UNIT 5 TRANSPORTATION PROBLEM

Objectives

After studying this unit, you should be able to :

- formulate a transportation problem
- locate a basic feasible solution of a transportation problem by various methods
- ascertain minimum transportation cost schedule by Modified Distribution (MODI) method
- explain Stepping Stone Method to find the minimum transportation cost schedule
- discuss appropriate method to make unbalanced transportation problems balanced
- deal with degenerate, transportation problems
- formulate and solve transhipment problems .
- discuss suitable method when the problem is to maximise the objective function instead of minimising it.

Structure

5.1 INTRODUCTION

The **transportation problem** is a special type of linear programming problem where the objective is to **minimise the cost** of distributing a product from a number of **sources** or **origins** to a number of **destinations.** Because of a special structure present the usual simplex method is unsuitable for solving transportation problems. These problems require a special method of solution. The special features of a transportation problem are illustrated with the help of the following example.

Example 1

Consider a manufacturer who operates three factories and despatches his products to five different retail shops. The Table below indicates the capacities of the three factories, the quantity of products required at the various retail shops and the cost of shipping one unit of the product from each of three factories to each of the five retail shops.

	Retail Shops					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	70	50	40	40	300

The Table usually referred to as **Transportation Table** provides the basic data regarding the transportation problem. The capacity of factories 1, 2, 3, is 50, 100 and 150 respectively. The requirement at retail shops 1, 2, 3, 4,, 5 is 100, 70, 50, 40 and 40 respectively. The quantities inside the bordered rectangle are known as **unit transportation cost.** The cost of transportation of one unit from factory 1 to retail shop 1 is 1, factory 1 to retail shop 2 is 9 and so on. A transportation problem can be formulated as a linear programming problem using **variables with two subscripts.** Let

x_{11} = Amount to be transported from factory 1 to retail shop 1

x_{12} = Amount to be transported from factory 1 to retail shop 2

....

x_{35} = Amount to be transported from factory 3 to retail shop 5.

Let the unit transportation costs be denoted by $C_{11}, C_{12}, \dots C_{35}$ i.e. $C_{11} = 1$, $C_{12} = 9$ and so on. Let the capacities of the three factories be denoted by $a_1 = 50$, $a_2 = 100$, $a_3 = 150$. The requirement of the retail shops are $b_1 = 100$, $b_2 = 70$, $b_3 = 50$, $b_4 = 40$, $b_5 = 40$.

Then the transportation problem can be formulated as

Minimise $C_{11}X_{11} + C_{12}X_{12} + \dots\dots\dots + C_{35}X_{35}$

Subject to :

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = a_1$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = a_2$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = a_3$$
$$x_{11} + x_{21} + x_{31} = b_1$$
$$x_{12} + x_{22} + x_{32} = b_2$$
$$x_{13} + x_{23} + x_{33} = b_3$$
$$x_{14} + x_{24} + x_{34} = b_4$$
$$x_{15} + x_{25} + x_{35} = b_5$$
$$x_{11} \geq 0,\ x_{12} \geq 0,\ \dots\dots,\ x_{35} \geq 0.$$

The problem thus has 8 constraints and 15 variables. It will be unwise if not impossible to solve such a problem using simplex method. This is why a special computational procedure is necessary to solve transportation-problem.

In the next section a number procedures will be presented to derive **an initial basic feasible solution of the problem.**

Activity 1

Fill up the blanks

i) The adjective of a transportation problem is to .... the transportation cost.

ii) The constraints of a transportation problem are ....

iii) If a transportation problem has m factories and n retail shops the number variables is and the number of constraints is

5.2 BASIC FEASIBLE SOLUTION OF A TRANSPORTATION PROBLEM

We illustrate with the example introduced in the last section the computation of an initial feasible solution of a transportation problem. Although the problem has eight constraints and fifteen variables one of the constraints can be eliminated since $a_l + a_2 + a_3 = b_t + b_2 + b_3 + b_4 + b_5$. Thus the problem has in fact **seven** constraints and **fifteen** variables. Any basic feasible solution thus has at most seven non zero

In general, any basic feasible solution of a transportation problem with **m origins** (such as factories) and **n destinations** (such as retail shops) has at most m + n - 1 non

Zero X_{ij} The following methods are available for the computation of an initial basic feasible solution.

