UNIT 8 INTEGER PROGRAMMING

Objectives

After reading this unit, you should be able to:

- Discuss the importance of integer programming models in supporting decisions.
- Explain the use of integer variables for providing modelling capabilities beyond those available in linear programming.
- Describe the rationale behind Cutting Plane and Branch and Bound methods used for solving integer programming models.

Structure

8.1 Introduction.

8.2 Some Integer Programming Formulation Techniques

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8.1 INTRODUCTION

In practice ,we quite often face situations where the variables of interest have to be integers. Consider for example, the product-mix problem, where a Company, operating within the existing departmental capacities, has to decide on the number of units of each product to be manufactured, m0 as to maximize the profit. Under certain assumptions, it is possible to formulate the situation as a Linear program and to solve it by using any linear programming (L.P.) technique. The optimal solution in such cases may result in fractional values of the decision variables. It is obvious that such fractional values do not make any sense in practice, and as such, one is . tempted to round-off these values to the nearest integers and use them for action. Rounding-off, however, may result in sub-optimal or infeasible solutions. It is possible to take care of such eventualities in the initial L.P. formulation by specifying that the decision variables can take only positive integer values. Such a linear program with decision variables restricted to integer values is called a linear integer program. While the integer restrictions on decision variables in a L.P. framework may be inherent in the problem situation, there are also situations, where these restrictions may be imposed or engineered by the analyst. Integer variables, specifically those which can take a value of zero or one, have been used in many problem situations to provide modelling capabilities beyond those available in L.P. Consider for example, the capital budgeting problem, where a company is to choose m among n projects. As each project has an associated return and uses a specific quantity of one or more resources, the problem is to choose the projects so as to maximize return within the given resource constraint. Analysts have used integer variables to formulate this problem. The decision variables are defined as x_i corresponding to each project $_j$ and x_j is allowed to take a value of one or zero depending on whether the project j is chosen or not. Once again, if the integer restrictions are relaxed and the problem is solved by L.P. technique, fractional values of x_js' may result, which will be meaningless in this case.

In this unit we will be interested in looking at some formulations and solution techniques for linear integer programming models; and in all our future references to such models we will use the term "integer programming (I.P.) models", Accordingly, in the next section, we present some I.P. formulations to help you appreciate the application of such models. Before discussing the methods of solving, I.P. models, it is necessary to understand the conditions under which L.P. techniques give integer solutions of I.P. problems. Unimodularity property of a matrix is useful in this context and is presented in the subsequent section In the last two sections, we present two methods for solving I.P. models.

Activity I

Give three problem situations where integer restrictions are inherent in the problems.

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Activity 2

Solve the L.P. given below and round-off the values of all the decision variables to the nearest integers. Check as to whether the integer solution thus obtained is a feasible solution to the problem. What inference would you draw from the results of this check?

$$\begin{aligned}\text{Maximize} \quad & x_1 + 2x_2 \\ \text{Subject to} \quad & -x_1 + 3x_2 \leq 1 \\ & 4x_1 - 3x_2 \leq 2 \\ & x_1, x_2 \geq 0\end{aligned}$$

8.2 SOME LP. FORMULATION TECHNIQUES

I.P. formulations of problem situations do not pose problem when the variables inherently are discrete in nature. Problems arise in situations where direct L.P. or I.P. formulations are not possible. Codification of variables are helpful in such situations. We first present a direct I.P. formulation of a problem situation. This is followed by two other situations that require codification and transformation.

Situation 1

You have landed in a treasure island full of three types of valuable stones, emerald(E), ruby(R) and topaz(T). Each piece of E, R, and T weighs 3,2,2 kg., and is known to have a value of 4,3,1 crore respectively. You have got a bag with you that can carry a maximum of 11 kgs. Your problem is to decide on how many pieces of each type to carry, within the capacity of your bag, so as to maximize the total value carried. The stones cannot be broken.

We start the formulation exercise by defining the decision variables. From the problem situation it is apparent that we want to determine the number of each type of stone to be carried. Hence we define variables corresponding to each type of stone, as follows:

X_1= Number of emeralds to be carried

X_2 = Number of rubies to be carried

X_3 = Number of topaz to be carried.

You may note that the pieces of a type of stone are uniform so far as their weights and values are concerned.

We now express the Objective function and the constraint in terms of the decision variables. The total value carried as a consequence of X_1 of E, X_2 of R and X_3 of T, under the assumption of linearity, is $4 X_1 + 3 X_2 + X3$, and as per the problem this has to be maximized.

Under similar assumption, we can express the total weight to be carried, as 3 X1 + 2 X_2 + 2 X_3, and the capacity of the bag being 11 kgs., the weight carried has to be less than 11,

Finally, we note the fact that stones cannot be broken, suggests that the variables have to take discrete values. Thus we have the following formulation:

Maximize $(4 X_1 + 3 X_2 + 1 X_3)$

Subject to $3 X_1 + 2 X_2 + 2 X_3 \leq 11$

X_1, X_2, X_3, are all non-negative integers

The above model with a single constraint is known as the Knapsack problem in O.R. literature, and is an example of a direct formulation as the decision variables here are integers by definition. Solution of the above I.P. is given in a later section; meanwhile, you may try your own method to solve the problem. Explore for example, by relaxing the integer restriction and solving the resulting L.P. Then try to generate different feasible solutions for the I.P.

