UNIT 9 DYNAMIC PROGRAMMING

Objectives

After reading this unit, you should be able to :

- Explain the relevance of dynamic programming in decision making.
- Discuss the rationale behind dynamic programming methodology.
- Formulate and solve some standard problems using dynamic programming.

Structure

9.1 INTRODUCTION

In practice, many problems involve taking decisions over several stages in a sequence. Consider, for example, a production planning situation, where a Company has to .decide on the production plan of an item for the next three months, so as to meet the demands in different months at minimum cost. Assume that it does not have any stock to start with. The different months for which the production is to be decided, constitute the stages. Taking the best decision, month by month, may not be the optimal policy in such cases. Say, the relevant costs are the fixed cost of production in any period, which is incurred only if production is undertaken in that period and the inventory carrying cost per unit per period on any inventory left at the end of a period. Then, if the fixed production cost is very high compared to the inventory carrying cost, it is always better to produce all the demands in the first period itself. A period by period decision, in such cases, can never suggest carrying inventory to be optimal, and hence can never lead to the optimal plan. Linear Programming formulations of such problems are possible, provided the objective function and the constraints can be expressed as linear functions. In the given situation, the objective function is not linear; and in Unit 8, we have shown how it can be formulated as an integer program. Dynamic Programming (D.P.) provides us with an alternative methodology for solving such multistage problems, involving decisions over several stages in a sequence.

By the very nature of any planning problem, decisions are called for in every period, so that the "multistage" aspect is essentially inherent in such problems. Associated with each stage or period are input data, decision variables(s), output parameter(s), and objective function. The stages can be thought of to be connected in series, with the output of a stage forming one of the inputs to the immediately succeeding stage. Finally, for determining the effectiveness of the decisions taken, the objective function of the problem is arrived at from the individual objective functions of the stages. In the example above, for each month, the input data are the beginning inventory and the demand, the decision variable is the production quantity, the output is the end inventory, and the objective function is the cost expressed in terms of the decision variable. The stages are essentially linked in series, as the end inventory of any period is the beginning inventory of the immediately succeeding period. Once such characterization is achieved for a problem situation, it is possible to apply D.P. for formulation and solution.

Other situations also exist, where different decisions are to be taken, not necessarily in a sequence. Imposing an arbitrary sequence in such a situation, does not chang6 the original problem, and helps us in applying D.P. methodology. Consider, for example, a problem of allocating some resource among three competing activities Obviously, there being no time dimension, the allocation is to be done simultaneously. If we arbitrarily now, number the activities as 1,2 and 3, and assume that we will be allocating in that sequence, it does not make any difference so far as the original problem is concerned. In each stage now, we can conceive of allocating to each activity. The input data to each stage would be the resource requirements per unit of each activity and the amount of each resource available for allocation at the start of the allocation process. The decision variable is obviously the amount that is allocated at that stage, and the output will be the amount left for allocation to any stage or activity, is nothing but the amount available for allocation beginning of the next stage. These provide the linkages between two stages. The objective functions for each stage could be the return one would get by the allocation in the particular stage or activity. Thus, a variety or situations may be amenable to D.P. methodology by being characterized is multistage problems.

The objective of this unit is to present the D.P. methodology. First, an example is presented to demonstrate the methodology. The definitions of stage, state, and the principle of optimality, are then presented, and the notations specific to D.P. introduced. Finally, some more applications are presented.

Activity 1

Consider a L.P. with n variables and m constraints. Is it possible to characterize it as a multistage problem? Justify your answer.

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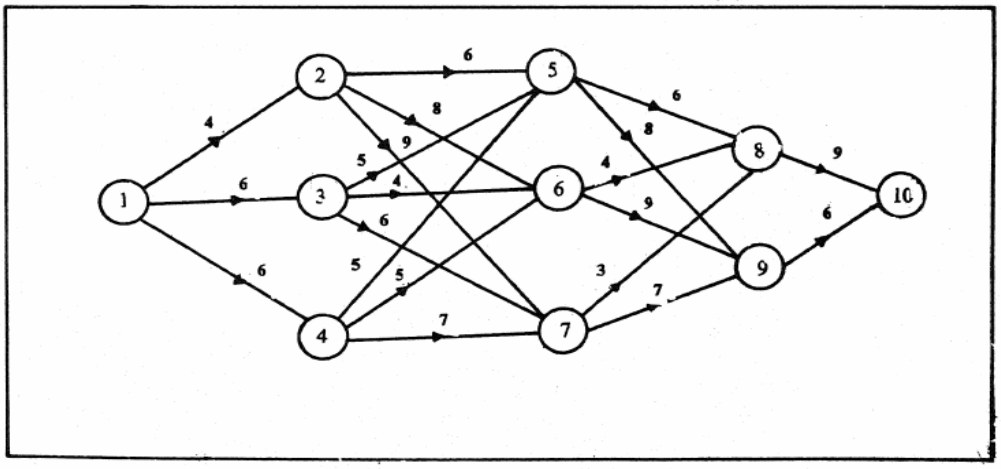
9.2 D.P.METHODOLOGY -- AN EXAMPLE

Consider the following diagram where circles denote cities, and lines between two such circles represent highways connecting the cities. The numbers inside the circles represent city numbers, and those given beside the lines denote the distances between the cities connected by the lines. Suppose you start from city 1 and would like to go to 10. You would like to know the shortest route from city 1 to 10.