1) **The North West Corner Rule**

In the North West corner rule allocations are made starting from the North West (upper left) corner **completely disregarding the transportation cost.** By applying North West corner rule to the transportation problem of Section 5.1, we obtain $x_{11} = 50$ as the capacity of factory 1 is 50. Eliminating the first row as the first' factory is unable to supply any more the reduced transportation Table becomes

Factories	Retail Shops					Capacity
	1	2	3	4	5	
2	24	12	16	20	1	100
3	14	33	1	23	20	150
Requirement	50	70	50	40	40	250

The value of b_1 is reduced to 50 in the revised transportation Table as 50 units have already been supplied in retail shop 1 from factory 1. We now allocate 50 units to the north west corner *of the revised* transportation Table. Thus $x_{21}= 50$.

Prodeeding in this way we obtain $x_{22}=50$, $x_{32} = 20$, $x_{33} = 50$, $x_{34} = 40$, $x_{35} = 40$. The corresponding transportation cost is given by

$1 \times 50 + 24 \times 50 + 12 \times 50 + 35 \times 20 + 1 \times 50 + 23 \times 40 + 26 \times 40 = 4520.$

It is clear mat as soon as a value of x_{ij} is determined a row or a column is eliminated from further consideration. The last value of x_{ij} eliminates both a row and a column. Hence a feasible solution computed by the north-west corner rule can have at most $m + n - 1$ positive x_{ij} if the transportation problem has m origins and n destinations. Thus the solution is a basic feasible solution. In the present problem $m = 3$, $n = 5$. Hence using the north west corner rule we have derived a basic feasible solution with seven non zero x_{ij}.

2) **Matrix Minimum Method**

We look for the row and the column corresponding to which C_{ij} is **minimum** in the entire transportation Table. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We choose the value of the corresponding x_{ij} as much as possible subject to capacity and requirement constraints. A row or a column is dropped and the same procedure is repeated with the reduced transportation cost matrix. The method is illustrated with the help of the transportation problem presented in Section 5.1.

We observe that $C_{11} = 1$ which is the minimum transportation cost in the entire transportation Table. Hence $x_{11} = 50$ and the first row is eliminated from any further allocation. The reduced Transportation matrix is

Factories	Retail Shops					Capacity
	1	2	3	4	5	
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	50	70	50	40	40	250

$C_{25} = 1$ is the minimum transportation cost in the reduced transportation table. So $x_{25} = 40$. Proceeding in this way we observe that $x_{33} = 50$, $x_{22} = 60$, $x_{31} = 50$, $x_{32} = 10$, $x3_4 = 40$. The basic feasible solution developed by the matrix minimum method has a transportation cost

$1 \times 50 + 1 \times 40 + 1 \times 50 + 12 \times 60 + 14 \times 50 + 33 \times 10 + 23 \times 40 = 2810$

The minimum transportation cost obtained by using matrix minimum method is much lower than the corresponding cost of the solution derived by using north west corner rule. This is to be expected as the matrix minimum method takes into account the unit transportation cost while choosing the values of the basic variables.

3) Vogel Approximation Method (VAM)

Vogel Approximation method for finding a basic feasible solution involves the following steps.

i) From the transportation Table we determine the **penalty** for each row and column. The penalties are calculated for each row (column) by subtracting the lowest cost element in that row (column) from the next lowest cost element in the same row (column).

ii) We identify the row or column with the **largest penalty** among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column.

iii) We select $\mathbf{X_{ij}}$ as a basic **variable** if C_{ij} is the **minimum cost** in the row or column with **largest penalty.** We choose the numerical value of x_{ij}, as high as possible subject to the row and the column constraints. Depending upon whether a_i or b_j is **the smaller of the two, ith row or jth column is eliminated.**

iv) The step (ii) is now performed on the reduced matrix until all the basic variables have **been** identified.

Example 2

Consider the transportation problem presented in the following Table.

	Destination				
Origin	**1**	**2**	**3**	**4**	a_i
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

The following Table shows the computation of **penalty** for various **rows** and **columns**.