Situation 2

A Company has to decide on the production plan of an item for the next three periods, so as to meet the demands in different periods at minimum cost. The relevant costs are the production and inventory carrying costs. In any period, only if production is undertaken then a cost of Rs. 5 is incurred. The variable cost of production is Rs. 5 per unit. The inventory carrying cost is Rs. 1 per unit per period, and is levied on the inventory at the end of any period. Assume that the demands for the upcoming periods are d_1, d_2, d_3, and the beginning inventory is known to be I,,.

Here, our interest is to find the number of units of the item to be manufactured in different periods. We also note, that as demands are known figures, the inventories get determined simultaneously with the production. Hence, we have the following decision variables:

X_1 = number of units to be produced in period 1

X_2 = number of units to be produced in period 2

X_3 = number of units to be produced in period 3

I_1 = number of units in inventory at the end of period 1

I_2 number of units in inventory at the end of period 2

I_3 = number of units in inventory at the end of period 3

In general, we can write X_t = number of units to be produced in period t and I_t = end inventory of period t, where t = 1,2,3.

The problem in formulation becomes evident once we try to write the production cost function in period t.

$$\text{Production cost in period t, } C_t \text{ (say)} = S + v X_t, \text{ if } X_t > 0$$
$$= 0, \text{ otherwise}$$

Clearly, the function as above cannot be incorporated in the objective function; hence we take recourse to codification. We define three binary variables Y_1, Y_2, Y_3 for each period, where Y, (t = 1, 2, 3) in general, will take a value of 1 if there is production in that period (i.e. X, > 0), and a value of 0 otherwise.

The objective function can now be expressed as :

$$\text{Minimize } S(Y_1 + Y_2 + Y_3) + v(X_1 + X_2 + X_3) + i(I_1 + I_2 + I_3) \quad \dots (1)$$

The constraints are essentially the material balance for each period, given by: opening stock + production − closing stock = demand, and can be expressed as follows:

$$I_0 + X_1 - I_1 = d_1 \quad \dots (2)$$
$$I_1 + X_2 - I_2 = d_2 \quad \dots (3)$$
$$I_2 + X_3 - I_3 = d_3 \quad \dots (4)$$

The other conditions are with respect to the variables Y_t and X_t :

X_t, for all t, are non-negative integers ... (5)

$Y_t = 1$, if $X_t > 0$

$= 0$, otherwise, for all t ... (6)

The above completes the formulation, however, the conditions in (6) is still not in an acceptable I.P. format. Standard transformation of (6) in the form of a linear constraint is given below:

$X_t \leq M\, Y_t$, where M is a large positive number and

$Y_t = 0$ or 1 ... (7)

Equation (1) to (5) and (7) give the required formulation.

Situation

Consider a tailor facing the problem of sequencing four stitching jobs given by different people. All jobs have arrived simultaneously; and he has only one stitching machine. From his experience, it is possible for him to make a fairly accurate estimate of the stitching time of the jobs. Each customer has specified a date by which the respective job is to be delivered. The tailor wants to determine a sequence so as to minimize the average tardiness of all the jobs, where, tardiness of any job is given by Maximum (0, completion time-delivery time). It is also agreed that once a job is started on a machine, it cannot he taken out before completion.

As the problem of sequencing is solved by specifying the starting time for each job, we define the decision variables accordingly. Assuming a datum of zero time for starting the work, and an arbitrary numbering of the jobs as 1 to 4, let X_I denote the starting time for job I from the datum (I = 1 to 4). It the processing time for job I (known) is denoted by P_I, then the completion time for job I = $X_1 + P_I$. If D_1 denote the given delivery date for job I from the same datum, then by definition, tardiness for Job I = Max $[0, X_i + P_I - D_I]$. Hence, we can write down the objective functions as:

Minimize Max $[O, X_I + P_I - D_I]$ ... (1)

To convert the above into linear form, we define $Y_I = \text{Max}\,[O, X_I + P_I - D_I]$

Then (1) is equivalent to :

Minimize Y_I

Subject to $Y_I \geq 0$ ... (2)

$Y_I \geq X_I + P_I - D_I$

The constraints of the problem are implicit, and is given by the fact that two jobs cannot be undertaken simultaneously. This would imply that for any pair of job I and J, either I precedes J, or J precedes I. To express this in terms of the decision variables, we note that if,l precedes J, then completion time of I has to be less than the starting time for J, i.e.,

$X_I + P_I \leq X_J$ ... (3)

Similary, if J precedes I, we can write:

$X_J + P_J \leq X_I$ ... (4)

Either (3) or (4) should be active, and in our problem as there are 4 jobs, $4C_2$ such pairs of constraints will be there. Once again, this either or condition being not linear, we take recourse to integer variables S_{IJ}, where

$S_{IJ} = 1$, if job i precedes j

$= 0$, if job j precedes i

Then Either (3) or (4)" can be replaced by :

$M\, S_{IJ} + (X_I - X_J) \geq P_J$ ... (5)

$M\,(1 - S_{IJ}) + (X_J - X_I) \geq P_I$ ... (6)

where M is a very large number. Equations (2), (5), (6) together with the restriction on variables X_I and S_{IJ}, give the required formulation.

