UNIT 2 POWER SERIES SOLUTIONS

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2.1 INTRODUCTION

In Unit 1, we recalled the methods of finding explicit solutions of linear differential equations with constant coefficients and Euler-Cauchy equations which are reducible to equations with constant coefficients by a change of independent variable and illustrated these methods through various examples. The class of functions, appearing as solution of these equations involved elementary functions, i.e., exponential, trigonometric, logarithmic and polynomial functions, all of which can be written in terms of exponential functions. In many engineering, mathematical and physical problems, the equations governing the physical situations turn out to be homogeneous, linear differential equations with variable coefficients. Solutions of such equations cannot be expressed in terms of elementary functions. But since elementary functions can be expressed in terms of power series; this gives us an idea that for a general homogeneous, linear differential equation with variable coefficients, we might try a function defined by a general power series

$$y = \sum_{n=0}^{\infty} a_n \, (x - x_0)^n$$

as a solution, where the series on the right being meaningful only when it is convergent. We shall see that this gives a powerful method of solving linear homogeneous differential equations with variable coefficients and leads to the introduction of some new 'standard functions', commonly known as 'higher transcendental functions' or 'special functions' such as Legendre, Bessel, Hypergeometric, Hermite functions etc. The appearance of these special functions as solutions of differential equations describing various phenomena in physics and engineering indicates that these functions are as 'standard' as our earlier 'elementary functions'. In this unit we shall discuss methods of finding power series solutions of linear homogeneous differential equations with variable coefficients.

In Sec.2.2, we have started the discussion with the introduction of power series and its radius of convergence. We have discussed ordinary and singular points of a differential equation in Sec.2.3. In Sec.2.4 we have discussed power series solutions of a differential equation about an ordinary point and series solutions about a regular singular point – in particular, the Frobenius method. We shall be discussing the applications of these power series methods to various physical situations, in Units 3 and 4.

Objectives

After studying this unit you should be able to

- describe the need for a series solution of homogeneous linear differential equation with variable coefficients;
- define the radius of convergence and interval of convergence, of a power series;
- distinguish between an ordinary point, a regular singular point and an irregular singular point of a differential equation;

- obtain a power series solution of a given differential equation about an ordinary point;
- obtain a series solution of a given differential equation about a regular singular point using Frobenius method.

2.2 POWER SERIES AND RADIUS OF CONVERGENCE

You are already familiar with the series of **constant** terms i.e., series of the form,

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

If the terms of a series are **variable**, say, function of a variable x, then they assume definite values when x is given a fixed value. The basic idea of the power series method for solving differential equations, which is our concern in this unit, is very simple and natural. Given a differential equation, we represent all the functions in the equation by a power series. This process involves various operations like differentiation, addition and multiplication of power series. Therefore, to justify the method let us first consider the theoretical basis of these operations.

A power series in powers of $(x - a)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \qquad (1)$$

where x is a variable, $a_0, a_1 \ldots$ are constants, called the **coefficients**, and a is a constant, called the **centre** of the series. For discussion in this unit we shall assume that all variables and constants are real.

In particular, if $a = 0$ then we get a power series in powers of x as

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (2)$$

Remark: The term 'power series' usually refers to series of the form (1), including the case (2), but does not include series of negative powers of x such as $c_1 x^{-1} + c_2 x^{-2} + \cdots$ or, series involving fractional powers of x.

The convergence behaviour of a power series can be characterized in a simple way. We know from d'Alembert's ratio test for absolute convergence of a series that series (2) is absolutely convergent or divergent according as

$$\lim_{n\to\infty} \frac{|a_n x^n|}{|a_{n+1} x^{n+1}|} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \left| \frac{1}{x} \right|$$

is greater or less than one, provided the above limit exists.

Let $$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|, \qquad (3)$$

then subject to the existence of this limit, series (2) converges absolutely if

$$\frac{R}{|x|} > 1,\ x \neq 0, \text{ i.e., } |x| < R$$

and diverges if , $\frac{R}{|x|} < 1$, i.e., $|x| > R$

Thus the region of convergence is the interval $-R < x < R$ within which the series (2) is absolutely convergent and outside this interval it is divergent. Accordingly, series (1) converges in the interval.

$$a - R < x < a + R \qquad (4)$$

with centre $x = a$. The number R is known as the **radius of convergence** of series (1) and interval (4) is known as its **interval of convergence.** Here R is always non-negative and may assume the value 0 or ∞, or, any value in between.

Usually we put R equal to 0 when the series converges only for $x = 0$ and equal to ∞ when it converges for all x. This convention allows us to cover all possibilities in a single statement: each power series in x has a radius of convergence R, where $0 \le R \le \infty$ with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. You may note that if $R = 0$ then no x satisfies $|x| < R$, and if $R = \infty$ then no x satisfies $|x| > R$. We can thus say that all power series in x fall into one, or, another, or three major categories which are typified by the following examples

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots \qquad (5)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (6)$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (7)$$

Series(5) diverges (i.e. fails to converge) for all $x \neq 0$. Series (6) converges for all x and series (7) converges for $|x| < 1$ and diverges for $|x| > 1$. How R is calculated for the above three series will be clear to you after going through the following examples.

Example 1: Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} n^n x^n = 1 + x + 2^2 x^2 + 3^3 x^3 + \cdots$$

Solution: The given series has radius of convergence R given by

$$R = \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n} \right| = 0$$

Hence the series does not converge for any non-zero x. However, it obviously converges for $x = 0$.

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Example 2: Find the interval of convergence of the series $\sum_{n=0}^{\infty} n\,x^n$.

Solution: The radius of convergence of the given series is

$$R = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| \frac{1}{1+\frac{1}{n}} \right| = 1$$

Thus the given series converges for $|x| < 1$, i.e., $-1 < x < 1$ and diverges for $|x| > 1$.

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Let us now suppose that series (2) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$ then ,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (8)$$

Now $f(x)$ is obviously continuous and has derivatives of all orders for $|x| < R$. Also the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} n\,a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 3\ 2a_2 x + \cdots$$

and so on, and each of the resulting series converges for $|x| < R$. The formula linking the a_n to $f(x)$ and its derivatives can be obtained from these successively differentiated series as follows:

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{9}$$

Just as we have differentiated series (8) it can also be integrated termwise provided the limits of integration lie inside the interval of convergence. Further, if we have a second power series in x that converges to a function $g(x)$ for $|x| < R$, so that

$$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots \tag{10}$$

then series (8) and (10) an be added or subtracted termwise as

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1)x + \cdots$$

They can also be multiplied like polynomials, such that

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. If series (8) and (10) converge to the same functions, so that $f(x) = g(x)$ for $|x| < R$, then according to formula (8) they have the same coefficients: $a_0 = b_0$, $a_1 = b_1, \ldots$

In particular, if $f(x) = 0$ for $|x| < R$ then $a_0 = 0, a_1 = 0, \ldots$.

Now let us suppose that $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$ with $R > 0$. Then you may wonder whether $f(x)$ can be represented by a power series or not. If we use formula (9) to define the a_n then we can hope that the expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots \tag{11}$$

will hold throughout the interval. This is often true but sometimes it is false. To check the validity of this expansion for a specific point x in the interval we can use Taylor's formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} + R_n(x)$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

for some point ξ between 0 and x. To verify (11), it suffices to show that $R_n(x) \to 0$ as $n \to \infty$. The above procedure can be used quite easily to obtain the following familiar expansions, which are valid for all x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{12}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{31} + \frac{x^5}{5!} \cdots \tag{13}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots \tag{14}$$

You may now try the following exercises.

E1) Verify that $R = 0$, $R = \infty$ and $R = 1$, for the series (5), (6) and (7), respectively.

E2) Discuss the convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

E3) Obtain the radius of convergence of the following series

(a) $1 - 2x + 3x^2 - 4x^3 + \cdots$

(b) $\sum_{n=0}^{\infty} \frac{(kn)!}{(n!)^k} x^n$, k is a positive integer.

As we have mentioned earlier our aim here is to find the general solution of a linear homogeneous differential equation with variable coefficients. In particular, we shall try to find the solution of the equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0, \quad a_0(x) \neq 0.$$

or,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \tag{15}$$

where, $P(x) = \frac{a_1(x)}{a_0(x)}$, $Q(x) = \frac{a_2(x)}{a_0(x)}$ and $a_0(x), a_1(x)$ and $a_2(x)$ are continuous functions of x on some interval I. You have studied some properties of Eqn.(15) in Unit 1 when a_0, a_1, a_2 are constants and obtained solutions of such equations in terms of elementary functions. But for equations with variable coefficients when solutions cannot be obtained in terms of these elementary functions, power series solutions come to our rescue. For studying the power series method we need to study some definitions, which we shall take up next.

2.3 ORDINARY AND SINGULAR POINTS OF AN EQUATION

Here we start with recalling the definition of an analytic function which you have studied in your complex analysis course (MMT-005).

Analytic Function: A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{16}$$

is valid in some neighbourhood of the point a is said to be **analytic** at a. In this case the a_n's are given by

$$a_n = \frac{f^{(n)}(a)}{n!}, \quad n = 0, 1, 2, \ldots$$

and Eqn.(16) is called the Taylor series of $f(x)$ at $x = a$. Thus, Eqns. (12), (13) and (14) show that e^x, $\sin x$ and $\cos x$ are analytic at $x = 0$, and the given series are the Taylor series of these functions at this point. You must also be familiar with the following facts about analyticity:

i) Polynomials and the functions e^x, $\sin x$ and $\cos x$ are analytic at all points.

ii) If $f(x)$ and $g(x)$ are analytic at a, then $f(x) + g(x), f(x)g(x)$, and $f(x)/g(x)$ [if $g(a) \neq 0$] are also analytic at a.

iii) The sum of a power series is analytic at all points inside the interval of convergence.

We shall now use this definition to introduce some more definitions which will be of use for our further discussions.

Ordinary Point: A point $a \in I$ is called an **ordinary point** of Eqn.(15) if , $P(x)$ and $Q(x)$ are analytic at $x = a$. Ordinary point is also called a **regular point** of the equation.

Equation $y'' - 2xy' + 2py = 0$, where p is a constant is called Hermite's equation and has an ordinary point at $x = 0$, since $-2x$ and $2p$ are analytic at $x = 0$.

The point $x = 2$ is not an ordinary point for the equation $(x-2)y'' + y = 0$, because the function $\frac{1}{x-2}$ does not admit a power series around 2.

Singular Point: Any point of I which is not an ordinary point of Eqn.(15) is said to be a **singular point** of it. Thus for $x = a$ to be a singular point of Eqn.(15), at least one of the coefficients $P(x), Q(x)$ must fail to be analytic at $x = a$.

Regular Singular Point: A singular point a of Eqn.(15) viz.,

$$y'' + \frac{a_1(x)}{a_0(x)} y' + \frac{a_2(x)}{a_0(x)} y = 0, \quad a_0(x) \neq 0,$$

is said to be regular singular point, if $(x-a)\frac{a_1(x)}{a_0(x)}$ and $(x-a)^2 \frac{a_2(x)}{a_0(x)}$ are analytic at $x = a$.

In other words, we say that in such cases, the singular points are 'weak', in the sense that the coefficient functions are only mildly non-analytic and, with simple modifications, can be made analytic and can have a Taylor series expansion about the singular point.

A singular point, which is not regular singular point is called an **irregular singular point.**

As a simple example, the origin is clearly a regular singular point of the equation

$$y'' + \frac{2}{x} y' - \frac{2}{x^2} y = 0.$$

Consider the equation

$$(1 - x^2) y'' - 2xy' + p(p+1) y = 0,$$

where p is a constant. This equation is Legendre's equation about which we shall be studying in detail in the next unit. The equation can be put in the form

$$y'' - \frac{2x}{1-x^2} y' + \frac{p(p+1)}{1-x^2} y = 0.$$

It is clear that $x = 1$ and $x = -1$ are its singular points. The point $x = 1$ is regular because

$$(x-1)P(x) = (x-1)\left[-\frac{2x}{1-x^2}\right] = \frac{2x}{x+1},$$

and $(x-1)^2 Q(x) = (x-1)^2 \left[\frac{p(p+1)}{1-x^2}\right] = -\frac{(x-1)\ p(p+1)}{x+1}$,

are analytic at $x = 1$. For similar reasons $x = -1$ is also a regular singular point.

As another example, consider Bessel's equation of order p, where p is a non-negative constant

$$x^2 y'' + x y' + (x^2 - p^2) y = 0.$$

Writing the equations in the form

$$y'' + \frac{1}{x} y' + \frac{x^2 - p^2}{x^2} y = 0,$$

it is apparent that origin is a regular singular point of the equation because

$$x\,P(x) = x\,\frac{1}{x} = 1 \quad \text{and} \quad x^2Q(x) = x^2, \frac{x^2 - p^2}{x^2} = x^2 - p^2,$$

are analytic at $x = 0$.

You may now try the following exercises.

E4) Find the regular and singular points of the following DEs. Also, specify the nature of singular points.

(a) $y'' + 4x\,y' + y = 0$.

(b) $x^3(x-2)\,y'' + x^3y' + 6y = 0$.

(c) $(x+1)\,y''' + x\,y'' + x^2y' + x^3\,y = 0$.

(d) $x^4y'' + x^2y' + 2y = 0$.

(e) $x\,y'' + (\sin x)\,y' + e^x\,y = 0$.

In the next section we shall discuss the series solution methods for solving differential equations with variable coefficients about an ordinary point and about a regular singular point.

2.4 SERIES SOLUTION METHODS

Let us consider Eqn.(15) viz.,

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

where, $P(x) = \frac{a_1(x)}{a_0(x)}$, $Q(x) = \frac{a_2(x)}{a_0(x)}$ and $a_0(x), a_1(x)$ and $a_2(x)$ are continuous functions of x over some interval I and $a_0(x) \neq 0$.

The series solution methods of solving such equations are classified into two categories: Power series method and general series solution method-in particular, the Frobenius method. We first discuss the power series solution of Eqn.(15) about an ordinary point.

2.4.1 Power Series Solution About an Ordinary Point

We start with a **sufficient condition** for the existence of a power series solution of Eqn.(15).

