UNIT 1 FIRST AND HIGHER ORDER EQUATIONS

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1.1 INTRODUCTION

From your knowledge of differential equations which you must have studied at the undergraduate level, you are familiar with various methods of solving first/second or even higher order linear, ordinary differential equations (ODEs) with constant coefficients. Also, for linear ODEs upto second order with variable coefficients you know certain transformations of the dependent and independent variables, which reduce the equation to a form, which is solvable by the known methods. However, in all the cases, solutions to ODEs were obtained in terms of known elementary functions. To this day, no method of getting a solution of general first order initial value problem (IVP) of the form $\frac{dy}{dx} = f(x, y)$, in closed form or, in terms of elementary functions exists. In this unit we shall give a method of finding an approximate solution to this general first order IVP.

In Sec.1.2 we shall start with defining the initial and boundary value problems and illustrate them through various examples giving geometrical interpretation of their solutions. In Sec.1.3, we have discussed the Picard's method of successive approximation for solving a first order initial value problem (IVP), $y' = f(x, y)$ subject to $y(x_0) = y_0$. We have also stated and proved Picard's theorem on existence and uniqueness of solution of IVP here. We have shown, through examples, how the nature of solutions of first order IVPs can be examined even without solving them. In Sec 1.4, we shall do a quick recap of methods of solving linear differential equations of any order with constant coefficients and illustrate them through a number of examples.

Objectives

After studying this unit you should be able to

- identify initial value problems and boundary value problems;
- obtain an approximate solution of an IVP of the first order;
- describe the nature of solution of first order differential equation with reference to the existence and uniqueness of solutions;
- solve linear differential equations of first/second/higher orders with constant coefficients.

1.2 INITIAL AND BOUNDARY VALUE PROBLEMS

Consider the first order linear differential equation

$$y' = 2x\ (1 + y^2)$$

By the usual process of integration, it can be seen that the solution of the given differential equations is

$$y(x) = \tan(x^2 + c),$$

where c is an arbitrary constant. Since c is arbitrary, for each value of c, we get a distinct solution of the given equation. Hence, a one-parameter family consisting of an infinite number of solutions is obtained. Geometrically, this equation represents a one-parameter family of curves, called integral curves of the given equation. Each integral curve is the geometric representation of the corresponding solution of the differential equation. In physical applications, we often require a specific solution out of this family representing a particular physical phenomenon. Specifying a particular solution is equivalent to picking out a particular integral curve from the one-parameter family. It is usually convenient to do this by prescribing a point (x_0, y_0) through which the integral curve must pass; that is, we seek a solution such that

$$y(x_0) = y_0.$$

Such a condition is called an **initial condition.** A first order differential equation together with an initial condition form an **initial value problem**. This terminology is suggested by the fact that the independent variable often denotes time, the initial condition defines the situation at some fixed instant, and the solution of the initial value problem describes what happens later. In the case of the equation considered above we may be interested in a solution which satisfies the requirement

$$y(0) = 0.$$

This condition means that we obtain that solution $y(x)$ which passes through the point $(0, 0)$. Thus,

$$y(0) = \tan(0 + c) = \tan c.$$

and $y(0) = 0$ gives $c = 0$.

The desired solution is given by

$$y(x) = \tan x^2, \ (0 < x < \sqrt{\pi/2}).$$

Next consider the differential equation of order two, namely,

$$y'' + 5y' + 6y = 0. \qquad (y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}).$$

It can be verified that for the above equation

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$$

is a solution, existing on $-\infty < x < \infty$. Here c_1 and c_2 are arbitrary constants. For each choice of c_1 and c_2, we get a distinct solution. Hence, $c_1 e^{-2x} + c_2 e^{-3x}$ is a two parameter family of solutions. To find a specific solution of this family, we require two conditions, the value of the solution $y(x_0)$ and its derivative $y'(x_0)$ at a point x_0. Thus to determine uniquely an integral curve of a second order equation it is necessary to specify not only a point through which it passes, but also the slope of the curve at that point. Suppose for the above second order equation the conditions are

$$y(0) = 0 \text{ and } y'(0) = 1$$

Then, it follows that

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = -2c_1 - 3c_2 = 1$$

These are simultaneous equations in c_1 and c_2 and solving them we get $c_1 = 1, \ c_2 = -1$. The specific solution is, therefore, given by

$$y(x) = e^{-2x} - e^{-3x}, -\infty < x < \infty.$$

We thus observe that to determine a particular solution of a first order equation, we need one condition while in the case of a second order equation, we need two conditions at the same value of x. Similarly, we conclude that for the n^{th} order equation of the form

$$P_0(x)\, y^{(n)}(x) + P_1(x)\, y^{(n-1)}(x) + \cdots + P_{n-1}(x)\, y' + P_n(x)\, y = G(x)$$

where the functions $P_0, P_1, \ldots P_n$ and G are functions of x only on some interval $x_1 < x < x_2$, we need n conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, \ldots, y^{(n-1)}(a) = \alpha_{n-1},$$

where, $x_1 < a < x_2$ and $\alpha_0, \alpha_1 \ldots, \alpha_{n-1}$ is any set of prescribed real constants. These conditions are referred to as **initial conditions**.

A differential equation together with initial conditions is called an Initial Value Problem (IVP).

You may **observe** that an IVP is determined by the differential equation and the initial conditions. A change in either of them would mean a different IVP.

Next, consider a second order differential equation

$$y'' + y = 0$$

It can be verified that solutions of this equations is given by

$$y(x) = c_1 \cos x + c_2 \sin x, \qquad -\infty < x < \infty.$$

where c_1 and c_2 are arbitrary constants. Let us assume that the following pairs of conditions are required to be satisfied by the solutions

(i) $y(0) = 0, \quad y(\pi/2) = 0$

(ii) $y(0) = 0, \quad y'(\pi/2) = 0$

(iii) $y(0) = 0, \quad y'(\pi/2) = 1$

Since the general solution involves two constants c_1 and c_2, two conditions are required to determine them. It may be noted that in this case the two conditions in each of the pair above are given at two different points, namely, $x = 0$ and $x = \frac{\pi}{2}$. The conditions given in (i), (ii) and (iii) are referred to as boundary conditions stated at $x = 0$ and $x = \frac{\pi}{2}$.

A differential equation together with boundary conditions is called Boundary-Value Problem (BVP).

If you apply these pair of conditions to the general solution $y(x) = c_1 \cos x + c_2 \sin x$, then you would see that corresponding solutions for the three cases are given respectively, by

(i) $y(x) = 0$ (ii) $y(x) = c_2 \sin x$ (iii) No solution, existing on $-\infty < x < \infty$.

You may **note** that in (i) above there is only one solution viz., trivial solution $y = 0$. In (ii), since c_2 in an arbitrary constant, we get an infinite number of solutions. In (iii), $y(0) = 0$ gives $c_1 = 0$ and $y'(\pi/2) = 1$ gives $1 = c_2 0 = 0$, which is a contradiction and hence no solution. A boundary-value problem may thus have infinite solution, single solution or no solution at all.

Similarly, for an initial value problem (IVP) the following questions may come to your mind

i) Does an initial value problem always have a solution?
ii) Can it have more than one solution?
iii) Is the solution valid for all x, or only for some restricted interval about the initial point?

Such questions are answered by the existence and uniqueness theorem for the solution of an IVP which we shall be discussing in the next section, but before that you may try this exercise:

E1) Discuss the solution of the boundary value problem

$$y'' + \lambda y = 0$$

with boundary conditions $y(0) = 0$ and $y(\pi) = 0$.

Let us now discuss the existence and uniqueness of solutions of IVPs.

1.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR INITIAL VALUE PROBLEMS

We know that only a few simple type of differential equations can be solved explicitly in terms of known elementary functions. Some of these types are discussed in Blocks 1 and 2 of MTE-08. However, many differential equations fall outside this category and nothing has been done so far to suggest a procedure that might work in such cases. Let us examine the initial value problem (IVP).

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_{0,} \tag{1}$$

where $f(x, y)$ is an arbitrary function defined and continuous in some neighborhood of the point (x_0, y_0). Geometrically, we want to obtain a function $y = y(x)$ satisfying the differential equation $y' = f(x, y)$ in some neighborhood of x_0 and whose graph passes through the point (x_0, y_0) (see. Fig.1).

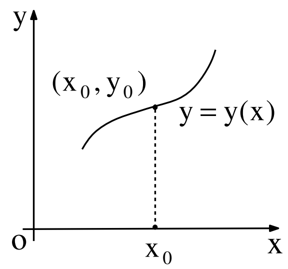

Fig.1

We know that elementary procedures will not work here. We will have to evolve a process to construct a sequence of functions that converges to a limit function satisfying the IVP. This process is provided by **Picard's method of successive approximations**.

Picard's methods for solving differential equations is quite different from any method that you might have studied so far. This method consists in replacing the initial value problem (1) by the equivalent integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\, dt \tag{2}$$

Eqn.(2) is called an i**ntegral equation** because the unknown function occurs under the integral sign.

It can be easily seen that Eqns.(1) and (2) are equivalent. For if $y(x)$ is a solution of Eqn.(1), then $y(x)$ is continuous and so is the right hand side of

$$y'(x) = f[x, y(x)] \tag{3}$$

Now we integrate Eqn.(3) w.r.t. x, from x_0 to x and obtain

$$y(x) = C + \int_{x_0}^{x} f[t, y(t)]\, dt.$$

Using the initial condition $y(x_0) = y_0$, we obtain

$$y(x_0) = C + \int_{x_0}^{x_0} f[t, y(t)]\,dt = y_0$$

implies $C = y_0$. Thus,

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\,dt,$$

which is our Eqn.(2). **Note** that t is a dummy variable here.
Thus any solution of Eqn.(1) is a continuous solution of Eqn.(2).

Conversely, if $y(x)$ is a continuous solution of Eqn.(2), then $y(x_0) = y_0$ because the integral vanishes when $x = x_0$. Also by differentiation of Eqn.(2) w.r.t. x we recover the differential equation

$$y'(x) = f[x, y(x)].$$

We can thus say that the integral Eqn.(2) and the IVP(1) are equivalent.

We now try to solve Eqn.(2) by a method of successive approximation i.e., we begin with a guess or approximation to the solution of Eqn.(2) and improve it step-by-step by applying a repeatable operation, which we hope will bring us as close as we please to an exact solution. The primary advantage that Eqn.(2) has over Eqn.(1) is that the integral equation provides a convenient mechanism for carrying out this process as we see below.

The first or a rough approximation to a solution of Eqn.(2) is given by the constant function $y(x_0) = y_0$, which is simply a horizontal straight line through the point (x_0, y_0). We put this approximation in the right hand side of Eqn.(2) to obtain a new approximation $y_1(x)$ as follows:

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t,\ y_0(t)]\ dt.$$

We hope that this new function is a better approximation to the solution. We then use $y_1(x)$ to generate another and perhaps even better approximation $y_2(x)$ in the same way and obtain

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t,\ y_1(t)]\ dt.$$

In this manner we obtain the iterants i.e., a sequence of functions $y_1(x), y_2(x), y_3(x), \ldots$ whose n^{th} term is defined by the relation

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]\,dt, \qquad n = 1, 2, 3\ldots \tag{4}$$

The repetitive use of formula (4) is known as **Picard's method of successive approximations/iterations.**

We shall now show how this method works by means of an example.

Example 1: Apply Picard's iteration method to the initial value problem

$$y' = x^2 y, \quad y(0) = 1$$

and compare with the exact solution, if possible.

Solution : For an initial value problem

$$y' = x^2 y, \quad y(0) = 1,$$

the corresponding integral equation is

$$y(x) = 1 + \int_{x_0}^{x} t^2\ y(t)\ dt$$

Picard's iterations are

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x t^2\ 1dt = 1 + \frac{x^3}{3}$$

$$y_2(x) = 1 + \int_0^x t^2\left(1 + \frac{t^3}{3}\right)dt = 1 + \frac{x^3}{3} + \frac{x^6}{18} = 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2$$

$$y_3(x) = 1 + \int_0^x t^2\left(1 + \frac{t^3}{3} + \frac{t^6}{18}\right)dt = 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{1}{18}.\frac{x^9}{9}$$

$$= 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(\frac{x^3}{3}\right)^3$$

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$$y_n(x) = 1 + \frac{x^3}{3} + \frac{1}{2!}\left(\frac{x^3}{3}\right)^2 + \frac{1}{3!}\left(\frac{x^3}{3}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{x^3}{3}\right)^n$$

The exact solution of the given problem can easily be obtained as

$$y = e^{\left(\frac{x^3}{3}\right)},$$

to which the above approximate solution converges when $n \to \infty$.