Example I

Consider a Company which would like to take up three projects. However, because of budget limitations, not all the projects can be selected. It is known that project j has a present value of C_3,;and would require an investment of a_{it} in period t. The capital available in period t is h_t. The problem is to choose projects so as to maximize present value, without violating the budget constraints. Formulate the problem as an I.P.

Solution

For choice situations of "yes-no", "go-no go" type, where the objective is to determine whether or not a particular activity is to be undertaken, integer binary variables that can take a value of 0 or 1, can be used to represent the decision variables. Here, we find that for each project, we want to find out whether it should be taken up or not, as such we define three decision variables X_j (j =1, 2, 3) corresponding to each project, and

X_j = 1, if project j is selected

= 0, otherwise

Then, the objective function and the constraints can be expressed in terms of the decision variables, to give the required formulation :

Maximize $\Sigma C_J X_J$

Subject to $\Sigma a_{jt} X_J \leq b_t$, for all period t.

$X_J = 1$ or 0, for all projects.

Activity 3

Consider the example given above, suppose the projects are arbitrarily numbered as 1, 2 and 3. Also, over and above the budget constraints for different periods, it has been specified that between projects 1 and 2 only one can be selected. Show, how you will incorporate this constraints in the above formulation.

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Example 2

Consider equation (7) as given in "situation 2" : $X_t \leq M Y_t$, where M is a large positive number and $Y_t = 0$ or 1. Show how it is equivalent to the condition stated in equation (6) of the situation. Recall that equation (6) states that

$Y_t = 1$, if $X_t > 0$

= 0, otherwise.

Solution

Form equation (7), we can write, $Y_t \geq X_t/M$......(1)

Therefore, if $X_t > 0$, they $Y_t \geq$ a positive quantity, greater than zero. As equation (7) also states that $Y_t = 0$ or 1, it follows that Y_t has to be 1, if X_t is greater than zero.

On the other hand, if $X_t = 0$, then from (1), we find $Y_t \geq 0$, i.e. 0 or 1, instead of 0 as stated in equation (6). Apparently equivalence seems to be violated. This is resolved if we inspect the objective function of the problem situation. In the objective function, Y_t are multiplied by positive coefficients S, and as the problem is of minimisation type, the lower of the two values of Y_t, namely zero, must be chosen for optimization.

Activity 4

Consider equations (5) and (6) as given in "situation 3":

$$M S_{IJ} + (X_I - X_J) \geq P_J \quad \ldots (5)$$

$$M (1 - S_{IJ}) + (X_J - X_I) \geq P_I \quad \ldots (6)$$

a) Show how they are equivalent to "either (3) or (4)", as given in the situation.

b) Present a problem situation where "either-or" type constraint may arise within the L.P. framework. Show how you will convert it to I.P.

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8.3 UNIMODULARITY

We have noted in the introduction that in general, an I.P. model cannot be solved as an L.P., because if fractional values are obtained in the solution, rounding-off may not be possible or desirable. It is worthwhile examining in this context, conditions, it any, under which I.P. models can be solved as L.Ps, to yield integer solutions. Existence of such conditions in a problem situation or formulation would make our task simpler as we do not have to apply any new methods for solution. The objective of this section is to examine such conditions.

For the purpose, let us consider the following I.P. problem :

P1 : Maximize $c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

Subject to $a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$

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$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m$

$x_1, x_2, \ldots\ldots, x_n$ are all positive integers.

Without loss of generality, we can assume that all the technological coefficient a_{11}, a_{12}, ... a_{mn}, and all the right hand side constraints b_1, b_2 ….. b_m are integers. (Even if they are not integers, note that it is always possible to convert them to integers by suitable multiplication on both the sides of the constraints.)

If the integer restrictions on the variables are relaxed, and we specify that the variables can take any value greater than or equal to zero, then the above becomes a L.P. problem. From L.P. theory we know that, given m equations and n unknowns (m less than n), we can set (n-m)'variables to zero and solve for the rest of the m unknowns, provided a solution exists. The (n-m) variables that are set to zero are called nonbasic variables, and the rest m variables are called basic variables. For example, without loss of generality, the first m variables x_i, x_2 ,..., x_m can be the basic variables, and the remaining x_{m+i} + …... X_n, the nonbasic variables. If we now set x_{m+i}…. x_n to zero, the constraint set of the above problem can be expressed as follows :

P2

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1m}x_m = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2m}x_m = b_2$

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$a_{m1} + x_1 + a_{m2}\, x_2 + \ldots\ldots + a_{mm}x_m = b_m$

It may be noted here that m equations in m variables now constitute the constraint set of P2. The number of different ways in which the m basic variables can be chosen from among n variables is known to be ${}^nC_m$. This implies that a total of ${}^nC_m$ combinations of constraint sets as in P2 can arise. However, a solution may not exist for all such sets. When a solution exists it is referred to as basic solution. The clads of basic solutions with all non-negative values of the m variables constitute a basic feasible solution (bfs). Recall that L.P. technique essentially gives us a systematic approach of exploring the different basic feasible solutions to arrive at the optimum. It does not test for or guarantee integer solution. To understand when the bfs will be integer, we first note that the nonbasic variables being all set to zero, are integers and as such do not concern us. Thus, we are interested in finding out the condition under which all the basic variables are integers. For our purpose we first state the conditions in the context of the problem Pl. The proof of the condition requires familiarity with matrix algebra and is presented in the appendix of this section for those who are interested.