The above problem can definitely be solved by enumerating all possible routes from city 1 to 10, and calculating the total distance for each route. The route that gives the minimum distance would be the required answer. However, this involves too much of computation, hence it may be useful to look for other methods. We attempt this in the following paragraphs.

We first note from the diagram that in order to travel from 1 to 10, one has to necessarily cover four highways, i.e., any route has to contain four lines. Thus, the problem may be thought of as one of deciding on these four lines or highways that should constitute the shortest route. The first line has to be chosen from among 1-2, 1-3, 1-4, as 1 is the starting city. Similarly, it is apparent from the diagram that the second highway has to emanate from 2,3 or 4, the third has to be from 5, 6 or 7, and the fourth from 8 or 9. If we now define a stage as 4 point where a decision is called for, thee, city 1 can be conceived to be in stage 1, cities, 2, 3, 4 in stage, 2, cities 5, 6. 7 in stage 3, dies 8, 9 in stage 4, with city 10 as the destination. 'The stages are connected. At the beginning of stage 1, we are in city l, and as soon as the decision at this stage is taken, it marks the end of stage 1, and it implies beginning of stage 2, and so on. We also realize that beginning of any stage, we can be in any one of the cities that constitute the stage. We refer to these cities as states. Thus, beginning of stage 2, we can be in city 2, 3 or 4, which form the different states at stage 2. In our effort to reduce computation, we now look at the problem stage by stage, starting from the last stage.

Beginning of stage 4, we can be in either 8 or 9 (states). We note that if during the course of our travel, we are at 8 ever, the only highway we can choose is 8-10. If we would have reached 9 instead, then the only choice would have been highway 9-10. We summarize this information in the format below

Table 1 N = 1

States	Decision	Distance
8	10	9
9	10	6

In the above table, N = 1, denotes one more stage to go, i.e., we are at stage 4, The states we can be in at sage 4, are 8 and 9, as shown in column L the decision implies the destination from the states, which is 10 here, Finally, the distances from the states to the destination are listed in the last column.

We now go back one stage. Beginning of stage 3, there are two more stages to go (i.e., N = 2), and we can be in states 5, 6 or 7. Suppose, we now try to answer the following question:

If, at any stage of our travel, we are at state 5, what will be the best course of action for reaching the destination?

Unlike the earlier case, we find that for any state in stage 3, it is possible for us to go at 8 or 9. From 5, for example, if we choose 5-8, then the distance covered would be 6, and from 8, the only way to the destination requires a distance of 9 units to be covered, as already summarized in the table 1. Thus, if the decision is to go to 8, then the corresponding distance would be 15 (= 6 + 9). Instead, if we would have chosen-to go to 9, then the length of the highway 5-9 added to the distance corresponding to state 9 in the table, would give us the total distance to be covered. This works out to 14 (= 8 + 6). Thus, in the course of our travel if we are ever at 5, out of the two possible decisions at the stage, namely, going to 8 or to 9, it is better to choose 9, as-this will involve lesser total distance. This answers our question. Similar exercise can be done for the other states in the stage, and the information can be summarized as shown in the table below.

Table 2 N = 2

Decision State	8	9	Best. Decision	Best Distance
5	6+9=15	8+6=14	9	14
6	4+9= 13	9+6= 15	8	13
7	3+9=12	7+6=13	8	12

The above gives us the best courses of actions to be followed, if we are at any of the states 5, 6 or 7 in the course of our travel. Together with the best decision, the distance corresponding to the best action is also listed.

Activity 2

Suppose, at the beginning of stage 3, you are at state 6. Comment on. the following alternative argument, for determining the best course of action : "The possible decisions are either to go to 8, or to go to 9. The distances of 6-8 and 6-9 are 4 and 9 respectively. As 61 is the shorter of the two, it is best to go to 8, if you are ever at 6, in the course of your travel." You may note that we have reached the same answer.

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It is worthwhile noting at this point that, if our problem consisted of only two stages, namely 3 and 4, then we have found all the possible shortest routes, corresponding to all the possible starting points 5, 6 or 7. In the process so far, we have in fact enumerated all routes, and not saved on computation,

We now proceed backward to stage 2. Beginning of stage 2, there are three states possible, 2, 3 or 4. We are interested in finding and' tabulating the best possible decision for reaching the destination (10), if we ever are at any of these states in the course of our travel. It should be obvious by now, that the best decision that w are looking for, corresponds to all the stages that we may have to travel through to reach 10, and doe; not merely refer to the best for this stage. From 2, there are three possible decisions, namely, to go*to 5, 6 or 7. If we decide to go to 5, then tile minimum distance we have to cover to reach 10, will be the sum of the distance of 2-5 and the best distance from 5 onwards, The. former is given in the diagram as 6, while the latter information as already summarized in the table 2, is 14. The overall effect of choosing 5 9 thus 19. The effect of the other decisions from 2 can be found out in the same way, and the best decision would be the one that gives the minimum overall distance. The exercise can be repeated for the other states as well. The outcome is summarized in the table below.

Table 3 N = 3·

Decision State	5	6	7	Best Decision	Best Distance
2	6+14=20	8+13=21	9+12=21	5	20
3	5+14=19	4+13=17	1+12=18	6	17
4	6+14=19	5+13=18	7+12=19	6	18

Proceeding in the same fashion, we move to the beginning of stage 1. At this stage, there is only one state (city 1), and three possible decisions. Using the same method for creating a table, we create table 4 as shown below.

