution of Eqn.(62). Substituting

$$J_0'(x)=\sum_{m=1}^{\infty}\frac{(-1)^m 2m\,x^{2m-1}}{(m!)^2\,2^{2m}}=\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{m!(m-1)!2^{2m-1}}=\frac{-x}{2}+\frac{x^3}{(2!)\,2^3}-\frac{x^5}{(3!)\,(2!)\,2^5}+\cdots$$

in Eqn.(63), we obtain

$$\left[-x+\frac{x^3}{(2!)2^2}-\frac{x^5}{(3!)\,(2!)2^4}+\cdots\right]+[c_1+4c_2x+(c_1+9c_3)x^2+(c_2+16c_4)x^4+\cdots]=0$$

Equating the coefficients of even power of x to zero, we obtain

$$c_1=0, c_1+9c_3=0\Rightarrow c_3=0,\ldots,c_{2m-1}+(2m+1)^2c_{2m+1}=0, m=1,2,\ldots$$

Therefore, $c_1=0=c_3=c_5=\cdots$

Coefficient of x equated to zero yields,

$$-1+4c_2=0,\ \text{or}, \ c_2=\frac{1}{4}=\frac{1}{2^2}.$$

Equating the coefficient of x^{2m+1} to zero, we obtain

$$\frac{(-1)^{m+1}}{m!(m+1)!2^{2m}}+c_{2m}+(2m+2)^2\,c_{m+2}=0,\ \ m=1,2,\ldots$$

For $m=1$, we get

$$\frac{1}{(2!)2^2}+c_2+16c_4=0,\ \text{or}\ \ c_4=\frac{-1}{16}\left[\frac{1}{(2!)2^2}+\frac{1}{2^2}\right]=\frac{-1}{2^2 4^2}\left[1+\frac{1}{2}\right]$$

For $m=2$, we have

$$\frac{-1}{(3!)\,(2!)\,2^4}+c_4+36c_6=0$$

or, $$c_6=\frac{1}{6^2}\left[\frac{1}{(3!)\,(2!)\,2^4}+\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)\right]=\frac{1}{2^2.4^2.6^2}\left[1+\frac{1}{2}+\frac{1}{3}\right],\ldots$$

Therefore, the second linearly independent solution of Eqn.(62) is

$$y_2(x)=J_0(x)\ln x+\left[\frac{1}{2^2}x^2-\frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4+\frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6-\cdots\right]$$

and is called the **Neumann function of order zero.** However, any other linear combination of $y_1(x)$ and $y_2(x)$ can also be taken as the second linearly independent solution. In practice, it is convenient and standard to use

$$y_2^*(x)=\frac{2}{\pi}[\,y_2(x)+(\gamma-\ln 2)\,J_0\,(x)\,] \tag{64}$$

where γ is the **Euler constant** $(\gamma=0.5772157)$, as the second linearly independent solution. This solution is called the **Bessel function of the second kind of order zero**. It is denoted by $Y_0(x)$.

Hence,

$$Y_0(x)=\frac{2}{\pi}\left[J_0(x)\ln\left(\frac{x}{2}+\gamma\right)+\frac{x^2}{2^2}-\left(1+\frac{1}{2}\right)\frac{x^4}{2^4(2!)^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^6}{2^6(3!)^2}-\cdots\right] \tag{65}$$

It can be seen that $Y_0(x)\sim\frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2}+\gamma\right)$ as $x\to 0$. However, since $J_0(x)\sim 1$ and $|\ln x|>>\gamma-\ln 2$ for small x we deduce that $Y_0(x)$ has the logarithmic behaviour $Y_0(x)\sim\frac{2}{\pi}\ln x$ as $x\to 0$ (see Fig.3).

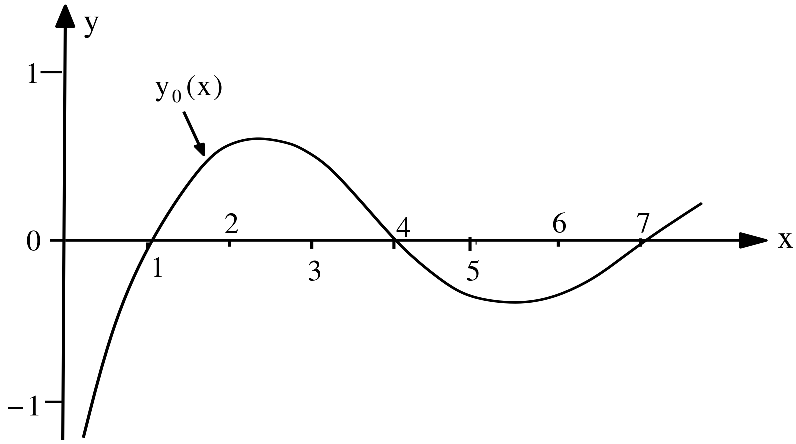

Fig.3: Bessel function of the second kind $Y_0(x)$

For $n = 1,2,\ldots$ the solution can be obtained in a similar manner. It can be seen that the indicial roots $k = \pm\, n$ differ by an integer and the second linearly independent solution always contains a logarithmic term (see Case 2, Sec.2.4.2, Unit 2).

This solution can be written as

$$Y_n(x) = J_n(x)\ln x - \frac{1}{2}\sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j-n}$$
$$-\frac{1}{2}\sum_{j=0}^{\infty} \frac{(-1)^j[\phi(j)+\phi(j+n)]}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n} \qquad (66)$$

where, $\phi(0) = 0$ and $\phi(j) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$, for $j \geq 1$.