* * *

We would like to **remark** here that you should not be deceived by the relative ease with which the iterants $y_n(x)$ were obtained in Example-1. In general, it is seen that the integration involved in generating each $y_n(x)$ can become complicated very quickly. Even if it is possible to generate $y_n(x)$, it may not always converge to an explicit function or the exact solution. Let us consider the following example.

Example 2: Apply Picard's method of successive approximations to the initial value problem.

$$y' = x\ (y - x^2 + 2), \qquad y(0) = 1.$$

Solution : The integral equation, equivalent to the given initial value problem is

$$y(x) = 1 + \int_0^x t(y(t) - t^2 + 2)\,dt.$$

The approximate solutions are

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x t(3 - t^2)\,dt = 1 + \frac{3}{2}x^2 - \frac{1}{4}x^4$$

$$y_2(x) = 1 + \int_0^x t(3 + \frac{t^2}{2} - \frac{t^4}{4})\,dt = 1 + \frac{3x^2}{2} + \frac{x^4}{8} - \frac{x^6}{24}$$

$$y_3(x) = 1 + \int_0^x t(3 + \frac{t^2}{2} + \frac{t^4}{8} - \frac{t^6}{24})\,dt$$

$$= 1 + \frac{3x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{192}$$

and so on.

The exact solution of the given problem can be easily obtained as

$$y = x^2 + e^{\frac{x^2}{2}}$$

$$= x^2 + 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \frac{x^8}{192} + \cdots$$

The approximate solution in this case differs from the exact solution.

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After going through Examples 1 and 2, you might ask the question:
Is Picard's method a practical means of solving a first order equation $y' = f(x, y)$ subject to $y(x_0) = y_0$? In most cases the answer is no. The question then arises, what is it good for? Picard's methods of iteration is a theoretical tool used in the consideration of existences and uniqueness of solutions of differential equations. Under certain conditions on $f(x, y)$ it can be shown that as $n \to \infty$, the sequence $\{y_n(x)\}$ defined by Eqn.(4) converges to a function $y(x)$ that satisfies the integral Eqn.(2) and hence the IVP (1) and the solution obtained is unique. The theorem that makes precise assertions of this kind is called Existence and Uniqueness Theorem which we shall discuss now but before that you may try the following exercise.

E2) Apply Picard's iteration method to the following initial value problems and compare the results obtained with the exact solutions wherever possible. Perform at least three iterations.

(a) $y' = x + y,\ y(0) = 1$ (b) $y' = x - y^2,\ y(0) = \frac{1}{2}$

(c) $y' = 2x(1+y),\ y(0) = 0$ (d) $y' = y,\ y(0) = 1$

(e) $y' = y^2,\ y(0) = 1.$

We shall now state and prove **Picard's theorem on existence and uniqueness of solution of IVP.**

Theorem 1: Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes (see. Fig.2). If (x_0, y_0) is an interior point of R, then there exists a number $h > 0$ with the property that the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0 \tag{5}$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h.$

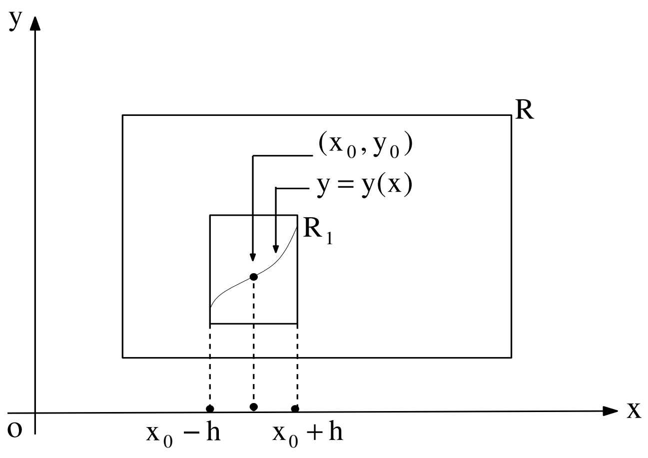

Fig.2

Proof: We know that every solution of IVP (1) is also a continuous solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t,\ y(t)]\,dt \tag{6}$$

and conversely. We can thus say that Eqn.(5) has a unique solution on an interval $|x - x_0| \le h$**,** if and only if, Eqn.(6) has a unique continuous solution on the same interval.

By Picard's method of successive approximations, the sequence of functions $y_n(x)$ defined by

$$y_0(x) = y_0$$

$$y_1(x) = y_0 + \int_{x_0}^{x} f[t,\ y_0(t)]\,dt$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t,\ y_1(t)]\,dt$$

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$$y_n(x) = y_0 + \int_{x_0}^{x} f[t,\ y_{n-1}(t)]\,dt. \tag{7}$$

may converge to a solution of Eqn.(6). Here you may observe that $y_n(x)$ is the n[th] partial sum of the series of functions

$$y_0(x)+\sum_{n=1}^{\infty}[y_n(x)-y_{n-1}(x)] = y_0(x)+[y_1(x)-y_0(x)]+[y_2(x)-y_1(x)] + \cdots + [y_n(x)-y_{n-1}(x)]+\cdots \tag{8}$$

and thus the convergence of the sequence (7) is equivalent to the convergence of series (8). In order to prove Theorem 1, we are now required to produce a number $h>0$ defined on the interval $|x-x_0|\le h$, and show that on this interval, the following statements are true

(i) the series (8) converges to a function y(x)
(ii) y(x) is a continuous solution of Eqn.(6)
(iii) y(x) is the only continuous solution of Eqn.(6).

We shall now prove the above three statements one by one but first of all let us try to produce a positive number h.

From the hypothesis of Theorem 1 we know that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions on the rectangle R. Now since R is closed (as it includes its boundary) and bounded, each of these functions is necessarily bounded on R. Thus there exist constants M and N such that

$$|f(x,y)| \le M \tag{9}$$

and

$$\left|\frac{\partial}{\partial y} f(x,y)\right| \le N \tag{10}$$

for all points (x,y) is R. Now if (x,y_1) and (x,y_2) are two distinct points in R with the same x-coordinate, then the mean value theorem guarantees that

$$|f(x,y_1)-f(x,y_2)| = \left|\frac{\partial}{\partial y} f(x,y^*)\right| |y_1-y_2| \tag{11}$$

for some number y^* between y_1 and y_2. It is clear from Eqns.(10) and (11) that

$$|f(x,y_1)-f(x,y_2)| \le N|y_1-y_2| \tag{12}$$

for any points (x,y_1) and (x,y_2) in R lying on the same vertical line. Let us choose h to be any positive number such that

$$Nh<1 \tag{13}$$

and the rectangle R_1 defined by the inequalities $|x-x_0|\le h$ and $|y-y_0|\le Mh$ is contained in R. As (x_0,y_0) is an interior point of R, we can say that such an h exists.

Now after choosing h we shall now prove statements (i) – (iii) above. Here onwards, we shall confine our attention to the interval $|x-x_0|\le h$.

In order to prove (i), it is sufficient to show that the series

$$|y_0(x)|+|y_1(x)-y_0(x)|+|y_2(x)-y_1(x)|+\cdots+|y_n(x)-y_{n-1}(x)|+\cdots \tag{14}$$

converges and to achieve this, we estimate the terms $|y_n(x)-y_{n-1}(x)|$. First we show that the graph of each of the functions $y_n(x)$ lies in R_1 and hence in R. Obviously, $y_0(x)=y_0$, and so the point $[t,y_0(t)]$ lies in R_1. Eqn.(9) gives $|f(t,y_0(t))|\le M$, and

$$|y_1(x)-y_0| = \left|\int_{x_0}^{x} f[t,y_0(t)]\,dt\right| \le Mh,$$

which proves that the point $y_1(x)$ lies in R_1. It follows consequently from this inequality, that the point $[t, y_1(t)]$ is in R_1, so that $\left| f[t, y_1(t)] \right| \le M$ and

$$|y_2(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_1(t)]\, dt \right| \le Mh.$$

Similarly, we can say that

$$|y_3(x) - y_0| = \left| \int_{x_0}^{x} f[t, y_2(t)]\, dt \right| \le Mh.$$

and so on. Now we have estimated the terms $|y_n(x) - y_{n-1}(x)|$.

We know that a continuous function has a maximum on a closed interval. Since $y_1(x)$ is continuous, we can define a constant 'α' by $\alpha = \max |y_1(x) - y_0|$ and write

$$|y_1(x) - y_0| \le \alpha.$$

Next, the point $[t, y_1(t)]$ and $[t, y_0(t)]$ lie in R_1 and so relation (12) yields

$$\left| f[t, y_1(t)] - f[t, y_0(t)] \right| \le N \left| y_1(t) - y_0(t) \right| \le N\alpha,$$

and we have

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} \left\{ f[t, y_1(t)] - f[t, y_0(t)] \right\} dt \right| \le N\alpha h = \alpha(Nh)$$

Similarly, we obtain

$$|f[t, y_2(t)] - f[t, y_1(t)]| \le N\, |y_2(t) - y_1(t)| \le N^2 \alpha h$$

and

$$|y_3(x) - y_2(x)| = \left| \int_{x_0}^{x} \left\{ f[t, y_2(t)] - f[t, y_1(t)] \right\} dt \right|$$

$$\le (N^2 \alpha h)\, h = \alpha (Nh)^2$$

Continuing this way, we find that

$$\left| y_n(x) - y_{n-1}(x) \right| \le \alpha (Nh)^{n-1},$$

for every $n = 1, 2, \ldots$. You can thus observe that each term of the series (14) is less than or equal to the corresponding term of the series of constants

$$\left| y_0 \right| + \alpha + \alpha\, (Nh) + \alpha\, (Nh)^2 + \cdots + \alpha\, (Nh)^{n-1} + \cdots$$

But relation (13) guarantees that this series converges. So series (14) converges by the comparison test and hence Eqn.(8) converges to a sum, which we denote by $y(x)$, and thus $y_n(x) \to y(x)$. Also, since the graph of each $y_n(x)$ lies in R_1, the graph of $y(x)$ also has this property. This completes the proof of statement (i).

Next, we prove statement (ii). From the proof of (i) it is clear that not only $y_n(x)$ converges to $y(x)$ in the interval, but also that this convergence is uniform. This means that by choosing n to be sufficiently large, we can make $y_n(x)$ as close as we please to $y(x)$ for all x in the interval. This means that if we are given $\in > 0$, then there exists a positive integer n_0 such that, for $n \ge n_0$, we have $\left| y(x) - y_n(x) \right| < \in$, for all x in the interval. Since each $y_n(x)$ is continuous, this uniformity of the convergence implies that the limit function $y(x)$ is also continuous. To prove that $y(x)$ is actually a solution of Eqn.(6), we must show that

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]\, dt = 0, \qquad (15)$$

From relations (7) we know that

$$y_n(x) - y_0 - \int_{x_0}^{x} f[t, y_{n-1}(t)]dt = 0, \quad (16)$$

Subtracting the left side of Eqn.(16) from the left side of Eqn.(15) we get

$$y(x) - y_n(x) + \int_{x_0}^{x} \{ f[t, y_{n-1}(t)] - f[t, y(t)]\}dt = 0,$$

We can thus write

$$y(x) - y_0 - \int_{x_0}^{x} f[t, y(t)]dt = y(x) - y_n(x) + \int_{x_0}^{x} \{f[t, y_{n-1}(t)] - f[t, y(t)]\} dt.$$

Taking modulus of both sides, we obtain

$$\left| y(x) - y_0 - \int_{x_0}^{x} f(t, (t))\,dt \right| \le \left| y(x) - y_n(x) \right| + \left| \int_{x_0}^{x} \{f(t, y_{n-1}(t)) - f(t, y(t))\}dt \right|$$

Since the graph of $y(x)$ lies in R_1 and hence in R, using relations (12) we can write

$$| y(x) - y_0 - \int_{x_0}^{x} f(t, y(t))dt | \le | y(x) - y_n(x) | + Nh.\max | y_{n-1}(x) - y(x) | \quad (17)$$

The uniform convergence of $y_n(x)$ to $y(x)$ implies that the right side of inequality (17) can be made as small as we please by taking n large enough. The left side of inequality (17) must, therefore, be equal to zero and this completes the proof of statement (ii).

To prove statement (iii), let us assume that $\bar{y}(x)$ is also a continuous solution of Eqn.(6) on the interval $| x - x_0 | \le h$. We shall then be through, if, we show that $\bar{y}(x) = y(x)$ for every x in the interval. As a first step we shall try to show that $\bar{y}(x)$ lies in R_1 and hence in R. If possible, let us suppose that the graph of $\bar{y}(x)$ leaves R_1 see Fig. 3.