Table 4 N = 4

Decision State	2	3	4	Best Decision	Best Distance
1	4+20=24	6+17=23	6+18=24	3	23

The above tells us that, if we are at 1, then the best decision for reaching the destination would be to go to city3, and the overall distance to be covered would be 23. The value 23 thus gives us the total distance corresponding to the shortest route. To obtain the highways that lead to this shortest route, we note from table 4, that 1-3 is to be included. We now go back to table 3 and note that if ever we reach city 3, then the best decision is to go to city 6, hence 3-6 is to be included. Similarly, table 2 tells us that the best decision from 6 is to go to 8, therefore 6-8 is also there on the shortest route. Finally, we know that from 8, we can go only to 10, which is shown in table 1 as well. Thus, we have the following answer to the example.

To shortest route from 1 to 10 is given by 1-3-6-8-10, and the distance to be covered is 23.

While solving the problem, we have used the concepts of stage and state. Moreover, the problem has been solved stage by stage, and to ensure that suboptimal solution does not result, we have cumulated the objective function value in a particular way. Working backwards, for every stage, we have found the decisions in that stage that will allow us to reach the final destination optimally, starting from each of the states of the stage. It may be noted that, these decisions could be taken optimally, without the knowledge of how we actually reach the different states. This has been stated as the "principle of optimality" in the D.P. literature. The concepts and the notations are formalized in the next section.

Activity 3

Consider the above example. We have noted at the end of table 2 that there has been no saving in computation. What is your inference on savings in computation at the end of table 4? You may like to enumerate all the routes for the problem and see for yourself, where, if at all, savings occur.

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9.3 DEFINI'T'IONS NOTATIONS

We have noted in the introduction that D.P. provides us with an alternative methodology for solving multistage problems. Identification of stage and state corresponding to a problem situation forms the first step in this direction. The method is based on the principle of optimality. The principle essentially recognizes the fact that period by period optimization is myopic. It gives a systematic and efficient way of reaching the optimal, and also reveals the conditions under which a multistage problem can be solved by D.P. In the last section, the methodology has been illustrated through an example. Here, the objective is to formalize the procedure, to help you in gaining insight into the D.P. methodology. Definitions and notations are presented first, and are followed by discussion.

Stage

The points at which decisions are called for are referred to as stages. Each stage can be thought of having a beginning and an end. The different stages come. in a sequence, with the ending of a stage marking the beginning of the immediately succeeding stage.

State

The variable that links up two stages is called a state variable. At any stage, the status of the problem can be described by the values the state variable can take. These values are referred to as states.

Principle of Optimality

The principle states that the optimal decision from any state in a stage to the end, is independent of how one actually arrives at that state.

The above have been used in the example given in the preceding section. Notations specific to D.P. are now introduced.

Let fN(S) denote the overall optimal objective function value with N more stages to go and S as the starting state.

Let J denote the decisions or possible alternatives at S.

Let d_{sj} denote the effect of starting from state S and deciding on alternative J. This is the objective function corresponding to the stage we are in.

Let fO(D) denote the overall optimal objective function value with no more stages to go and D, the destination as the starting stage; fO(D) = 0.

For a better understanding of the notations, we have the help of the example of the preceding section. A look at the tables created there tells us that the fN(S) values as defined here are stored in the last column of every table.

Thus, from table 1, with N = 1, f1(8) = 9, and f1(9) = 6.

(Note that the initial condition are given by f0(10) = 0)

Now, in order to compute f2(S), for S = 5, 6 and 7, note that for each of the states, there are two possible alternatives, which would imply an effect of d_{S8} and d_{S9} respectively. f2(S) being the overall optimal from S to 10, it can be expressed as :

f2(S) = Minimum [d_{S8} + f1(8), d_{S9} + f1(9)], for S = 5, 6, 7.

It should be apparent that in our example d_{SJ} is the distance between the city S and J, and the same computation has been done in table 2.

Using the same notations, table 3 and 4 computations can be expressed as :

f3(S) = Min [d_{S5} + f2(5), d_{S6} + f2(6), d_{S7} + f2(7)], for S = 2, 3, 4.

f4(S) = Min [d_{S2} + f3(2), d_{S3} + f3(3), d_{S4} + f3(4)], for S = 1.

In general, we can express the above as follows :

fN(S) = Minimum [d_{SJ} + fN−1(J)], N = 1 to 4, where, f0(10) = 0.

The objective function fN(S) is thus computed in D.P. through an equation that is a function of at least two stages. This equation is referred to as recursive equation of simply recursion.

D.P. thus involves a stage by stage solution procedure. Starting from the last stage, given the associated objective function of that stage, we can write it down as a function of the decision variable pertaining to that stage and the statge variable beginning of the stage. This function gives the effect of starting with a particular state and taking a decision, if the problem consisted only of the last stage. Next we go to the last but one stage. Here also the associated stage objective function is found in terms of the decision variable, and the beginning state variable. This gives the effect of the decision pertaining to this stage only. We now note that, once we start with a beginning state and take a decision, we attain an ending state. This ending state is nothing but the beginning state of the next stage, for which we have already found the effect. Thus, in order to ascertain the overall effect of starting this period with a particular state and taking a decision, we need to add the former and the latter. It should be apparent that in the given example once we know the particular state we are in, we can always find the optimal from that point to the final destination. It is irrelevant how we reach the state, so far as optimality is concerned. This has been enunciated as the principle of optimality. It should be noted however, that the principle applies only if certain conditions are fulfilled.