Theorem 1: Let $x = a$ be an ordinary point (regular point) of Eqn.(15). Then, every solution of the equation is analytic at $x = a$ and hence has a power series expansion about the point $x = a$ of the form

$$y(x) = \sum_{m=0}^{\infty} c_m (x-a)^m \tag{17}$$

where c_0, c_1 are constants and c_k for $k \geq 2$ are obtained in terms of c_0 and c_1.

The **proof** is obvious since $a_0(x) \neq 0$, $P(x)$ and $Q(x)$ are analytic at $x = a$. Hence $y'(a), y''(a), y'''(a), \ldots$ exist and the Taylor expansion of $y(x)$, that is, power series solution about $x = a$ exists. Further, note that every function which is analytic in the region $|x - a| < R$ admits a converging power series representation $\sum_{m=0}^{\infty} c_m (x-a)^m$ in the region.

Let us for the sake of convenience, restrict our argument to the case in which $a = 0$. This allows us to work with power series in x rather than $x - a$, and involves no real loss of generality. With this slight simplification, Eqn.(15) will now have solution of the form

$$y(x)=\sum_{m=0}^{\infty} c_m x^m \tag{18}$$

where c_m's are constants to be determined. Also now $P(x)$ and $Q(x)$ are analytic at the origin and therefore have power series expansions of the form

$$P(x)=\sum_{m=0}^{\infty} p_m x^m \tag{19}$$

and, $$Q(x)=\sum_{m=0}^{\infty} q_m x^m \tag{20}$$

where p_m's and q_m's are known constants. The above expansion will converge on an interval $|x|< R$ for some $R > 0$.

In order to find a solution of Eqn.(15), term by term differentiation of Eqn.(18) yields

$$y'=\sum_{m=1}^{\infty} mc_m x^{m-1}=\sum_{m=0}^{\infty}(m+1)c_{m+1}x^m = c_1+2c_2x+3c_3x^2+\cdots$$

and $$y''=\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2}=\sum_{m=0}^{\infty}(m+1)(m+2)c_{m+2}x^m$$

$$=2c_2+2.\,3c_3x+3.4c_4x^2+\cdots \tag{21}$$

From the rule for multiplying power series, it follows that

$$P(x)y'=\left(\sum_{m=0}^{\infty} p_m x^m\right)\left[\sum_{m=0}^{\infty}(m+1)c_{m+1}x^m\right]$$

$$=\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m} p_{m-k}(k+1)c_{k+1}\right]x^m \tag{22}$$

and $$Q(x)\,y=\left(\sum_{m=0}^{\infty} q_m x^m\right)\left(\sum_{m=0}^{\infty} c_m x^m\right)$$

$$=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} q_{m-k}c_k\right)x^k \tag{23}$$

On substituting Eqns(21), (22) and (23) into Eqn. (15) and adding the series term by term, we obtain

$$\sum_{m=0}^{\infty}\left[(m+1)(m+2)c_{m+2}+\sum_{k=0}^{m} p_{m-k}(k+1)c_{k+1}+\sum_{k=0}^{m} q_{m-k}c_k\right]x^m=0.$$

Thus we have the following recursion formula for c_m:

$$(m+1)\ (m+2)c_{m+2}=-\sum_{k=0}^{m}[(k+1)\,p_{m-k}c_{k+1}+q_{m-k}c_k\,] \tag{24}$$

For $m = 0,1,2,\ldots$ this formula becomes

$$2c_2=-(p_0c_1+q_0c_0)$$

$$2.3c_3=-(p_1c_1+2p_0c_2+q_1c_0+q_0c_1)$$

$$3.4c_4=-(p_2c_1+2p_1c_2+3p_0c_2+q_2c_0+q_1c_1+q_0c_2)$$

These formulas determine $c_2, c_3,\ldots$ in terms of c_0 and c_1, and the series (18), which satisfies Eqn.(15), is uniquely determined.

We illustrate the above method by means of the following examples.

Example 3: Solve $y''-x\,y'+3y=0$ (25)

Solution: We observe that Eqn.(25) is a linear equation and its coefficients are analytic everywhere. In particular, $x = 0$ is an ordinary point of Eqn.(25) and hence we try power series solution

$$y(x)=\sum_{m=0}^{\infty} c_m x^m \tag{26}$$

where the unknown constants c_m's are to be determined. Here

$$y'(x)=\sum_{m=1}^{\infty} m\, c_m x^{m-1}$$
$$y''(x)=\sum_{m=2}^{\infty} m\,(m-1)\; c_m x^{m-2} \tag{27}$$

Substituting from Eqns.(26) and (27) the values of y, y', y'' into Eqn.(25), we get

$$\sum_{m=2}^{\infty} m(m-1)\, c_m x^{m-2} - x\sum_{m=1}^{\infty} mc_m x^{m-1} + 3\sum_{m=0}^{\infty} c_m x^m = 0\,.$$

or, $$\sum_{m=0}^{\infty}(m+2)\,(m+1)c_{m+2}x^m - \sum_{m=1}^{\infty} m\, c_m x^m + 3\sum_{m=0}^{\infty} c_m x^m = 0 \tag{28}$$

Equating coefficients of various powers of x on both sides of Eqn.(28), we get,

$$2.1\, c_2 + 3c_0 = 0 \tag{29}$$

and, $$(m+2)\,(m+1)\, c_{m+2} - (m-3)\, c_m = 0,\; m \ge 1 \tag{30}$$

From Eqns.(29) and (30), we get

$$c_2 = -(3/2)\, c_0 \tag{31}$$

and, $$c_{m+2} = [\,(m-3)\,/(m+1)\;(m+2)\,]\, c_m \;\text{ for }\; m \ge 1 \tag{32}$$

Setting $m = 1, 2, 3, \ldots,$ we get

$$c_3 = -(1/3)\; c_1 \tag{33}$$

$$c_4 = -\frac{1}{4.3}c_2 = -\frac{1}{4.3}\left(-\frac{3}{2}\right)c_0 = \frac{1}{8}c_0 \tag{34}$$

$$c_5 = 0 \tag{35}$$

We observe that once $c_5 = 0$, all the coefficients

$$c_7 = c_9 = \cdots = 0\,. \tag{36}$$

Remaining even indexed coefficients $c_6, c_8, \ldots$ can be determined in terms of c_0 using Eqn.(32). Recurrence relation (32) yields

$$c_{2r+4} = \frac{2r-1}{(2r+4)\,(2r+3)}c_{2r+2}$$
$$= \frac{(2r-1)}{(2r+4)\,(2r+3)}.\frac{(2r-3)}{(2r+2)\,(2r+1)}c_{2r}$$
$$= \frac{(2r-1)\,(2r-3)}{(2r+4)\,(2r+3)\,(2r+2)\,(2r+1)}\;\frac{(2r-5)}{2r\,(2r-1)}c_{2r-2} = \cdots\cdots$$
$$= \frac{(2r-1)\,(2r-3)\cdots.3.1}{(2r+4)\,(2r+3)\cdots 7.6.5.}c_4$$
$$= \frac{(2r-1)\,(2r-3)\cdots 3.1}{(2r+4)\,(2r+3)\cdots 7.6.5}.\frac{1}{4.3}.\frac{3}{2}c_0$$
$$= \frac{3(2r-1)\,(2r-3)\cdots.3.1}{(2r+4)!}c_0, \text{for } r = 1, 2, \ldots. \tag{37}$$

Thus from Eqn.(26), the solution becomes

$$y(x) = c_0 + c_2 x^2 + c_4 x^4 + \sum_{r=1}^{\infty}\frac{3(2r-1)\,(2r-3)\ldots.3.1}{(2r+4)!}c_0 x^{2r+4} + c_1 x + c_3 x^3$$
$$= c_0\left[1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + 3\sum_{r=1}^{\infty}\frac{(2r-1)\,(2r-3)\ldots 3.1}{(2r+4)!}x^{2r+4}\right] + c_1\left(x - \frac{1}{3}x^3\right) \tag{38}$$

The radius of convergence R of the series, obtained by using ratio test, is

$$R = \lim_{r\to\infty}\left|\frac{(2r-1)(2r-3)\ldots 3.1}{(2r+4)!}\cdot\frac{(2r+6)!}{(2r+1)(2r-1)\ldots .3.1}\right|$$

$$= \lim_{r\to\infty}\left|\frac{(2r+5)\ (2r+6)}{(2r+1)}\right| \to \infty$$

Therefore the series in Eqn.(38) converges for all x and the solution is valid for all x.

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Example 4: Find the power series solution about the origin of the equation

$$(1-x^2)\,y'' - 4x\,y' + 2y = 0 \qquad (39)$$

Solution: We know that $x = 0$ is an ordinary point of Eqn.(39) and therefore the power series solution exists. Substituting the expressions for y, y', y'' from Eqns.(26) and (27) in Eqn.(39), we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)\,c_m x^{m-2} - 4x\sum_{m=1}^{\infty} m\,c_m x^{m-1} + 2\sum_{m=0}^{\infty} c_m x^m = 0.$$

or, $$\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)c_m x^m - 4\sum_{m=1}^{\infty} mc_m x^m + 2\sum_{m=0}^{\infty} c_m x^m = 0$$

or, $$2c_2 + 6c_3 x + \sum_{m=4}^{\infty} m(m-1)c_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)c_m x^m - 4c_1 x$$

$$-4\sum_{m=2}^{\infty} mc_m x^m + 2(c_0 + c_1 x) + 2\sum_{m=2}^{\infty} c_m x^m = 0 \qquad (40)$$

In the third term of Eqn.(40), setting $m-2 = t$ or, $m = t+2$, we get

$$2(c_2 + c_0) + 2(3c_3 - c_1)\,x + \sum_{t=2}^{\infty}(t+2)(t+1)\,c_{t+2}\,x^t$$

$$-\sum_{m=2}^{\infty} m(m-1)\,c_m x^m - 4\sum_{m=2}^{\infty} m\,c_m x^m + 2\sum_{m=2}^{\infty} c_m x^m = 0$$

Since t is dummy variable, we can write

$$2(c_2 + c_0) + 2(3c_3 - c_1)x + \sum_{m=2}^{\infty}[\,(m+2)(m+1)\,c_{m+2} - (m^2 + 3m - 2)\,c_m\,]x^m = 0$$

Setting the coefficients of successive powers of x to zero, we obtain

$$c_2 + c_0 = 0,\ 3c_3 - c_1 = 0,\ (m+2)(m+1)\,c_{m+2} = (m^2 + 3m - 2)c_m,\ m \ge 2$$

Solving, we obtain

$$c_2 = -c_0,\ c_3 = \frac{1}{3}c_1,\ c_{m+2} = \frac{(m^2 + 3m - 2)}{(m+2)(m+1)}c_m,\ m \ge 2$$

where c_0 and c_1 are arbitrary constants. We thus have

$$c_4 = \frac{2}{3}c_2 = \frac{-2}{3}c_0,\ c_5 = \frac{4}{5}c_3 = \frac{4}{15}c_1,$$

$$c_6 = \frac{13}{15}c_4 = \frac{-26}{45}c_0,\ c_7 = \frac{19}{21}c_5 = \frac{76}{315}c_1,\ldots$$

The power series solution is

$$y(x) = c_0\left[1 - x^2 - \frac{2}{3}x^4 - \frac{26}{45}x^6 - \cdots\right] + c_1\left[x + \frac{x^3}{5} + \frac{4}{15}x^5 + \frac{76}{315}x^7 + \cdots\right] \qquad (41)$$

***

Remarks: In Example 4, $x = \pm 1$ are the singular points of Eqn.(39). Thus, the power series solution cannot exist in any interval I which contains $+1$ or -1. Therefore the interval of largest length, in which the power series solution (41) holds is $]-1,1[$

which is its interval of convergence. Further, since the radius of convergence R is the distance between the point $x = a$, about which the power series solution is sought and the nearest singularity of the equation therefore, in this case, $R = 1$.

You may now try some exercises.

E5) Solve $(x^2 - 2x + 2)\, y'' - 4(x - 1)\, y' + 6y = 0$, about the point $x = 1$.

E6) Solve $y'' + x\, y' + 3y = 0$, by power series method about the point $x = 0$.

E7) Find power series solution in powers of $(x - 1)$ of the initial value problem

$$\begin{aligned} x\, y'' + y' + 2y = 0 \\ y(1) = 1,\ y'(1) = 2 \end{aligned}.$$

E8) Find the power series solution of the equation $y'' + xy' + y = 0$, about $x = 2$.

E9) The equation $y'' - 2xy' + 2ny = 0$, where n is a positive integer, is called Hermite's equation of order n. Using the power series method show that if

$$z_n(x) = \sum_{k=0}^{\infty} a_k x^k$$

is a solution of Hermite's equation, then

$$z_n(x) = a_0 \left[1 + \sum_{k=1}^{\infty} \frac{2^k (-n)(-n+2)\cdots(-n+2k-2)}{(2k)!} x^{2k} \right]$$
$$+ a_1 \left[x + \sum_{k=1}^{\infty} \frac{2^k (1-n)(1-n+2)\cdots(1-n+2k-2)}{(2k+1)!} x^{2k+1} \right].$$

So far, we have discussed a method of finding series solution of a differential equation about an ordinary point, that is, power series solution of the equation. Power series is inadequate to represent a solution about a regular singular point $x = a$, as the solution may also contain negative powers and/or fractional powers of $(x - a)$. We shall now discuss the Frobenius method for obtaining a series solution about a regular singular point of the equation.

2.4.2 Series Solution about a Regular Singular Point: Frobenius Method

The method we are going to discuss here is due to Ferdinand George Frobenius (1849-1917), a German mathematician who made several valuable contribution to the theory of elliptic functions and differential equations. We shall be restricting ourselves to second order equations only because most of the important equations, in physical science and engineering, such as Bessel's equation, Legendre's equation, Hermite's equation or, the hypergeometric equations for which the method works are of second order only. Also, to simplify the matter we shall assume that the singular point a is located at the origin, for if it is not, then we can always move it to the origin by changing the independent variable from x to $x - a$.

Method of Frobenius

Consider linear differential equation of order two of the form

$$f(x)\frac{d^2y}{dx^2} + g(x)\frac{dy}{dx} + r(x)\, y = 0 \qquad (42)$$

Let $x = 0$, be a regular singular point of Eqn.(42).