Thus, we have two different general solutions for Eqn.(1), depending on whether υ is an integer or not.

$y(x) = A\,J_\upsilon(x) + B\,J_{-\upsilon}(x)$, if υ is not an integer

and $y(x) = A\,J_n(x) + B\,Y_n(x)$, if $\upsilon = n$, an integer.

Without proof we state the result and define the **Bessel function of the second kind of order υ** as

$$Y_\upsilon(x) = \frac{(\cos\upsilon\pi)\,J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\upsilon\pi} \qquad (67)$$

and $Y_n(x) = \lim_{\upsilon\to n} Y_\upsilon(x)$

that is, the limit of $Y_\upsilon(x)$ as $\upsilon \to n$ $(n = 0,1,2,\ldots)$ give the same result as (66). Further, you may observe that for $\upsilon > 0$, $J_\upsilon(x) \sim 0$ for $x \to 0$ and thus

$$Y_\upsilon(x) \sim \frac{-J_{-\upsilon}(x)}{\sin\upsilon\pi}$$
$$\sim \frac{-(x/2)^{-\upsilon}}{\Gamma(1-\upsilon)\sin\upsilon\pi},\quad \upsilon > 0,\quad x \to 0$$

But since $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$, we finally obtain the asymptotic formula

$$Y_\upsilon(x) \sim -\frac{\Gamma(\upsilon)}{\pi}\left(\frac{2}{x}\right)^{\upsilon},\quad \upsilon > 0,\ x \to 0 \qquad (68)$$

Also since $J_\upsilon(x)$ and $Y_\upsilon(x)$ are linearly independent, the **general solution of the Bessel's differential equation for all values of** υ, is

$$\mathbf{y(x) = A\,J_\upsilon(x) + B\,Y_\upsilon(x)} \qquad (69)$$

where $Y_\upsilon(x)$ is defined by Eqn.(67).

Let us look at the following examples.

Example 12: Find the solution of the differential equation

$$x^2 y'' + xy' + (x^2 - 16)\, y = 0$$

Solution: The given equation is the Bessel's equation with $\upsilon = 4$. Its general solution is thus written as

$$y(x) = A\, J_4(x) + B\, Y_4(x)\,.$$

***

Example 13: Using the substitution $z = 2\sqrt{x}$, obtain the solution of the differential equation $xy'' + y' + y = 0$.

Solution: We have for $z = 2\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2} - \frac{1}{2x^{3/2}}\frac{dy}{dz}$$

and the given equation reduces to

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + zy = 0$$

This is the Bessel's equation of order zero. Its general solution is

$$y(z) = A\, J_0(z) + B\, Y_0(z)$$

or, $$y(x) = A\, J_0(2\sqrt{x}) + B\, Y_0(2\sqrt{x})$$

***

You may now try the following exercise.

E14) Using the substitutions given in E10) reduce the following equations to the Bessel's equations and hence find their solution

a) $x^2 y'' + xy' + \frac{1}{4}(x - n^2)\; y = 0$

b) $xy'' + y = 0$

c) $xy'' + y' + y/4 = 0$.

We shall now consider some applications of Bessel functions.

4.3 APPLICATIONS OF BESSEL FUNCTIONS

In this section we shall illustrate the role played by the Bessel functions in getting solutions of problems related to physical situations.

Vibrating Membrane

One of the simplest physical applications of Bessel functions occurs in Euler's theory of the vibration of a circular membrane. In this context a membrane is understood to be a uniform thin sheet of flexible material pulled taut into a state of uniform tension and clamped along a given closed curve. When this membrane is slightly displaced from its equilibrium position and then released, the restoring forces due to the deformation cause it to vibrate and the problem is to analyze this vibrational motion. The governing equation for a wide variety of applications involving vibration phenomena is the wave equation

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c^{-2}\frac{\partial^2 u}{\partial t^2} \qquad (70)$$

Here we consider the case of a circular membrane.

The Circular Membrane

We begin by determining the small displacements u of a thin circular membrane of unit radius whose edge is rigidly fixed (see. Fig.4). The governing equation for this problem is the wave Eqn.(70), where c is a physical constant having the dimension of velocity. We shall not be going into the details of formulating Eqn.(70) here. However, we mention that several simplifying assumptions lead to the formulation of partial differential Eqn.(70), and we hope that this equation describes the motion with a reasonable degree of accuracy.

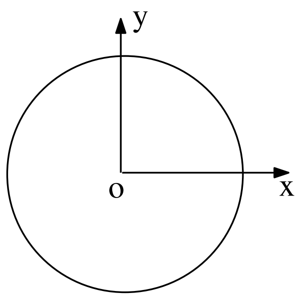

Fig.4

Since we are considering the case of circular membrane, it is natural to use polar coordinates with the origin located at the centre. If the displacement u depends only upon the radial distance r from the centre of the membrane and on time t, then Eqn.(70) reduces to the radial symmetric form of the wave equation given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=c^{-2}\frac{\partial^2 u}{\partial t^2}, \quad 0<r<1, \quad t>0 \tag{71}$$

where $u=u(r,\ t)$. Since we have assumed that the membrane is rigidly fixed on the boundary, we impose the boundary condition

$$u(1,\ t)=0 \ , \quad t>0 \tag{72}$$

If the membrane is initially deflected to the shape $f(r)$ with velocity $g(r)$, we also prescribe the initial conditions

$$u(r,0)=f(r)\ , \quad \frac{\partial u}{\partial t}(r,0)=g(r), \quad 0<r<1 \tag{73}$$