We have seen that the procedure involves finding the decision at each stage that optimises some function of the individual stage returns. The example solved in the preceding section can be formulated as a mathematical program with X_{ij} = 1, if the fine between city I and J is in the shortest route.

= 0, otherwise.

The objective function f(.) may be expressed as 11 d_{ij}, X_{ij}. This function is an addition of the effects of the individual stages. The nature of this function is important for applying the stage by stage procedure. The function is required to be separable and monotonous. Additive functions are separable, hence we could apply D.P. in the given example. Multiplicative functions are also separable, thus, if our problem would have been to find the route from I to 10 that maximizes or minimizes the product of all the distances of the lines in the route, then also D.P. could have been applied. If the individual effect or objective function of a stage i is denoted in general by E,(.), then, while E_i and E_l- are separable, f(.) = [E_1 + E_2] [E_3 + E_4] is not separable. Monotonicity, on the other hand, implies that if the effect in any individual stage i for a decision X_i = a (say) in that stage, is more compared to a decision of X_i = b. then the same should be presented in the function f(.), i.e., the value of 4E_1, E-, ... E_i (X_i = a), ... E_N] is -more than f (E_1, E2,…..E_i (X_i = b), ... EN].

Activity 4

Suppose you are playing a game, and you have been given the figure shown in the example of the preceding section. The objective is once again to start from l and reach 10, so as to maximize your total points. The figures beside the lines now denote the points you collect by traversing the lines. The total points of a route is given by the product of the points of the lines in the route. Find the required route. Suppose, the total points in a route is given as ,the addition of the points of the lanes in a route, will the answer change? Give justifications for your answer.

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Example 1

Consider the problem of finding the shortest route as given in the example in the preceding section. Suppose you are interested now to find the longest route, is it possible to apply D.P. for solution? If your answer is yes, then solve the problem. If the answer is no, then give your reasons for the answer.

Solution

It is apparent that the only change in the overall objective function is that, instead of is minimisation problem, it now becomes a maximization problem. The overall function still is an additive function of the individual stage functions, and as such is separable. The monotonicity property is also preserved, hence D.P. can be applied. Going exactly in the same way as earlier, the general recursion can be written as follows:

fN(S) = Maximum [d_{sj} + fN-1(1)], N = 1 to 4, where, f0(10) = 0.

The recursion can be applied for N = 1 to 4, to create the four tables. The final route is then obtained as usual by working back from table 4 to table 1, as demonstrated earlier. The tables and the result are presented below.

Table 1 N = 1

States	Decision	Distance
8	10	9
9	10	6

Table 2 N = 2

Decision State	8	9	Best Decision	Best Distance
5	6.+9= 15	8+6= 14	8	15
6	4+9=13	9+6=15	9	15
7	3+9= 12	7+6= 13	9	13

Table 3 N = 3

Decision State	5	6	7	Best Decision	Best Distance
2	6+15=21	8+15=23	9+13=22	6	23
3	5+15=20	4+15=29	6+13=19	5	20
4	5+15=20	5+15=20	7+13=20	5,6,7	20

Table 4 N = 4

Decision State	2	3	4	Best Decision	Best Distance
1	4+23=27	6+20=26	6+20=26	2	27

The required route is 1-2-6-9-10 with a total distance of 27.

Example 2

A company has to decide on the production of an item for the next four periods, so as to meet the demands at minimum cost. The relevant costs are the production and inventory carrying costs. In any period, only if production is undertaken then a cost of Rs. 10 is incurred. The variable cost of production is Rs. 2 per unit. The inventory carrying cost is Rs. 2 per unit per period, and is levied on the inventory at the end of any period. Assume that the demands for the upcoming periods are 4, 1,1 and 3 units respectively,, and the beginning inventory is 0.

Solution

Every period a decision is called for, hence periods are equivalent to stages. The beginning or entering inventory of any stage is the same as the end4nventory of the preceding stage, as such, it constitutes the state variable for our problem. We now proceed in the usual way, starting from the last stage.

One more period to go, i.e., N 7 1, implies that we are at the beginning of the last stage. As the demand for the last period is 3, the maximum number of units with which we can start this period is also 3. If we start with more than 3, then obviously we will be left with some stock at the end of the planning horizon, which is undesirable because of the associated inventory cost. Thus, the entering inventory or the state variable, in the last stage can take values of 0, 1, 2 and 3. There is only one possible decision for each of these values. For example, if we start with no inventory, as demand has to be met, and as no inventory should be there at the end of the period, the best decision is to produce 3 units. Similarly, if we start with I unit, then the only decision is to produce 2.units. This information together with the cost corresponding to a decision can be tabulated as follows

Table 1 N = 1

States: entering inventory	Decision: Production	Cost
0	3	16
1	2	14
2	1	12
3	0	0

We go backwards by one stage. Beginning of the third period implies N = 2. As the tool demand for the last two periods is 4 units, the beginning inventory of this stage can take any value between 0 to 4. If we start the stage with no stock, it is possible for us to decide on a production ranging from 1 to 4 units. Less than I unit will be infeasible as the demand for the current period will not be met, and more than 4 is not desirable as stock will be left at the end of the planning horizon. Thus, unlike the last stage, here there are several decisions possible with respect to a particular state. This can also be verified by the following procedure.