The method of Frobenius to solve Eqn.(42) consist of the following steps:-

Step I: Take a trial solution of Eqn.(42) of the form

$$y(x) = x^k (c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m + \cdots)$$

i.e., $$y(x) = x^k \sum_{m=0}^{\infty} c_m x^m .$$

$$= \sum_{m=0}^{\infty} c_m x^{m+k} \text{, where k is any real number and } c_0 \neq 0 \quad (43)$$

Step II: Differentiate relation (43) and obtain

$$\left.\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+k) c_m x^{m+k-1} \\ \text{and} \quad y''(x) &= \sum_{m=0}^{\infty} (m+k)\ (m+k-1) c_m x^{m+k-2} \end{aligned}\right\} \quad (44)$$

Using Eqns.(43) and (44), Eqn.(42) reduces to an identity.

Step III: Equate to zero the coefficient of the smallest power of x in the identity in Step II above. This would yield a quadratic equation in k. The quadratic equation so obtained is called the **indicial equation.** The two roots of the indicial equation are called **exponents** of Eqn.(42) and are the only possible values of k in Eqn.(43).

Step IV: Solve the indicial equation. The following cases are possible:

1) Indicial roots are distinct and do not differ by an integer.
2) Indicial roots are distinct and differ by an integer.
3) Indicial roots are equal.

Step V: Equate to zero the coefficient of general power of x (e.g. x^{k+m} or x^{k+m-1}, whichever may be the lowest) in the identity obtained in Step II. The equation so obtained is called the **recurrence relation** because it connects the coefficients c_m, c_{m-2} **or,** c_m, c_{m-1}, etc.

Step VI: If the recurrence relation connects c_m and c_{m-2}, then, in general, determine c_1 by equating to zero the coefficient of the term having next higher power in x i.e., the term in x next to the term already used for getting the indicial equation. On the other hand, if the recurrence relation connects c_m and c_{m-1}, this step may be omitted.

Step VII: Obtain all the coefficients with the help of Steps V and VI above. Substitute them in relation (43) and obtain the required solutions corresponding to two roots of the indicial equation.

We shall illustrate various steps involved in the method through examples. Depending upon the nature of the indicial roots listed in Step IV, different types of solution procedures are followed. We shall now illustrate these procedures one-by-one. You may **notice** here that the two series solutions obtained in each case are linearly independent.

Case 1: Indicial roots are distinct and do not differ by an integer.

Example 5: Obtain the series solution about the origin of the equation

$$9x(1-x)\, y'' - 12y' + 4y = 0 \quad (45)$$

Solution: Evidently $x = 0$ is a regular singular point of Eqn.(45).

Let $$y(x) = \sum_{m=0}^{\infty} c_m x^{k+m},\ c_0 \neq 0 .$$

Differentiating $y(x)$ twice w.r.t. x, we get

$$y'(x) = \sum_{m=0}^{\infty} (k+m)\, c_m x^{k+m-1}$$

and $$y''(x) = \sum_{m=0}^{\infty} (k+m)\,(k+m-1)\, c_m x^{k+m-2}$$

Substituting the above values of y, y', y'' in Eqn.(45), we get

$$9x(1-x)\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2} - 12\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}$$
$$+4\sum_{m=0}^{\infty}c_m x^{k+m} = 0$$

or, $$9\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-1} - 9\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m}$$
$$-12\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + 4\sum_{m=0}^{\infty}c_m x^{k+m} = 0 \qquad (46)$$

Combining the first and third term and the second and fourth term of Eqn.(46), we get

$$3\sum_{m=0}^{\infty}(k+m)(3k+3m-7)c_m x^{k+m-1} - \sum_{m=0}^{\infty}[9(k+m)^2 - 9(k+m) - 4]c_m x^{k+m} = 0$$

$$\text{or, } 3\sum_{m=0}^{\infty}(k+m)(3k+3m-7)c_m x^{k+m-1} - \sum_{m=0}^{\infty}(3k+3m-4)(3k+3m+1)c_m x^{k+m} = 0$$

$$\text{or, } 3\sum_{m=-1}^{\infty}(k+m+1)(3k+3m-4)c_{m+1} x^{k+m} - \sum_{m=0}^{\infty}(3k+3m-4)(3k+3m+1)c_m x^{k+m} = 0$$

$$\text{or, } 3k(3k-7)c_0 x^{k-1} + \sum_{m=0}^{\infty} x^{k+m}\big[3(k+m+1)(3k+3m-4)c_{m+1}$$
$$-(3k+3m-4)(3k+3m+1)c_m\big] = 0 \qquad (47)$$

Equating to zero the coefficient of smallest power of x (i.e., x^{k-1}) in identity (47), we get **indicial equation** as

$$3k(3k-7) = 0 \Rightarrow k = 0,\ 7/3.$$

Clearly roots of indicial equation are distinct, not differing by an integer. Equating to zero the coefficient of general power of x, i.e., x^{k+m}, in identity (47), we get the recurrence relation as

$$3(k+m+1)(3k+3m-4)c_{m+1} = (3k+3m-4)(3k+3m+1)c_m, \text{for } m \ge 0$$

$$\text{i.e., } c_{m+1} = \frac{(3k+3m-4)(3k+3m+1)}{3(k+m+1)(3k+3m-4)}c_m = \frac{3k+3m+1}{3(k+m+1)}c_m \qquad (48)$$

For $k = 0$, $c_{m+1} = \dfrac{3m+1}{3(m+1)}c_m,\ m \ge 0$

Putting $m = 0, 1, 2, 3, \ldots$

$$c_1 = \frac{1}{3}c_0, c_2 = \frac{4}{6}.c_1 = \frac{1.4}{3.6}c_0, c_3 = \frac{7}{9}c_2 = \frac{1.4.7}{3.6.9}c_0, \cdots$$

Therefore, series solution corresponding to $k = 0$ is

$$y_1(x) = c_0\left(1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \cdots\right), \qquad (49)$$

For $k = 7/3$, we get from Eqn.(48),

$$c_{m+1} = \frac{3m+8}{3m+10}C_m, \text{ for } m \ge 0$$

Putting $m = 0, 1, 2, \ldots,$, we get

$$c_1 = \frac{8}{10}c_0, c_2 = \frac{11}{13}c_1 = \frac{11.8}{13.10}c_0, c_3 = \frac{14}{16}c_2 = \frac{14.11.8}{16.13.10}c_0, \cdots$$

Therefore, series solution corresponding to $k = 7/3$ is

$$y_2(x) = c_0 x^{7/3}\left(1+\frac{8}{10}x+\frac{8.11}{10.13}x^2+\frac{8.11.14}{10.13.16}x^3+\cdots\right) \quad (50)$$

Since the solutions $y_1(x)$ and $y_2(x)$ given by Eqns.(49) and (50) are linearly independent, the general solution of Eqn.(45) may be written as

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x),$$

where α_1 and α_2 are arbitrary constants.

***

Note: In general, if indicial equation has two distinct roots, differing not by an integer, two linearly independent solutions are obtained by substituting the values of k in the series for $y(x)$.

Let us look at another example.

Example 6: Find a series solution about $x = 0$, of the equation $2x^2y'' - xy' + (1-x^2)y = 0$.

Solution: Observe that the given equation

$$2x^2y'' - x\,y' + (1-x^2)\,y = 0 \quad (51)$$

has $x = 0$ as a regular singular point.

Let $$y(x) = x^k(c_0 + c_1x + c_2x^2 + \cdots) = \sum_{m=0}^{\infty} c_m x^{m+k} \quad (52)$$

be a solution of Eqn.(51).

Differentiating Eqn.(52) w.r.t. x, we get

$$y'(x) = \sum_{m=0}^{\infty}(k+m)c_m x^{m+k-1} \quad (53)$$

and $$y''(x) = \sum_{m=0}^{\infty}(k+m)(k+m-1)\,c_m x^{k+m-2} \quad (54)$$

Putting the values of y, y', y'' from Eqns.(52), (53) and (54) in Eqn.(51), we get

$$2x^2\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2} - x\sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + (1-x^2)\sum_{m=0}^{\infty}c_m x^{k+m} = 0$$

$$\text{or, } \sum_{m=0}^{\infty}2(k+m)(k+m-1)c_m x^{k+m} - \sum_{m=0}^{\infty}(k+m)c_m x^{k+m} + \sum_{m=0}^{\infty}c_m x^{k+m} - \sum_{m=0}^{\infty}c_m x^{k+m+2} = 0 \quad (55)$$

Equating the coefficient of lowest power of x, i.e., of x^k to zero in identity (55), we get the **indicial equation** as

$$2c_0k(k-1) - k\,c_0 + c_0 = 0, \text{ i.e., } c_0[2k(k-1) - k + 1] = 0$$

i.e., $c_0(k-1)\ (2k-1) = 0 \Rightarrow k = 1,\ 1/2$, i.e., roots of indicial equation are distinct, but difference is not an integer.

To get a recurrence relation, we equate the coefficients of general power of x, i.e., x^{k+m} in identity (55) equal to zero and obtain

$$[2(k+m)(k+m-1) - (k+m) + 1]\,c_m = c_{m-2}$$

i.e., $$c_m = \frac{c_{m-2}}{(k+m-1)(2k+2m-1)} \text{ for all } m \geq 2. \quad (56)$$

Again, equating the coefficient of x^{k+1} in Eqn.(55) to zero , we get

$$2(k+1)kc_1 - (k+1)c_1 + c_1 = 0 \Rightarrow c_1[2k(k+1) - (k+1) + 1] = 0$$
$$\Rightarrow c_1\, k(2k+1) = 0 .$$

Now for $k = 1, 1/2,\; k(2k+1) \neq 0$, hence $c_1 = 0$ (57)

From Eqns.(56) and (57), we get

$$c_1 = 0 = c_3 = c_5 = c_7 = \ldots, = c_{2m+1} = \ldots$$

Putting $m = 2,4,6,\ldots$ in relation (56), we get

$$c_2 = \frac{1}{(2k+3)(k+1)} c_0 .$$

$$c_4 = \frac{1}{(k+3)(2k+7)} c_2 = \frac{1}{(2k+3)(2k+7)(k+1)(k+3)} c_0 .$$

$$c_6 = \frac{1}{(k+5)(2k+11)} = \frac{1}{(2k+3)(2k+7)(2k+11)(k+1)(k+3)(k+5)} c_0 .$$

....

Putting these values of $c_0, c_1, \ldots$ in Eqn.(52), we get

$$y(x) = x^k \left[c_0 + \frac{c_0}{(2k+3)(k+1)} x^2 + \frac{c_0}{(2k+3)(2k+7)(k+1)(k+3)} x^4 + \right.$$
$$\left. + \frac{c_0}{[(2k+3)(2k+7)(2k+11)(k+1)(k+3)(k+5)]} x^6 + \cdots \right].$$

Putting $k = 1$, we get

$$y_1 = x\, c_0 \left[1 + \frac{x^2}{2.5} + \frac{x^4}{2.4.5.9} + \frac{x^6}{2.4.6.5.9.13} + \cdots \right] = c_0 u \quad (58)$$

Putting $k = 1/2$ in $y(x)$, we get

$$y_2 = c_0 x^{1/2} \left[1 + \frac{x^2}{2.3} + \frac{x^4}{2.3.4.7} + \frac{x^6}{3.7.11.2.4.6} + \cdots \right] = c_0 v, \quad (59)$$

Hence the general solution of Eqn.(51) is

$$\begin{aligned} y(x) &= A_1 y_1(x) + B_1 y_2(x) \\ &= Au + Bv \end{aligned} \quad (60)$$

where $A = A_1 c_0$ and $B = B_1 c_0$ are arbitrary constants and

$$u = x\left[1 + \frac{1}{2.5} x^2 + \frac{1}{2.4.5.9} x^4 + \frac{1}{2.4.6.5.9.13} x^6 + \cdots \right] \quad (61)$$

$$v = x^{1/2}\left[1 + \frac{1}{2.3} x^2 + \frac{1}{2.4.3.7} x^4 + \frac{1}{2.4.6.3.7.11} x^6 + \cdots \right] \quad (62)$$

***

You may now try the following exercises.

E10) Find a series solution about $x = 0$, of the following equations

a) $2x(1-x)y'' + (1-x)y' + 3y = 0$.

b) $(2x + x^2)y'' - y' - 6xy = 0$.

Case 2: Indicial roots are distinct and differ by an integer

Let the roots of the indicial equation be k_1, k_2 such that $k_1 - k_2 = n$, an integer. In this case two types of problems arise. In **one type** two linearly independent solutions are obtained by substituting $k = k_1$ and $k = k_2$ in $y(x)$ as in Case 1. This case is illustrated through the following example.

Example 7: Find a series solution about $x = 0$, of the equation

$$x^2 y'' + x^3 y' + (x^2 - 2)\,y = 0 \tag{63}$$

Solution: Evidently $x = 0$, is a regular singular point of Eqn.(63).

Let $$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \tag{64}$$

$$\therefore y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2}$$

Putting the values of y, y', y'' in Eqn.(63), we get

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2} + x^3 \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} + (x^2 - 2) \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

i.e., $$\sum_{m=0}^{\infty} c_m \left[(k+m)(k+m-1) - 2\right] x^{k+m} + \sum_{m=0}^{\infty} c_m (k+m+1)\; x^{k+m+2} = 0. \tag{65}$$

Equating to zero the coefficients of smallest power of x, namely x^k, Eqn.(65) gives

$$c_0[k(k-1) - 2] = 0 \Rightarrow k^2 - k - 2 = (k-2)\;(k+1) = 0, \text{ or } k = 2, -1,\; c_0 \neq 0.$$

These are distinct roots of **indicial equation**, which differ by an integer. Equating to zero the coefficient of next power of x, namely x^{k+1}, we get

$$[(k+1)k - 2]\; c_1 = 0, \text{ or } c_1 = 0 \text{ for } k = 2, -1.$$

The remaing terms are given by

$$\sum_{m=2}^{\infty} [\,(k+m)\;(k+m-1) - 2]\; c_m\; x^{k+m} + \sum_{m=0}^{\infty} (k+m+1) c_m x^{k+m+2} = 0$$

From here we get the recurrence relation

$$[(m+k+1)\;(m+k+2) - 2] c_{m+2} + (m+k+1) c_m = 0, \; m \geq 0$$

or, $$c_{m+2} = -\frac{(m+k+1)\; c_m}{(m+k+1)\;(m+k+2)}, \; m \geq 0 \tag{66}$$

Putting $m = 1, 2, 3, \ldots,$ we get

$$c_2 = \frac{-(k+1) c_0}{(k+1)\;(k+2) - 2}, \; c_3 = \frac{-(k+2) c_1}{(k+2)\;(k+2) - 2} = 0, \; c_4 = \frac{-(k+3) c_2}{(k+3)\;(k+4) - 2}, \cdots.$$

Therefore from Eqn.(64), we get

$$y(x) = c_0 x^k \left[1 - \frac{(k+1)}{(k+1)\;(k+2) - 2} x^2 + \frac{(k+3)\;(k+1)}{[(k+1)\;(k+2) - 2]\;[(k+3)(k+4) - 2\,]} x^4 - \cdots\right] \tag{67}$$

For $k = 2$, we get

$$y(x) = c_0 x^2 \left[1 - \frac{3}{10} x^2 + \frac{3}{56} x^4 - \cdots\right] = y_1$$

For $k = -1$, coeffieicents $c_2 = 0,\; c_3 = 0,\; c_4 = 0, \ldots$ and

$$y(x) = \frac{c_0}{x} = y_2\,.$$

Hence the derived solution is $y = a\,y_1 + b\,y_2$, where a and b are arbitrary constants.