We now apply the method of separation of variables and look for a particular solution of the form

$$u(r,\ t)=R(r)\ T(t) \tag{74}$$

When values of u and its derivatives are inserted in Eqn.(71), we obtain

$$\frac{R''+\left(\frac{1}{r}\right)R'}{R}=\frac{T''}{c^2 T}=-\lambda \tag{75}$$

where $-\lambda$ is the separation constant (the choice of the negative sign is conventional, not necessary). Hence Eqn.(71) is equivalent to the system of ordinary differential equations

$$rR''+R'+\lambda rR=0\ , \quad 0<r<1 \tag{76}$$

$$T''+\lambda c^2 T=0 \ , \quad t>0 \tag{77}$$

Under the assumption $u(r,\ t)=R(r)T(t)$, the boundary condition(72) becomes

$$u(1,t)=R(1)\,T(t)=0$$

from which we deduce that

$$R(1)=0 \tag{78}$$

If you look at Eqn.(76), you will notice that it is a generalised form of Bessel's equation of order zero. By writing $\lambda=k^2>0$, the general solution is obtained as

$$R(r)=C_1\,J_0(kr)+C_2 Y_0(kr)\,, \tag{79}$$

where C_1 and C_2 are constants.

Note that for $\lambda\le 0$, Eqn.(76) has no bounded non-trivial solutions satisfying condition (78).

To maintain finite displacements of the membrane at $r=0$, we must set $C_2=0$, since Y_0 becomes unbounded when the argument is zero. The remaining solution $R(r)=C_1 J_0(kr)$ must then satisfy the boundary condition (78), i.e.

$$R(1)=C_1 J_0(k)=0 \tag{80}$$

You know that the Bessel function J_0 has infinitely many zeros on the positive axis, but they are not evenly spaced. Thus for $C_1 \neq 0$, we can satisfy Eqn.(80) by selecting k as one of the zeros of J_0, which we denote by $k_n\,(n = 1, 2, 3, \ldots)$. With k so restricted, we set $\lambda = k_n^2\,(n = 1, 2, \ldots)$ in Eqn.(77) to obtain

$$T'' + k_n^2\, c^2\, T^2 = 0$$

which has the general solution

$$T_n(t) = a_n \cos k_n ct + b_n \sin k_n ct, \quad n = 1, 2, \ldots \tag{81}$$

where the a's and b's are arbitrary constants.
Combining our result, we have the family of solutions

$$u_n(r,t) = (a_n \cos k_n ct + b_n \sin k_n ct)\, J_0(k_n r), \; n = 1, 2, \ldots \tag{82}$$

These solutions are called **standing waves**, since each can be viewed as having fixed shape $J_0(k_n r)$ with varying amplitude $T_n(t)$. The zeros of a standing wave, i.e. curves for which $J_0(k_n r) = 0$, are referred to as **nodal lines.** Clearly the number of nodal lines depends on the value of n. For example, when $n = 1$, there is no nodal line for $0 < r < 1$. There is one nodal line when $n = 2$; there are two nodal lines when $n = 3$, and so on (see Fig.5).

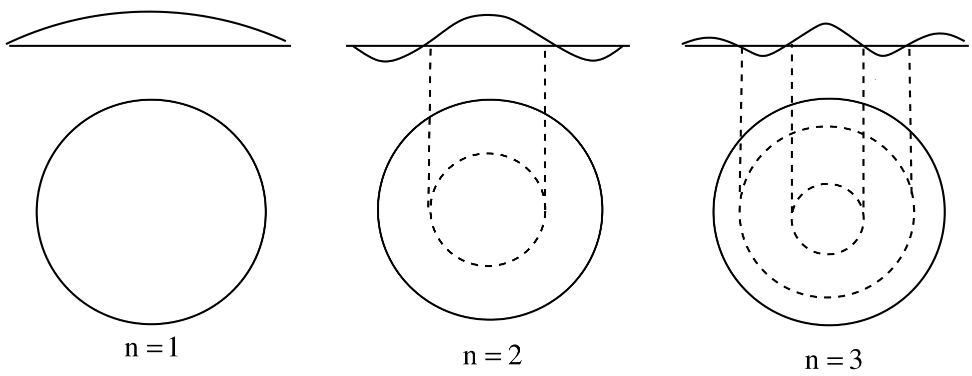

Fig.5: Nodal lines for a circular membrane

By forming a linear combination of the solution (82) through the superposition principle, we obtain

$$u(r,t) = \sum_{n=1}^{\infty} (a_n \cos k_n ct + b_n \sin k_n ct)\; J_0(k_n r) \tag{83}$$

The constants a_n and b_n are selected in such a way that the initial conditions (73) are satisfied. Therefore, we find

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} a_n J_0(k_n r)\,, \quad 0 < r < 1 \tag{84}$$

and

$$\frac{\partial u}{\partial t}(r,0) = g(r) = \sum_{n=1}^{\infty} k_n c\, b_n\, J_0(k_n r), \; 0 < r < 1 \tag{85}$$

Eqns.(84) and (85) are the **Bessel series** for $f(r)$ and $g(r)$, respectively, where

$$a_n = \frac{2}{[J_1(k_n)]^2} \int_0^1 r\, f(r)\, J_0(k_n r)\; dr, \; n = 1, 2, \ldots \tag{86}$$

and

$$k_n c\, b_n = \frac{2}{[J_1(k_n)]^2} \int_0^1 r\; g(r)\, J_0(k_n r)\, dr, \quad n = 1, 2, \ldots \tag{87}$$

give the coefficients of the series (83) and the required solution is completely determined.