Let the decision in a stage (the production quantity for that period) be denoted by x. Let the state variable be denoted by the symbol i. For any period, we know that the beginning inventory plus the production, minus the demand should be equal to the end inventory, i.e., end inventory = i + x - d. As demand for the current stage

is 1 unit, we have, x = end inventory + I - i. Thus, for different values of i, and the different end inventories possible, as given in table 1, it is possible to find the range of x. For example, for i = 0, x = i + end inventory. As end inventory is equivalent to the beginning inventory of the next stage, which in this case can take values from 0 to 3. Hence x can take values between 1 and 4 as we have found earlier.

We now try to answer the following question

If at any stage, we are at state i, what will be the best course of action to minimise the total cost?

We found that for any state in stage 3, there are more than one course of action. Starting with no stock (i = 0), for "ample, if we choose to produce 1 unit, then the end inventory is zero and the cost or effect corresponding to this stage is 12 (production cost of 1 unit + inventory carrying cost =10 + V 1 + 0). End inventory of 0 implies that we start the last stage with i = 0, for which we know from table 1, that the only decision is to produce 3 units at a cost of 16. Hence, if we start stage 3 with 0 units, the total cost of the decision of producing 1 unit is 28. If we would have chosen to produce 2 units, then the end inventory would have been 1. The cost for this stage and the last stage would have been 16 and 14 respectively, giving us a total cost of 30 for the decision. Similarly, the cost corresponding to other decisions can also be found. The best course of action is given by the decision associated with the minimum cost. The above can be expressed in the form of a recursion.

Let fN(i) denote the overall optimal objective function value with N more stages to go and i as the starting state.

Let, C(x) = cost of producing x units = 10 + 2* x.

With N-periods to go, if we start with entering inventory i, and produce x units, then, given a demand of d units, the inventory at the end of the stage is i + i - d. This is nothing but the beginning state with N-1 periods to go, and the best decision from that point onwards can be expressed as f N-1 (i + x - d).

The inventory carrying cost for the period is on the end inventory, and can be expressed as : 2 * (i + x - d).

Hence, we have the following recursion

$$fN(i) = \underset{x}{\text{Min}}\,[C(x) + 2^{*}(i + x - d) + fN{-}1\,(i + x - d)],\ N = 1 \text{ to } 4, \text{ and } f0\,(0) = 0$$

The recursion can now be applied to create table 2 to 4. Table 2 N = 2

x 0 i	1	2	3	4	Min Cost	Best Deci-sion
0 --	12+0+16 =28	14+2+14 =30	16+4+12 =32	18+6+0 ==24	24	4
1 0+0+16 =16	12+2+14 =28	14+4+12 =30	16+6+0 =22	--	16	0
2 0+14+2 =16	12+4+12 =28	14+0+6 =20	-	-	16	0
3 0+12+4 =16	12+6+0 1-8	-	-	-	16	0
4 0+6+0 =6	-	-	-	-	6	0

Table 3 N = 3

X	6	1	2	3	4	5	Min Cost	Best Decision
0	-	36	32	36	40	34	32	2
1	24	30	34	38	32	-	24-	0
2	18	32	36	30	--	-	18	0
	20	34	28	-	-	-	20	0
4	22	26	--	-	-	-	22	0
5	19	-	-	-	-	-	19	0

Table 4 N = 4

x	4	5	6	7	8	9	Min. Cost	Best Deci-sion
i=0	18+0+ 32=50	20+2+ 24=46	22+4+ 18=44	24+6+ 20=50	26+8+ 22=56	28+10+ 14=52	44	6

The required production plan (found by working backwards from table 4), is to produce 6 units in the first period and 3 units in the last period, and the resulting cost is Rs. 44.

Activity 5

In the example solved above, give the break-up of the total cost for each cell of table 3. Also describe the procedure for arriving at the production plan by working backwards from table 4.

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9.4 D.P. APPLICATIONS

In the foregoing sections we have illustrated how Dynamic Programming can be useful for solving a class of multistage problems. Our presentation has been based on two example situations. The first situation consisted in finding the shortest route, and is known as the Stage Coach problem in Operations Research literature. The second situation involved finding the production plan, given a nonlinear post function. In both the situations, decisions were called for in a sequence, and this sequence was built-in in the problem structure. Many applications of D.P. however exist where such sequences are not apparent.-In this section, we present one situation and an example, -and the reader is referred to the readings cited in the last section, for more such applications.

Situation

We take up the familiar example on Emeralds, Rubies and Topaz as described in Unit 8 on Integer Programming:

You have landed in a treasure island full of three types of valuable stones, Emerald(E), ruby(R) and topaz(T). Each piece of E, R and T weighs 3, 2, 2 kg., and is known to have a value of " 4,3,1 crore respectively. You have got a bag with you that can carry a maximum of 11 kgs. Your problem is to decide on how many pieces of each type to carry, within the capacity of your bag, so as to maximize the total value carried. The stones cannot be broken.

The above does not involve decisions to be made in a sequence. However, without changing the problem, it is possible, to impose an arbitrary sequence. For example, we may assume that decision on the number of Emeralds are to be taken first, then the decision on the number of Rubies, and finally, the decision on the number of

Topaz. These then will form the three stages of the problem. As we have already seen, if we decide independently in each stage, we may not get a feasible or an optimal solution. Such stage-wise decisions are possible, provided we link up properly the stages, to preserve the overall optimality. The linkage between a stage and all the subsequent stages, is provided by the total amount of weight that is left to be allocated among these stages. Specifically, the state -variable corresponding to stage i may be identified as the amount of weight that is left to be allocated for stage i to stage n. Finally, recall that the situation was formulated as an I.P. in the last unit :

Maximize $(4 X_1 + 3 X_2 + 1 X_3)$
Subject to $3 X_1 + 2 X_2 + 2 X_3 \leq 11$
X_1, X_2, X_3 are all nonnegative integers.