Condition: If the matrix of the technological coefficients denoted by A with elements a_te, is totally unimodular, then every basic solution of P1 is integer. We note that in P1, the matrix A is given by

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}$$

We also note that as all the coefficients a_{12}, ... a_{mn} are integers, A is integer. We now specify the rule for detecting unimodularity of A.

Rule : An integer matrix A with all elements = 0, 1, - 1 is totally unimodular if :

a) Each column does not have more than two nonzeros.

b) The rows can be classified into two subsets such that:

   i) if a column contains two nonzero elements with the same sign, one element is in each of the subsets.

   ii) if a column contains two nonzero elements of different sign, both elements are in the same subset.

It is apparent from the above condition that if we detect unimodularity in the A matrix of an I.P., then we can use any L.P. technique to arrive at a solution. This is because of the fact that all basic solutions being integers, all basic feasible solutions if they exist will also be integers and hence the optimal solution, if it exists will have to be integer.

Appendix

The problems P1 and P2 can be expressed in matrix notation as:

P1	**Maximize CX**	**P2 Maximize $C_m X_m$**
	Subject to AX = b	**Subject to $B X_m = b$**

where, $C_m = (c_1, c_2 \ldots. c_m)$, represents the row vector of objective function coefficients corresponding to the basic variables.

C represents the row vector of objective function coefficients of all the variables.

X_m is the column vector representing the basic variables $x_1 \ldots x_m$.

X is the column vector of all the variables.

b is the integer column vector representing the right hand side coefficients $b_1 \ldots b_m$.

A is the integer technological coefficient matrix associated with all the variables.

B is the integer technological coefficient matrix associated with the basic variables.

From $X_m = B^{-1} b$ we can see that as b is an integer matrix, a sufficient condition for Xm to be an integer vector is that B^{-1} has to be integer. The condition that B^{-1} is integer is not a necessary condition because it is possible to have X_m integer even if B^{-1} is not integer. Now, B^{-1} is given by: $B^{-1}=B^*/(\det B)$, where, B^*, the adjoint matrix, is known to be integer as B is integer, and (det B) denoting the determinant of B is not equal to zero. Hence, from the expression of B^{-1} it is clear that if determinant of B is equal to 1 then B^{-1} is integer, which in turn gives us the condition for a bfs to be integer.

The definition of Unimodularity is relevant in this context and is presented below:

A square integer matrix B is said to be =modular if determinant of B is equal to 1. An (m x n) integer matrix (as the matrix A in P1) is said to be totally unimodular if every square, nonsingular submatrix of A is unimodular.

Recall that we have obtained B by arbitrarily choosing the first m of the n variables as basic variables. Thus B is one of the several combinations of the n columns of the matrix A. Hence if A is totally unimodular, then all such nonsingular B matrices are unimodular, the solutions corresponding to these B matrices give the set of basic solutions, which are then integers. If basic feasible solutions exist, then these solutions being a subset of the basic solutions, have to be also integers. As the optimal solution has to be one of the basic feasible solutions only, it has to be integer also. Thus we have the following condition under which an I.P. model solved by a L.P. technique gives an integer solution :

If the integer technological coefficient matrix A of an integer program is totally unimodular, then the optimal solution (if it exists) of the corresponding linear program is integer.

Example 3

Consider the problem of assigning three jobs to three men. Each man is capable of doing all the jobs, however, the time taken by the different men on each job is different and can be assumed to be known. The assignment has to be done so that each job is assigned only once, each men gets only one job and the total time taken by all the jobs is minimized.

Formulate the problem as an I.P. with the decision variables defined as:

X_{ij} = 1, if the ith man is assigned to job j

= 0, otherwise.

Comment on the solution procedure that can be adopted for the problem.

Solution

Let T_{ij} denote the time taken by the i^{th} man to perform the j^{th}job We can then formulate the problem as follows :

Minimize $\sum_i \sum_j T_{it} X_{ij}$

Subject to $\sum_j X_{ij} = 1$, for i = 1,2,3 (each man gets one job) ... (1)

$\sum_i X_{ij} = 1$, for j = 1,2,3 (each job gets allotted) ... (2)

$X_{ij} = 0$ or 1, for all i and j ... (3)

Tijs form the date of the problem and is given in the matrix form. As both i and j can take three values each, the number of Xijs (unknowns) are nine in number . The number of constraints are six, three each corresponding to equations (2) and (3). Thus, the A matrix of the problem will have 6 rows and 9 columns as follows.