Before we consider another application of Bessel functions you may try the following exercises.

E15) Show that if $u = u(r, t)$, then in polar coordinate system Eqn.(70) reduces to Eqn.(71).

E16) If the initial conditions (73) are given by

$$u(r, 0) = A\ J_0(k_3 r) \quad (A\ \text{constant})$$

$$\frac{\partial u}{\partial r}(r, 0) = 0$$

where $J_0(k_3) = 0$, show that the subsequent displacements of the membrane are described by

$$u(r, t) = A\, J_0(k_3 r) \cos k_3 ct .$$

E17) If the initial conditions (73) are given by $f(r) = 0$ and $g(r) = 1$, show that the subsequent displacement of the membrane are described by

$$u(r, t) = \frac{2}{c} \sum_{n=1}^{\infty} \frac{\sin k_n ct}{k_n^2\, J_1(k_n)}\, J_0(k_n r)$$

where, $J_0(k_n) = 0 \quad (n = 1, 2, \ldots)$.

Let us consider another application of the Bessel functions.

Buckling of a Long Column

One of the oldest engineering problems concerns the buckling of vertical columns under a compressive load. The first mathematical model to accurately predict the **critical compressive** load that a column can withstand before deformation or buckling takes place was developed by Euler (1707 – 1783).

Let us consider a long column or rod of length b that is simply supported at each end and is subject to an axial compressive load P applied at the top, as shown in Fig.6.

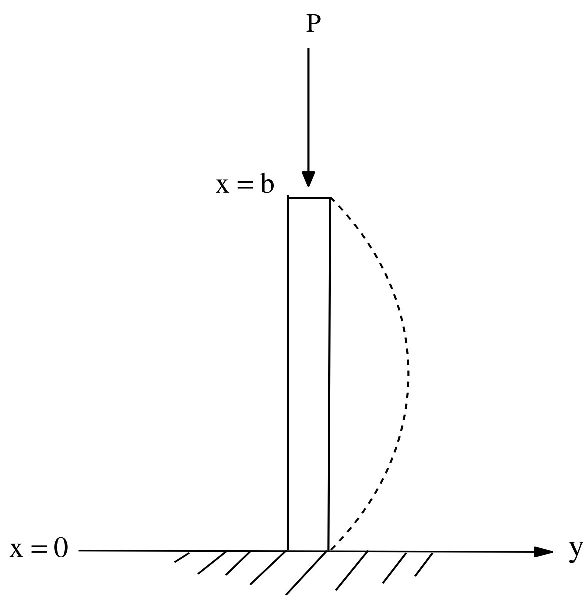

Fig.6: Buckling Column

By 'long' we mean that the length of the column is much greater than the largest dimension in its cross section. From the elementary theory of small deflections of beams and columns, the departure y from the vertical x-axis for such a beam or column is governed by the equation

$$EI\,y'' = M \tag{88}$$

Where M is the **bending moment**, E is the **modulus of elasticity**, and I is the **moment of inertia** of the cross section of the column. When the column is deflected a small amount y from the vertical due to the loading P, the bending moment is $M = -Py$. Since the column is assumed to be simply supported, there can be no displacement at the ends. Thus Eqn.(88) together with the boundary conditions takes the form

$$EI\,y'' + Py = 0\ ,\quad y(0) = 0\ ,\quad y(b) = 0 \tag{89}$$

The problem described by Eqn.(89), clearly has the trivial solution $y = 0$, corresponding to the column not bending away from the x-axis. However, if P is large enough, there may exist non-trivial solution of Eqn.(89). That is, if P is sufficiently large, the column may suddenly come out of its equilibrium state, called a state of **buckling**. The smallest value of P that leads to buckling is called the **Euler critical load**, and the corresponding deflection mode is called the **fundamental buckling mode.**

The classical example of a buckling column involves the case when E, I, and P are all constant, and then it can be easily shown that the Euler critical load P_1, and corresponding deflection mode are given, respectively, by

$$P_1 = \frac{\pi^2 EI}{b}\ ,\ y = C\sin\frac{\pi x}{b} \tag{90}$$

where C is an arbitrary constant. The constant C remains undetermined here as the model is linear.

We now consider the case where the column is taperred so the moment of inertia is not constant, but is given by $I(x) = \alpha x$, where $\alpha > 0$. In this case Eqn.(89) becomes

$$E\alpha x\,y'' + Py = 0\ ,\ y(0) = 0\ ,\ y(b) = 0 \tag{91}$$

We rewrite the above equation in the form

$$x^2 y'' + k^2 xy = 0 \tag{92}$$

where $k^2 = P/E\alpha$. Further it can be easily verified that first substituting $y = \sqrt{x}\,z$, and then $t = \sqrt{x}$ Eqn.(92) reduces to the form

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (4k^2t^2 - 1)\ z = 0 \tag{93}$$

We see that Eqn.(93) is the Bessel's equation of order one and its solution can be written as

$$z(t) = C_1 J_1(2kt) + C_2 Y_1(2kt)$$

or, $$y(x) = \sqrt{x}\ [C_1 J_1(2k\sqrt{x}) + C_2 Y_1(2k\sqrt{x})] \tag{94}$$

where C_1 and C_2 are arbitrary constants.