***

In the **second type** of problems again two cases arise. We now discuss them as I and II and give the rules for solving such problems.

I: Indicial roots are distinct, differ by an integer and make a coefficient of y indeterminate:

In this situation the following rule is used to obtain the two linearly independent solutions of the given problem.

Rule: Let the indicial equation has two roots k_1 and k_2 such that $k_2 - k_1 = n$, an integer. Now if one of the coefficients of y becomes indeterminate when $k = k_1$ (say), the complete solution is given by putting $k = k_1$ in y, which then contains two arbitrary constants. The result of putting $k = k_2$ in y merely gives a numerical multiple of one of the series contained in the first solution hence producing a linearly dependent solution, which is rejected.

We illustrate the method for this case through the following example:

Example 8: Obtain a series solution about $x = 0$, of the equation

$$x^2 y'' + 6xy' + (6 + x^2)\, y = 0 \tag{68}$$

Solution: We find $x = 0$ is a regular singular point of Eqn.(68).

$$\text{Let } y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \quad c_0 \neq 0. \tag{69}$$

Substituting for y, y', y'' in Eqn.(68), we get

$$x^2 \sum_{m=0}^{\infty} (m+k)\,(m+k-1)\, c_m x^{m+k-2} + 6x \sum_{m=0}^{\infty} (m+k)\, c_m x^{m+k-1} + (6 + x^2) \sum_{m=0}^{\infty} c_m x^{m+k} = 0$$

or,

$$\sum_{m=0}^{\infty} (m+k)\,(m+k-1)\, c_m x^{m+k} + 6\sum_{m=0}^{\infty} (m+k)\, c_m x^{m+k} + 6\sum_{m=0}^{\infty} c_m x^{m+k} + \sum_{m=0}^{\infty} c_m x^{m+k+2} = 0.$$

or,

$$\sum_{m=0}^{\infty} \left[(m+k)\,(m+k+5) + 6\right] c_m x^{m+k} + \sum_{m=0}^{\infty} c_m x^{m+k+2} = 0. \tag{70}$$

Setting the coefficient of lowest degree term, namely x^k to zero, we obtain

$$[k(k+5)+6]\, c_0 = 0, \text{ or } (k+2)\,(k+3) = 0, \; c_0 \neq 0 \tag{71}$$

The **indicial** roots are $k = -2, -3$. Setting the coefficient of x^{k+1} to zero, we get

$$[\,(k+1)\,(k+6) + 6\,]\, c_1 = 0.$$

For $k = -2$, $c_1 = 0$ and for $k = -3$, c_1 is **indeterminate**. Therefore, $k = -3$ gives the complete solution, which contains two arbitrary constants. Equating the coefficients of general term to zero, we get

$$c_{m+2} = \frac{-c_m}{(m+k+2)\,(m+k+7) + 6}, \quad m \geq 0 \tag{72}$$

Putting $m = 1, 2, 3, \ldots$ in Eqn.(72), we get

$$c_2 = \frac{-c_0}{(k+2)\,(k+7)+6} \qquad c_3 = \frac{-c_1}{(k+3)\,(k+8)+6}$$

$$c_4 = \frac{-c_2}{(k+4)\ (k+9)+6} = \frac{c_0}{[(k+2)\ (k+7)+6]\ [(k+4)\ (k+9)+6]}$$

and so on.

Therefore from Eqn.(69), we get

$$y(x) = c_0 x^k \left[1 - \frac{1}{(k+2)\ (k+7)+6} x^2 + \frac{1}{[(k+2)\ (k+7)+6]\ [(k+4)\ (k+9)+6]} x^4 - \cdots\right]$$
$$+ c_1 x^k \left[x - \frac{1}{(k+3)\ (k+8)+6} x^3 + \frac{1}{[(k+3)\ (k+8)+6]\ [(k+5)\ (k+10)+6} x^5 - \cdots\right]$$

For $k = -3$, we get

$$y(x) = c_0 x^{-3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + c_1 x^{-3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$
$$= c_0\left(\frac{\cos x}{x^3}\right) + c_1\left(\frac{\sin x}{x^3}\right) = c_0 y_1(x) + c_1 y_2(x) \qquad (73)$$

For $k = -2$, we know $c_1 = 0 = c_3 = c_5$ and so on.

$$\therefore \quad y(x) = c_0 x^{-2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right] = c_0 x^{-3}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = c_0 y_2(x) \qquad (74)$$

which is a linearly dependent solution.

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You may now attempt a few exercises.

E11) Obtain the series solution about $x = 0$, of the equation $(2 + x^2)y'' + xy' + (1 + x)\,y = 0$.

E12) Find a series solution about $x = 0$, of the differential equation $(1 - x^2)\,y'' + 2xy' + y = 0$.

We, now take up II for the second type of problem.

II: Indicial roots are distinct, differ by an integer and make a coefficient of y infinite:

In this case, the following rule is used:

Rule: Let the indicial equation has two roots k_1 and k_2 such that $k_1 - k_2 = n$, an integer. Now if one of the coefficients of y becomes infinite for $k = k_2$ (say), then the form of y in Eqn.(43) is modified by replacing c_0 by $d_0(k - k_2)$ where $d_0 \neq 0$. Two linearly independent solutions are then obtain by substituting $k = k_2$ in y and $\frac{\partial y}{\partial k}$. The solution corresponding to $\left.\frac{\partial y}{\partial k}\right)_{k=k_2}$ will always contain a logarithmic term. The result of putting $k = k_1$ in y merely gives a solution that is a numerical multiple of the one obtained for $k = k_2$, leading to a linearly dependent solution of the given equation.

Let us take up an example to understand this case.

Example 9: Find the series solution about $x = 0$, of the differential equation

$$(x + x^2 + x^3)\, y'' + 3x^2 y' - 2y = 0 \tag{75}$$

Solution: Evidently, $x = 0$, is a regular singular point of Eqn.(75).

$$\text{Let} \quad y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0. \tag{76}$$

$$\text{Then } y' = \sum_{m=0}^{\infty} (k+m) c_m x^{k+m-1} \text{ and } y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) c_m x^{k+m-2} \tag{77}$$

Substituting for y, y', y'' in Eqn.(75), we get

$$(x + x^2 + x^3) \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-2} + 3x^2 \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$$

$$-2 \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

i.e., $\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m+1} + \sum_{m=0}^{\infty} c_m 3(k+m) x^{k+m+1}$

$$+ \sum_{m=0}^{\infty} c_m [(k+m)(k+m-1) - 2]\, x^{k+m} + \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)\, x^{k+m-1} = 0.$$

i.e., $\sum_{m=0}^{\infty} c_m (k+m)(k+m+2) x^{k+m+1} + \sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1) - 2\} x^{k+m}$

$$+ \sum_{m=-1}^{\infty} c_{m+1} (k+m+1)(k+m) x^{k+m} = 0$$

i.e., $\sum_{m=1}^{\infty} c_{m-1} (k+m-1)(k+m+1) x^{k+m} + \sum_{m=0}^{\infty} \{(k+m)(k+m-1) - 2\} c_m x^{k+m}$

$$+ \sum_{m=-1}^{\infty} (k+m+1)(k+m)\, c_{m+1} x^{m+k} = 0$$

i.e., $k(k-1) c_0 x^{k-1} + [\,(k+1) k\, c_1 + \{\,k(k-1) - 2\} c_0\,]\; x^k$

$$+ \sum_{m=1}^{\infty} [\,(k+m+1)(k+m)\, c_{m+1} + (m+k-1)(m+k+1)\, c_{m-1}$$

$$+ \{(k+m)(k+m-1) - 2\} c_m\,] x^{m+k} = 0 \tag{78}$$

This is an identity. Here the **indicial equation** obtained by equating to zero the coefficients of x^{k-1} is

$$c_0 k(k-1) = 0 \Rightarrow k = 0, 1 \quad (\because c_0 \neq 0). \tag{79}$$

Equating to zero the coefficients of x^k and x^{m+k} in Eqn.(78), we get

$$(k+1) k\, c_1 + \{k(k-1) - 2\} c_0 = 0 \tag{80}$$

and,

$$(k+m+1)(k+m)\, c_{m+1} + (m+k+1)(m+k-1)\, c_{m-1} + \{\,(k+m)(k+m-1) - 2\} c_m = 0, \text{ for } m \geq 1 \tag{81}$$

From Eqn.(80), we get $c_1 = -\dfrac{k(k-1) - 2}{k(k+1)} c_0$

Putting $m = 1, 2, 3, 4, \ldots$ in Eqn.(81), we get the following:

For $m = 1$

$$(k+2)(k+1)\, c_2 + k(k+2)\, c_0 + \{k(k+1) - 2\} c_1 = 0$$

i.e., $$c_2 = -\frac{k}{k+1}c_0 + \frac{[k(k-1)-2][k(k+1)-2]}{k(k+1)^2(k+2)}c_0 = -\frac{3k-2}{k(k+1)}c_0$$

For $m = 2$

$$(k+3)(k+2)c_3 + (k+3)(k+1)c_1 + \{(k+2)(k+1)-2\}\ c_2 = 0$$

i.e., $$c_3 = -\frac{k+1}{k+2}c_1 - \frac{k^2+3k}{(k+3)(k+2)}c_2 = -\frac{k+1}{k+2}c_1 + \frac{(3k-2)}{(k+1)(k+2)}c_0$$

$$= \left[\frac{k(k-1)-2}{k(k+2)} + \frac{3k-2}{k+1}\right]c_0 = \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}c_0$$

and so on.

Substituting the values of $c_1, c_2, c_3, \ldots$ in Eqn.(76), we get

$$y(x) = c_0 x^k\left[1 - \frac{k^2-k-2}{k(k+1)}x - \frac{3k-2}{k(k+1)}x^2 + \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}x^3 - \cdots - \right] \qquad (82)$$

It is seen that **for $k = 0$**, the coefficients in solution (82) become infinite. So we, replace c_0 by $k\,r_0\ (r_0 \neq 0)$. Thus Eqn(82) reduces to

$$y(x) = r_0 x^k\left[k - \frac{k^2-k-2}{k+1}x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 - \cdots - \right] \qquad (83)$$

Substituting from Eqn.(83) in L.H.S. of Eqn.(75), we get

$$(x + x^2 + x^3)y'' + 3x^2y' - 2y = r_0k^2(k-1)x^{k-1}$$

(where the right hand side is simply the indicial equation). Here the presence of k^2 shows that y and $\frac{\partial y}{\partial k}$ satisfy Eqn.(75) for $k = 0$.

From Eqn.(83), we get

$$\frac{\partial y}{\partial k} = r_0x^k \ln x\left[k - \frac{k^2-k-2}{k+1}x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 - \cdots\right]$$

$$+ r_0x^k\left[1 - \left\{\frac{2k-1}{k+1} - \frac{k^2-k-2}{(k+1)^2}\right\}x - \left\{\frac{3}{k+1} - \frac{3k-2}{(k+1)^2}\right\}x^2\right.$$

$$\left. + \left\{\frac{3k^2+6k-5}{(k+1)(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)^2(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)(k+2)^2}\right\}x^3 - \cdots\right] \qquad (84)$$

Putting $k = 0$ in Eqns.(83) and (84), we get

$$y = r_0[2x + 2x^2 - x^3 - \cdots] = y_1, \text{say} \qquad (85)$$

and $$\frac{\partial y}{\partial k} = (\ln x)y_1 + r_0[1 - x - 5x^2 - x^3 - \cdots] = y_2, \text{say} \qquad (86)$$

Putting $k = 1$ in Eqn.(83), we get

$$y = r_0x\left[1 + x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \cdots\right] = y_3, \text{say} \qquad (87)$$

Evidently, $2y_3 = y_1$.

Hence, out of three solutions y_1, y_2 and y_3 obtained above, only two are linearly independent. Hence general solution is

$$y = \alpha y_1 + \beta y_2,$$

where α and β are arbitrary constants.

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You may now try the following exercise.

E13) Find a series solution of the following differential equations about $x = 0$.

a) $xy'' + (3 - x)y' + y = 0$.

b) $x(1 - x)y'' - 3xy' - y = 0$.

We shall now take up the case when the roots of indicial equation are equal.

Case 3: Indicial roots are equal

In this case, the following rule works:

Rule: If indicial equation has two equal roots $k_1 = k_2$, obtain two linearly independent solution by substituting this value of k in y and $\frac{\partial y}{\partial k}$.

We illustrate this rule through an example.