1	1	1	0	0	0	0	0	0
0	0	0	1	1	1	0	0	0
0	0	0	0	0	0	1	1	1
1	0	0	1	0	0	1	0	0
.0	1	0	0	1	0	0	1	0
0•	0	1	0	0	1	0	0	1

From the rule given for detection of unimodularity, it is apparent that each column of A has two nonzeros, also rows I to 3 can be one set and 4 to 6 another set , as given the rule, hence A is totally unimodular.

The problem thus can be solved by any L.P. procedure to yield integer solution. This problem is known as the Assignment problem in the O.R. I iterate -e. '

Activity

Consider the standard transportation pr6blem. Check whether the matrix A is unimodular. What inference would you draw from the above check.

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8.4 CUTTING PLANE METHOD

Historically, the first method for solving I.P. problems was the cutting plane method. The basic steps involved in the method are as follows

i) Solve a L.P.; to start with take the I.P. problem with integer restriction relaxed.

ii) Check whether the optimal solution is integer. If yes, stop. The resulting solution is the required one. If no, go to step iii.

iii) Generate a new constraint from the fractional optimal solution obtained in i. The constraint is created so as to rule out the fractional values obtained, without ruling out any integer solutions to the problem. Add the constraint to the L.P. in i; and go back to step i.

Thus, the method involves solving a L. P. at every iteration. The approach is to reduce or cut the solution space in every successive iteration, ruling out the current fractional solution, while ensuring that no integer solutions are excluded in the process. The process is continued till an integer solution is found or infeasibility occurs. As the constraints are essentially used to cut the solution space, these are referred to as cuts, and the method is referred to as the cutting plane method. The original method as stated above has been found to. be inefficient computationally, however, it has helped in developing valuable insights to I.P. solution procedure, and subsequently has helped in generating other methods. For our purpose, we present here the original method, useful for solving an I.P. with all integer variables. Generation of a cut, the rationale and the mechanics, is central to the method and is presented below.

Activity 6

Read carefully the steps given above. The steps do not mention anything regarding infeasibility. Modify the steps to account for infeasibility. (Hint :. If at any iteration the L.P. is infeasible, what can you say about the I.P. solution?)

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We now consider the following problem :

We now consider the following problem :

Maximize $6X_1 + 2X_2$ ... (1)

Subject to $3X_1 + 2X_2 = 5$ ... (2)

X_1, X_2 are integers ≥ 0 ... (3)

It is easy to verify that L.P. optimal for the above problem is $X_1 = 5/3$, $X_2 = 0$. The solution being fractional, is not acceptable. Suppose now, we include the constraint $X_1 \leq 1$ ... (4), in the original L.P., and solve the new L.P. with (1), (2) and (4). Then the new L.P. optimal with $X_1 = 1$, $X_2 = 1$ is integer and hence the required solution. In this case, it is possible to infer by inspection that (2) and (3) together imply that X_1 has to be less than or equal to 1. Inclusion of this constraint (4) ruled out the current solution of $X_1 = 5/3$, $X_2 = 0$, without ruling out any integer solution.

Hence, (4) is an example of a, cut as described earlier. The example, however does not give us a systematic way of generating a cut. We take this up in the following paragraphs.

Consider the general I.P. problem,

Maximize $\sum_J C_J X_J$

Subject to $\sum_J A_{IJ} X_J = b_L \ \forall I.$

X_J nonnegative integers.

Assume that as given in the first step in cutting plane method, we solve the above as a L.P., and some of the variables in the optimal solution are not integers. Obviously, such variables have to be among the basic variables. We now need to generate a cut. We take any one of the non integer basic variables, say the I^{th} one. Let the equation .corresponding to this variable in the final simplex tableau be represented as:

$$\sum_J A_{IJ} X_J = b_I \qquad \ldots (4)$$

Where, A_{IJ} 's are the technological coefficients in the final tableau.

b_I is the optimal value of the ^{Ith} basic variable, and is assumed to be non integer.

It may be noted that though the summation in (4) has been written over all variables, actually in the optimal tableau, it will be for all the nonbasic variables, and only one basic variable X, will have a coefficient of 1.

Let, $A_{IJ} = [A_{IJ}] + f_{IJ}$, where $[A_{IJ}]$ is the integer part and f_{IJ} the fractional part of A_{IJ}, $0 < f_{IJ} < 1$.

Let, $b_I = [b_I] + f_I$, where $[b_I]$ is the integer part and f_I the fractional part of b_I, $0 < f_I < 1$.

(In general, the symbol [a] is assumed to denote the largest integer that is less than a (1 for example, [7−6] = 7, [−7.6] = −8, [7.4] = 7 etc.)

Equation (4) can now be expressed as :

$$\Sigma [A_{IJ}] X_J + \Sigma f_{IJ} X_J = [b_I] + f_I \qquad \ldots (5)$$

As X_J's are to be integers, we can write from above $\Sigma [A_{IJ}] X_J \leq [b_I]$.

or,

$$\Sigma [A_{IJ}] X_J + S_I = [b_I] \qquad \ldots (6)$$

where S_I is a nonnegative integer variable.