To apply the first boundary condition $y(0) = 0$, we use the asymptotic forms for both J_1 and Y_1; hence using formulas (27) and (68), we obtain

$$y(0) = \lim_{x\to 0}\sqrt{x}\ [C_1 J_1(2k\sqrt{x}) + C_2\,Y_1(2k\sqrt{x})]$$

$$= \lim_{x\to 0}\sqrt{x}\left[C_1\,k\sqrt{x} - C_2\frac{1}{\pi k\sqrt{x}}\right]$$

$$= -C_2\frac{1}{\pi k} \tag{95}$$

which can vanish only if $C_2 = 0$. The second boundary condition then requires that

$$y(b) = C_1\sqrt{b}\,J_1(2k\sqrt{b}) = 0 \tag{96}$$

There are infinitely many solutions of $J_1(2k\sqrt{b}) = 0$, the first of which yields the Euler critical load P_1. That is, if we let k_1 denote the smallest value of k for which Eqn.(96) is satisfied, the corresponding Euler critical load is

$$P_1 = E\alpha k_1^2 \tag{97}$$

leading to the fundamental deflection mode

$$y = C_1\sqrt{x}\ J_1(2k_1\sqrt{x}) \tag{98}$$

where C_1 remains undetermined.

E18) Show that the solution of the boundary value problem (89) for constant E, I and P gives the critical load $P_1 = \pi^2 \dfrac{EI}{b^2}$ and corresponding deflection $y = C\sin\dfrac{\pi x}{b}$, where C is an arbitrary constant.

E19) Verify that on substituting $y = \sqrt{x}z$ and $t = \sqrt{x}$, Eqn.(92) reduces to Eqn.(93).

We now end this unit by giving a summary of what we have covered in it.

4.4 SUMMARY

In this unit we have learnt the following points:

1. The differential equation $x^2y'' + xy' + (x^2 - \upsilon^2)y = 0$, where υ is a non-negative real number is known as **Bessel's equation of order** υ and its solutions are known as **Bessel functions.**
2. When υ **is not an integer,** one of the linearly independent series solution of Bessel's equation is the **Bessel function of the first kind of order** υ and is denoted by $J_\upsilon(x)$ where
$$J_\upsilon(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)}\left(\frac{x}{2}\right)^{2m+\upsilon}$$
3. The second linearly independent solution of Bessel's equation is $J_{-\upsilon}(x)$ and thus the **general solution of Bessel's equation when** υ **is not an integer** is $y(x) = A\,J_\upsilon(x) + B\,J_{-\upsilon}(x)$, where A and B are arbitrary constants.
4. When υ**= n is an integer** then $J_n(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{m!(m+n)!}\left(\dfrac{x}{2}\right)^{2m+n}$ is one linearly independent solution of the Bessel's equation and the second linearly independent solution is denoted by $Y_n(x)$, and given by
$$Y_n(x) = J_n(x)\ln x - \frac{1}{2}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{x}{2}\right)^{2j-n} - \frac{1}{2}\sum_{j=0}^{\infty}(-1)^j\frac{[\phi(j)+\phi)(j+n)]}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n}$$
where $\phi(0) = 0$ and $\phi(j) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{j}$ for $j \ge 1$.
5. The **Bessel function of the second kind of order** υ is defined as
$$Y_\upsilon(x) = \frac{(\cos\upsilon x)\ J_\upsilon(x) - J_{-\upsilon}(x)}{\sin\ \upsilon x}$$
where $Y_n(x) = \lim_{\upsilon\to n} Y_\upsilon(x)$ and the general **solution of the Bessel's differential equation for all values of** υ**, whether** υ **is an integer or not,** is $y(x) = AJ_\upsilon(x) + B\,Y_\upsilon(x)$.
6. Some of the **recurrence relations** for Bessel functions are

$$\frac{d}{dx}[x^{\upsilon}J_{\upsilon}(x)] = x^{\upsilon}J_{\upsilon-1}(x)$$
$$\frac{d}{dx}[x^{-\upsilon}J_{\upsilon}(x)] = -x^{-\upsilon}J_{\upsilon+1}(x)$$
$$x\,J_{\upsilon}'(x) + \upsilon J_{\upsilon}(x) = x\,J_{\upsilon-1}(x)$$
$$x\,J_{\upsilon}'(x) - \upsilon J_{\upsilon}(x) = -x\,J_{\upsilon+1}(x)$$
$$2\,J_{\upsilon}'(x) = J_{\upsilon-1}(x) - J_{\upsilon+1}(x)$$
$$2\upsilon J_{\upsilon}(x) = xJ_{\upsilon-1}(x) + xJ_{\upsilon+1}(x)$$

7. The function on the left side of equation

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \ \ t \neq 0$$

is called the **generating function** of the Bessel function $J_n(x)$.

8. **Orthogonality property** of Bessel function $J_{\upsilon}(x)$ is

$$\int_0^1 x\,J_{\upsilon}(\lambda_m x)\,J_{\upsilon}(\lambda_n x)\,dx = \begin{cases} 0 & , \text{ if } \ m \neq n \\ \dfrac{1}{2}J_{\upsilon+1}^2(\lambda_n), & \text{if } \ m = n \end{cases}$$

where λ_n and λ_m are the positive zeros of Bessel functions $J_{\upsilon}(x)$ with $\upsilon \geq 0$.