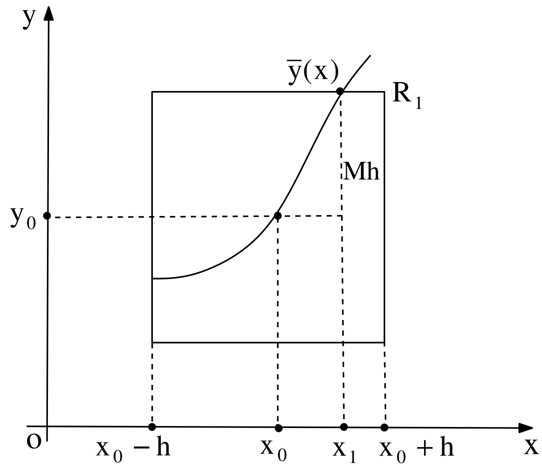

Fig.3

Now since $\bar{y}(x)$, is continuous and the fact that $\bar{y}(x_0) = y_0$ (being a solution) implies that there exists an x_1 such that $| x_1 - x_0 | < h$, $| \bar{y}(x_1) - y_0 | = Mh$ and $| \bar{y}(x) - y_0 | < Mh$ if $| x - x_0 | < | x_1 - x_0 |$. For such an x_1, it follows that

$$\frac{| \bar{y}(x_1) - y_0 |}{| x_1 - x_0 |} = \frac{Mh}{| x_1 - x_0 |} > \frac{Mh}{h} = M.$$

Thus by the mean value theorem, there exists a number x^* between x_0 and x_1, such that

$$\frac{| \bar{y}(x_1) - y_0 |}{| x_1 - x_0 |} = | \bar{y}'(x^*) |$$

But, $|\bar{y}'(x^*)| = \left| f[x^*, \bar{y}(x^*)] \right| \leq M$, since $[x^*, \bar{y}(x^*)]$ lies in R_1.

$$\therefore \quad \frac{|\bar{y}(x_1) - y_0|}{|x_1 - x_0|} \leq M$$

which is a contradiction. This shows that no such point x_1 exists and the graph of $\bar{y}(x)$ lies in R_1. Finally, to show that the two solutions $y(x)$ and $\bar{y}(x)$ of Eqn.(6) are equal we write

$$|\bar{y}(x) - y(x)| = \left| \int_{x_0}^{x} \{ f[t, \bar{y}(t)] - f[t, y(t)] \} dt \right|$$

Since the graphs of $\bar{y}(x)$ and $y(x)$ both lie in R_1, relation (12) yields

$$|\bar{y}(x) - y(x)| \leq Nh \quad \max \left| \bar{y}(x) - y(x) \right|.$$

$$\therefore \quad \max \ |\bar{y}(x) - y(x)| \leq Nh \ \max \ |\bar{y}(x) - y(x)|$$

This implies that max. $|\bar{y}(x) - y(x)| = 0$, for otherwise we would have $1 \leq Nh$ in contradiction to relation (13). This proves that $\bar{y}(x) = y(x)$ for every x in the interval $|x - x_0| \leq h$, and Picard's Theorem is completely proved.

We will now make the following remarks.

Remark 1: Picard's theorem can be strengthened in various ways by weakening its hypotheses. For example, our assumption that $\frac{\partial f}{\partial y}$ is continuous on R is stronger than the proof requires, and is used only to obtain the inequality (12). We can, therefore, introduce this inequality (12) into the theorem as an assumption which replaces the assumption of continuity of $\frac{\partial f}{\partial y}$ on R. This would lead us to a stronger form of the theorem as in practice, there are number of functions that lack a continuous partial derivative but satisfy relation (12) for some constant N. This inequality viz; $|f(x, y_1) - f(x, y_2)| \leq N |y_1 - y_2|$, which says that the difference quotient

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}$$

is bounded on R is called a **Lipschitz condition in the variable y.** The constant N is called the **Lipschitz constant.**

Remark 2: If we drop the Lipschitz condition, and assume only that $f(x, y)$ is continuous on R, then it is still possible to prove that the IVP (5) has a solution. This result is known as **Peano's theorem** named after Italian logician and mathematician Guiseppe Peano (1858 – 1932). But with only this assumption the solution whose existence this theorem guarantees may not necessarily be unique. Consider for example, the problem $y' = 3y^{\frac{2}{3}}$, $y(0) = 0$ and let R be the rectangle $|x| \leq 1, |y| \leq 1$. Here $f(x, y) = 3y^{\frac{2}{3}}$ is continuous on R. Also $y_1(x) = x^3$ and $y_2(x) = 0$ are two different solution of this problem valid for all x. The given problem therefore has a solution but it is not unique. The reason being that $f(x, y)$ does not satisfy the Lipschitz condition on the rectangle R, because the difference quotient

$$\frac{f(0, y) - f(0, 0)}{y - 0} = \frac{3y^{\frac{2}{3}}}{y} = \frac{3}{y^{\frac{1}{3}}}$$

is unbounded in every neighborhood of the origin.

Remark 3: Theorem 1, is called a **local** existence and uniqueness theorem because it guarantees the existence of a unique solution only on some interval $\left| x - x_0 \right| \leq h$, where h may be very small. However, there are several important cases in which this restriction can be removed. For example, consider the first order linear equation

$$y' + P(x)\,y = Q(x),$$

where $P(x)$ and $Q(x)$ are defined and continuous on an interval $a \le x \le b$.
If we compare this equation with the IVP(5), then we have

$$f(x, y) = -P(x)\,y + Q(x),$$

and if $N = \max|P(x)|$ for $a \le x \le b$, it is clear that

$$|f(x, y_1) - f(x, y_2)| = |-P(x)(y_1 - y_2)| \le N|y_1 - y_2|$$

The function $f(x, y)$ is therefore continuous and satisfies a Lipshitz condition with **Lipshitz constant** $N = \max|P(x)|$, on the infinite vertical strip defined by $a \le x \le b$ and $-\infty < y < \infty$. Under these circumstances, the IVP

$$y' + P(x)\,y = Q(x), \quad (x_0) = y_0$$

has a unique solution on the entire interval $a \le x \le b$.

We shall now take up a few examples to illustrate the existence and uniqueness theorem.

Example 3: Show that $f(x, y) = x^2|y|$ satisfies a Lipschitz condition on the rectangle $|x| \le 1$ and $|y| \le 1$, but $\dfrac{\partial f}{\partial y}$ fails to exist at many points of this rectangle.

Solution: Here rectangle R is defined by $R : |x| \le 1$ and $|y| \le 1$.
Now,

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |x^2\,|y_1| - x^2\,|y_2|| \\ &= |x^2|\;||y_1| - |y_2|| \\ &\le |x^2|\,|y_1 - y_2| \\ &\le |y_1 - y_2| \text{ in } R, \end{aligned}$$

Hence $f(x, y)$ satisfies a Lipschitz condition, with Lipschitz constant as 1.
Let $(a, 0)$ be any point in R for $a \ne 0$.

Here, $\lim\limits_{h \to 0} \dfrac{f(a, 0 + h) - f(a, 0)}{h} = \lim\limits_{h \to 0} \dfrac{a^2|h| - a^2 0}{h} = \lim\limits_{h \to 0} \dfrac{a^2|h|}{h}$, which does not exist.

Hence, $\dfrac{\partial f}{\partial y}$ does not exist for many points $(a, 0)$ in R for $a \ne 0$.

***

Example 4: Examine the IVP $y' = e^{-x^2} y^2 \sin x$, for Lipschitz condition on the region R where, R is the strip $0 \le y \le 2$ in the xy-plane.

Solution: The function $f(x, y) = e^{-x^2} y^2 \sin x$ is continuous for all x and y.
Let (x, y_1) and (x, y_2) be the two points of R, then

$$\begin{aligned} |f(x, y_2) - f(x, y_1)| &= \left|e^{-x^2} \sin x\right| |y_2 + y_1| |y_2 - y_1| \\ &\le 4|y_2 - y_1| \end{aligned}$$

because $\left|e^{-x^2} \sin x\right| \le 1$ for all x and $|y_2 + y_1| \le 4$, if y_1 and y_2 are both in the interval $[0, 2]$. Thus $f(x, y)$ satisfies the Lipschitz condition with $N = 4$ in the strip R.

***

Example 5: Discuss the nature of solutions of the initial value problem $y' = \sqrt{|y|}$, $y(0) = 0$, on the rectangle $|x| \le 1$ and $|y| \le 1$.

Solution: The function $f(x,y)=\sqrt{|y|}$ is continuous for all values of y. Taking $y_1=0$ and $y_2>0$, as two points, we get

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|}=\frac{\sqrt{|y_2|}}{|y_2|}=\frac{1}{\sqrt{y_2}}$$

Now $\frac{1}{\sqrt{y_2}}$ can be made as large as we please by choosing the sufficiently small values of y_2. Therefore, there exists no N, independent if x, y such that

$$|f(x,y_2)-f(x,y_1)|<N\,|y_2-y_1|.$$

$\Rightarrow$ Lipschitz condition is not satisfied in any region containing $y=0$.

It may be observed that given IVP has solutions $y(x)=0$ and $y(x)=\frac{x^2}{4}$. Thus continuity of $f(x,y)$ is not sufficient for the unique solution of the IVP.

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Example 6: Discuss the nature of solution of the IVP $y'=y^2,\ y(0)=b>0$ on the infinite strip R consisting of the points (x,y) for which $0\le x\le 1$ and y is arbitrary.

Solution: Function $f(x,y)=y^2$ is continuous everywhere. Taking (x,y_1) and (x,y_2) as two points of the region R, we get

$$\begin{aligned}\left|f(x,y_2)-f(x,y_1)\right|&=\left|y_2^2-y_1^2\right|\\&=\left|y_2+y_1\right|\left|y_2-y_1\right|\end{aligned}$$

Because $\left|y_2+y_1\right|$ can be made arbitrarily large, it follows that $f(x,y)$ does not satisfy Lipschitz condition on the infinite strip R. However, when we solve the given IVP by separation of variable, we get

$$y(x)=\frac{b}{1-bx}$$

Now because the denominator vanishes for $x=\frac{1}{b}$, the above solution of IVP is only valid for $x<\frac{1}{b}$. In particular, if b is large, then we have a solution only on a very small interval to the right of $x=0$.

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You may now try some exercises.

E3) For IVP

$$y'=\begin{cases}\frac{2y}{x}, & x>0 \\ 0, & x=0.\end{cases}\qquad y(0)=0$$

show that the Lipschitz condition is not satisfied in any closed rectangle containing (0, 0).

E4) Examine the existence and uniqueness of solution of IVP
$y'=f(x,y),\ y(0)=1,$
where, (a) $f(x,y)=x$
(b) $f(x,y)=-\left|y\right|$.

E5) Show that $f(x,y)=xy^2$
(a) satisfies a Lipschitz condition on any rectangle $a\le x\le b$ and $c\le y\le d$.
(b) does not satisfy a Lipschitz condition on any strip $a\le x\le b$ and $-\infty<y<\infty$.

E6) Show that $y' = \frac{(y-1)}{x}$, $y(0) = 1$ has an infinity of solutions.

E7) Examine IVP $y' = \begin{cases} \frac{4x^3 y}{x^4 + y^2} & \text{when x \& y are not both zero} \\ 0, & \text{when } x = y = 0 \end{cases}$, $y(0) = 0$

for uniqueness of solutions.

In the next section we shall quickly recall various properties of solutions of linear differential equations. We shall also give here a quick recap of various methods of finding solutions of these equations when the coefficients are constants. For details, you can refer Block-2 of MTE-08. The case when the coefficients are variable will be discussed in Unit 2.

1.4 LINEAR DIFFERENTIAL EQUATIONS

The most general **linear, non-homogeneous differential equation** of order n is of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad (18)$$

where $a_0(x) \neq 0$. Further, if the functions $a_0(x)$, $a_1(x), \ldots, a_n(x), f(x)$ are continuous on an interval $[a, b]$, where the interval $[a, b]$ can be of finite or infinite length or, it can be open or closed at either end, then a solution $y(x)$ of Eqn.(18) over $[a, b]$ together with its derivatives $y'(x), y''(x), \ldots, y^{(n-1)}(x)$ will be continuous on this interval. From Eqn.(18) it follows that $y^{(n)}(x)$ will also be continuous on the $[a, b]$ (ref. Unit 5, MTE-08).

We shall now state a theorem (without proof) which gives the conditions under which we can expect a solution of Eqn.(18).

Theorem 2: Let $x = x_0$ be a point of the interval $[a, b]$ and let $b_0, k_1, \ldots, k_{n-1}$ be arbitrary set of n real constants. If the functions $a_0(x), a_1(x), \ldots, a_n(x), f(x)$ are continuous on $[a, b]$ and $a_0(x)$ does not vanish at any point of the interval then there exists one and only one solution $y(x)$ of Eqn.(18) with the property

$$y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}.$$

Further, this solution is defined over the entire interval $[a, b]$.