From (4) and (5), we can write : $-\sum_J f_{IJ} X_J + S_I = -f_I \qquad \ldots (7)$

Equation (7) has been derived with the assumption that Xj ,s are integers, as such, it gives us the condition to be imposed, if we are to get integer solution. The linear constraint so derived is what we have called earlier a cut. At every iteration such a cut is to be generated from the optimal tableau, by taking any one equation corresponding to a basic variable that is not an integer. We further note that in equation (7), the summation is over all nonbasic variables, hence if this is added to the original L.P., $S_I = -f_I$, renders the current optimal solution infeasible. Under such situations the Dual Simplex method can be applied.

Example 4

Use the cutting plane method for solving the I.P. problem as given by equations (1) to (3).

Solution

Step 1: The optimal tableau obtained by solving the relaxed problem can be verified to be :

	X_1	X_2	RHS
X_1	1	21	5/3
Obj.fn. Z	0	2	10

Step 2 : As the basic variable X_1 is noninteger, we take the corresponding equation from the above table :

$X_1 + 2/3\ X_2 = 5/3$, which is the same as that depicted in equation(4) above.

We note that in the above equation X_1 does not have any fractional part, and X_2 does not have any integer part. The integer part of the RHS is 1. Hence, proceeding in the same way as shown in the text above, we can generate an equation similar to (6) of the text :

$$1\ X_1 + 0\ X_2 + S_1 = 1$$

Subtracting the earlier equation from the above, we get the required cut.

$$-2/3\ X_2 + S_1 = -2/3$$

The above can now be incorporated in the optimal tableau as shown below

	X_1	X_2	S_1	RHS
X_1	1	2/3	0	5/3
S_1	0	-2/3	1	-2/3
Z	0	2	0	10

Note from the above tableau that the second equation corresponds to the cut, and the current solution is optimal but infeasible. Dual Simplex can now be applied; the steps are as shown below.

i) Determine the leaving variable: The basic variable with the most negative value is chosen as the leaving variable. In our example, the basic variables are X_1 = 5/3, and S_1 = -2/3. Hence, S_1 is the leaving variable.

ii) Determine the entering variable: The ratios of the L.H.S. coefficients of the Z equation to the corresponding coefficients of the leaving variable equation is taken, and neglecting the ratios associated with positive or zero denominators, the nonbasic variable corresponding to the smallest ratio is chosen as the entering variable, if the problem is of minimization type; for a maximization problem, the variable corresponding to the smallest absolute value is chosen. For the given problem, the respective equations and the ratios are `shown below:

	X_1	X_2	S_1
Z Equation	0	2	0
S_1 Equation	0	-2/3	1
Ratios	-	-3	-

Thus X_2 is the entering variable. The optimal' solution is shown in the tableau below

	X_1	X_2	S_1	RHS
X_1	1	0	1	1
X_2	0	1	-3/2	1
Z	0	0	3	8

At the end of the second iteration, we find that the optimal solution is X, = 1, and X_2 = 1. The solution being integer, this is the required answer.

Activity 7

Assume that the following equation is taken from the optimal tableau of a L.P.:

$$X_5 + 3/4\ X_6 + 1/4\ X_7 = 3.5$$

Which basic variable in the final tableau does the equation correspond to? Generate a cut from the above.

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8.5 BRANCH AND BOUND

Branch and Bound is a generic method for solving a class of optimization problems. The optimal solution is selected by successive partitioning of the set of all possible solutions into two mutually exclusive and exhaustive subsets, and establishing limit for each set such that all solutions in the set are worse than the limit. The first part of the procedure involving partitioning is called branching, while the second part of establishing limit is referred to as bounding. In this section we will be restricting ourselves to the application of the procedure in solving I.P. problems. The method is illustrated with the help of an example.

Consider the example as given in section 8.2, situation 1 :

Maximize $(4 X_1 + 3 X_2 + 1 X_3)$

Subject to $3 X_1 + 2 X_2 + 2 X_3 \leq 11$

X_1, X_2, X_3 are all non-negative integers.

Suppose we relax the integer restriction and solve the resulting L.P. By now, you must have discovered that the solution is trivial. You have to only find the maximum of the ratios of objective function coefficients to the constraint coefficients, for each of the variables; and you should fill up your bag with the variable associated with the maximum ratio.

Thus, Max [4/3, 3/2, 1/2] = 3/2, and, as the maximum occurs for X_{2i} the bag should be filled up with this only, i.e., $X_2 = 11/2$, $X_1 = X_3 = 0$ is the L.P. optimal, with the value of the objective function, Z = 33/2. It is easy to verify here, that objective function value of any integer solution to the problem have to be less than 33/2. The observation is summarized below

If an I.P. problem of maximization type is solved as an L.P., the optimal objective function value so obtained will have to be an upper bound of any optimal integer solution.

Activity 8

With reference to the above example, give the rationale behind the following statements

a) The optimal solution to the L.P. is to fill up the bag with X2 (Ruby) only. (Note that L.P. assumes that the stones can be broken).

b) Any integer optimal solution cannot have Z > 33/2.

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The above illustrates the "bound" aspect of Branch and Bound. In general, it may be noted, the integer restriction can only lead to a worsening of the objective function value. As such, for a maximization problem, the L.P. optimal is an upper bound, while for a minimization problem it gives the lower bound.