4.5 SOLUTIONS/ANSWERS

E1) $$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\Gamma(r+3/2)} = \frac{(x/2)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{1!\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\Gamma(7/2)} - \cdots$$
$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}} - \frac{(x/2)^{5/2}}{1!(3/2)(1/2)\sqrt{\pi}} + \frac{(x/2)^{7/2}}{2!(5/2)(3/2)(1/2)\sqrt{\pi}} - \cdots$$
$$= \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\left\{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right\} = \frac{(x/2)^{1/2}}{(1/2)\sqrt{\pi}}\frac{\sin x}{x} = \sqrt{\frac{2}{\pi x}}\sin x$$
$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{1/2+2r}}{r!\Gamma(r+1/2)} = \frac{(x/2)^{-1/2}}{\Gamma(1/2)} - \frac{(x/2)^{3/2}}{1!\Gamma(3/2)} + \frac{(x/2)^{7/2}}{2!\Gamma(5/2)} - \cdots$$
$$= \frac{(x/2)^{-1/2}}{\sqrt{\pi}}\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right\} = \sqrt{\frac{2}{\pi x}}\cos x$$

E2) a) $y'' + k^2xy = 0$

Put $y = u\sqrt{x}$ and $z = \frac{2k}{3}x^{3/2}$

$$y' = \frac{d}{dx}\left(u\sqrt{x}\right) = \sqrt{x}\,u' + \frac{1}{2\sqrt{x}}u$$
$$y'' = \sqrt{x}\,u'' + \frac{1}{2\sqrt{x}}u' + \frac{1}{2\sqrt{x}}u' - \frac{1}{4(x)^{3/2}}\,u$$

Then , $y'' + k^2xy = 0 \Rightarrow \sqrt{x}\,u'' + \frac{1}{\sqrt{x}}u' - \frac{1}{4(x)^{3/2}}u + k^2(x)^{3/2}u = 0$

$$u' = \frac{du}{dx} = \frac{du}{dz}.\frac{dz}{dx} = \frac{2k}{3}.\frac{3}{2}.x^{1/2}.\frac{du}{dz} = kx^{1/2}\frac{du}{dz}$$
$$u'' = \frac{d}{dx}\left(kx^{1/2}\frac{du}{dz}\right) = \frac{k}{2\sqrt{x}}\frac{du}{dz} + kx^{1/2}\frac{d^2u}{dz^2}.\frac{dz}{dx} = \frac{k}{2\sqrt{x}}\frac{du}{dz} + k^2x\frac{d^2u}{dz^2}$$

The given equation reduces to

$$\sqrt{x}\left[k^2x\frac{d^2u}{dz^2} + \frac{k}{2\sqrt{x}}\frac{du}{dz}\right] + \frac{1}{\sqrt{x}}\left[k\sqrt{x}\frac{du}{dz}\right] + \left(k^2x^{3/2} - \frac{1}{4x^{3/2}}\right)u = 0$$

or, $\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left[1-\left(\frac{1}{3z}\right)^2\right]u=0$

or, $z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+\left(z^2-\left(\frac{1}{3}\right)^2\right)u=0$, which is Bessel's equation of order $1/3$ and its solution is

$u=J_{1/3}(z)$

$\Rightarrow y=\sqrt{x}\,u=\sqrt{x}\,J_{1/3}(z)=\sqrt{x}\;J_{1/3}\left(\frac{2k}{3}x^{3/2}\right)$.

b) Put $y=u\sqrt{x}$ and $z=\frac{kx^2}{2}$, then the given equation transforms to

$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+\left(z^2-\left(\frac{1}{4}\right)^2\right)u=0$, which is Bessel's equation of order $\frac{1}{4}$.

Hence, $y=\sqrt{x}\;u=\sqrt{x}\;J_{1/4}\left(\frac{kx^2}{2}\right)$.

c) Put $y=u\sqrt{x}$ and $z=\frac{kx^3}{3}$

Then given equation transforms to

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+\left[z^2-\left(\frac{1}{6}\right)^2\right]u=0,$$

which is Bessel's equation of order $1/6$ and its solution is

$$y=\sqrt{x}u=\sqrt{x}\;J_{1/6}(z)=\sqrt{x}\;J_{1/6}\left(\frac{kx^3}{3}\right).$$

E3) a) $z=\sqrt{x}\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\frac{dy}{dz},\ \frac{d^2y}{dx^2}=\frac{1}{4x}\frac{d^2y}{dz^2}-\frac{1}{4x^{3/2}}\frac{dy}{dz}$

Substituting these values in the given equation, we obtain

$x\frac{d^2y}{dz^2}+\sqrt{x}\frac{dy}{dz}+(x-\upsilon^2)\;y=0$

$\therefore y=A\,J_\upsilon(\sqrt{x})+B\,J_{-\upsilon}(\sqrt{x})$

b) $y=xu\Rightarrow y'=u+xu'$ and $y''=2u'+xu''$. The given equation reduces to $x^2u''+xu'+(x^2-1)\;u=0$