Table : Computation of penalty for VAM

	Destination					
Origin	**1**	**2**	**3**	**4**	a_i	**Penalty**
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Penalty	4	15	8	3		

The highest penalty occurs in the **second column.** The **minimum** c_{ij} in this column is $c_{12} = 22$. Hence $x_{12} = 40$ and the second column is eliminated. Proceeding in this way $x_{14} = 80$, $x_{23} = 30$, $x_{24} = 30$, $x_{21} = 10$, $x_{31} = 50$. The transportation cost corresponding to this choice of basic variables is

$$22 \times 40 + 4 \times 80 + 9 \times 30 + 7 \times 30 + 24 \times 10 + 32 \times 50 = 3520$$

The VAM provides a basic feasible solution whose cost is quite close to the minimum transportation cost.

Activity 2

Four factories (A, B, C, D) supply the requirements of three warehouses (E, F, G). The availability at the factories, the requirement of the warehouses and the unit transportation costs are presented in the following Table.

Factory	Warehouses E	F	G	Available
A	10	8	9	15
B	5	2	3	20
C	6	7	4	30
D	7	6	8	35
Required	25	26	49	100

Find an initial basic feasible solution of the transportation problem by using

i) North west corner rule

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ii) Matrix minimum method

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iii) Vogel approximation method

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5.3 MODIFIED DISTRIBUTION (MODI) METHOD

The modified distribution method, also known as MODI method or a-v method provides a minimum cost solution to the transportation problem. The steps involved in the Modified distribution method are as follows :

1) Find out a basic feasible solution of the transportation problem using one of the 'three methods described in the previous section.

2) We introduce dual variables corresponding to the row constraints and the column constraints. If there are in origins and n destinations then there will be m+n dual variables. The dual variables corresponding to the row constraints are denoted by

   u_i (I = 1, 2, ….., m) while the dual variables corresponding to column constraints are denoted by v_i (j=1, 2, …….., n). The values of the dual variables should be determined from the following equations.

$$u_i + v_j = c_{ij} \text{ if } x_{ij} > 0.$$

3) Any basic feasible solution of the transportation problem has m+n - 1

   $X_{ij} > 0$. Thus there will be m+n 1 equations to determine m+n dual variables.

   One of the dual variables can be chosen arbitrarily. It is to be also noted that as the primal constraints are equations, **the dual variables are unrestricted in sign.**

4) If $x_{ii} = 0$, the dual variables computed in 3 are compared with the c_{ij} values of this allocation as

$$c_{ij} - u_i - v_j$$

If all $c_{ij} - u_i - v_j \geq 0$, then by an application of **complementary slackness theorem** (see Unit 4) it can be shown that the corresponding solution of the transportation problem is optimum. If one or more of $c_{ij} - u_i - v_j < 0$, we choose the cell with least value of $c_{ij} - u_i - v_j$ and allocate as much as possible subject to the row and the column constraints. The allocation of a number of adjacent cell are adjusted so that a basic variable becomes non basic.

5) A fresh set of dual variables are computed and entire procedure is repeated.

Let us consider the following transportation problem given in Example I with a basic feasible solution computed by Matrix Minimum method,

	Destination					
Origin	**1**	**2**	**3**	**4**	**5**	**Availability**
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	70	50	40	40	300

1) The initial basic feasible solution by matrix minimum method is

$$x_{11} = 50,\ x_{22} = 60,\ x_{25} = 40,\ x_{31} = 50,$$
$$x_{32} = 10,\ x_{33} = 50,\ x_{34} = 40.$$

2) The dual variables u_1, u_2, u_3 and v_1, v_2, v_3, v_4, v_5 can be computed from the corresponding C_{ij} values

$$u_1 + v_1 = 1,\ u_2 + v_2 = 12,\ u_2 + v_5 = 1,\ u_3 + v_1 = 14$$
$$u_3 + v_2 = 33,\ u_3 + v_3 = 1,\ u_3 + v_4 = 23.$$

3) Since one of the dual variables can be chosen arbitrarily we take $u_3 = 0$ as it occurs most often in the equations. The values of the dual variables are $u_1 = -13,\ u_2 = -21,\ u_3 = 0,\ v_1 = 14,\ v_2 = 33,\ v_3 = 1,\ v_4 = 23,\ v_5 = 22.$

4) We now compute $c_{ij} - u_i - v_j$ values for all the cells where $x_{ij} = 0$. All the $c_{ij} - u_i - v_j \geqslant 0$ except for cell (1, 2) where $c_{12} - u_1 - v_2 = -11$.