The overall objective function is an additive function of the individual stage objective, as can be seen from the formulation. Thus we can apply D.P. for solution.

As usual, we star + t from the last stage. Beginning of last stage (N = 1), we are faced with the decision on the, number of Topaz to carry. Given that a maximum of 11 kg. can be carried, it is possible that at the starting of this stage, 0, 1, 2, 3, ... 11 kg. are still left to be filled up by Topaz. These are the possible states we can be in at N = 1. Suppose we start the stage with a weight W left to be allocated for carrying Topaz, then as each Topaz weight 2 kg., the maximum number of Topaz ($^X3$) We will be able to carry is, [w/2], where [a] denotes the largest integer less than or equal to a. As each Topaz has a value of I crore, the effect of the decision of carrying [W/2] Topaz will be Rs. [W/2]. This constitute the best decision if we start the last stage with Wk&- still left to be filled up Following our usual notation, with fN(W) denoting o the overall maximum with N more to go, and the starting state as W, we construct the table I below :

Table 1 N = 1

W	0	1	2	3	4	5	6	7	8	9	10	11
$^X3$	0	0	1	1	2	2	3	3	4	4	5	5
fl (W)	0	0	1	1	2	2	3	3	4	4	5	5

In the above table, VV denotes the state variable, $_{X3}$ denotes the best decision, and fl(W) denotes the overall objective function value corresponding to the starting state and the best decision. As already explained, $_{X3}$ values corresponding to each W are found as the largest integer less than equal to W/2. Also note that the second row and the, third row values are same because of the fact that each Topaz has a value of .1 crore.

We now go back by one stage, Beginning of this stage, we are yet to decide on the number of Rubies and Topaz to be carried. In this stage we are going to decide on the number of Rubies only. The question we need to answer, may be formulated as follows:

If W k& is still to be filled up, how many Rubies should we carry so as to maximize the overall value?

Suppose that beginning of this stage, 11 kg. are yet to be filled up. We are interested in finding the over all maximum value that is possible to achieve, and the decision in this stage that gives the value. By our notations, these are f2(11), and $_{X2}$ respectively. X2 denotes the number of Rubies; as each Ruby weighs 2 Kg.; if 11 Kg. is the amount that is yet to be filled up, $_{X2}$ can take any integer value between 0 to [^{11}/2], Le, between 0 to 5. Moreover, if we decide to carry $_{X2}$ Rubies, then the weight that will remain to be filled up at the beginning of the last stage is (11 - 2 X_2).. By definition; this will be the state value with one more -stage to go, and the overall best from the state till end has already been found and stored in table I as fl(11 - 2X2). To find the best decision, we have to add to this, the effect of the decision of this stage *(carrying X2 Rubies), which is given by 3 X_2. Thus, we have the following recursion

$$f2(11) = \underset{X_2 = 0, 1 \dots 5}{\text{Maximum}} [3 X_2 + f1(11 - 2 X_2)]$$
$$= \text{Max}\ [0 + f1(11), 3 + f1(9), 6 + f1(7), 9 + f1(5), 12 + f1(3), 15 + f1(1)]$$
$$= \text{Max}\ [0 + 5, 3 + 4, 6 + 3, 9 + 2, 12 + 1, 15 + 0] = 15 \text{ (corresponding to } X_2 = 5)$$

This implies that if we start sage 2 with 11 k& still to be filled up, then the best, decision is to fill up the bag with 5 Rubies.

Activity 6

Consider a modified version of the above problem. Suppose there were only two types of stones, Ruby and Topaz, instead of the three types mentioned above. All other data remaining unchanged, how many of each kind you will carry in your bag so as to maximize the total value carried?

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In general, with 2 stages to go and W as the starting state, the maximum value from this point to the end can be expressed as :

$$f2(W) = \text{Maximum } [3 X_2 + f1(W - 2X_2)]$$
$$X_2 = 0, 1 \ldots [W/2]$$

Using the recursion we can create table 2 as shown below.

Using the recursion we can create table 2 as shown below. Table 2 N = 2

W	0	1	2	3	4	5	6	7	8	9	10	11
X_2	0	0	1	1	2	2	3	3	4	4	5	5
f2(W)	0	0	3	3	6	6	9	9	12	12	15	15

We now go back by one more stage. This implies that we are at the beginning of the entire problem, and we have not decided on the number to be carried for any of the stones. As such the whole bag is to be filled up with a load of 11 Kg. Thus, with N = 3, we can be in only one state, that of W = 11. We are now interested in finding out the best decision (No. of Emeralds to be carried) pertaining to this stage that will maximize the overall value carried. The recursion can be written exactly in the same way as in the last stage

$$f3(l\ 1) = \text{Maximum } [4 X_1 - f2(11 - 3 X_1)]$$
$$X, = 0, 1 \ldots [11/3]$$

It may be noted that the effect corresponding to this stage is 4 X,, where X, represents the decision on the number of Emeralds, and the coefficient of 4 represent the value per Emerald. If 11 Kg. is to be filled up, then the maximum number of Emeralds that can be taken are [11/3] = 3, as each Emerald weighs 3 Kg. Once again, the best decision of this stage need not be the best overall decision, hence we allow X, to take values from 0 to 3. The balance, left to be filled up is equal to (11-3 X_j) and it constitutes the state of the system with N = 2. In table 2 we have already found out the maximum value corresponding to all the states the system can be in, beginning of stage 2. Hence for each value of X1, this is added with the effect of this stage to arrive at the overall optimum.