Its general solution is

$u=A\,J_1(x)+B\,J_{-1}(x)$, or, $y=x\,C\,J_1(x)$ (since $J_{-1}(x)=-J_1(x)$)

c) $y=u/x^k$, $y'=x^{-k}u'-kx^{-k-1}u$

$y''=x^{-k}u''-2kx^{-k-1}u'+k(k+1)\,x^{-k-2}u$

The given equation reduces to

$x^2u''+xu'+(x^2-k^2)\;u=0$

Its general solution is

$u=A\,J_k(x)+B\,J_{-k}(x)$

or, $y=x^{-k}\left[A\,J_k(x)+B\,J_{-k}(x)\right]$

E4) $J_n(-x)=(-1)^nJ_n(x)=J_n(x)$, for n even

and $J_n(-x)=(-1)^nJ_n(x)=-J_n(x)$, for n odd

E5) $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$

$$\therefore J_0'(x) = -\frac{x}{2} + \frac{4x^3}{2^4(2!)^2} - \frac{6x^5}{2^6(3!)^2} + \frac{8x^7}{2^8(4!)^2} - \cdots$$

$$= -\left[\frac{x}{2} - \frac{x^3}{2^3!2!} + \frac{x^5}{2^5 2!3!} - \frac{x^7}{2^7 3!4!} + \cdots\right] = -J_1(x)$$

E6) $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$, $J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

E7) a) Set $\upsilon = 2$ in Eqn.(32) and $\upsilon = 1$ in Eqn.(37) and simplify.
b) Take $\upsilon = n$ and differentiate Eqn.(36). Use Eqn.(36) again.
c) $2J_\upsilon(x)J_\upsilon'(x) = J_\upsilon(x)\ (J_{\upsilon-1}(x) - J_{\upsilon+1}(x))$. Using Eqns.(32) and (33), we get

$J_\upsilon = \frac{x}{2\upsilon}(J_{\upsilon-1} + J_{\upsilon+1})$; hence $2J_\upsilon(x)J_\upsilon'(x) = \frac{x}{\upsilon}(J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x))$

Or, $[J_\upsilon^2(x)]' = \frac{x}{2\upsilon}\ (J_{\upsilon-1}^2(x) - J_{\upsilon+1}^2(x))$

d) Using $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$ for $n = -1/2$ and values for $J_{1/2}(x)$ and $J_{-1/2}(x)$, required solution is obtained.
e) Use the expressions for $J_{1/2}(x)$ and $J_{-1/2}(x)$.

E8) We have $J_0'(x) = -J_1(x)$. Differentiating , we get

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}[J_0(x) - J_2(x)] \qquad \text{[using Eqn.(36)]}$$

Differentiating again, we get

$$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)] = -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}(J_1(x) - J_3(x))\right] \qquad \text{[(using Eqn.(36)]}$$

or, $4J_0'''(x) + 2J_0'(x) - J_1(x) + J_3(x) = 0$

or, $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$, since $J_1(x) = -J_0'(x)$.

E9) a) Use Eqn.(36) with $\upsilon = n$.
b) Write $J_2(x) = J_0(x) - 2J_1'(x)$ and integrate
c) Write $J_5(x) = x^4(x^{-4}\ J_5(x))$ and integrate. Use Eqn.(31) repeatedly.
d) Write $x^3 J_0 = x^2(x\ J_0)$ and use Eqn.(29). Then use Eqn.(37).
e) $\int x^5 J_0\, dx = x^5 J_1 - 4\int x^4 J_1\, dx$

$$= x^5 J_1 - 4x^4 J_2 + 8\int x^3 J_2\, dx = x^5 J_1 - 4x^4 J_2 + 8x^3 J_2 + C$$

f) Let $r\,x = t$ and use Eqn.(29).
g) Using Eqn.(29), we obtain

$$\int_0^1 (x - x^3) J_0\, dx = 2J_2(1) = 2[2J_1(1) - J_0(1)] \qquad \text{[using Eqn.(37)]}$$

h) Integrate by parts, use Eqn.(32) and again integrate by parts.

E10) Use the transformations $y = ux^a$, $t = x^c$ and $z = bt$, successively and get the required result.

E11) From Example 7, $J_n(x+y) = \sum_{k=-\infty}^{\infty} J_{n-k}(x) J_k(y)$

For $n = 0$, we have

$$J_0(x+y) = \sum_{k=-\infty}^{\infty} J_{-k}(x)\, J_k(y)$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} J_{-k}(x)J_k(y) + \sum_{k=1}^{\infty} J_k(x)J_{-k}(y)$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k [J_k(x)J_k(y) + J_k(x)J_k(y)]$$

$$= J_0(x)J_0(y) + \sum_{k=1}^{\infty} (-1)^k 2J_k(x)J_k(y)$$

or, $J_0(x+y) = J_0(x)J_0(y) - 2J_1(x)J_1(y) + 2J_2(x)J_2(y) - \cdots$

Replacing y by $-x$ and using the fact that $J_n(x)$ is even or odd according as n is even or odd, the required identity is obtained.

E12) We have, $a_n = \dfrac{2}{J_1^2(\lambda_n)} \displaystyle\int_0^1 x\, f(x)\, J_0(\lambda_n x)\, dx$

Now $\displaystyle\int_0^1 x\, f(x) J_0(\lambda_n x)\, dx = \int_0^{1/2} x\, J_0(\lambda_n x)\, dx = \frac{x}{\lambda_n} J_1(\lambda_n x)\Big|_0^{1/2} = \frac{1}{2\lambda_n} J_1(\lambda_n/2)$

$$\therefore a_n = \frac{1}{J_1^2(\lambda_n)} \frac{1}{\lambda_n} J_1(\lambda_n/2)$$

and it follows that, $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{J_1(\lambda_n/2)}{\lambda_n J_1^2(\lambda_n)} J_0(\lambda_n x)$