We next inspect the optimal solution of the L.P. obtained above. X2 is the only non integer variable in the solution with a value of 5.5. (In case more than one variables are non integer, we choose any one among these.) For an integer solution, a value of greater than 5: and less than 6 being ruled out, we can specify that either

X_2 is less than equal to -5, or it is greater than equal to 6. These can be used to partition the solution space of the original L. P. into two distinct problems as follows

P1 **Maximize** $(4X_1 + 3X_2 + 1X_3)$

Subject to $3X_1 + 2X_2 + 2X_3 \leq 11$

$X_2 \leq 5$

X_1, X_2, X_3 **are all non-negative**

P1 **Maximize** $(4X_1 + 3X_2 + 1X_3)$

Subject to $3X_1 + 2X_2 + 2X_3 \leq 11$

$X_2 \geq 6$

X_1, X_2, X_3 **are all non-negative**

It may be noted that by the above partitioning, we have not ruled out any integer solution to the original problem.

Suppose we now solve both the problems. The various possible outcomes are listed below:

i) Both the problems have feasible solution. Let the objective function values obtained for Pl and P2 be Zl and Z2 respectively. (Note that both these values have to be less than equal to Z, as the objective function value can never increase because of an additional constraint). The solutions in turn may indicate any of the three outcomes :

   a) Both PI and P2 have integer solution.

   b) Only one of the problems yields an integer solution.

   c) None of the problems yield integer solution.

   It (a) occurs, then we can find max (Z1, Z2), and the optimal solution is the one associated with the problem for which the maximum occurs. This is true, as the two partitions between them covers the total solution space.

   If (b) occurs, the integer solution gives us a feasible solution to the I. P. problem. Without loss of generally, let us assume that P1 gives us an integer solution. Thus, no further branching from Pl arises. To determine branching from P2, we first note that, the maximum objective function value, that any problem arising from P2 can have is Z2. Thus, if Zl. is already greater than or equal to Z2, we do not need to explore P2 further, as there cannot be any gain in doing so. In fact, the objective function value of any feasible integer solution obtained at any stage of solving, is a lower bound of the required optimal (an upper bound for a minimization problem), and can be used to discard inferior branches. The solution corresponding to P1 is the required optimal solution. However, if Z1 < Z2, then we have to branch from P2, as we cannon rule out the possibility of reaching a feasible integer solution with a objective function value > Zl Any one of the non integer variable may be chosen, and branching done based on it, as shown earlier.

   If (c) occurs, we need to branch from both. The method is exactly same as shown earlier. The new upper bounds for the branches from Pl and P2 are Zl and Z2 respectively.

ii) Both the problems are infeasible. As the whole solution space has been covered, we can say that the I.P. does not have any feasible solution.

iii) Only one of the problems is feasible. If the solution is integer, then it is the required optimal solution. If the solution is not integer, then we branch from that point, with the new upper bound as the objective function value of the feasible problem (P1 to P2).

In our case, P1 and P2 solutions are given below

P1 Objective function 16.33

X_2 5.00

X_1 0.33

P2 Infeasible

Thus, we have encountered outcome iii above. As the solution is not integer, we choose X_1 for branching. With $X_1 \leq 0$, and $X_1 \geq 1$, we create two branches, and call them problems P3 and P4 as shown below :

P3 Maximize $(4\,X_1 + 3\,X_2 + 1\,X_3)$

Subject to $3\,X_1 + 2\,X_2 + 2\,X_3 \leq 11$

$X_2 \leq 5$

$X_1 \leq 0$

X_1, X_2, X_3 are all non-negative

P3 Maximize $(4\,X_1 + 3\,X_2 + 1\,X_3)$

Subject to $3\,X_1 + 2\,X_2 + 2\,X_3 \leq 11$

$X_2 \leq 5$

$X_1 \geq 1$

X_1, X_2, X_3 are all non-negative

Note that the new problems (P3 & P4) that are created by branching are enlarged versions of the problem from which the branching is done (P2). The enlargement is in terms of an additional constraint for each problem ($X_1 \leq 0$ for P3, and $X_1 \geq 1$ for P4).

Once again, we solve P_3 and P_4. The solution is given below.

P3 Objective function 15.50

X_2 5.00

X_3 0.50

P4 Objective function 16.00

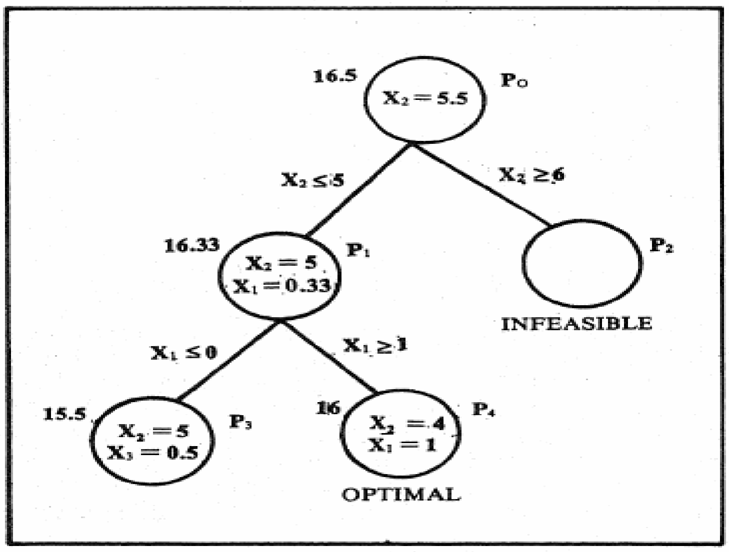
X_2 4.00

X_1 1.00

Both the problems are feasible, with P4 giving an integer solution. This is equivalent to the outcome i (b) as listed earlier. As the objective function value of P4 is greater than P3, we do not stand to gain by branching any further from P3. This is because of the fact (as argued earlier) that any problem arising from P3 can have an objective function value less than equal to 15.5.