For example, let the interval $[a, b]$ be $0 \le x \le 1$ and that $n = 3$. Then at the point $x = \frac{1}{3}$, we may prescribe $y = \sqrt{2}, y' = \pi^2, y'' = 10^{10}$. Then the theorem asserts the existence of one and only one solution $y(x)$ taking on these values at $x = \frac{1}{3}$. This solution will, further, be defined at every point of $0 \le x \le 1$.

If $f(x) \equiv 0$ on $[a, b]$ Eqn.(18) becomes

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \qquad (19)$$

Eqn.(19) is called a **homogeneous** equation. Before talking about the properties of the solutions of Eqn.(18) and (19) let us quickly recap linear dependence and independence of set of functions on a given interval.

Consider a set of n functions, namely $y_1(x), y_2(x), \ldots, y_n(x)$ of x defined over the interval $[a, b]$. These n functions are said to be **linearly dependent** in an interval $[a, b]$ if for every x in the interval, there exists a relation

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) = \sum_{i=1}^{n} c_i y_i(x) = 0, \qquad (20)$$

where the constants c_i's are not all zero.

If such a relation does not exist, then the functions are said to be **linearly independent** in $[a, b]$, i.e., none of the functions can be expressed as a linear combination of the others.

For example, the functions $\cosh x, e^x$ and e^{-x} are linearly dependent, since

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ or, $\cosh x - \frac{1}{2}e^x - \frac{1}{2}e^{-x} = 0$. Similarly, function $y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$ are linearly dependent on the interval $]\infty, \infty[$ since $c_1 \sin 2x + c_2 \sin x \cos x = 0$ is satisfied for every real x with $c_1 = \frac{1}{2}$ and $c_2 = -1$.

Note that in the consideration of linear dependence or linear independence, the interval on which the functions are defined is important. You can check this for yourself that the functions $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the interval $]-\infty, \infty[$ whereas, they are linearly dependent on $]0, \infty[$.

The above procedure of examining the linear dependence or independence of a set of functions appears to be quite involved. We, therefore, give below a **sufficient condition** for examining the linear independence of a set of n functions.

Suppose that $y_1(x), y_2(x), \ldots, y_n(x)$ are n functions defined on an interval $[a, b]$ possessing derivatives upto $(n-1)^{th}$ order. If the determinate

$$W(y_1(x), y_2(x), \ldots, y_n(x)) = \begin{vmatrix} y_1 & y_2 \cdots & y_n \\ y_1' & y_2' \cdots & y_n' \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} \cdots & y_n^{(n-1)} \end{vmatrix}$$

is not zero for at least one point in the interval $[a, b]$, then the functions $y_1(x), \ldots, y_n(x)$ are linearly independent on the interval. The determinant $W(y_1(x), y_2(x), \ldots, y_n(x))$ is called the **Wronskian** of the functions. It is named after a Polish mathematician Josef Maria Hoene Wronski (1778-1853).

The functions $y_1(x) = e^{kx}, y_2(x) = e^{-kx}$ and $y_3(x) = \sinh kx$, for instance are linearly dependent on $]-\infty, \infty[$ because

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{kx} & e^{-kx} & \sinh kx \\ ke^{kx} & -ke^{-kx} & k\cosh kx \\ k^2e^{kx} & k^2e^{-kx} & k^2 \sinh x \end{vmatrix} = 0.$$

Remember that in the above condition the non-vanishing of the Wronskian at a point in the interval provides only a sufficient condition. In other words, if $W(y_1, y_2, \ldots, y_n) = 0$ for some x in the interval $[a, b]$ then it does not necessarily mean that the functions are linearly dependent on the interval. For example, if $y_1(x) = x^2$ and $y_2(x) = x|x|$ then $W(y_1(x), y_2(x)) = 0$ for every real number whereas, $y_1(x)$ and $y_2(x)$ are linearly independent on $]-\infty, \infty[$.

With the above background let us now come back to the properties of solutions of Eqns.(18) and (19).

We can think of the forms of the solutions of Eqn.(19) by making use of the following elementary theorems.

Theorem 3: If $y = y_1$ is a solution of Eqn.(19) on an interval $[a, b]$, then $y = c y_1$ is also its solution on $[a, b]$, where c is any arbitrary constant.

Theorem 4: If $y = y_1, y_2, \ldots, y_n$ are n solutions of homogeneous differential Eqn.(19) on an interval $[a, b]$, then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is also a solution of Eqn.(19) on $[a, b]$, where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Theorem 4 is known as the **superpostion principle**. Theorems 3 and 4 represent properties that non-linear differential equations, in general, do not possess. This will become more clear to you after doing E8).

Let us now consider the following definition which involves a linear combination of solutions.

Definition: Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of homogenenous linear differential Eqn.(19) of order n on an interval $[a, b]$. Then

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_i, i = 1, 2, \ldots, n$ are arbitrary constants, is defined to be the general solution of Eqn.(19) on $[a, b]$.

The above definition generates our interest in knowing when n solutions $y_1, y_2, \ldots, y_n$ of the homogeneous differential Eqn.(19) are linearly independent. Surprisingly, the non-vanishing of the Wronskian of a set of n such solutions on an interval $[a, b]$ is both **necessary** and **sufficient** for linear independence. That is, if $y_1, y_2, \ldots, y_n$ be n solutions of homogeneous linear n^{rh} order differential Eqn.(19) on $[a, b]$, then the set of solutions is linearly independent on $[a, b]$, if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

For every x in the interval. Such a set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of Eqn.(19) on $[a, b]$ is said to be a **fundamental set of solutions** on the interval. For instance, the second order equation $\frac{d^2y}{dx^2} - k^2 y = 0$, has two solutions $y_1 = e^{kx}$ and $y_2 = e^{-kx}$ since

$$W(e^{kx}, e^{-kx}) = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{-kx} \end{vmatrix} = -2k \neq 0$$

for every value of x. Functions y_1 and y_2 form a fundamental set of solution on $]-\infty, \infty[$. The general solution of the differential equation on the interval is $y = c_1 e^{kx} + c_2 e^{-kx}$.

We now state one more theorem wich pertains to the solution of non-homogeneous linear differential Eqn.(18).

Theorem 5: If $y = Y_0(x)$ is any solution of differential Eqn.(18) on an interval $[a, b]$ and if $y = Y(x)$ is the general solution of the corresponding homogeneous Eqn.(19) on the interval, then

$$y = Y_0(x) + Y(x)$$

is the general solution of Eqn.(18) on the given interval.

We shall not prove Theorems 3, 4 and 5 here but illustrate them through various examples. If you are interested in the proofs of these theorems then you can refer to Unit 5, Block 2 of MTE-08.

Let us consider the following examples.

Example 7: Show that if $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the equation $x^3 y''' - 2xy' + 4y = 0$, on the interval $]0, \infty[$. Then $y = c_1 x^2 + c_2 x^2 \ln x$ is also a solution of the equation on the given interval.

Solution: We have $y = c_1x^2 + c_2x^2 \ln x$

$\therefore y' = 2c_1x + 2c_2x \ln x + c_2x,$

$y'' = 2c_1 + 2c_2 \ln x + 3c_2$, $y''' = \dfrac{2c_2}{x}$.

$$\therefore x^3y''' - 2xy' + 4y = x^3\left(\frac{2c_2}{x}\right) - 2x(2c_1x + 2c_2x\ln x + c_2x) + 4c_1x^2 + 4c_2x^2 \ln x$$

$$= 0$$

Thus $y = c_1x^2 + c_2x^2 \ln x$ is also a solution of the equation.

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Example 8: Show that linearly independent solutions of

$$y'' - 2y' + 2y = 0$$

on any interval are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2, y'(0) = -3$.

Solution : It can be verified that $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$ satisfy the given equation and hence, are solutions of given equation.
Here

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \cos x + e^x \sin x & e^x \cos x - e^x \sin x \end{vmatrix} = -e^x \neq 0.$$

Hence $e^x \sin x$ and $e^x \cos x$ are linearly independent solutions of the given equation. The general solutions can then be written as

$$y(x) = c_1 e^x \sin x + c_2 e^x \cos x.$$

where, c_1 and c_2 are arbitrary constants.

Here $y'(x) = c_1(e^x \sin x + e^x \cos x) + c_2(e^x \cos x - e^x \sin x)$

Now $y(0) = 2$ and $y'(0) = -3$ imply

$2 = c_2$ and $-3 = c_1 + c_2$

i.e, $c_1 = -5, c_2 = 2$

Hence, in this case the solution is $y(x) = -5e^x \sin x + 2e^x \cos x$.

* * *

You may now try the following exercises.

E8) Functions $y_1 = 1$ and $y_2 = \ln x$ are solutions of the non-linear differential equation $y'' + (y')^2 = 0$, on the interval $]0, \infty[$. Then

a) is $y_1 + y_2$ a solution of the equation?

b) is $c_1y_1 + c_2y_2$, a solution of the equation, where c_1 and c_2 are arbitrary constants?

E9) Show that $\sin ax$ and $\cos ax$ are linearly independent solutions of $y'' + a^2y = 0$, a being a positive constant on the interval $]-\infty, \infty[$. Obtain the general solution of the equation on the interval.

E10) Show that on the interval $0 < x < \infty, \sin\frac{1}{x}$ and $\cos\frac{1}{x}$ are linearly independent solutions of the equation $x^4 y'' + 2x^3y' + y = 0$. Find the solution $y(x)$ of the differential equation with the property $y\left(\frac{1}{\pi}\right) = 1$ and $y'\left(\frac{1}{\pi}\right) = -1$.

In practice, equations of the form (18) and (19), where coefficients are functions of x, do not usually have solutions expressible in terms of elementary functions, and even when they do, it is very difficult to find them. However, if coefficients in Eqns.(18) and (19) are constants then they are called **linear equations with constant coefficients**, and their solutions in terms of elementary functions can be obtained. In the next section we shall quickly recall some of the methods of finding these solutions.

1.4.1 Solutions of Homogeneous, Linear Differential Equations with Constant Coefficients

Consider the differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad (21)$$

where $a_0, a_1, \ldots, a_n$ are all constants and $a_0 \neq 0$.

Suppose that a possible solution of Eqn.(21) is

$$y = e^{mx} \qquad (22)$$

Differentiating y w.r.t. x, n times we get

$$\frac{dy}{dx} = m e^{mx}, \frac{d^2 y}{dx^2} = m^2 e^{mx}, \ldots, \frac{d^n y}{dx^n} = m^n e^{mx}.$$

With these values, Eqn.(21) takes the form

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0 \qquad (23)$$

Since $e^{mx} \neq 0$ for all m and x, we can divide Eqn.(23) by e^{mx} to obtain

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad (24)$$

Now each value of m, for which Eqn.(24) holds, will make $y = e^{mx}$ a solution of Eqn.(21). Eqn(24) is called the **auxiliary equation** (A.E.) or, the **characteristic equation** (C.E.) of Eqn.(21). **Observe** that Eqn.(24) is an algebraic equation in m of degree n and, therefore, by the fundamental theorem of algebra, it has at least one and not more than n distinct roots. We denote these roots by $m_1, m_2, \ldots, m_n$, where m's need not all be distinct or real.

The following three cases arise while solving the A.E. (24).

1. All the roots of A.E. (24) are **real** and **distinct.**
2. All the roots of A.E. (24) are **real** but **some** are **repeated.**
3. **Some** of the roots of A.E. (24) are **complex.**

We, now, discuss these three cases one by one.

Case I: If the n roots $m_1, m_2, \ldots, m_n$ of A.E.(24) are distinct, then n solutions of Eqn.(21) are

$$y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \ldots, y_n = e^{m_n x}$$

But these n solutions are different and linearly independent and, thus, the general solution of Eqn.(21) is

$$y_c = y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} \qquad (25)$$

Here y_c is known as the **complementary function.**

Let us look at an example now.

Example 9: Solve $\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} - 6\frac{dy}{dx} = 0.$

Solution: A.E. is

$$m^3 - m^2 - 6m = 0$$

i.e., $m(m^2 - m - 6) = 0$, or, $m = 0, -2, 3$

Thus the general solution is

$$y = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x} = c_1 + c_2 e^{-2x} + c_3 e^{3x.}$$

* * *

You may now try the following exercises:

E11) Solve $y''' + 6y'' + 11y' + 6y = 0$

E12) Solve $y'' - 3y' + 2y = 0$ with $y = 0$ and $y' = 0$ when $x = 0$.

Case II: If A.E. (24) has a root m_1, which repeats r times, then the part of the solution corresponding to $m = m_1$ is

$$\left(c_1 + x c_2 + x^2 c_3 + \cdots + c_r x^{r-1}\right)e^{m_1 x}.$$

Now if A.E.(24) has r roots each equal to m_1 and the remaining (n – r) roots are all distinct, then solution of Eqn.(21) is

$$y_c = y = \left(c_1 + c_2 x + c_3 x^2 + \cdots + c_r x^{r-1}\right)e^{m_1 x} + c_{r+1}e^{m_{r+1}x} + \cdots + c_n e^{m_n x}$$

We now illustrate this case with the help of the following examples.