Example 10: Solve, in series, the differential equation

$$xy'' + y' + xy = 0 \quad (88)$$

Solution: Here $x = 0$ is a regular singular point.
Let the series solution of Eqn.(88) be of the form

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad c_0 \neq 0 \quad (89)$$

Thus, $y' = \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1}$ and $y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-2}$ (90)

Putting these values of y, y', y'' in Eqn.(88), we get

$$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1) x^{k+m-1} + \sum_{m=0}^{\infty} c_m (k+m) x^{k+m-1} + \sum_{m=0}^{\infty} c_m x^{k+m+1} = 0$$

or, $$\sum_{m=0}^{\infty} c_m (k+m)^2 x^{k+m-1} + \sum_{m=0}^{\infty} c_m x^{k+m+1} = 0, \quad (91)$$

which is an identity. Equating to zero the coefficient of least power x, in identity (91) **indicial equation** is obtained as

$$c_0 k^2 = 0 \Rightarrow k = 0, 0 \quad (\because c_0 \neq 0) \quad (92)$$

Thus indicial equation has equal roots.
Again from Eqn.(91), the recurrence relation is obtained as

$$c_m (k+m)^2 + c_{m-2} = 0$$

i.e., $$c_m = -\frac{1}{(k+m)^2} c_{m-2} \quad (93)$$

Again equating the coefficient of x^k in Eqn.(91) to zero, we get

$$c_1 (k+1)^2 = 0, \text{but since } k = 0, k+1 \neq 0 \text{ and hence } c_1 = 0. \quad (94)$$

For $c_1 = 0$, we get from Eqn.(93), $c_1 = c_3 = 0 = c_5 = c_7 = \cdots$.
For $m = 2, 4, 6, \ldots$ Eqn.(93) yields

$$c_2 = -\frac{1}{(k+2)^2} c_0,$$

$$c_4 = -\frac{1}{(k+4)^2} c_2 = \frac{1}{(k+2)^2 (k+4)^2} c_0$$

$$c_6 = -\frac{1}{(k+2)^2 (k+4)^2 (k+6)^2} c_0$$

and so on.

Putting these values of c's in Eqn.(89), we get

$$y = c_0 x^k \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2 (k+4)^2} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2} + \cdots \right] \quad (95)$$

Putting $\mathbf{k = 0}$ in Eqn.(95), we get

$$y = c_0 \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \right) = c_0 u, \text{say} \quad (96)$$

To get another independent solution, substituting Eqn.(95) in L.H.S. of Eqn.(88) and simplifying, we get

$$x^2 y'' + y' + xy = c_0 k^2 x^{k-1} \quad (97)$$

Differentiating both sides of Eqn.(97) partially w.r.t. to k gives

$$\frac{\partial}{\partial k}\left[x^2 y'' + y' + xy\right] = \frac{c_0}{x} \frac{\partial}{\partial k}(k^2 x^k) = \frac{c_0}{x}\left[2kx^k + k^2 x^k \ln x\right] \quad (98)$$

The right hand side of Eqn.(98) vanishes for $k = 0$, this suggests that a second solution of the given equation is $\left.\frac{\partial y}{\partial k}\right)_{k=0}$, where y is given by Eqn.(95).

Differentiating Eqn.(95) partially w.r.t. to k, we get

$$\frac{\partial y}{\partial k} = c_0 x^k \ln x \left[1 - \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2 (k+4)^2} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2} \cdots \right]$$
$$+ c_0 x^k \left[- \frac{(-2)x^2}{(k+2)3} + \frac{x^4}{(k+2)^2 (k+3)^2} \left\{ \frac{-2}{k+2} - \frac{2}{k+4} \right\} - \frac{x^6}{(k+2)^2 (k+4)^2 (k+6)^2} \left\{ -\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6} \right\} + \cdots \right] \quad (99)$$

Putting $\mathbf{k = 0}$ in Eqn.(99), we get

$$\left.\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x \left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots \right)$$
$$+ c_0 \left[\frac{x^2}{2^2} - \frac{x^4}{2^2.4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots \right]$$
$$= c_0 \left[u \ln x + \left\{ \frac{x^2}{2^2} - \frac{x^4}{2^2.4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots \right. \right]$$
$$= c_0 v, \text{ say} \quad (100)$$

Thus the required general series solution of the given equation is,

$$y = A_1 c_0 u + B_1 c_0 v$$
$$= A u + B v$$

Where A_1, B_1, A and B are arbitrary constants, such that $A_1 c_0 = A$ and $B_1 c_0 = B$.

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How, about trying a few exercises now?

E14) Find a series solution about $x = 0$, of the differential equation $x(1-x)y'' + (1-x)y' - y = 0$.

E15) Solve in series, $xy'' + (p - x)y' - y = 0$ when,
(a) $p = 1$ (b) p is not an integer.

E16) Find a series solution about $x = 0$, of the following equations:
a) $xy'' + (1 + x)y' + 2y = 0$.
b) $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$.
c) $4(x^4 - x^2)y'' + 8x^3y' - y = 0$.

In the next subsection we discuss the case when for an equation of the type (15), we are required to find a series solution for large values of the independent variable.

2.4.3 Series Solution about a Point at Infinity

Often we come across physical situations where we need to find the solution of equations of the type

$$y'' + P(x)y' + Q(x)y = 0 \tag{101}$$

for large values of the independent variable x. For instance, if the variable is time , we may want to know how the physical system described by Eqn.(101) behaves in the distant future, when transient disturbances have faded out. In such cases, we can use our previous ideas and broaden them up by studying solutions near a **point at infinity**. If we change the independent variable in Eqn.(101) from x to t by the relation

$$t = \frac{1}{x},$$

then large x's correspond to small t's. The transformed equation can then be solved near $t = 0$ and then replacing t by $\frac{1}{x}$ in the solution obtained, we get solution of Eqn.(101) that are valid for large values of x. To carry out this, we need the formulas

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{-1}{x^2}\right) = -t^2\frac{dy}{dt} \tag{102}$$

and
$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right)(-t^2) \tag{103}$$

When these expressions for y' and y'' are substituted in Eqn.(101) then it becomes

$$y'' + \left[\frac{2}{t} - \frac{P(1/t)}{t^2}\right]y' + \frac{Q(1/t)}{t^4}y = 0 \tag{104}$$

where primes in Eqn.(104) denote derivatives w.r. to t.

We then say that Eqn.(101) has $x = \infty$ as an ordinary point, a regular singular point with exponents k_1 and k_2, or an irregular singular point, if the point $t = 0$ has the corresponding character for the transformed Eqn.(104).

We now illustrate the above procedure through an example.

Example 11: Obtain a series solution of the following equation for large values of x.

$$x^2y'' + (3x - 1)y' + y = 0. \tag{105}$$

Solution: Here $x = 0$, is an irregular singular point and Frobenius method can not be applied.

We change the independent variable x to t by putting $x = \frac{1}{t}$

Then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -t^2\frac{dy}{dt}$

and $\frac{d^2y}{dx^2} = -t^2\frac{d}{dt}\left(-t^2\frac{dy}{dt}\right) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$

Substituting the above values in the given Eqn.(105), we get

$$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}\right)+\left(\frac{3}{t}-1\right)\left(-t^2\frac{dy}{dt}\right)+y=0$$

i.e., $$t^2\frac{d^2y}{dt^2}-t(1-t)\frac{dy}{dt}+y=0 \qquad (106)$$

Now $t=0$ is a regular singular point of the above equation and we look for its series solution.

Let $$y(t)=\sum_{m=0}^{\infty}c_m t^{m+k},\ \ (c_o\neq 0) \qquad (107)$$

Then $$y'(t)=\sum_{m=0}^{\infty}(m+k)\,c_m t^{m+k-1} \qquad (108)$$

and $$y''(t)=\sum_{m=0}^{\infty}(m+k)\,c_m t^{m+k-2} \qquad (109)$$

On substituting the above expression for y, y' and y'' in Eqn.(106), we get

$$\sum_{m=0}^{\infty}(m+k-1)^2 c_m t^{m+k}+\sum_{m=1}^{\infty}(m+k-1)\,c_{m-1}t^{m+k}=0 \qquad (110)$$

The **indicial equation** is $c_0(k-1)^2=0\Rightarrow k=1,1$.

Equating coefficients of t^{m+k} in Eqn.(110) to zero, we get the recurrence relation as

$$c_m=\frac{(-1)\,c_{m-1}}{m+k-1},\ \text{ for }\ m\geq 1. \qquad (111)$$

$$\Rightarrow c_m=\frac{(-1)^m c_0}{k(k+1)\cdots(k+m-1)},\ \text{ for }\ m\geq 1.$$

Substituting the values of $c_0, c_1, \ldots$ etc. in Eqn.(107), we obtain

$$y(t)=c_0t^k+\sum_{m=1}^{\infty}\frac{(-1)^m c_0 t^{m+k}}{k(k+1)\cdots((k+m-1)}. \qquad (112)$$

Putting $k=1$ in the above expression for $y(t)$ we get one solution of Eqn.(106).
Since roots of indicial equation are equal, to find second linearly independent solution, we need to **put $k=1$** in

$$\frac{\partial y}{\partial k}=y\ln t-\sum_{m=1}^{\infty}\frac{(-1)^m\left\{\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{k+m-1}\right\}}{k(k+1)\cdots(k+m-1)}c_0t^{m+k} \qquad (113)$$

Thus, we get two linearly independent series solution of Eqn.(106) as

$$y(t)\Big|_{k=1}=c_0t+\sum_{m=1}^{\infty}c_0\frac{(-1)^m t^{m+1}}{(m)!}=u,\ \text{ say}$$

and $$\left.\frac{\partial y}{\partial k}\right)_{k=1}=u\ \ln t-\sum_{m=1}^{\infty}c_0\frac{(-1)^m H_m\, t^{m+1}}{(m)!}=v,\ \text{ say}$$

where, $H_m=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}=\sum_{p=1}^{m}\left(\frac{1}{p}\right).$

Therefore, two linearly independent solutions of Eqn.(105) are

$$u=\sum_{m=0}^{\infty}c_0\frac{(-1)^m}{(m)!}\frac{1}{x^{m+1}}=\frac{c_0}{x}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m)!}x^{-m}=\frac{c_0}{x}e^{\frac{1}{x}}$$

and $$v=u\ln\left(\frac{1}{x}\right)-\sum_{m=1}^{\infty}c_0\frac{(-1)^m H_m}{(m)!}.\frac{1}{x^{m+1}},\ \text{ where } H_m=\sum_{p=1}^{m}\left(\frac{1}{p}\right).$$

The general solution of Eqn.(105) is then given by $y(x) = au + bv$, where a and b are arbitrary constants.

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You may now try this exercise.

E17) Find a series solution of the following equations for large values of x.

a) $4x^3y'' + 6x^2y' + y = 0$.

b) $2x^2(1-x)y'' - 5x(1+x)y' + (5-x)y = 0$.

In this unit we discussed methods of finding series solutions of second order linear, homogeneous differential equations with variable coefficients. In the next two units, we shall apply these methods to solve some differential equations, which occur frequently in physical and engineering problems. In particular, we shall solve Legendre's, Bessel's, Laguerre's and Hermite's differential equations.

We now end this unit by giving a summary of what we have covered in it.

2.5 SUMMARY

In this unit, we have learnt the following points:

1. Second order linear homogeneous differential equation with variable coefficients cannot be, in general, solved by methods giving closed form solutions and there is a need for series solutions for such equations.
2. For a power series, $\sum_{m=0}^{\infty} c_m x^m$, if $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| \cdot \left|\frac{1}{x}\right|$ exists, then **the power series converges if** $|x| < R$ **and diverges if** $|x| > R$ and the **region of convergence is** $-R < x < R$**.** Here **R is** known as **radius of convergence** of the power series.
3. A power series can be differentiated term by term any number of times without affecting its radius of convergence.
4. For the power series $f(x) = \sum_{m=0}^{\infty} c_m (x-a)^m$, **the r-times differentiated series has the sum** $\frac{d^r[f(x)]}{dx^r}$, for each positive integer r and the coefficients c_m are completely determined by $c_m = \frac{f^{(m)}(a)}{(m)!}$.
5. A function $f(x)$ is said to be **analytic** at $x = a$ if, $f(x)$ can be expanded in a power series $\sum_{m=0}^{\infty} c_m (x-a)^m$ with a positive radius of convergence.
6. A point $x = a$ is called an **ordinary point** of the differential equations $y'' + d_1(x)y' + d_2(x) = b_1(x)$ if $d_1(x)$ and $d_2(x)$ are analytic at $x = a$.
7. If $x = a$ is not an ordinary point of differential equation $y'' + d_1(x)y' + d_2(x)y = 0$, then it is called a **singular point** of the differential equation. Further if $(x-a)d_1(x)$ and $(x-a)^2 d_2(x)$ are analytic at $x = a$, then $x = a$ is called a **regular singular point** of the differential equation. A singular point, which is not regular, is called an **irregular singular point.**
8. If $x = 0$ is an **ordinary point** of the differential equation

$$y'' + d_1(x)y' + d_2(x)y = 0,$$

then power series solution is assumed as $y(x) = \sum_{m=0}^{\infty} c_m x^m$. Expressions for y, y' and y'' are substituted in the given equation and $c_1, c_2 \ldots$ are determined by comparing the coefficient of different power of x.

9. If $x = 0$ is a **regular singular** point of the differential Eqn(42), viz.,

$$f(x)\,y''(x) + g(x)\,y'(x) + r(x)\,y = 0$$

then series solution of the equation is obtained by using the Frobenius method where the series solution assumed is of the form

$$y(x) = \sum_{m=0}^{\infty} c_m\, x^{m+k},\ c_0 \neq 0.$$

10. In the Frobenius method, equation obtained by equating to zero the coefficient of smallest power of x is a quadratic equation in k, called the **indicial equation**.

11. When the roots of indicial equation k_1, k_2 (say) are **unequal** and **do not differ by an integer** then two linearly independent series solutions of Eqn.(42) are obtained corresponding to two values of k viz., k_1, k_2 in the expression for $y(x)$.

12. When the roots of indicial equation k_1, k_2 (say) are **unequal**, **differ by an integer** and make a coefficient of y **indeterminate** for $k = k_2$ (say) then two linearly independent solutions of Eqn(42) are obtained by putting $k = k_2$ in $y(x)$, which then contains two arbitrary constants.

13. When the roots of indicial equation k_1, k_2 (say) are **unequal**, **differ by an integer** and make a coefficient of y **infinite** for $k = k_2$ (say), then the form of y is modified by replacing c_0 by $d_0(k - k_2)$. Two linearly independent solutions are then obtained by substituting $k = k_2$ in $y(x)$ and $\frac{\partial y}{\partial k}$.

14. When the roots of indicial equation are **equal**, then two linearly independent solution of Eqn.(42) are obtained by substituting this value of k in $y(x)$ and $\frac{\partial y}{\partial k}$.

2.6 SOLUTIONS/ANSWERS

E1) For series (5), $R = \lim\limits_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim\limits_{n\to\infty}\left|\frac{n}{n+1}\right| = 0$.