$$f3(11) = \text{Max } [0 + f2(11), 4 + f2(8), 8 + f2(5), 12 + f2(2)]$$
$$= \text{Max } [0 + 15, 4 + 12, 8 + 6, 12 + 3] = 16 \text{ (corresponding to } X_1=1)$$

Thus, the maximum value that can be carried is 16 crores, and the number of Emeralds to be carried is 1. The decisions on other stones can be found by working backwards. One emerald weighs 3 kg, hence beginning -of stage 2, we will have (11-3) = 8 kg. still to be filled up. Thus, with N = 2, we have W = 8. The corresponding best decision is tabulated in table 2, and is equal to 4. Thus, number of Rubies to be carried is 4. Finally, we are left with no more capacity, and obviously the best decision with N = 1 and W = 0 is listed in table 1 as 0, implying that no topaz is to be carried.

You may check that we obtained the same optimal solution in Unit 8.

Activity 7

In tie above example create the table corresponding to N 3.

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Example 3

Solve the following L.P. by using Dynamic Programming :

Maximize $5X + 9Y$

Subject to $-X + 5Y \leq 3$ ... (1)

$5X + 3Y \leq 27$ ... (2)

$X, Y \geq 0.$

Solution

It is useful to compare the given problem with the one on Emerald etc. that we have just solved. Note that the earlier problem was an integer programming (I.P.) problem. The differences between the two can be identified as follows:

i) The number of variables are two and three respectively in the L.P. and the Y.P.

ii) There are two constraints in the L.P., whereas there is only one constraint in the I.P.

iii) The variables are not restricted to integer in the L.P., whereas they are restricted to integers in the I.P.

In the I.P., there were three items and decision for each were taken in stages, in an aribitrary sequence. In the L.P., as there are only two variables, there can be only two stages. Assume that the decision on X corresponds to stage 1 and that of Y to stage 2. Thus, i, above is advantageous as it will involve lesser computation.

In the I. P., state is specified by the amount still left to be filled up in that stage and all the subsequent stages. The state variable is thus associated with the constraint of the problem. In the L.P. there are two constraints, as such, two state variables will be needed to describe the state of the system. For a better understanding, assume that the I.P. is now modified to accommodate an extra constraint specifying that the total number of stones ($X_I + X_2 + X_3$) that can be carried is less than or equal to 5. This implies that we need to keep track of not only the balance weight capacity W' be filled up, but also the balance number of stones that can be carried at the beginning of every stage. The state is now represented by a pair of numbers, corresponding to each state variable. The computation also increases as we have to now find the maximum over all the values of both the state variables.

The third difference brings out a point which has so far not arisen in our' exposition on the D.P. methodology. In all the examples that we have considered, the variables were restricted to nonnegative integers. L.P. allows for continuous values of variables and as such it will be useful to find how this is solved by D.P. methodology.

A physical representation of the L.P. formulation may be useful at this stage. Assume that we want to decide on the number of litres of two items, A and B to be carried, . so as to maximize the value. The value per litre of A and B are Rs.5 and Rs.9 and the weight per unit of A and B are.5 and 3 Kg. respectively. The total weight one is allowed to carry is 27 Kg. It is also known that each of item B requires 5 units of a special solid for preserving. A does not require this. However, for every litre of A carried, we will get one such solid free. Three units of the solids are in hand.

It is apparent that X and Y denote the quantities of A and B that are to be carried. The first constraint specifies that the requirement of the special solid should be within the available capacity. The requirement is 5Y for Y litres of B carried, and the availability is 3 units in hand plus X units we get free for carrying X litres of A. Thus. $5Y \leq 3 + X$. The second constraint is the typical weight constraint.

To solve the problem, we start from the last stage. One more stage to go implies that we are deciding on Y. Let the capacities available for allocation at the beginning of this stage be b_1 and b_2, assume that b_1 and b_2 are associated with the first and second constraint respectively. The maximum value of the capacities are specified in the R.H.S. of the two constraints as 3 and 27, It is evident from (1) that if we are deciding only on Y, and the R.H.S. (capacity) available is b_1, then 5Y has to be less than equal to b_1, i.e. $Y \leq b_1/5$. Similarly, from (2), we have, $Y \leq b_2/3$. Both together would mean that Y has to be less than or equal to the minimum of $b_1/5$ and $b_2/3$. As we are maximising, the maximum value of Y has to be equal to (and not less) the minimum of $b_1/5$ and $b_2/3$. Expressing this in terms of the usual notation:

$$f1(b_1, b_2) = \text{Maximum } [9Y] = 9 \text{ Minimum } [b_1/5, b_2/3] \quad \ldots (1)$$
$$Y \leq \min (b_1/5, b_2/3)$$

We now go backwards by one stage. Two more stages to go implies that we have at our disposal the whole capacities as indicated by the right hand sides of the two constraints. We want to decide on the X that will maximize the overall objective function. Given that we start this stage with 3 and 27 capacities or two different things, if we decide on a value of X, then we will be left with (3 + X) and (27 − 5X) respectively for allocation in the subsequent stages. This is because the coefficients of X in (1) and (2) are — 1 and 5 respectively. Thus with N = 2, our starting state can be represented by the pair (3, 27), and a decision of X leaves us with the ending state (the beginning stage with N = 1) represented by the pair (3 + X, 27 − 5X). The effect corresponding to this state, as given in the objective function is 5X. Finally from (1) and (2), putting Y = 0, we find that X can lie between −3 and 27/5, as negative values are not allowed, X can take any value between 0 and 27/5 (both inclusive). Expressing this in the usual notations, we have :

$$f2\,(3, 27) = \text{Maximum } [5x + f1\,(3 + X, 27-5X)] \quad \ldots (2)$$
$$0 \leq X \leq 27/5$$