Example 10: Solve $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2 y = 0$, a being a positive constant.

Solution: Here A.E. is

$$m^2 - 2am + a^2 = 0$$

and has a double root $m = a$. The general solution is

$$y(x) = (c_1 + c_2 x)e^{ax}.$$

* * *

Example 11: Solve $y''' - 3y' + 2y = 0$.

Solution: The A.E. is

$$m^3 - 3m + 2 = 0 \Rightarrow (m+2)(m^2 - 2m + 1) = 0$$

which gives $m = 1, 1, -2$ and the general solution is given by

$$y(x) = (c_1 + c_2 x)e^x + c_3 e^{-2x}$$

* * *

We now discuss the case when some of the roots of A.E.(24) are complex.

Case III: We know from the theory of equations that if all the coefficients of Eqn.(21) are real, then any complex root it may have must occur in conjugate pairs. Thus, if $\alpha + i\beta$ is one root, then $\alpha - i\beta$ must be another root. If $\alpha \pm i\beta$ are the two complex roots of an A.E.(24), then the corresponding part of the general solution for constants A and B is given by

$$A e^{(\alpha+i\beta)x} + B e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right] = e^{\alpha x}\left[c_1\cos\beta x + c_2\sin\beta x\right],$$

where $c_1 = A + B$ and $c_2 = i(A - B)$ are the new constants.

If $\alpha + i\beta$ and $\alpha - i\beta$ each occurs twice as a root of A.E. Eqn.(24), then the corresponding part of the general solution is

$$e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right].$$

Let us consider the following examples to illustrate this case.

Example 12: Solve $\left(\frac{dy}{dx} - y\right)^2\left(\frac{d^2y}{dx^2} + y\right)^2 = 0$

Solution: The A.E. is

$$(m-1)^2(m^2+1)^2 = 0,$$

which gives $m = 1, 1, \pm i, \pm i$. Then the general solution is

$$y(x) = (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x.$$

* * *

Example 13: Solve $\frac{d^3y}{dx^3} + y = 0.$

Solution : The A.E. is $m^3 + 1 = 0$, or, $(m+1)(m^2 - m + 1) = 0$

which gives $m = -1, \frac{1 \pm i\sqrt{3}}{2}$ and the solution is

$$y(x) = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right].$$

***

How about trying the following exercises now?

E13) Solve the following differential equations:

a) $16y'' + 24y' + 9y = 0$

b) $y^{(4)} - 2a^2 y'' + a^4 y = 0$, a being a constant.

c) $y''' + y'' = 0$ with $y(0) = 1, y'(0) = 0, y''(0) = 1.$

E14) Solve the following differential equations.

a) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

b) $y'' + 4y = 0$

c) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

d) $y^{(6)} + 9y^{(4)} + 24y^{(2)} + 16y = 0.$

We next take up the solution of the non-homogeneous linear DEs with constant coefficients.

1.4.2 Solution of Non-Homogeneous Linear Differential Equations with Constant Coefficients

The general solution of a non-homogeneous linear DE

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x), \qquad (26)$$

where $a_0 \neq 0, f(x) \neq 0$ and $a_0, a_1, \ldots, a_n$ are constants is $y(x) = y_c + y_p$.

Here y_c and y_p are respectively, known as the **complementary function** (C.F.) and **particular integral** (P.I.) or, particular solution. In Sec.1.4.1, we discussed, for $f(x) = 0$, the methods of finding the general solution $y = y_c$. It now remains to find the P.I. of a given differential equation.

Before going into the details of the procedures of finding the P.I. of a given DE, let us recall the definition of the differential operator.

Definition: A mathematical device by means of which we can convert one function into another is known as an **operator.**
The operation of differentiation is an operator as it converts a differentiable function $f(x)$ into a new function $f'(x)$. The letter D, which we shall use to denote the differentiation, is called the **differential operator**.

If y is an n^{th} order differentiable function, then

$$D^0 y = y, Dy = y', D^2 y = y'', \ldots, D^n y = y^{(n)} \qquad (27)$$

If $F(D)$ is a **polynomial operator** of order n defined by

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, a_0 \neq 0, \qquad (28)$$

and y is an n^{th} order differentiable function, then

$$F(D)\,y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)\,y$$
$$= a_0D^ny + c_1D^{n-1}y + \cdots + a_{n-1}Dy + a_ny$$
$$= a_0y^n + a_1y^{n-1} + \cdots + a_{n-1}y' + a_ny, \qquad (29)$$

and Eqn.(26) can be written as

$$F(D)y = f(x), a_0 \neq 0.$$

If $F(D)$ is defined by the relation (28), then

$$F(D) = a_0(D - m_1)(D - m_2)\ldots(D - m_n), \qquad (30)$$

where $m_1, m_2, \ldots, m_n$ are the real or complex roots of A.E. corresponding to $F(D)y = 0$. We can thus say that **a polynomial operator with constant coefficients can be factored just the way we factor an ordinary polynomial.**

Remark 1: $F(D)$ is a polynomial operator given by Eqn.(28) , then

$$F(D) = F_1(D).F_2(D),$$

where $F_1(D)$ and $F_2(D)$ may be composite factors of $F(D)$, i.e., F_1 and F_2 may be products of factors of Eqn.(30).

Remark 2: The polynomial operator satisfies the commutative law for multiplication, viz; $(D - m_1)(D - m_2) = (D - m_2)(D - m_1)$.

Remark 3: If $F(D)$ is a polynomial operator defined by Eqn.(28) and $g(x)$ is an n^{th} order differentiable function of x , then

$$F(D)[g(x)e^{ax}] = e^{ax}F(D + a)g(x),$$

where 'a' is a constant.
We are now in a position to give a method for solving Eqn(26) by making use of the polynomial operator. The procedure is illustrated by means of the following example

Example 14: Solve the differential equation $y''' + 2y'' - y' - 2y = e^{2x}$

Solution : In the operator notation, the given DE can be written as

$$(D^3 + 2D^2 - D - 2)y = e^{2x}$$

or,
$$(D - 1)(D + 1)(D + 2)\,y = e^{2x} \qquad (31)$$

Let
$$u = (D + 1)(D + 2)\,y \qquad (32)$$

Then Eqn.(31) becomes

$$(D - 1)\,u = e^{2x}, \qquad (33)$$

which is a linear differential equation and its solution is

$$u = e^{2x} + c_1e^x.$$

Putting this value of u in Eqn.(32), we get

$$(D + 1)(D + 2)\,y = e^{2x} + c_1e^x \qquad (34)$$

Let
$$v = (D + 2)\,y \qquad (35)$$

Then Eqn.(34) becomes

$$(D + 1)\,v = e^{2x} + c_1e^x$$

which is a linear equation and its solution is

$$v = \frac{1}{3}e^{2x} + \frac{c_1}{2}e^x + c_2e^{-x}.$$

putting the value of v in Eqn.(35), we obtain

$$Dy + 2y = \frac{1}{3}e^{2x} + \frac{c_1}{2}e^x + c_2e^{-x},$$

which is again, a linear equation with the solution as

$$y = \frac{1}{12}e^{2x} + \frac{c_1}{6}e^x + c_2e^{-x} + c_3e^{-2x}$$

Here $\frac{1}{12}e^{2x}$ is a P.I. (free from constant) and $\frac{c_1}{6}e^x + c_2e^{-x} + c_3e^{-2x}$ is C.F.

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In general, if the non-homogeneous linear differential equation

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_{n-1} y + a_n = f(x), a_0 \neq 0,$$

of order n is written as

$$(D - m_1)(D - m_2)\ldots(D - m_n) y = f_1(x), \tag{36}$$

where $m_1, m_2, \ldots, m_n$ are the roots of A.E., then a general solution can be obtained as follows:

$$\text{Let} \quad u = (D - m_2)(D - m_3)\ldots(D - m_n) y \tag{37}$$

Then Eqn.(36) takes the form

$$(D - m_1) u = f_1(x), \tag{38}$$

which is a linear equation in u. Obtain its solution and put it in Eqn.(37) to get

$$(D - m_2)(D - m_3)\ldots(D - m_n) y = u(x) \tag{39}$$

Now, let $v = (D - m_3)(D - m_4)\ldots(D - m_n) y$ (40)

Then Eqn.(39) becomes

$$(D - m_2) v = u(x), \tag{41}$$

which is linear in v. Obtain its solution and put it in Eqn.(40) to get

$$(D - m_3)\ (D - m_4)\ldots(D - m_n) y = v(x)$$

Repeat this process $(n - 2)$ times to get the solution for y.

We shall now define the inverse operator and use it to find the P.I. We shall also give some shortcut methods to find P.I. when the right hand side, i.e., the non-homogeneous term of the given equation is a polynomial in x, an exponential function, sine or cosine function or, combination of these functions.

Inverse Operation

We start with the definition of inverse operator

Definition : Let $F(D)y = f(x)$, where $F(D)$ is the polynomial operator defined as

$$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

and $f(x)$ is a function of x. The **inverse operator of F(D)**, written as $F^{-1}(D)$ or $\frac{1}{F(D)}$**,** is then defined as an operator, which when operated on $f(x)$, gives the P.I. y_p of $F(D) = f(x)$**,** i.e.,

$$F^{-1}(D) f(x) = y_p \quad \text{or} \quad y_p = \frac{1}{F(D)} f(x)$$

Remark 1: From the above definition, we conclude that $D^{-n} f(x)$ will mean integration of $f(x)$ n-times by ignoring constants of integration.

Remark 2: If $F(D)\, y = 0$, then

$$y_p = \frac{1}{F(D)}(0) = 0.$$

Remark 3: From the above definition and general method stated above,

$$\frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)} f(x) = e^{m_{n-1}} \int e^{(m_{n-1} - m_n)x} \left(\int e^{(m_{n-2} - m_{n-1})x} \int \ldots \left(\int e^{(m_1 - m_2)x} \left(\int e^{-m_1 x} f(x) dx \right) \cdots dx \right) dx$$

Remark 4: If we write P.I. of $F(D)y = f(x)$ as

$y_p = \frac{1}{F(D)} f(x) = \frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)} f(x)$**,** then following the ordinary rules of algebra, for distinct $m_1, m_2, \ldots m_n$**,** we can write

$$y_p = \left(\frac{\alpha_1}{D - m_1} + \frac{\alpha_2}{D - m_2} + \cdots + \frac{\alpha_n}{D - m_n} \right) f(x)$$

$$= \alpha_1 e^{m_1 x} \int e^{-m_1 x} f(x) dx + \alpha_2 e^{m_2 x} \int e^{-m_2 x} f(x) dx + \cdots \alpha_n e^{m_n x} \int e^{-m_n x} f(x) dx, \text{ where}$$

coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be obtained by a simple algebraic manipulations (as in partial fractions).

In case a root m_1 of A.E. is repeated r times, the corresponding partial fraction will be

$$\frac{\alpha_1}{D - m_1} + \frac{\alpha_2}{(D - m_1)^2} + \cdots + \frac{\alpha_r}{(D - m_1)^r}$$

and the corresponding terms in the integral will be

$$\alpha_1 e^{m_1 x} \int e^{-m_1 x} f(x) dx + \alpha_2 e^{m_1 x} \iint e^{-m_1 x} f(x)(dx)^2 + \cdots$$

$$\cdots + \alpha_r e^{m_1 x} \iint \ldots \int e^{-m_1 x} f(x)(dx)^r.$$

We illustrate the above theory and remarks through the following examples:

Example 15: Find the particular integral of $\frac{d^2 y}{dx^2} + y = \sec^2 x.$

Solution : Here,

$$\text{P.I.} = y_p = \frac{1}{D^2 + 1} \sec^2 x = \frac{1}{(D+i)(D-i)} \sec^2 x$$

$$= \frac{1}{2i} \left[\frac{1}{D - i} - \frac{1}{D + i} \right] \sec^2 x$$

$$= \frac{1}{2i} \left[e^{ix} \int e^{-ix} \sec^2 x \, dx - e^{-ix} \int e^{ix} \sec^2 x \, dx \right]$$

$$= \frac{1}{2i} \left[e^{ix} \int \frac{\cos x - i \sin x}{\cos^2 x} dx - e^{ix} \int \frac{\cos x + i \sin x}{\cos^2 x} dx \right]$$

$$= \frac{1}{2i} \left[e^{ix} \int (\sec x - i \sec x \tan x) dx - e^{ix} \int (\sec x + i \sec x \tan x) dx \right]$$

$$= \frac{1}{2i} \left[\left(e^{ix} - e^{-ix}\right) \int \sec x \, dx - i \left(e^{ix} + e^{-ix}\right) \int \sec x \tan x \, dx \right]$$

$$= \frac{1}{2i} \left[(2i \sin x) \ln |\sec x + \tan x| - (2i \cos x) \sec x \right]$$

$$= \sin x \ln |\sec x + \tan x| - 1$$

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Example 16: Find the particular integral of $(D-1)^2 (D+1)^2 y = e^x$.