For series (6), $R = \lim\limits_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim\limits_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim\limits_{n\to\infty}|n+1| \to \infty$

For series (7), $R = \lim\limits_{n\to\infty}\left|\frac{1}{1}\right| = 1$.

E2) For the series $\sum \frac{x^n}{n!}$, $\quad R = \lim\limits_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim\limits_{n\to\infty}(n+1) \to \infty$.

Hence the given series converges for all x.

E3) a) We have, $1 - 2x + 3x^2 - 4x^3 + \cdots = \sum (-1)^n (n+1)\,x^n$

$$R = \lim_{n\to\infty}\left|\frac{(-1)^n (n+1)}{(-1)^{n+1}(n+2)}\right| = \lim_{n\to\infty}\left|\frac{-\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)}\right| = 1.$$

b) We have, $R = \lim_{n\to\infty}\left|\frac{(kn)!}{(n!)^k}\cdot\frac{[(n+1)!]^k}{[k(n+1)]!}\right|$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^k}{k(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{k-1}}{k}\right| \to \infty,\ \text{for } k>1$$

$$\text{and}\quad \lim_{n\to\infty}\left|\frac{(n+1)^{k-1}}{k}\right| = 1\ \text{ for}, k=1.$$

E4) Comparing the given equations with $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$,

a) Here $a_0(x)=1,\ a_1(x)=4x,\ \ a_2(x)=1$

Now $(x-a)\frac{a_1(x)}{a_0(x)} = (x-a)4x$ and $(x-a)^2\frac{a_2(x)}{a_0(x)} = (x-a)^2$, are analytic at $x=a$. Hence $x=a$ is an ordinary point of given differential equation for all real finite a, including $a=0$.

b) Here $a_0(x)=x^3(x-2),\ a_1(x)=x^3,\ a_2(x)=6$

Now $(x-a).\frac{a_1(x)}{a_0(x)} = \frac{(x-a)x^3}{x^3(x-2)}$, is analytic for $a=2$

Also, $(x-a)^2.\frac{a_2(x)}{a_0(x)} = \frac{(x-a)^2.6}{x^3(x-2)}$ is analytic for $a=2$ and singular for $x=0$.

Hence $x=0$ is an irregular singular point, and $x=2$ is a regular singular point for the given differential equation.

c) $x=-1$, is singular point.

d) $x=0$, is an irregular singular point, since $x^2\frac{a_2(x)}{a_0(x)} = x^2.\frac{2}{x^4} \to \infty$ at $x=0$.

e) $x=0$, is a regular singular point, since $x\frac{\sin x}{x}$ and $x^2.\frac{e^x}{x}$ are analytic at $x=0$.

E5) The given D.E. is

$$(x^2-2x+2)y'' - 4(x-1)y' + 6y = 0$$

In this case, we change the independent variable x to u by the relation $x-1=u$, the given D.E. becomes

$$(u^2+1)y'' - 4uy' + 6y = 0, \qquad (114)$$

where y' and y'' now denote the derivatives of y w.r.t. u. We are now required to solve Eqn.(114) about the point $u=0$.

We see that the coefficients of y' and y, i.e., $-\frac{4u}{u^2+1}$ and $\frac{6}{u^2+1}$ are analytic everywhere. Hence we assume solution of Eqn.(114) as

$$y = \sum_{m=0}^{\infty} c_m u^m \qquad (115)$$

Substituting for expressions for y, y', y'' from (115) into Eqn.(114), we get

$$(u^2+1)\sum_{m=2}^{\infty} c_m m(m-1)u^{m-2} - 4u\sum_{m=1}^{\infty} c_m m u^{m-1} + 6\sum_{m=0}^{\infty} c_m u^m = 0$$

or,

$$(2c_2+6c_0) + (2c_1+6c_3)u + \sum_{m=2}^{\infty}\left[(m+2)(m+1)c_{m+2} + (m^2-5m+6)c_m\right]u^m = 0$$

Setting the coefficients of successive powers of u to zero, we get $2c_2+6c_0=0,\ 2c_1+6c_3=0$ and $(m+2)(m+1)c_{m+2} + (m-2)(m-3)c_m = 0$ for $m \geq 2$.

where c_0, c_1 are arbitrary constants. Solving, we get

$$c_2 = -3c_0,\ c_3 = \frac{-c_1}{3} \text{ and } c_{m+2} = -\frac{(m-2)(m-3)}{(m+2)(m+1)} c_m, \text{ for } m \geq 2.$$

$$\Rightarrow c_4 = 0,\ c_5 = 0,\ c_6 = -\frac{2.1}{6.5} c_4 = 0,\ c_7 = -\frac{3.2}{7.6} c_5 = 0$$

$\therefore$ The solution of given D.E. is

$$y = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = c_0(1-3u^2) + c_1\left(u - \frac{1}{3}u^3\right)$$

$$= c_0\left[1 - 3(x-1)^2\right] + c_1\left[(x-1) - \frac{1}{3}(x-1)^3\right].$$

E6) The given D.E. is

$$y'' + xy' + 3y = 0 \qquad (116)$$

For the given equations, $x = 0$ is an ordinary point and we shall obtain the solution about $x = 0$.

Let $\quad y(x) = \sum_{m=0}^{\infty} c_m x^m$.

Then Eqn.(116) becomes

$$\sum_{m=2}^{\infty} m(m-1)c_m x^{m-2} + x\sum_{m=1}^{\infty} m c_m x^{m-1} + 3\sum_{m=0}^{\infty} c_m x^m = 0$$

or, $\quad (2c_2 + 3c_0) + \sum_{m=1}^{\infty}\{(m+2)(m+1)c_{m+2} + (3+m)c_{m=0}\}x^m$

Equating the coefficients of similar powers of x, we get

$$2c_2 + 3c_0 = 0 \Rightarrow c_2 = \frac{-2}{3}c_0$$

$$6c_3 + 4c_1 = 0 \Rightarrow c_3 = \frac{-2}{3}c_1$$

Indeed, $(m+2)(m+1)c_{m+2} + (m+3)c_m = 0$, for $m \geq 1$.

$$\therefore \quad c_4 = -\frac{5}{4.3}c_2 = (-1)^2 . \frac{5.2}{4.3.3}c_0$$

$$c_5 = -\frac{6}{5.4}c_3 = (-1)^2 \frac{2.6}{3.4.5}c_1$$

$$c_6 = -\frac{7}{6.5}c_4 = (-1)^3 \frac{7.5.2}{6.5.4.3.2}c_0$$

$$c_7 = -\frac{8}{7.6}c_5 = (-1)^3 \frac{8.6.2}{7.6.5.4.3}c_1$$

$$\cdots\cdots\cdots\cdots$$

where c_0 and c_1 are arbitrary constants.

Hence the general solution is

$$y(x) = c_o\left[1 - \frac{2}{3}x + (-1)^2\frac{5}{4.3}\left(\frac{2}{3}\right)x^2 + (-1)^3\frac{7.5}{6.5.4.3}\left(\frac{2}{3}\right)x^3 + \cdots\right]$$

$$+ c_1\left[x - \left(\frac{2}{3}\right)x^3 + (-1)^2\frac{6}{4.5}\left(\frac{2}{3}\right)x^4 + (-1)^3\frac{8.6}{7.6.5.4}\left(\frac{2}{3}\right)x^5 + \cdots\right]$$

E7) The given initial value problem is

$$\left.\begin{aligned} xy'' + y' + 2y = 0 \\ y(1) = 1 \\ y'(1) = 2 \end{aligned}\right\} \qquad (117)$$

Evidently, $x = 1$ is an ordinary point.

Let $\quad y = \sum_{m=0}^{\infty} c_m (x-1)^m \qquad (118)$

Differentiating twice in succession, we get

$$y' = \sum_{m=1}^{\infty} m c_m (x-1)^{m-1} \qquad (119)$$

and $\quad y'' = \sum_{m=2}^{\infty} m(m-1) c_m (x-1)^{m-2} \qquad (120)$

Substituting for y, y', y'' in Eqn.(117), we get

$$x \sum_{m=2}^{\infty} m(m-1)c_m (x-1)^{m-2} + \sum_{m=1}^{\infty} m c_m (x-1)^{m-1} + 2\sum_{m=0}^{\infty} c_m (x-1)^m = 0$$

or, $\quad (2c_2 + c_1 + 2c_0) + \sum_{m=1}^{\infty} \{m(m+1)c_{m+1} + (m+2)(m+1)c_{m+2}$

$$+ (m+1)c_{m+1} + 2c_m\}(x-1)^m = 0$$

$\therefore \quad 2c_2 + c_1 + 2c_0 = 0$ i.e., $c_2 = -\left[c_0 + \frac{c_1}{2}\right] \qquad (121)$

and $(m+1)mc_{m+1} + (m+2)(m+1)c_{m+2} + (m+1)c_{m+1} + 2c_m = 0$, for $m \ge 1$

i.e., $(m+1)(m+2)c_{m+2} + (m+1)^2 c_{m+1} + 2c_m = 0$, for $m \ge 1 \qquad (122)$

using initial conditions given by (117) in relations (118) and (119), we get

$c_0 = 1$ and $c_1 = 2$

From (121) and above, $c_2 = -2$

For $m = 1, 2, 3, \ldots$ in Eqn.(122) and using above values, we get

$6c_3 + 4c_2 + 2c_1 = 0 \Rightarrow c_3 = 2/3$

$12c_4 + 9c_3 + 2c_2 = 0 \Rightarrow c_4 = -1/6$

$20c_5 + 16c_4 + 2c_3 = 0 \Rightarrow c_5 = 1/15.$

and so on.

Hence $y = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots.$

E8) The given D.E. is $y'' + xy' + y = 0 \qquad (123)$

Here $x = 2$, is an ordinary point.

Let $y = \sum_{m=0}^{\infty} c_m (x-2)^m \qquad (124)$

Substituting for y, y', y'' in Eqn.(123), we get

$$\sum_{m=2}^{\infty} c_m (x-2)^{m-2} m(m-1) + x\sum_{m=1}^{\infty} m c_m (x-2)^{m-1} + \sum_{m=0}^{\infty} c_m (x-2)^m = 0$$

or, $\quad (2.1.c_2 + 2.1.c_1 + c_0) + \sum_{m=1}^{\infty}\{(m+2)(m+1)c_{m+2} + m c_m$

$$+ 2(m+1)c_{m+1} + c_m\}(x-2)^m = 0.$$

$\therefore \quad 2c_2 + 2c_1 + c_0 = 0 \Rightarrow c_2 = -c_1 - \frac{1}{2}c_0$

and $(m+2)(m+1)c_{m+2} + 2(m+1)c_{m+1} + (m+1)c_m = 0$, for $m \ge 1 \qquad (125)$

For $m = 1, 2, 3, \ldots$ and using Eqn.(125), we get

$3.2.c_3 + 2.2c_2 + 2c_1 = 0 \Rightarrow c_3 = \frac{1}{3}(c_1 + c_0)$

$4.3.c_4 + 2.3c_3 + 3c_2 = 0 \Rightarrow c_4 = \frac{1}{3.4}c_1 - \frac{1}{2.4.3}c_0$

$$5.4.c_5 + 2.4\,c_4 + 4\,c_3 = 0 \Rightarrow c_5 = \frac{1}{5.6}c_1 + \frac{1}{12}c_0$$

and so on. Hence Eqn.(124) yields

$$y(x) = c_0\left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{24}(x-2)^4 + \frac{1}{12}(x-2)^5 + \cdots\right]$$
$$+ c_1\left[(x-2) - (x-2)^2 + \frac{1}{3}(x-2)^3 + \frac{1}{12}(x-2)^4 + \frac{1}{30}(x-2)^5 + \cdots\right].$$

E9) The given Hermite equation of order n is

$$y'' - 2xy' + 2ny = 0 \qquad (126)$$

If $z_n(x) = \sum_{k=0}^{\infty} a_k x^k$ is a solution of Eqn.(126), then substituting for y, y', y'' in Eqn.(126), we get

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} - 2x\sum_{k=1}^{\infty} k\,a_k x^{k-1} + 2n\sum_{k=0}^{\infty} a_k x^k = 0$$

or, $$2.1.a_2 + 2n\,a_0 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2} - 2k\,a_k + 2n\,a_k\right]x^k = 0$$

Equating coefficients of successing powers of x, we get

$$2a_2 + 2n\,a_0 = 0 \Rightarrow a_2 = -n\,a_0$$

and $(k+2)(k+1)a_{k+2} - 2(k-n)a_k = 0$, for $k \ge 1$

putting $k = 1, 2, 3, \ldots$ in the above relation, we get

$$3.2.a_3 - 2(1-n)a_1 = 0 \Rightarrow a_3 = \frac{2(1-n)}{2.3}a_1$$

$$4.3\,a_4 - 2(2-n)a_2 = 0 \Rightarrow a_4 = \frac{2(2-n)}{4.3}a_2 = -\frac{2n(2-n)}{4.3}a_0$$

$$5.4\,a_5 - 2(3-n)a_3 = 0 \Rightarrow a_5 = \frac{2(3-n)}{5.4}a_3 = \frac{2^2(1-n)(3-n)}{2.3.4.5}a_1$$

$$6.5\,a_6 - 2(4-n)a_4 = 0 \Rightarrow a_6 = \frac{2(4-n)}{6.5}a_4 = -\frac{2^2 n(2-n)(4-n)}{6.5.4.3}a_0$$

and so on.