From (1), we know that $f1(b_1, b_2) = 9 \text{ Minimum } [b_1/5, b_2/3]$

Therefore, $f1\,(3 + X, 27-5X) = 9 \text{ Minimum } [(3+X)/5, (27-5X)/3]$

Thus, if $(3+X)/5$ is $\leq (27-5X)/3$, then $f1\,(3+X, 27-5X) = 9\,(3+X)/5$ ... (3)

otherwise,

$$f1\,(3+X, 27-5X) = 9\,(27-5X)/3 \quad \ldots (4)$$

We now find the range of X for which $(3+X)/5 < (27-5X)/3$.

Verify that to satisfy the condition, X should be less than 4.5. Replacing (3) and (4) in (2), we have :

$$f2\,(3, 27) = \text{Maximum } [5X + 9\,(3+X)/5], \text{ if } X \leq 4.5$$
$$= \text{Maximum } [5X + 9\,(27-5X)/3], \text{ if } X > 4.5$$

From the above, it is easy to verify that the maximum occurs at X = 4.5. The corresponding value of f2 (3, 27) is the value of the objective function. The value of Y may be obtained by working backwards.

X = 4.5 implies $f1\,(3+X, 27-5X) = f1\,(7.5, 4.5)$.

From (1), we know that the optimal Y = Minimum (7.5/5, 4.5/3) = 1.5

Hence the required answer is X = 4.5 and Y = 1.5.

Activity 8

Suppose in the example 3 given above, the decision variables are restricted to nonnegative integers, i.e., the problem is now converted from a L.P. to an I.P. Solve the resulting problem by using D.P. methodology.

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9.5 SUMMARY

Dynamic Programming provides us with an alternative methodology for solving a class of multistage problems. In this unit, the concepts and method of D.P. have been presented Several problem situations, which are otherwise I.P. or L.P., have been solved to demonstrate the method. We have tried to point out the savings in computation that can result by using D.P., when only one state variable is involved. As the number of state variables increases, the method becomes computationally inefficient.

Unlike other mathematical programming formulation like I.P., L.P., which are somewhat mechanical in nature, D.P. formulations remain an art and are often characterized by the presence of the analysts' ingenuity.

It is expected that by going through this unit you will be able to use D.P. for formulating and solving certain standard situation, similar to those given here. More important, the unit will help you in understanding the exposition on several other situations that are presented in the references given in 9.7.

9.6 SELF-.ASSESSMENT EXERCISES

1) What is dynamic programming? In. what areas of management can it be applied successfully?

2) Discuss briefly

   a) The general similarities between dynamic programming and linear programming.

   b) How dynamic programming differs conceptually from linear programming?

3) What is the dynamic recursive relation? Describe the general process of backward recursion.

4) A truck can carry a total of 10 tons of commodity. Three types (A, B and C) of commodities are to be carried. These commodities have the characteristics *as shown in the following table :

Commodity	Unit Weight (tons)	Profit per Ton (Rs)
A	4	100
B	5	130
C	3	180

Given the total allowable weight, the problem is to determine the number of units of each of the three commodities to carry -so as to maximise the total profit.

5) The XYZ company manufactures two types of television sets, regular and colour. Production takes place on two assembly lines. Regular television sets are assembled on assembly line I and colour television sets are assembled on assembly line II. Because of the limitation of the assembly line capacities, the daily production is limited to no more than 80 regular television sets on assembly line I and 60 colour television sets on assembly line II. The production of both types of television sets requires electronic components. The production of each of these television sets requires 5 units and 6 units of electronic component respectively. The electronic components are supplied by another manufacturer, and the supply is limited to 600 units per day. The company has 160 employees. The production of one unit of regular television set requires 1 man-day of labour whereas 2 men-days of labour are required for a colour television set. Each unit of these televisions is sold in the market at a profit of Rs:. 50 and Rs. 80 respectively. How many units of regular and colour television sets should the company produce in order to obtain a maximum profit?

6) The product manager of Apollo Corporation finds that the budgeted amount for advertising during the next quarter permits 8 insertions in the three newspapers. The Times of India, Bombay; The Hindu, Madras; The Statesman, Calcutta. The Company solicits enquiries from prospective customers through the advertisements and the number of enquiries received is used as a measure of the efficacy of the advertisements. The average estimated number of enquiries per insertion for various numbers of insertions in the three newspapers is given below:

Estimated Number of Enquiries			
Number of insertions	**The Times of India**	**The Hindu**	**The Statesmen**
0	150	160	200
1	165	170	240
2	190	210	270
3	230	245	320
4	260	260	340
5	290	300	350
6	330	310	355
7	350	315	360
8	360	315	360

It will be noticed that in each of the newspapers, after a certain number of insertions the response remains static. This is because the response does not increase in direct proportion to the extent of advertising and reaches a saturation level. What should be the allocation of insertions in each of the newspapers in order to maximise the total number of enquiries.

9.7 FURTHER READINGS

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