Solution: The particular integral is

$$y_p = \frac{1}{(D-1)^2 (D+1)^2} e^x = \frac{1}{4} \left[\frac{-1}{D-1} + \frac{1}{(D-1)^2} + \frac{1}{D+1} + \frac{1}{(D+1)^2} \right] e^x$$

$$= \frac{1}{4} \left[-e^x \int e^{-x} e^x dx + e^x \int \left(\int e^{-x} e^x dx \right) dx + e^{-x} \int e^x e^x dx + e^{-x} \int \left(\int e^x e^x dx \right) dx \right]$$

$$= \frac{1}{4} \left[-x e^x + \frac{x^2}{2} e^x + \frac{1}{2} e^x + \frac{1}{4} e^x \right].$$

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You may now try the following exercises.

E15) Find the complete solution of the following equations:

a) $y''' + y' = x^3 + \cos x$

b) $(D^3 + 3D^2 - D - 3) y = \cosh x$

c) $(D^3 + D^2 + 4D + 4) y = \sin 2x$

d) $y'' + n^2 y = \sec nx$

e) $(D^3 - D^2 - 8D + 12)y = X(x)$.

The general method of computing a particular integral as given above is quite laborious and requires a lot of calculations as can be seen from Examples 15 and 16 above. However, in certain cases, the P.I. can be obtained by shorter methods. We shall now consider these shorter methods.

Short Methods of Finding Particular Integrals

Consider the general n^{th} order linear DE, namely,

$F(D)y = (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = f(x),\ a_0 \neq 0,$

where $a_0, a_1, \ldots, a_n$ are constants.

For certain particular forms of the non-homogeneous term $f(x)$, shorter methods of finding P.I. are available which we are stating below.

Case I: When $f(x) = e^{\alpha x}$, α constant.

$$\frac{1}{F(D)}e^{\alpha x} = \frac{1}{F(\alpha)}e^{\alpha x},\ F(\alpha) \neq 0$$

When, $F(\alpha) = 0$ and $F(D) = (D - \alpha)^p \phi(D)$, $\phi(\alpha) \neq 0$ for some p, then

$$\frac{1}{F(D)}e^{\alpha x} = \frac{1}{(D-\alpha)^p \phi(D)}e^{\alpha x},\ \phi(\alpha) \neq 0$$

$$= \frac{x^p}{p!}.\frac{1}{\phi(\alpha)}e^{\alpha x}.$$

Case II: when $f(x) = \cos(ax + b)$, or, $\sin(ax + b)$.

If $f(D) = \phi(D^2)$,

then $\dfrac{1}{\phi(D^2)}\cos(ax+b) = \dfrac{1}{\phi(-a^2)}\cos(ax+b)$, if $\phi(-a^2) \neq 0$

and, $\dfrac{1}{\phi(D^2)}\sin(ax+b) = \dfrac{1}{\phi(-a^2)}\sin(ax+b)$, if $\phi(-a^2) \neq 0$

If, $\phi(-a^2) = 0$ and $\phi(D^2) = (D^2 + a^2)^p \psi(D^2)$ for some p and $\psi(-a^2) \neq 0$

$$\text{Then, } \frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{(D^2+a^2)^p \psi(D^2)}\cos(ax+b)$$

$$= \frac{1}{\psi(-a^2)}\left[\frac{1}{(D^2+a^2)^p}\cos(ax+b)\right] \text{if, } \psi(-a^2) \neq 0.$$

$$\text{and, } \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{(D^2+a^2)^p \psi(D^2)}\sin(ax+b)$$

$$= \frac{1}{\psi(-a^2)}\left[\frac{1}{(D^2+a^2)^p}\sin(ax+b)\right] \text{if, } \psi(-a^2) \neq 0.$$

Note that the terms in the brackets above are evaluated by the general method. Cosine and sine functions in Case II above can also be dealt as exponential functions by writing them in the form:

$$\cos(ax+b) = \text{Re } e^{i(ax+b)},$$

and $\quad \sin(ax+b) = \text{Im } e^{i(ax+b)}$

Symbols Re and Im are read as 'real part of' and 'imaginary part of' respectively.

Case III: When $f(x)$ is a polynomial in x.

Let $F(D)$ be a polynomial in D of degree n and let $f(x)$ be a polynomial in x of degree say, K. Then

$$\frac{1}{F(D)}f(x)=\frac{1}{a_0D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n}f(x)$$

$$=\frac{1}{a_n}\left(1+\frac{a_{n-1}}{a_n}D+\cdots\frac{a_1}{a_n}D^{n-1}+\frac{a_0}{a_n}D^n\right)^{-1}f(x)$$

$$=\left(c_0+c_1D+c_2D^2+\cdots+c_kD^k\right)f(x)+0(D^{k+1})f(x)$$

$=\left(c_0+c_1D+c_2D^2+\cdots+c_kD^k\right)f(x)$, the second term on r.h.s. is zero.

Case IV: When $f(x)=e^{\alpha x}\ V(x)$, α constant.

$$\frac{1}{F(D)}\left[e^{\alpha x}\ V(x)\right]=e^{\alpha x}\frac{1}{F(D+\alpha)}V(x)$$

This result is known as **shifting theorem**.

Remark 1: We may remark that Euler's equation

$$(a_0x^nD^n+x^{n-1}a_1D^{n-1}+\cdots+a_{n-1}x\,D+a_n)y=f(x)$$

can be reduced to an equation with constant coefficients with the help of substitution $x=e^z$.

Remark 2: Differential Equation

$$[a_0(ax+b)^nD^n+a_1(ax+b)^{n-1}D^{n-1}+\cdots+a_{n-1}(ax+b)D+a_n]y=f(x)$$

can be either reduced to Euler's equation by the substitution $ax+b=z$, or, it can be reduced to an equation with constant coefficients by the transformation $ax+b=e^z$.

We, now, take up some examples to illustrate the shorter methods of finding P.I. stated above.

Example 17: Find P.I. of $y'''+y''+y'+y=x^4+2x+1$

Solution: Here P.I. $=\dfrac{1}{1+D+D^2+D^3}(x^4+2x+1)$

$$=\frac{1-D}{1-D^4}(x^4+2x+1)$$

$$=(1-D^4)^{-1}\left[x^4+2x+1-4x^3-2\right]$$

$$=(1+D^4+D^8+\cdots)(x^4-4x^3+2x-1)$$

$$=(x^4-4x^3+2x-1)+24$$

$$=x^4-4x^3+2x+23$$

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Example 18: Find P.I. of $(D^4-2D^3+D^2)y=x$

Solution : Here P.I. $=\dfrac{1}{D^4-2D^3+D^2}x$

$$=\frac{1}{D^2}[1-(2D-D^2)]^{-1}\ x$$

$$=\frac{1}{D^2}\left[1+2D-D^2+\cdots\right]x$$

$$=\frac{1}{D^2}\left[x+2\right]$$

$$=\frac{1}{D}\left(\frac{x^2}{2}+2x\right)$$

$$=\frac{x^3}{6}+x^2.$$

***

Example 19: Solve $(D^2-2D+1)y = x \sin x$

Solution : A.E. is $m^2-2m+1=0 \Rightarrow m=1,1$

$$\therefore \text{C.F.} = y_c = (c_1+c_2 x)e^x$$

$$\text{P.I.} = \frac{1}{D^2-2D+1} x e^x \sin x = e^x \frac{1}{(D+1)^2-2(D+1)+1} x \sin x$$

$$= e^x D^{-2}(x \sin x)$$

$$= e^x[-x \sin x - 2\cos x]$$

$\therefore$General solution is, $y(x) = (c_1+c_2x)e^x + e^x(-x\sin x - 2\cos x)$.

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Example 20: Find the particular integral of $(D^2+1)y = x^2 \sin 2x$.

Solution : Here P.I. $= \frac{1}{D^2+1} x^2 \sin 2x = \text{Im} \frac{1}{D^2+1}(x^2 e^{2ix})$

$$= \text{Im of } e^{2ix} \frac{1}{(D+2i)^2+1} x^2 = \text{Im of } e^{2ix} \frac{1}{D^2+4iD-4+1} x^2$$

$$= \text{Im of } e^{2ix} \frac{1}{-3}\left[1-\left(\frac{4iD+D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{Im of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD+D^2}{3}-\frac{16}{9}D^2+\cdots\right]x^2$$

$$= \text{Im of } \frac{e^{2ix}}{-3}\left[1+\frac{4i}{3}D-\frac{13}{9}D^2+0(D^3)\right]x^2 = \text{Im of } \frac{e^{2ix}}{-3}\left[x^2+\frac{8i}{3}x-\frac{13}{9}.2\right]$$

$$= \text{Im of } \left[\left(\frac{\cos 2x + i \sin 2x}{-3}\right)\left(x^2-\frac{26}{9}+i\frac{8}{3}x\right)\right] = -\frac{8}{9}x\cos 2x - \frac{1}{3}\left(x^2-\frac{26}{9}\right)\sin 2x$$

$$= -\frac{1}{27}\left[24x \cos 2x + (9x^2-26)\sin 2x\right].$$

***

Example 21: Find P.I. of $(D^4-1)y = \sin x$.

Solution: Here P.I. $= \frac{1}{D^4-1} \sin x = \frac{1}{(D^2-1)(D^2+1)} \sin x$

$$= \frac{1}{(-1-1)} \frac{1}{D^2+1} \sin x$$

$$= -\frac{1}{2}\left[-\frac{x}{2}\cos x\right] = \frac{x}{4}\cos x$$

***

Example 22: Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\ln (1+x)]$

Solution: Let $1+x = e^t, t = \ln (1+x)$ and $D \equiv \frac{d}{dt}$. Then the given equation becomes

$$[D(D-1)+D+1]y = 2\sin t, \text{ i.e.,}$$

$$(D^2+1)y = 2\sin t,$$

which is a linear equation with constant coefficients with

$$\text{C.F.} = y_c = c_1 \cos t + c_2 \sin t$$

$$\text{and P.I.} = y_p = \frac{1}{D^2+1} 2\sin t = -t \cos t$$

Hence, the complete solution is

$$y = y_c + y_p = c_1 \cos t + c_2 \sin t - t\cos t$$
$$= c_1 \cos[\ln(1+x)] + c_2 \sin[\ln(1+x)] - \ln(1+x)\cos[\ln(1+x)].$$

***

Example 23: Solve $x^3D^3y + 3x^2D^2y + x\,Dy + y = x + \ln x$

Solution: Let $x = e^t, \frac{d}{dt} \equiv D_1$, so that the given equation becomes

$$[D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) + D_1 + 1]\ y = e^t + t$$

i.e., $(D_1^3 + 1)y = e^t + t$

A.E. is, $m^3 + 1 = 0$, which has the roots $m = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus,

$$\text{C.F.} = y_c = c_1e^{-t} + e^{\frac{t}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}t + c_3 \sin\frac{\sqrt{3}}{2}t\right]$$
$$= c_1x^{-1} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$$

$$\text{P.I.} = y_p = \frac{1}{D_1^3+1}e^t + \frac{1}{D_1^3+1}t = \frac{1}{1+1}e^t + (1 - D_1^3 + \cdots)t$$
$$= \frac{1}{2}e^t + t = \frac{1}{2}x + \ln x$$

Therefore, the required solution is

$$y = y_c + y_p = c_1x^{-1} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + \frac{1}{2}x + \ln x.$$

***

You may now try the following exercises.

E16) Solve the following differential equations

a) $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$

b) $(D^4 + 2D^3 - 3D^2 + 4D + 4)y = 0$

c) $(D^2 - 2D + 5)y = 0$, given that $y = 0$ and $\frac{dy}{dx} = 4$, when $x = 0$.

E17) Obtain the general solution of the following differential equations

a) $y''' + 3y'' + 3y' + y = e^{-x}(2 - x^2)$.

b) $(D^2 + 1)(D^2 + 4)y = \cos\frac{x}{2}\ \cos\frac{3x}{2}$.

c) $(D^5 - D)y = 12e^x + 8\sin x - 2x$.

d) $(D^2 - 1)y = e^{-x} + \cos x + x^3 + e^x \cos x$.

e) $(x+3)^2 y'' - 4(x+3)y' + 6y = \ln(x+3)$.

f) $(D^2 + a^2)y = \tan\ ax$

We now end this unit by giving a summary of what we have covered in it.

1.5 SUMMARY

In this unit, we have shown you the following:

1. The differential equation together with the initial conditions is called an initial value problem (IVP).

2. The differential equation together with the boundary conditions is called boundary value problem (BVP).

3. IVP, $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt.$$

4. Picard's Method of successive Approximations for IVP,
$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$, is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]dt,\ n = 1, 2, 3, \ldots.$$

5. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R and $f(x_0, y_0)$ is an interior point of R, then there exists a number $h > 0$ with the property that the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h$. It is called the **Picard's theorem on existence and uniqueness** of solution of IVP.