Hence the solution is

$$z_n(x) = y(x) = a_0\left[1 + (-n)x^2 + \frac{2(-n)(2-n)}{4.3}x^4 + \frac{2^2(-n)(2-n)(4-n)}{6.5.4.3}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2(1-n)}{2.3}x^3 + \frac{2^2(1-n)(3-n)}{2.3.4.5}x^5 + \cdots\right]$$
$$= a_0\left[1 + \frac{2(-n)}{2}x^2 + \frac{2^2(-n)(2-n)}{4.3.2}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6.5.4.3.2}x^6 + \cdots\right]$$
$$+ a_1\left[x + \frac{2(1-n)}{2.3}x^3 + \frac{2^2(1-n)(3-n)}{2.3.4.5}x^5 + \cdots\right]$$
$$= a_0\left[1 + \sum_{k=1}^{\infty}\frac{2^k(-n)(-n+2)\ldots(-n+2k-2)}{(2k)!}x^{2k}\right]$$
$$+ a_1\left[x + \sum_{k=1}^{\infty}\frac{2^k(1-n)(1-n+2)\ldots(1-n+2k-2)}{(2k+1)!}x^{2k+1}\right]$$

E10) a) The given equation

$$2x(1-x)y'' + (1-x)y' + 3y = 0 \qquad (127)$$

has $x = 0$, as a regular singular point

Let $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$

Putting the values of y, y', y'' in Eqn.(127), we get

$$\sum_{m=0}^{\infty} \{3-(k+m)-2(k+m)(k+m-1)\}c_m x^{k+m}$$

$$+\sum_{m=0}^{\infty}\{2(k+m-1)+1\}(k+m)c_m x^{k+m-1}=0 \qquad (128)$$

Equating the coefficients of lowest power of x to zero in Eqn.(128), we get the **indicial equation** as $c_0(2k-2+1)k=0$ i.e., $c_0k(2k-1)=0 \Rightarrow k=0$ and $k=1/2$, i.e., the roots of indicial equation are **distinct**, and do not differ by an integer. To get the recurrence relation, we equate coefficients x^{k+m} in Eqn.(128) to zero and obtain

$$[3-(k+m)-2(k+m)(k+m-1)]c_m+\{2(k+m+1-1)+1\}(k+m+1)c_{m+1}=0$$

i.e., $$c_{m+1}=\frac{2(k+m)(k+m-1)+(k+m)-3}{(k+m+1)(2k+2m+1)}c_m$$

Putting $m=0,1,2,3,\ldots$

$$c_1=\frac{2k^2-k-3}{(k+1)(2k+1)}c_0$$

$$c_2=\frac{\{2(k+1)k+(k+1)-3\}\{2k(k-1)+k-3\}}{(k+1)(k+2)(2k+1)(2k+3)}c_0$$

$$c_3=\frac{\{2(k+2)(k+1)+(k+2)-3\}\{2(k+1)k+(k+1)-3\}\{2k(k-1)+k-3\}}{(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)}c_0$$

Putting values of $c_1, c_2, c_3, \ldots$ in $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$, we get

$$y(x)=x^k\left[c_0+\frac{2k^2-k-3}{(k+1)(2k+1)}c_0x+\frac{(2k^2+3k-2)(2k^2-k-3)}{(k+1)(k+2)(2k+1)(2k+3)}c_0x^2\right.$$

$$\left.+\frac{(2k^2+7k+3)(2k^2+3k-2)(2k^2-k-3)}{(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)}c_0x^3+\cdots\right] \qquad (129)$$

Putting $k=0$ in Eqn.(129), we get

$$y_1=c_0\left[1-3x+\frac{(-2)(-3)}{1.2.1.3}x^2+\frac{3(-2)(-3)}{1.2.3.1.3.5}x^3+\cdots\right]$$

$$=c_0\left[1-3x+\frac{3}{1.3}x^2+\frac{3}{1.3.5}x^3+\cdots\right]=c_0u\text{, say} \qquad (130)$$

Putting $k=1/2$ in Eqn.(129), we get

$$y_2=c_0x^{1/2}[1+(-x)+0+0+\cdots]=c_0x^{1/2}(1-x)=c_0v\text{, say} \qquad (131)$$

Hence the general solution is $y(x)=A_1y+B_1y_2=Au+Bv$

where $A=A_1c_0$, $B=B_1c_0$ are arbitrary constants and u and v are given by Eqn.(130) and Eqn.(131) respectively.

b) The given equation

$$(2x+x^2)y''-y'-6xy=0 \qquad (132)$$

has $x=0$ as a regular singular point.

Let $y(x)=\sum_{m=0}^{\infty} c_m x^{k+m}$ be a solution of Eqn.(132). Putting for y, y', y'' in Eqn.(132), we get

$$-6\sum_{m=0}^{\infty} c_m x^{k+m+1} + \sum_{m=0}^{\infty} (k+m)\ (k+m-1)\, x^{k+m}$$

$$+\sum_{m=0}^{\infty} (k+m)(2k+2m-3)\, c_m x^{k+m-1} = 0 \qquad (133)$$

Equating the coefficient of lowest power of x to zero in Eqn.(133), we get the **indicial equations** as $k(2k-3)c_0 = 0 \Rightarrow k=0$ and $k=3/2$, i.e., the roots of indicial equation are **distinct**, and do not differ by an integer. To get the recurrence relation, equate the coefficient of x^{k+m+1} to zero is relation (133) and obtain

$$-6c_m + (k+m+1)(k+m)c_{m+1} + (k+m+2)(2k+2m+1)c_{m+2} = 0 \quad (134)$$

Equating coefficient of x^k in (133) to zero, we get

$$k(k-1)c_0 + (k+1)(2k-1)c_1 = 0 \Rightarrow c_1 = -\frac{k(k-1)}{(k+1)(2k-1)}c_0$$

Putting $m = 0,1,2,3,\ldots$ in Eqn.(134), we get

$$c_2 = \frac{6(2k-1)+k^2(k-1)}{(k+2)(2k-1)(2k+1)}c_0$$

$$c_3 = -\frac{6k(k-1)}{(k+1)(2k-1)(k+3)(2k+3)}c_0 - \frac{(k+1)[k^3-k^2+12k-6]}{(k+3)(2k-1)(2k+1)(2k+3)}c_0$$

and so on.

For $k=0$, we get

$$c_1 = 0, c_2 = \frac{-6}{2(-1)(1)}c_0 = 3c_0, c_3 = -\frac{1(-6)}{3(-1).1.3}c_0 = -\frac{2}{3}c_0, \cdots$$

For $k = 3/2$, we get

$$c_1 = \frac{3}{4.5}c_0,\ c_2 = \frac{15}{32}c_0, \cdots$$

Hence $y_1 = c_0\left[1 + 3x - \frac{2}{3}x^2 + \cdots\right] = c_0\, u$, say (135)

and $y_2 = c_0 x^{3/2}\left[1 + \frac{3}{4.5}x + \frac{15}{32}x^2 + \cdots\right] = c_0\ v$, say (136)

Hence the general solution is $y(x) = A_1 y_1 + B_1 y_2 = Au + Bv$, where $A = c_0 A_1$ and $B = c_0 B_1$ are arbitrary constant and u and v are given by relations (135) and (136) respectively.

E11) The given D.E. is

$$(2+x^2)y'' + xy' + (1+x)y = 0$$

Let $y = \sum_{m=0}^{\infty} c_m x^{k+m}, c_0 \neq 0$, be a series solution of the given equation. Then putting the values of y, y', y'' in the given equation, we get

$$(2+x^2)\sum_{m=2}^{\infty} (m+k)(m+k-1)c_m x^{k+m-2} + x\sum_{m=1}^{\infty} (m+k)c_m x^{m+k-1}$$

$$+ (1+x)\sum_{m=0}^{\infty} c_m x^{m+k} = 0$$

$$\Rightarrow 2c_0 k(k-1)x^{k-2} + 2c_1 k(k+1)x^{k-1}\left[2c_2(k+2)(k+1) + c_0 k(k-1) + c_0 k + c_0\right]x^k$$

$$+\sum_{m=3}^{\infty}\left[2c_m(k+m)(k+m-1) + c_{m-2}(k+m-2)(k+m-3)\right.$$

$$\left. + c_{m-2}(k+m-2) + c_{m-2} + c_{m-3}\right]x^{k+m-3} = 0$$

Equating coefficients of lowest powers of x, we get **indicial equation** as

$$2c_0k(k-1)=0 \Rightarrow k=0,1 \text{ as } c_0 \neq 0$$

Equating to zero coefficients of other powers of x, we get

$$2c_1(k+1)k=0$$

This shows that c_1 becomes **indeterminate for $k=0$**.

$\therefore \quad 2c_2(k+2)(k+1)+c_0k(k-1)+c_0k+c_0=0$

$$c_2=-\frac{k^2+1}{2(k+1)(k+2)}c_0$$

The general recurrences relation is

$$2c_m(k+m)(k+m-1)+c_{m-2}\left[(k+m-2)^2+1\right]+c_{m-3}=0 \text{ for } m\geq 3$$

Putting $m=3,4,5,\ldots$, we get

$$c_3=-\frac{[(k+1)^2+1]c_1}{2(k+3)(k+2)}-\frac{c_0}{2(k+3)(k+2)}$$

$$c_4=-\frac{c_1}{2(k+3)(k+4)}+(-1)^2\frac{(k^2+1)(k^2+4k+5)c_0}{2^2(k+1)(k+2)(k+3)(k+4)}$$

and so on.

Putting these values in the assumed solution, we get

$$y=x^k\left[c_0+c_1x-\frac{(k^2+1)c_0}{2(k+1)(k+2)}x^2-\left\{\frac{c_0}{2(k+2)(k+3)}+\frac{\{(k+1)^2+1\}c_1}{2(k+3)(k+2)}\right\}x^3\right.$$
$$\left.+\left\{-\frac{c_1}{2(k+4)(k+3)}+\frac{(k^2+1)\{(k+2)^2+1\}}{2^2(k+1)(k+2)(k+3)(k+4)}c_0\right\}x^4+\cdots\right]$$

Putting $k=0$, we get

$$y=c_0\left[1-\frac{1}{2}x^2-\frac{1}{2.2.3}x^3+\frac{5}{2^2.1.2.3.4}x^4+\cdots\right]$$
$$+c_1\left[x-\frac{1}{6}x^3-\frac{1}{2.3.4}x^4+\cdots\right]$$

as the required solution, where c_0 and c_1 are arbitrary constants.

E12) The given equation is $(1-x^2)y''+2xy'+y=0$.
Let the series solution of given equation be

$$y(x)=\sum_{m=0}^{\infty}c_mx^{m+k},\ c_0\neq 0 \tag{137}$$

substituting for y, y', y'' in the given equation, we get

$$c_0k(k-1)x^{k-2}+c_1k(k-1)x^{k-1}+\sum_{m=0}^{\infty}\left[-(m+k)(m+k-1)c_m+2(m+k)c_m+\right.$$
$$\left.c_m+(m+2+k)(m+k+1)c_{m+2}\right]x^{m+k}=0$$

The **indicial equation** is $c_0k(k-1)=0 \Rightarrow k=0,1$ as $c_0\neq 0$. (138)

Thus indicial roots are **distinct** and **differ by an integer**.
Equating the coefficients of x^{k-1} to zero, we get

$$c_1k(k+1)=0 \tag{139}$$

When $k=0, c_1$ becomes **indeterminate** $(0/0)$. But in this case, we get the identity $c_1.0=0$, which is satisfied by every values of c_1. Thus, we can also take c_1 as arbitrary constant.
The recurrence relation is

$$(m+k+2)(m+k+1)c_{m+2}+\left[1-(m+k)(m+k-1)+2(m+k)\right]c_m=0$$

$$\Rightarrow c_{m+2} = \frac{(m+k)^2 - 3(m+k) - 1}{(m+k+2)(m+k+1)} c_m = \frac{(m+k)(m+k-3) - 1}{(m+k+2)(m+k+1)} c_m$$

Putting $m = 0, 1, 2, \ldots$, we get

$$c_2 = \frac{k^2 - 3k - 1}{(k+2)(k+1)} c_0 = \frac{k(k-3) - 1}{(k+2)(k+1)} c_0$$

$$c_3 = \frac{(k+1)(k-2) - 1}{(k+3)(k+2)} c_1$$

$$c_4 = \frac{\{(k+2)(k-1) - 1\}\{k(k-3) - 1\}}{(k+4)(k+2)^2(k+1)} c_0$$

$$c_5 = \frac{\{(k+3)(k) - 1\}\{(k+1)(k-2) - 1\}}{(k+5)(k+3)^2(k+2)} c_1$$

....

Substituting these values of coefficients in series for $y(x)$, we get

$$y = x^k \Bigg[c_0 + c_1 x + \frac{k(k-3) - 1}{(k+2)(k+1)} c_0 x^2 + \frac{(k+1)(k-2) - 1}{(k+3)(k+2)} c_1 x^3 + \frac{\{(k+2)(k-1) - 1\}\{k(k-3) - 1\}}{(k+4)(k+2)^2(k+1)} c_o x^4 + \frac{\{(k+3)k - 1\}\{(k+1)(k-2) - 1\}}{(k+5)(k+3)^2(k+2)} c_1 x^5 + \cdots \Bigg] \quad (140)$$

Substituting $\mathbf{k = 0}$ in this, we get

$$y = c_0 \left[1 - \frac{1}{2} x^2 - \frac{3}{16} x^4 + \cdots \right] + c_1 \left[x - \frac{1}{2} x^3 + \frac{(-3)}{90} x^5 - \cdots \right] \quad (141)$$

where c_0 and c_1 are arbitrary constants.

This is the required general series solution.

Note that if we substitute $k = 1$ in the series for $y(x)$, we get from relation (139) that $c_1 = 0$ and hence $c_3, c_5, \ldots$ all will vanish and then relation (140) gives

$$y = x \left[c_0 + \left(-\frac{1}{2} \right) x^2 c_0 + \left(-\frac{3}{90} \right) x^4 c_0 + \cdots \right] \quad (142)$$

which is the first series in Eqn.(141) and hence it is not a new series.