Thus in the next iteration x_{12} will be a basic variable changing one of the present basic variables non basic. We also observe that for allocating one unit in cell (1, 2) we have to reduce one unit in cells (3, 2) and (1, 1) and increase one unit in cell (3, 1). The net reduction in the transportation cost for each unit of such reallocation is

$$-33 - 1 + 9 + 14 = -11.$$

The maximum that can be allocated to cell (1, 2) is 10 otherwise the allocation in cell (3, 2) will be negative. The revised basic feasible solution is

$$x_{11} = 40,\ x_{12} = 10,\ x_{22} = 60,\ x_{25} = 40.$$
$$x_{31} = 60,\ x_{33} = 50,\ x_{34} = 40.$$

It can be verified that the new set of dual variables satisfy the optimality condition. Thus the minimum cost transportation schedule is

$$x_{11} = 40,\ x_{12} = 10,\ x_{22} = 60,\ x_{25} = 40,\ x_{31} = 60,$$

$x_{33} = 50,\ x_{34} = 40$. The corresponding transportation cost is 2700 which is about 3% less than the transportation cost arrived at by matrix minimum method.

Activity 3

Compute the dual variables of the second iteration in the above example verify that the solution presented is the optimum solution.

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5.4 STEPPING STONE METHOD

Stepping Stone Method is another method for finding the optimum solution of the transportation problem. The various steps necessary in the stepping stone method are given below :

1) Find an initial basic feasic feasible solution of the transportation problem.
2) Next check for **degeneracy.** A basic feasible solution with m origins and n destinations is said to be degenerate if the number of non zero basic variables is less than m + n - 1. When •a transportation problem is degenerate it has to be properly modified. This is included in Section 5.6'.
3) Each empty (non allocated) cell is now examined for a possible decrease in the transportation cast. One unit is allocated to an. empty cell. **A number of adjacent cells are balanced so that the row and the column constraints are not violated. If** the net result of such readjustment is a decrease in the transportation cost we include as many units as possible in the selected empty cell and carry out the necessary readjustment with other cells.
4) Step 3 is performed with all the empty cells till no further reduction in the transportation cost is possible. If there-is another allocation with zero increase or decrease in the transportation cost than the transportation problem has **multiple solutions.**

The stepping stone method is illustrated with the help of the following example.

Example 3

Consider the following transportation problem (cost in rupees)

	Depot				
Factory	D	E	F	G	Capacity
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

1) We compute an initial basic feasible solution of the problem by North-West corner rule in Table 1.

Table 1

	Depot				
Factory	D	E	F	G	Capacity
A	4(400)	6(300)	8	6	700
B	3	5(150)	2(250)	5	400
C	3	9	6(100)	5(500)	600
Requirement	400	450	350	500	1700

The figures in the parenthesis indicate the allocation in the corresponding cells.

1) The solution is not degenerate as the number of non zero basic variables is m+n-1 = 6.
2) The cell BD is empty. The result of allocating one unit along with the necessary adjustment in the adjacent cells is indicated in Table 2.

Table 2

Factory	Depot D	E	F	G	Capacity
A	4(399)	6(301)	8	6	700
B	3(+1)	5(149)	2(250)	5	400
C	3	9	6(100)	5(500)	600
Requirement	400	450	350	500	1700

The increase in the transportation cost per unit quantity of reallocation is

+3+6-4-5=0.

This indicates that every unit allocated to route BD will neither increase nor .decrease the transportation cost. Thus, such a reallocation is unnecessary.

4) The result of reallocating one unit to cell' CD is indicated in Table 3.

Table 3

Factory	Depot D	E	F	G	Capacity
A	4(399)	6(301)	8	6	700
B	3	5(149)	2(251)	5	400
C	3(+1)	9	6(99)	5(500)	600
Requirement	400	450	350	500	1700

The net increase in the transportation cost per unit of reallocation is

+3+6+2-4-5-6 = -4.