6. The inequality $\left|\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}\right| < N$ (a constant), is called a **Lipschitz condition** in the variable y and the constant N is called a Lipschitz constants.

7. Continuity of $f(x, y)$ guarantees a solution of IVP, $y' = f(x, y), y(x_0) = y_0$, whereas, Lipschitz condition or, continuity of $\frac{\partial f}{\partial y}$ ensures a unique solution.

8. For a linear differential equation of order n of the form
$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x), a_0(x) \ne 0$,
let $x = x_0$ be a point of interval $[a, b]$ and let $k_0, k_1, \ldots, k_{n-1}$ be an arbitrary set of n constants. Then there exists one and only one solution $y(x)$ of the equation in $[a, b]$, with the property

$$y(x_0) = k_0, y'(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1,}$$

where continuity of $a_0(x), a_1(x), \ldots, a_n(x)$ in $[a, b]$ has been assumed.

9. The 'n' functions $y_1(x), \ldots, y_n(x)$ defined on $[a, b]$ are said to be **linearly dependent** in [a, b] if, for all x in that interval, there exists a relation

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

in which the c_i are constants and not all are zero. If such a relation does not exist, then the functions are said to be **linearly independent** in [a, b].

10. If $a_0(x), a_1(x), \ldots, a_n(x), f(x)$ are continuous functions of x on [a, b] and $a_0(x) \ne 0$, then

a) for a homogeneous equation
$a_0(x)y^{(n)} + a_1(x)y^{(n-`1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$,
if, $y_1(x), y_2(x), \ldots, y_n(x)$ are n linearly independent solutions of the equation in [a, b] then $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$ is the general solution of the equation for constants $c_1, c_2, \ldots, c_n$.

b) for a non-homogeneous equation
$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$
If $Y(x)$ is a particular integral and $y_1, y_2, \ldots, y_n$ are n linearly independent

solutions of the corresponding homogeneous equation then,
$y(x) = Y(x) + c_1 y_1(x) + \cdots + c_n y_n(x)$ is the general solution for constants $c_1, c_2, \ldots, c_n$

11. Solution y, of an n^{th} order linear DE
$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, a_0 \neq 0,$
with constant coefficients $a_0, a_1, \ldots, a_n$ having n roots $m_1, m_2, \ldots, m_n$ when
   a) roots are all real and distinct, is
   $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$.
   b) roots are real and repeated, say $m_1 = m_2 = \cdots = m_r$, is
   $y = (c_1 + c_2 x + \cdots + c_r x^{r-1}) e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \cdots + c_n e^{m_n x}$.
   c) roots are complex and one such pair is $\alpha \pm i\beta$, then corresponding part of solution is $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.
12. For a non-homogeneous equation
$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x), a_0 \neq 0$, with constant coefficients $a_0, a_1, \ldots, a_n$
   a) the complete solution of the corresponding homogeneous part is called its complementary function (C.F.).
   b) particular solution of the non-homogeneous part involving no arbitrary constant is called its particular integral (P.I.).
   c) complementary function and particular integral together constitute its general solution.
13. A mathematical device by means of which we can convert one function into another is known as an **operator**, e.g., $D = \frac{d}{dx}$ is differential operator.
14. If $F(D)$ is a **polynomial operator** of order n, then it is defined as
$F(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, a_0 \neq 0.$
A polynomial operator with constant coefficients can be factored like an ordinary polynomial.
15. Euler's equation $(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \cdots + a_{n-1} x D + a_n) y = f(x)$
can be reduced to an equation with constant coefficients by substituting $x = e^z$.
16. D.E. $[a_0 (ax + b)^n D^n + a_1 (ax + b)^{n-1} D^{n-1} + \cdots + a_{n-1}(ax + b)d + a_n]\ y = f(x)$
can be either reduced to Euler's equation by the substitution $ax + b = z$ or it can be reduced to an equation with constant coefficients by using $ax + b = e^z$.

1.6 SOLUTIONS/ANSWERS

E1) If λ is negative say $\lambda = -\alpha$ then the solution is
$$y = c_1 e^{-\sqrt{-\alpha} x} + c_2 e^{\sqrt{-\alpha} x}$$
when $y(0) = 0$ and $y(\pi) = 0$, we get $c_2 = 0 = c_1$
Hence only solution of given problem is the trivial solution $y = 0$.
If $\lambda = 0$, then general solution is $y(x) = c_1 x + c_2$
Again $y(0) = 0$ and $y(\pi) = 0$ give the trivial solution $y = 0$.
If λ is positive, the general solution is
$$y(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x.$$
For $y(0) = 0$, we get $0 = c_2$ and the solution reduces to $y(x) = c_1 \sin \sqrt{\lambda} x$.
For the second b.c. $y(\pi) = 0$, we get $0 = c_1 \sin \sqrt{\lambda} \pi$
For the non-trivial solution, we must have $\sin \sqrt{\lambda} \pi = \sin n\pi$ for some positive integer n, so that $\lambda = n^2$, i.e, λ must be one of the numbers $1, 4, 9, \ldots$.
Hence the non-trivial solution of given problem are

$y(x) = a_1 \sin x$, or, $a_2 \sin 2x$, or, $a_3 \sin 3x$, or ....

And the solution vanishes at the end points 0 and π of interval $[0, \pi]$.

E2) a) Here $\frac{dy}{dx} = x + y, \quad y(0) = 1$. Thus $x_0 = 0, \quad y_0 = 1$, $f(x, y) = x + y$

$$\therefore y_1 = y_0 + \int_0^x f(x, y_0)\, dx = 1 + \int_0^x (x+1)\, dx = 1 + x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\, dx = 1 + \int_0^x \left(x + 1 + x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

$$y_3 = y_0 + \int_0^x f(x, y_2) = 1 + \int_0^x \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

b) $\frac{dy}{dx} = x - y^2, \quad y(0) = \frac{1}{2}$. Here $x_0 = 0, \quad y_0 = \frac{1}{2}, \quad f(x, y) = x - y^2$

$$\therefore y_1 = y_0 + \int_0^x (x - y_0^2)\, dx = \frac{1}{2} + \int_0^x \left(x - \frac{1}{4} \right) dx = \frac{1}{2} - \frac{1}{4} x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_0^x (x - y_1^2)\, dx = \frac{1}{2} + \int_0^x \left\{ x - \left(\frac{1}{4} + \frac{1}{16} x + \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{4} x - \frac{1}{4} x^3 \right) \right\} dx$$

$$= \frac{1}{2} - \frac{1}{4} x + \frac{19}{32} x^2 - \frac{x^3}{6} + \frac{1}{16} x^4 - \frac{x^5}{20}$$

$$y_3 = y_0 + \int_0^x (x - y_2^2)\, dx$$

$$= \frac{1}{2} - \frac{1}{4} x + \frac{5}{8} x^2 - \frac{21}{96} x^3 + \frac{89}{768} x^4 - \frac{1531}{3072} . \frac{x^5}{5} + \frac{67}{1440} x^6 - \frac{1303}{80640} x^7$$

$$+ \frac{77}{7480} x^8 - \frac{77}{8640} x^9 + \frac{1}{1600} x^{10} - \frac{x^{11}}{4400} .$$

c) $\frac{dy}{dx} = 2x(1 + y), \quad y(0) = 0$. Here $x_0 = 0, \; y_0 = 0, \; f(x, y) = 2x(1 + y)$

$$\therefore y_1 = y_0 + \int_0^x f(x, y)\, dx = 0 + \int_0^x 2x\, dx = x^2$$

$$y_2 = y_0 + \int_0^x 2x(1 + x^2)\, dx = 0 + x^2 + \frac{x^4}{2}$$

$$y_3 = y_0 + \int_0^x 2x \left(1 + x^2 + \frac{x^4}{2} \right) dx = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$$

d) $\frac{dy}{dx} = y, \quad y(0) = 1$. Here $x_0 = 0, \; y_0 = 1, \; f(x, y) = y$

$$y_1 = y_0 + \int_0^x f(x, y_0)\, dx = 1 + \int_0^x dx = 1 + x$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\, dx = 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2}$$

$$y_3 = y_0 + \int_0^x f(x, y_2)\, dx = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} \right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} .$$

e) $\frac{dy}{dx} = y^2, \quad y(0) = 1$. Here $x_0 = 0, \quad y_0 = 1, \quad f(x, y) = y^2$

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx = 1 + x$$

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx = 1 + x + x^2 + \frac{x^3}{3}$$

$$y_3 = y_0 + \int_0^x f(x, y_2)\,dx = 1 + x + x^2 + x^3 + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{x^6}{9} + \frac{x^7}{63}.$$

E3) Here $f(x, y) = \begin{cases} \frac{2y}{x}, \text{for } x > 0, & y(0) = 0 \\ 0, \text{for } x = 0 \end{cases}$

Thus, $\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\left|\frac{2}{x}\right| |y_1 - y_2|}{|y_1 - y_2|} = \left|\frac{2}{x}\right|.$

Hence $f(x, y)$ doesn't satisfy Lipschitz conditions in any closed rectangle containing $(0, 0)$.

E4) a) $f(x, y) = x$ and $\frac{\partial f}{\partial y} = 0$ are continuous on a whole plane and in every rectangle. Thus $f(x, y)$ satisfies the Lipschitz condition in every such rectangle and there exists a unique solution $y = \frac{x^2}{2} + 1$.

b) $f(x, y) = -|y|$ is not continuous and solution does not exist at any point in the neighbourhood of origin.

E5) Here $f(x,) = xy^2$, $\frac{\partial f}{\partial y} = 2xy$ and,

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{|xy_1^2 - xy_2^2|}{|y_1 - y_2|} = \frac{|x(y_1 - y_2)(y_1 + y_2)|}{|y_1 - y_2|} = |x||y_1 + y_2|$$

and the right hand side is $\leq N$ for, $a \leq x \leq b, \; c \leq y \leq d$ but is not less than equal to some finite constant for $a \leq x \leq b, \; -\infty < y < \infty$. Hence Lipschitz's condition is satisfied for any rectangle $a \leq x \leq b, \; c \leq y \leq d$ but is not satisfied for any strip $a \leq x \leq b, \; -\infty < y < \infty$.

E6) Here $f(x, y) = \frac{y - 1}{x}$. Thus $\frac{\partial f}{\partial y} = \frac{1}{x}$, which does not exist at the origin. Also

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\left|\frac{y_1 - 1}{x} - \frac{y_2 - 1}{x}\right|}{|y_1 - y_2|} = \frac{\left|\frac{1}{x}\right| |y_1 - y_2|}{|y_1 - y_2|} = \left|\frac{1}{x}\right|$$, which is

unbounded in every neighbourhood of origin.
Hence the Lipschitz condition is not satisified. The given problem may have an infinity of solutions.

E7) Here $f(x, y) = \frac{4x^2 y}{x^4 + y^2}$, when x and y are not both zero

$= 0$, when $x = y = 0$

Hence function $f(x, y)$ is continuous function of x and y (prove it.)
On the other hand,

$$|f(x, y_1) - f(x, y)| = \left| \frac{4x^3 (x^4 - y_1 y)}{(x^4 + y^2)(x^4 + y_1^2)} (y_1 - y) \right|$$

$$= \left| \frac{4(1-pq)}{(1+p^2)(1+q^2)} \frac{(y_1 - y)}{x} \right|, \text{ where, } y = px^2, y_1 = qx^2$$

$$= 4 \frac{|1-pq|}{|(1+p^2)(1+q^2)|} \cdot \frac{|y_1 - y|}{|x|}$$

and therefore the Lipschitz condition is not satisfied in any region containing the origin. The equation admits the solution $y = c^2 - \sqrt{x^4 + c^4}$, where c is an arbitrary real constant. Thus there is an infinity of solutions satisfying the intial conditions $x = 0,\ y = 0$.

E8) a) $y_1 + y_2$ is a solution of the equation

b) $c_1 y_1 + c_2 y_2$ (for arbitrary c_1, c_2) is not a solution of the equation when $c_2 \neq 0,1$.

E9) It can be easily verified that $= \sin ax$ and $y_2 = \cos ax$ satisfy the equation

$$y'' + a^2 y = 0$$

and, hence, are the solutions of the given equation on the interval $]-\infty, \infty[$.

Here $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin ax & \cos ax \\ a\cos ax & -a\sin ax \end{vmatrix}$

$$= -a\sin^2 ax - a\cos^2 ax = -a \neq 0.$$

Hence $\sin ax$ and $\cos ax$ are linearly independent solutions of the given equation. The general solution can then be written as $y = c_1 \sin ax + c_2 \cos ax$.