Thus P4 gives us the required optimal solution with 1 emerald and 4 rubies to be carried, giving a total value of 16 crore.

It is useful to smmarize the above procedure in terms of a -diagram. The problems are represented by nodes, with the solutions written inside, and the objective function values denoting the bounds, written beside the nodes. Denoting the first L.P. (the relaxed version of the given I.P.) by PO, we have the following diagram:



If after branching at any node, we find that both the problems are feasible and non integer, it may be useful to take any one node and branch. This way we go exploring depth-wise to identify a feasible integer solution, if it exists. Once a feasible integer solution with objective function value of Z has been found, the unexplored* nodes that are having their objective function values less than or equal to Z need not be explored further. From the rest of the nodes, any one can be chosen and the same strategy may be followed. Thus all nodes are not required to be solved. In our case - we did not have to explore problem P3.

Activity 9

Solve the following problem by using Branch and Bound :

Maximize $5X + 9Y$

Subject to $-X + 5Y \leq 3$

$5X + 3Y \leq 27$

X, Y are non-negative integers.

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8.6 SUMMARY

A linear program with decision variables restricted to integer values 2 called a linear integer program. For our purpose, we have referred to this as "integer program". Such integer restrictions on decision variables may either be inherent in the problem situation, or these may be imposed by the analyst. In the former type, the decision variables, by their very nature, have to take discrete values, The latter type, on the other hand, results from the analyst's ingenuity in modelling problem situations. 0-1 integer variables used for modelling "go-no go" decisions is but one example of this type. In either of the Aftes, relaxing the integer restrictions and hence solving a L.P. may not yield an integer optimal solution. Rounding-off to the nearest integer may not be possible, or may be even meaningless. This justifies the study of integer programming, for developing an understanding of the modelling capabilities available in 0-1 variables, and learning the methods for solving.

Some I.P. formulations has been presented first, to help you develop an insight to the use of 0-1 variables. The unimodularity property of a matrix has been introduced subsequently, to help you detect the I.P. formulations that can be solved by using L.P. techniques only. Finally, the cutting plane and the branch and bound Methods for solving an I.P. have been presented.

8.7 SELF-ASSESSMENT EXERCISES

1) The ABC Electric Appliance Company produces two products: refrigerators and ranges — Production takes place in two separate departments. Refrigerators are produced in Department I and ranges are produced in Department II. The company's two products are produced and sold on a weekly basis. The weekly production cannot exceed 25 refrigerators in Department I and 35 ranges in Department II, because of the limited available facilities in these two departments. The company regularly employs a total of 50 workers in the two departments. A refrigerator requires 2 man-weeks of labour, while a range requires 1 man-week of labour. A refrigerator contributes a profit of Rs. 300 and a range contributes a profit of Rs. 200. How many units of refrigerators and ranges should the company produce to realise a maximum profit?

2) What is an integer linear programming problem? How does the optimal solution of an integer programming problem compared with that of the linear programming problem?

3) Maximise $Z = 7x_1 + 9x_2$

Subject to $-x_1 + 3x_2 \leq 6$

$7x_1 + x_2 \leq 35$

and $x_1, x_2 \geq 0$ and are integers.

4) There are four projects consideration. Assume that the project run into three years. Total available funds are Rs. 75,000 (to be used at the rate of Rs. 25,000 each year). The expected profit and cost break-up is as follows:

Project	Expected Profit	Cost Year 1	Cost Year 2	Cost Year 3
1	90,000	8,000	10,000	12,000
2	60,000	2,000	5,000	8,000
3	1,80,000	15,000	10,000	5,000
4	1,00,000	10,000	5,000	5,000

Formulate the capital budgeting problem as a zero-one integer linear programming problem for the given data.

5) A company produces two products A and B. Each unit of product A requires 1 hour of engineering service and 5 hours of machine time. To produce 1 unit B requires 2 hours of engineering and 8 hours of machine time. There are 100 hours of engineering and 400 hours of machine time available. The cost of production is a non-linear function of the quantity produced as given in the following table:

Product A		Product B	
Production (units)	Unit Cost (Rs.)	Production (units)	Unit Cost (Rs.)
0 – 50	10	0 – 40	7
50 – 100	8	40 – 100	3

The unit selling price or product A is Rs. 12 and the unit selling price of product B is Rs. 14. The company would like a production plan which gives the number of units of A and the number of units of B to be produced that will maximize profit. Formulate an integer programming model for this problem.

8.8 FURTHER READINGS

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