E13) We have, $a_n = \dfrac{2}{J_{\upsilon+1}^2(\lambda_n)} \displaystyle\int_0^1 x^{\upsilon+1} J_\upsilon(\lambda_n x)\, dx$

Now $\displaystyle\int_0^1 x^{\upsilon+1} J_\upsilon(\lambda_n x)\, dx = \frac{1}{\lambda_n} x^{\upsilon+1} J_{\upsilon+1}(\lambda_n x)\Big|_0^1 = \frac{1}{\lambda_n} J_{\upsilon+1}(\lambda_n)$

$$\therefore a_n = \frac{2}{J_{\upsilon+1}^2(\lambda_n)} \frac{1}{\lambda_n} J_{\upsilon+1}(\lambda_n)$$

and $x^\upsilon = \displaystyle\sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{\upsilon+1}(\lambda_n)} J_\upsilon(\lambda_n x)$

E14) a) Comparing with E10), we get $a = 0,\ c = \dfrac{1}{2},\ b = 1$, therefore

$y = u, t = \sqrt{x}, z = t$ and $y = A\, J_n(\sqrt{x}) + B\, J_{-n}(\sqrt{x})$ or,

$A\, J_n(\sqrt{x}) + B\, Y_n(\sqrt{x})$, depending on n being a non-integer or an integer.

b) Comparing with E10), we get $a = 1/2,\ c = 1/2,\ b = 2,\ n = 1$ or

$y = u\sqrt{x},\ t = \sqrt{x},\ z = 2t,\ y = \sqrt{x}\,[A\, J_1(2\sqrt{x}) + B\, Y_1(2\sqrt{x})]$.

c) Comparing with E10), we get $y = u,\ t = \sqrt{x},\ z = t$,

$y = A\, J_0(\sqrt{x}) + B\ Y_0(\sqrt{x})$.

E15) Define $x = r\cos\theta,\ y = r\sin\theta$

Use the chain rule to show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

since u does not depend on θ, deduce that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}$$

E16) We have the solution given by Eqn.(83) as

$u(r,t)=\sum_{n=1}^{\infty}(a_n\cos k_n ct+b_n\sin k_n ct)J_0(k_n r)$. We use the given initial conditions to obtain a_n's and b_n's.

$$u(r,0)=A\,J_0(k_3 r)=\sum_{n=1}^{\infty}a_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\frac{\partial u}{\partial t}(r,0)=0=\sum_{n=1}^{\infty}k_n c b_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\Rightarrow b_n=0,\ n=1,2,\ldots$$

and a_n's are given by

$$a_n=\frac{2}{J_1^2(k_n)}\int_0^1 r\ AJ_0(k_3 r)J_0(k_n r)dr,\quad 0<r<1$$

Now since, $\int_0^1 rJ_0(k_3 r)J_0(k_n r)\,dr=0$, if $n\neq 3$

$$=\frac{1}{2}J_1^2(k_3),\text{ for } n=3$$

$\therefore\ a_n=0$ for $n\neq 3$

$$=\frac{2A}{J_1^2(k_3)}\frac{1}{2}J_1^2(k_3)=A\ \text{ for } n=3$$

$\therefore$ solution (83) reduces to

$$u(r,t)=AJ_0(k_3 r)\cos k_3 ct\ .$$

E17) The given initial conditions are

$$u(r,0)=0\text{ and }\frac{\partial u}{\partial t}(r,0)=1$$

Therefore, we have from Eqn.(83)

$$0=\sum_{n=1}^{\infty}a_n J_0(k_n r)\ ,\quad 0<r<1$$

$$\Rightarrow a_n=0\ ,\ n=1,2,\ldots$$

and $1=\sum_{n=1}^{\infty}k_n c b_n J_0(k_n r)\ ,\ 0<r<1$

$$\therefore k_n c b_n=\frac{2}{J_1^2(k_n)}\int_0^1 r\,J_o(k_n r)\ dr$$

Now $\int_0^1 r\,J_0(k_n r)\,dr=\frac{1}{k_n}r\,J_1(k_n r)\Big|_0^1=\frac{1}{k_n}J_1(k_n)$

$$\therefore b_n=\frac{2}{c}\frac{1}{k_n^2\,J_1(k_n)}$$

and Eqn.(83) reduces to

$$u(r,t)=\frac{2}{c}\sum\frac{\sin k_n ct}{k_n^2\,J_1(k_n)}\ J_0(k_n r)$$

E18) Eqn.(89) gives $y'' + \frac{P}{EI} y = 0$, its solution is given by

$$y(x) = B\cos\sqrt{\frac{P}{EI}}\,x + C\sin\sqrt{\frac{P}{EI}}\,x,\ B \text{ and } C \text{ arbitrary constants}$$

$$y(0) = 0 \Rightarrow B = 0, \text{ therefore } \quad y = C\sin\sqrt{\frac{P}{EI}}\,x$$

$$y(b) = C\sin\sqrt{\frac{P}{EI}}\,b = 0, \text{ if } C \neq 0 \text{ then } \sqrt{\frac{P}{EI}}\,b = \pi \text{ or, } P = \frac{EI\pi^2}{b^2}$$

$$\therefore \quad y(x) = C\sin\frac{\pi x}{b}$$

E19) Substituting $y = \sqrt{x}\,z$ in Eqn.(92) it reduces to

$$x^2 z'' + xz' + \left(k^2 x - \frac{1}{4} z\right) = 0$$

Putting $t = \sqrt{x}$ in the above equation, Eqn.(93) is obtained.

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