E10) Let $y_1 = \sin\frac{1}{x}$ and $y_2 = \cos\frac{1}{x}, y_1' = -\frac{1}{x^2}\cos\frac{1}{x}$, $y_2' = \frac{1}{x^2}\sin\frac{1}{x}$

$$y_1'' = -\frac{1}{x^4}\sin\frac{1}{x} + \frac{2}{x^3}\cos\frac{1}{x}, \quad y_2'' = \frac{-1}{x^4}\cos\frac{1}{x} - \frac{2}{x^3}\sin\frac{1}{x}$$

Here $x^4 y_1'' + 2x^3 y_1' + y_1$

$$= x^4\left(-\frac{1}{x^4}\sin\frac{1}{x} + \frac{2}{x^3}\cos\frac{1}{x}\right) + 2x^3\left(-\frac{1}{x^2}\cos\frac{1}{x}\right) + \sin\frac{1}{x} = 0$$

Hence y_1 satisfies given equation for $0 < x < \infty$.

Similarly $x^4 y_2'' + 2x^3 y_2' + y_2 = 0$, thus y_2 is also a solution of given equation.

Here $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin\frac{1}{x} & \cos\frac{1}{x} \\ -\frac{1}{x^2}\cos\frac{1}{x} & \frac{1}{x^2}\sin\frac{1}{x} \end{vmatrix} = \frac{1}{x^2} \neq 0$, for $0 < x < \infty$.

Hence y_1 and y_2 are linearly independent solutions of given equation and the general solution can be written as

$$y = c_1 \sin\frac{1}{x} + c_2 \cos\frac{1}{x} \text{ and } y' = -c_1 \frac{1}{x^2}\cos\frac{1}{x} + c_2 \frac{1}{x^2}\sin\frac{1}{x}$$

Now $y\left(\frac{1}{\pi}\right) = 1 \Rightarrow 1 = c_1 \sin\pi + c_2 \cos\pi = -c_2 \Rightarrow c_2 = -1$

$$y'\left(\frac{1}{\pi}\right) = -1 \Rightarrow -1 = -c_1\pi^2 \cos\pi + c_2\pi^2 \sin\pi \Rightarrow -1 = c_1\pi^2 \Rightarrow c_1 = -\frac{1}{\pi^2}$$

Hence the required solution is $y = -\frac{1}{\pi^2}\sin\frac{1}{x} - \cos\frac{1}{x}$.

E11) A.E. is $m^3 + 6m^2 + 11m + 6 = 0$, $m = -1, -2, -3$

Thus the general solution is, $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$.

E12) A.E. is $m^2 - 3m + 2 = 0$, $m = 1,2$

Thus the general solution is, $y = c_1e^{x} + c_2e^{2x}$

Satisfying the conditions $y = 0$ and $y' = 1$ when $x = 0$, we get $c_1 = -1,\ c_2 = 1$

Hence the solution of given problem is $y = -e^{x} + e^{2x}$

E13) a) A.E. is $16m^2 + 24m + 9 = 0$, $m = -3/4,\ -3/4$

$y = (c_1 + c_2x)e^{\frac{-3}{4}x}$

b) A.E. is $m^4 - 2a^2m^2 + a^4 = 0$, $m = a, a, -a, -a$.

Hence the solution is

$y = (c_1 + c_2x)e^{ax} + (c_3 + c_1x)e^{-ax}$.

c) A.E. is $m^3 + m^2 = 0$, $m = 0, 0, -1$

Thus, solution is $y = (c_1 + c_2x) + c_3e^{-x}$

Initial conditions $y(0) = 1, y'(0) = 0,\ y''(0) = 1$ give $c_1 = 0,\ c_2 = 1,\ c_3 = 1$

Hence required solution is $y = x + e^{-x}$.

E14) a) A.E. is $m^4 + 4m^2 + 16 = 0$, $m = 2i,\ 2i,\ -2i,\ -2i$

Hence the solution is

$y = (c_1 + xc_2)\cos 2x + (c_3 + xc_4)\sin 2x$.

b) $y = c_1 \cos 2x + c_2 \sin 2x$.

c) $y = c_1e^{-2x} + e^{x}(c_2 \cos x + c_3 \sin x)$

d) A.E. is $m^6 + 9m^4 + 24m^2 + 16 = 0$, $m = \pm\, i,\ \pm\, 2i,\ \pm\, 2i$

Hence the solution is

$y = (c_1 \cos x + c_2 \sin x) + (c_3 + xc_4)\cos 2x + (c_5 + xc_6)\sin 2x$.

E15) a) A.E. is $m^3 + m = 0$, $m = 0,\ m = \pm\, i$

C.F. $= c_1 + c_2 \cos x + c_3 \sin x$

$$\text{P.I.} = \frac{1}{D^3 + D}(x^3 + \cos x) = \frac{1}{D(D+i)(D-i)}(x^3 + \cos x)$$

$$= \left(\frac{1}{D} + \frac{1}{2}.\frac{1}{D-i} - \frac{1}{2}.\frac{1}{D+i}\right)(x^3 + \cos x)$$

$$= \int (x^3 + \cos x)\, dx + \frac{1}{2}e^{ix}\int e^{-ix}(x^3 + \cos x)\,dx - \frac{1}{2}e^{-ix}\int e^{ix}(x^3 + \cos x)\,dx$$

$$= \frac{x^4}{4} - 3x^2 - \frac{1}{2}x\cos x$$

Hence $y = c_1 + c_2 \cos x + c_3 \sin x + \frac{x^4}{4} - 3x^2 + x\cos x$

b) C.F. $= c_1e^{x} + c2\ e^{-x} + c_3\ e^{-3x}$

$$\text{P.I.} = \frac{1}{(D-1)\ (D+1)\ (D+3)}\cosh x = \left(\frac{1}{8}.\frac{1}{D-1} - \frac{1}{4}.\frac{1}{D+1} + \frac{1}{8}.\frac{1}{D+3}\right).\frac{1}{2}(e^{x} + e^{-x})$$

$$= \frac{1}{16}e^{x}\int e^{-x}(e^{x} + e^{-x})\,dx - \frac{1}{8}e^{-x}\int e^{x}(e^{x} + e^{-x})\,dx + \frac{1}{16}e^{-3x}\int e^{3x}(e^{x} + e^{-x})\,dx$$

$$= \frac{1}{16}x\,e^{x} - \frac{1}{8}x\,e^{-x} - \frac{3}{64}e^{x}$$

Hence $y = c_1e^{x} + c_2e^{-x} + c_3e^{-3x} + \frac{1}{16}x\,e^{x} - \frac{1}{8}x\,e^{-x}$

c) $\text{C.F.} = c_1e^{-x} + c_2\cos 2x + c_3\sin 2x$

$$\text{P.I.} = \frac{1}{(D+1)(D^2+4)}\sin 2x = \left[\frac{1}{5}.\frac{1}{D+1} - \frac{1}{4(i+2)}.\frac{1}{D+2i} + \frac{1}{4(i-2)}\frac{1}{D-2i}\right]\sin 2x$$

$$= \frac{1}{5}e^{-x}\int e^{x}\sin 2x\,dx - \frac{1}{4(i+2)}e^{-2ix}\int e^{2ix}\sin 2x\,dx + \frac{1}{4(i-2)}.e^{2ix}\int e^{-2ix}\sin 2x\,dx$$

$$= \frac{1}{5}e^{-x}\left[\frac{1}{5}e^{x}\sin 2x - \frac{2}{5}e^{x}\cos 2x\right] - \frac{1}{8i(i+2)}e^{2ix}\int e^{2ix}(e^{2ix} - e^{-2ix})\,dx$$
$$+ \frac{1}{8i(i-2)}e^{2ix}\int e^{-2ix} - e^{-2ix})\,dx$$

$$= \frac{-x}{20}(\cos 2x + 2\sin 2x)$$

d) $y = c_1\cos nx + c_2\sin nx + \dfrac{x}{n}\sin(nx) + \dfrac{\ln\cos(nx)}{n^2}.(\cos nx)$

e) $\text{C.F.} = (c_1 + c_2x)e^{2x} + c_3e^{-3x}$

$$\text{P.I.} = \frac{1}{(D-2)^2(D+3)}X(x)$$

$$= \left(-\frac{1}{25}.\frac{1}{D-2} + \frac{1}{5}.\frac{1}{(D-2)^2} + \frac{1}{25}.\frac{1}{D+3}\right)X$$

$$= -\frac{1}{25}e^{2x}\int e^{-2x}X\,dx + \frac{1}{5}e^{2x}\iint e^{-2x}X(dx)^2 + \frac{1}{25}e^{-3x}\int e^{3x}X\,dx$$

Hence, $y = (c_1 + c_2x)e^{2x} + c_3e^{-3x} - \dfrac{1}{25}e^{2x}\int e^{-2x}X dx + \dfrac{1}{5}e^{2x}\iint e^{-2x}X(dx)^2$
$+ \dfrac{1}{25}e^{-3x}\int e^{3x}X\ dx.$

E16) a) $y = (c_1 + c_2x)e^{x} + c_3\cos x + c_4\sin x$

b) $y = (c_1 + c_2x)e^{x} + (c_3 + c_4x)e^{-2x}$

c) $y = 2e^{x}\sin 2x$

E17) a) $\text{C.F.} = (c_1 + c_2x + c_3x^2)e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^3}e^{-x}(2 - x^2)$$

$$= e^{-x}\frac{1}{[(D-1)+1]^3}(2 - x^3) = e^{-x}\frac{1}{D^3}(2 - x^2)$$

$$= e^{-x}\frac{1}{D^2}\left(2x - \frac{x^3}{3}\right) = e^{-x}\frac{1}{D}\left(x^2 - \frac{x^4}{12}\right) = e^{-x}\left(\frac{x^3}{3} - \frac{x^5}{60}\right)$$

Hence, $y = (c_1 + c_2x + c_3x^2)e^{-x} + e^{-x}\left(\dfrac{x^3}{3} - \dfrac{x^5}{60}\right).$

b) $\text{C.F.} = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x$

$$\text{P.I.} = \frac{x}{12}\left(\sin x - \frac{1}{2}\sin 2x\right)$$

c) $y = c_1 + c_2e^{-x} + c_3e^{x} + c_4\cos x + c_5\sin x + 2x^2 + 2x\sin x + 3xe^{x}$.

d) $y = c_1e^{-x} + c_2e^{x} - \dfrac{x}{2}e^{-x} - \dfrac{1}{2}\cos x - x^3 - 6x - \dfrac{1}{5}e^{x}(\cos x - 2\sin x)$

e) Let $x+3=e^t$, $\frac{d}{dt}=D$, then the given equation becomes

$$(D^2-5D+6)\,y=t$$

A.E. is $m^2-5m+6=0 \Rightarrow m=2,\ 3$

$$\text{C.F.}=c_1\,e^{2t}+c_3\,e^{3t}=c_1(x+3)^2+c_2(x+3)^3$$

$$\text{P.I.}=\frac{1}{6-5D+D^2}\,t=\frac{1}{6}(1-\frac{5}{6}D+\frac{1}{6}D^2)^{-1}\,t$$

$$=\frac{1}{6}(1+\frac{5}{6}D+\cdots)\,t$$

$$=\frac{1}{6}\left(t+\frac{5}{6}\right)=\frac{1}{6}t+\frac{5}{36}=\frac{1}{6}\ln(x+3)+\frac{5}{36}$$

Hence $y=c_1(x+3)^2+c_2(x+3)^3+\frac{5}{6}+\frac{1}{6}\ln(x+3)$

f) $\text{C.F.}=c_1\cos ax+c_2\sin a\,x$

$$\text{P.I.}=\frac{1}{D^2+a^2}\tan ax=\frac{1}{2ia}\left(\frac{1}{D-i\,a}-\frac{1}{D+i\,a}\right)\tan ax$$

$$=\frac{1}{2ia}.e^{iax}\int e^{-iax}\tan ax\,dx-\frac{1}{2ia}e^{-iax}\int e^{iax}\tan ax\,dx$$

$$=\frac{1}{2ia}.e^{iax}\left[\int(\sin ax-i\sec ax+i\cos ax)\,dx\right]$$

$$-\frac{1}{2ia}e^{-iax}\left[\int(\sin ax+i\sec ax-i\cos ax)\,dx\right]$$

$$=-\frac{1}{a^2}\sin ax\cos ax+\frac{1}{a^2}\sin ax\cos ax-\frac{1}{a^2}\cos ax\ln\left|\sec ax+\tan ax\right|$$

$$=-\frac{1}{a^2}\cos\ (ax).\ln\left|\sec(ax)+\tan(ax)\right|$$

Hence $y=c_1\cos ax+c_2\sin ax-\frac{1}{a^2}\cos(ax).\ln\left|\sec(ax)+\tan(ax)\right|$.

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