E13) a) The given equation is

$$xy'' + (3 - x)y' + y = 0$$

Here $x = 0$, is a regular singular point.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, \; c_0 \neq 0$. (143)

Substituting for y, y', y'' in the given equation, we get

$$[k(k-1) + 3k] c_0 x^{k-1} + \sum_{m=0}^{\infty} [\{(m+k+1)(m+k) + 3(m+k+1)\} c_{m+1} - (m+k-1) c_m] x^{m+k} = 0$$

Indicial equation is

$$k(k-1) + 3k = 0, \text{ as } c_0 \neq 0, \; k = 0, -2$$

The recurrence relation is

$$[(m+k+1)(m+k) + 3(m+k+1)] c_{m+1} - (m+k-1) c_m = 0, \; m \geq 0$$

or, $$c_{m+1} = \frac{m+k-1}{(m+k+1)(m+k+3)} c_m, \quad m \geq 0$$

Putting $m = 0, 1, 2, 3, \ldots$

$$c_1 = \frac{k-1}{(k+1)(k+3)} c_0$$

$$c_2 = \frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)} c_0$$

$$c_3 = \frac{(k-1)k}{(k+2)(k+3)^2(k+4)(k+5)} c_0$$

$$c_4 = \frac{(k-1)k}{(k+3)^2(k+4)^2(k+5)(k+6)} c_0$$

and so on.

$$\therefore y(x) = c_0 x^k \left[1 + \frac{k-1}{(k+1))k+3)} x + \frac{(k-1)k}{(k+1)(k+2)(k+3)(k+4)} x^2 + \frac{(k-1)k}{(k+2)(k+3)^2(k+4)(k+5)} x^3 + \frac{(k-1)k}{(k+3)^2(k+4)^2(k+5)(k+6)} x^4 + \cdots \right]$$

For $k = -2$, coefficients of x^2 and x^3 become **infinite**. Replacing c_0 by $a(k+2)$, the above series solution takes the form

$$y(x) = a x^k \left[(k+2) + \frac{(k-1)(k+2)}{(k+1)(k+3)} x + \frac{(k-1)k}{(k+1)(k+3)(k+4)} x^2 + \frac{(k-1)k}{(k+3)^2(k+4)(k+5)} x^3 + \frac{(k-1)k(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)} x^4 + \cdots \right] \quad (144)$$

$$\therefore \frac{\partial y}{\partial k} = a x^k \ln x \left[(k+2) + \frac{(k-1)(k+2)}{(k+1)(k+3)} x + \cdots \right]$$
$$+ a x^k \left[1 + \left\{ \frac{2k+1}{(k+1)(k+3)} - \frac{(k-1)(k+2)}{(k+1)(k+3)} \left(\frac{1}{k+1} + \frac{1}{k+3} \right) \right\} x \right.$$
$$+ \left\{ \frac{2k-1}{(k+1)(k+3)(k+4)} - \frac{k(k-1)}{(k+3)(k+4)(k+1)} \left(\frac{1}{k+1} + \frac{1}{k+3} + \frac{1}{k+4} \right) \right\} x^2$$
$$\left. + \left\{ \frac{2k-1}{(k+3)^2(k+4)(k+5)} - \frac{k(k-1)}{(k+3)^2(k+4)(k+5)} \left(\frac{2}{k+3} + \frac{1}{k+4} + \frac{1}{k+5} \right) \right\} x^3 + \cdots \right] \quad (145)$$

Putting $k = -2$ in Eqns.(144) and (145), we get

$$y = a x^{-2} [-3x^2 + x^3] = a\, y_1, \text{ say}$$

$$\frac{\partial y}{\partial k} = a(\ln x) y_1 + a x^{-2} \left[1 + 3x + 4x^2 - \frac{11}{3} x^3 + \frac{1}{8} x^4 - \cdots \right] = a y_2, \text{ say}$$

$\therefore$ The general solution is

$$y(x) = c_1 y_1 + c_2 y_2 .$$

b) Given equation is

$$x(1-x)y'' - 3xy' - y = 0$$

Here $x = 0$, is a regular singular point.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}$, $c_0 \neq 0$

Substituting for y, y', y'' in the given equation, we get

$$\sum_{m=0}^{\infty} c_m (m+k)(m+k-1) x^{m+k-1} - \sum_{m=0}^{\infty} (m+k+1)^2 c_m x^{m+k} = 0$$

Equating to zero the coefficients of lowest power of x, i.e., x^{k-1}, the **indicial equation** is

$$c_0 k(k-1) = 0 \text{ or } k = 0,1 \ (\because c_0 \neq 0)$$

There are **unequal roots**, differing by an integer.

The recurrence relation is

$$c_m (m+k+1)^2 - c_{m+1}(m+k+1)(m+k) = 0 \quad \text{for } m \geq 0$$

$$\therefore c_{m+1} = \frac{m+k+1}{m+k} c_m \quad \text{for } m \geq 0$$

Putting $m = 0,1,2,3,\ldots$, we get

$$c_1 = \frac{k+1}{k} c_0 \ , \quad c_2 = \frac{k+2}{k} c_0 \ , \ c_3 = \frac{k+3}{k} c_0$$

and so on.

Putting these values in the series solution for $y(x)$, we get

$$y(x) = x^k \left[c_0 + \frac{k+1}{k} c_0 x + \frac{k+2}{k} c_0 x^2 + \frac{k+3}{k} c_0 x^3 + \cdots \right]$$

If we put $\mathbf{k = 0}$, we find that coefficients of $x, x^2, x^3, \ldots$ become **infinite**.

To remove this difficulty, we put $c_0 = k\, d_0$ in the above series, which then becomes $y(x) = d_0 x^k [1 + (k+1)x + (k+2)x^2 + (k+3)x^3 + \cdots]$.

Putting $\mathbf{k = 0}$, and replacing d_0 by a, we get

$$y(x) = a(x + 2x^2 + 3x^3 + \cdots) = a\, u_1 \text{, say.}$$

If we put $k = 1$ to obtain second solution, we obtain

$y(x) = c_0 (x + 2x^2 + 3x^3 + \cdots)$, which is not distinct from above solution.

In such a case, second independent solution is given by $\left(\frac{\partial y}{\partial k}\right)_{k=0}$.

Now $\frac{\partial y}{\partial k} = d_0 x^k \ln x \{k + (k+1)x + (k+2)x^2 + \cdots\} + d_0 x^k [1 + x + x^2 + \cdots]$

Putting $\mathbf{k = 0}$ and replacing d_0 by b in the above, we get

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b[\ln x (x + 2x^2 + 3x^3 + \cdots) + (1 + x + x^2 + \cdots)]$$

$$= b[u \ln x + (1 + x + x^2 + \cdots)] = bv \text{, say}$$

and required solution is $y(x) = au + bv$, where a and b are arbitrary constants.

E14) The given differential equation is

$$x(1-x)y'' + ((1-x)y' - y = 0 \tag{146}$$

Here $x = 0$, is a regular singular point of the above equation.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}$, $c_0 \neq 0$.

Substituting for y, y', y'' in the given equation, we get

$$k^2 c_0 x^{k-1} + \sum_{m=0}^{\infty} \left[(m+k+1)^2 c_{m+1} - \{(m+k)^2 + 1\} c_m \right] x^{m+k} = 0$$

Equating coefficients of least power of x, the **indicial equation** is

$$c_0 k^2 = 0 \Rightarrow k = 0, 0 \quad (\because c_0 \neq 0).$$

The recurrence relation is

$$c_{m+1} = \frac{(m+k)^2 + 1}{(m+k+1)^2} c_m \ , \ m \geq 0.$$

Hence $\quad c_1 = \frac{k^2+1}{(k+1)^2} c_0$

$$c_2 = \frac{(k^2+1)[(k+1)^2+1]}{(k+1)^2 (k+2)^2} c_0$$

$$c_3 = \frac{(k^2+1)[(k+1)^2+1]\,[(k+2)^2+1]}{(k+1)^2 (k+2)^2 (k+3)^2} c_0$$

and so on.

$$\therefore y(x) = c_0 x^k \left[1 + \frac{k^2+1}{(k+1)^2} x + \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2\,(k+2)^2} x^2 + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2\,(k+2)^2\,(k+3)^2} x^3 + \cdots \right] \quad (147)$$

Hence solution **for $k = 0$** is

$$[y(x)]_{k=0} = c_0 x \left[1 + x + \frac{1.2}{1^2.2^2} x^2 + \frac{1.2.5}{1^2.2^2.3^2} x^3 + \cdots \right] = y_1, \text{ say} \quad (148)$$

with y given by Eqn.(147), left hand side of Eqn.(146) becomes $k^2 c_0 x^{k-1}$

$$\therefore \frac{\partial}{\partial k}\left[x(1-x)y'' + (1-x)y' - y \right] = 2k\, c_0 x^{k-1} + c_0 k^2 x^{k-1} \ln x$$

or, $\left[x(1-x)\frac{d^2}{dx^2} + (1-x)\frac{d}{dx} - 1 \right] \frac{\partial y}{\partial k} = 2kc_0 x^{k-1} + c_0 k^2 x^{k-1} \ln x$

Therefore, for $k = 0, \frac{\partial y}{\partial k}$ is a second solution of Eqn.(146).

From Eqn.(147), we get

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x \left[1 + \frac{1}{1^2} x + \frac{1.2}{1^2.2^2} x^2 + \frac{1.2.5}{1^2.2^2.3^2} x^3 + \cdots \right] + c_0 \left[-2x - x^2 - \cdots \right]$$

$$= y_1 \ln x + c_0 \left[-2x - x^2 - \cdots \right] = y_2, \text{ say}$$

since y_1 and y_2 are linearly independent, hence the general solution is

$$y = \alpha_1 y_1 + \alpha_2 y_2,$$

where α_1, α_2 are arbitrary constants.

E15) a) Proceeding as in E14) when $p = 1$, $y = au + bv$

where, $u = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)$

and $v = u \ln x - \left[\frac{x}{1} + \frac{x^2}{2!}\left(1 + \frac{1}{2}\right) + \frac{x^3}{3!}\left(1 + \frac{1}{2} + \frac{2}{3}\right) + \cdots \right].$

b) When p is not a integer, $y = AU + BV$

where, $U = 1 + \frac{x}{p} + \frac{x^2}{p(p+1)} + \frac{x^3}{p(p+1)(p+2)} + \cdots$

$$V = x^{1-p}\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right].$$

E16) a) Proceeding as in E14), the solution is $y = au + bv$, where, $u = 1 - 2x - \frac{3}{2}x^2 - \frac{4}{3}x^3 - \cdots$

and $v = u \ln x + \left[2\left(2 - \frac{1}{2}\right)x - \frac{3}{2!}\left(2 + \frac{1}{2} - \frac{1}{3}\right)x^2 - \frac{4}{3!}\left(2 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)x^3 + \cdots\right]$

b) The given equation is

$$(x - x^2)y'' + (1 - 5x)y' - 4y = 0$$

Evidently $x = 0$ is a regular singular point of the equation.

Let $y(x) = \sum_{m=0}^{\infty} c_m x^{m+k}, c_0 \neq 0$. (149)

Substituting for y, y', y'' in the given equation, we get

$$\sum_{m=0}^{\infty} c_m (m+k)^2 x^{m+k-1} - \sum_{m=0}^{\infty} c_m (m+k+2)^2 x^{m+k} = 0,$$

Equating to zero, the coefficients of x^{k-1}, the indicial equation is

$$c_0 k^2 = 0 \text{ or } k = 0, 0 \quad (\because c_0 \neq 0).$$

The recurrence relation is $c_m = \frac{(m+k+1)^2}{(m+k)^2} c_{m-1}$, for $m \geq 1$.

Putting $m = 1, 2, 3, \ldots$, we get $c_1 = \frac{(k+2)^2}{(k+1)^2} c_0$, $c_2 = \frac{(k+3)^2}{(k+1)^2} c_0$

$$c_3 = \frac{(k+4)^2}{(k+1)^2} c_0, \; c_4 = \frac{(k+5)^2}{(k+1)^2} c_0, \ldots$$

Putting the values of coefficients $c_0, c_1, c_2, \ldots$ in solution (149), we get

$$y(x) = x^k c_0 \left[1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + \cdots\right] \quad (150)$$

Putting $k = 0$ in Eqn.(150), we get

$$y(x)\big|_{k=0} = c_0\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right] = c_0 u, \text{ (say)} \quad (151)$$

Differentiating Eqn.(150) partially, w.r.t. k and putting $k = 0$, the second solution is obtained as

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = c_0 \ln x\left[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots\right]$$

$$+ c_0\left[2^2 x(1-2) + 3^2 x^2\left(\frac{2}{3} - 2\right) + 4^2 x^3\left(\frac{2}{3} - 2\right) + \cdots\right]$$

$$= c_0 u \ln x - c_0\left[1.2x + 2.3x^2 + 3.4x^3 + \cdots\right] = c_0 v, \text{ (say)} \quad (152)$$

Hence the required general solution is

$$y = au + bv.$$

c) Proceeding as in E14), the general solution is $y = au + bv$,

where, $u = x^{1/2}\left\{1 + \frac{1.3}{4^2} x^2 + \frac{1.3.5.7}{4^2.8^2} x^4 + \cdots\right\}$

and $v = u \ln x + 2x^{1/2}\left\{\frac{1.3}{4^2}\left(1 + \frac{1}{3} - \frac{1}{2}\right)x^2 + \frac{1.3.5.7}{4^2.8^2}\right.$

$$\left.\left(1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right)x^4 + \cdots\right\}$$

E17) a) The given equation is

$$4x^3y'' + 6x^2y' + y = 0 \qquad (153)$$

Here $x = 0$, is an irregular singular point. We change variable x to t by $x = \frac{1}{t}$. Then

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \qquad (154)$$

Here $t = 0$, is a regular singular point.

Assuming $y = \sum_{m=0}^{\infty} c_m t^{m+k}$ and using Frobenius method, the solution of D.E. (154) is obtained as

$$y = a\left(1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots\right) + bt^{1/2}\left(1 - \frac{1}{3!}t + \frac{1}{5!}t^2 - \frac{1}{7!}t^3 + \cdots\right)$$

Putting $t = \frac{1}{x}$ in the above solution, we obtain

$$y(x) = a\left(1 - \frac{x^{-1}}{2!} + \frac{x^{-2}}{4!} - \cdots\right) + bx^{-1/2}\left(1 - \frac{x^{-1}}{1!} + \frac{x^{-2}}{5!} - \frac{x^{-3}}{71} + \cdots\right),$$

which is the required series solution of the given equation.

b) The given equation is

$$2x^2(1-x)y'' - 5x(1+x)y' + (5-x)y = 0 \qquad (155)$$

Since we have to find solution for large values of x, putting $x = \frac{1}{t}$, Eqn.(155) becomes

$$2t^2(t-1)\frac{d^2y}{dt^2} + t(9t+1)\frac{dy}{dt} + (5t-1)y = 0. \qquad (156)$$

Clearly $t = 0$ is a regular singular point. Using Frobenius method, the solution of Eqn.(156) will be

$$y = a\left[\sum_{n=0}^{\infty} (n+1)(2n+3)(2n+5)t^{n+1}\right] + b\left[\sum_{n=0}^{\infty} (n+1)(n+2)(2n+1)t^{n+1/2}\right]$$

Hence solution of given equation (155) is

$$y = a\sum_{n=0}^{\infty} (n+1)(2n+3)(2n+5)x^{-n-1} + b\sum_{n=0}^{\infty} (n+1)(n+2)(2n+1)x^{-n-1/2}.$$

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