Thus the new route would be beneficial to the company. The maximum amount that can be allocated in CD is 100 and this will make the current basic variable corresponding to cell CF non basic.

Table 4 shows the transportation table after the reallocation.

Table 4

Factory	Depot D	E	F	G	Capacity
A	4(300)	6(400)	8	6	700
B	3	5(50)	2(350)	5	400
C	3(100)	9	6	5(500)	600
Requirement	400	450	350	500	1700

This procedure was repeated with remaining empty cells CE, AF, CF, AG, BG**.** The results are summarised in the following Table.

Table 5

Unoccupied Cells	Increase in cost per unit of reallocation
CE	$9+4-6-3=4$
AF	$8+5-6-2=5$
CF	$6-2+5-6+4-3=4$
AG	$6-5+3-4=0$
BG	$5-5+6-4+3-5=0$

Since reallocation in any other unoccupied cell can not decrease the transportation cost the present allocation is optimum. The minimum transportation cost is thus

$x_{11} = 300 \quad x_{12} = 400 \quad x_{22} = 50 \quad x_{23} = 350 \quad x_{31} = 100 \quad x_{34} = 500$

The minimum transportation cost is $4 \times 300 + 6 \times 400 + 5 \times 50 + 2 \times 350 + 3 \times 100 + 5 \times 500 = 7350$.

The transportation schedule is, however, not unique as there are a number of unoccupied cells with zero increase in transportation cost.

Activity 4

Solve the •following transportation problem by stepping stone method

	Distributor			
Factory	1	2	3	Inventory
1	2	1	5	10
2	7	3	4	25
3	6	5	3	20
Order	15	22	18	55

5.5 UNBALANCED TRANSPORTATION PROBLEM

We solved the various transportation problems with the assumption that the total supply at the origins is equal to the total requirement at the destinations. If they are unequal the problem is known as an **unbalanced transportation problem.** If the total supply is more than the total demand we introduce an additional column which will indicate the surplus supply with transportation cost zero. Likewise, if the total demand is more than the total supply an additional row is introduced in the Table which represents unsatisfied demand with transportation cost zero. The balancing of an unbalanced transportation problem is illustrated in the following example.

Example 4

The total requirement is 1100 whereas the total supply 800. Thus we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

	Warehouses			
Plant	W_1	W_2	W_3	Available
A	25	17	25	300
B	15	10	18	500
Requirement	300	300	500	

The total requirement is 1100 whereas the total supply 800. Thus we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

	Warehouses			
Plant	W_1	W_2	W_3	Available
A	25	17	25	300
B	15	10	18	500
Unsatisfied demand	0	0	0	300
Requirement	300	300	500	1100

Now the problem can be worked out as discussed in previous sections.

5.6 DEGENERATE TRANSPORTATION PROBLEM

If a basic feasible solution of a transportation problem with an origins and n destinations has fewer than in + n - 1 positive x_{ij} (occupied cells) the problem is said to be a **degenerate transportation problem.**

While in the simple computation degeneracy does not cause any serious difficulty, it can cause computational problem in a transportation problem. If we apply modified distribution method, then the dual variables u_i and v_j are obtained from the c_{ij} values of the basic variables. If the problem is degenerate, you will be unable to locate one

or more c_{ij} value which should be equated to corresponding $u_i + v_j$. Computational difficulty will also arise while applying stepping stone method to a degenerate transportation problem.

It is thus necessary to identify a degenerate transportation problem at the very beginning and take appropriate step to avoid any computational difficulty. The degeneracy in a transportation problem can be identified through the following result (Mustafi, 1988) :

A degenerate basic feasible solution in a transportation problem exists if and only if some partial sum of availabilities (row) is equal to a partial sum of requirements (column).

As for illustration the transportation problem presented in Section5.5 3 degenerate as

$$a_1, = 300 = b_1$$

$$a_2 + a_3 = 800 = b2 + b_3.$$

Perturbation Technique

The degenerate basic solutions of the transportation problem can be avoided if we ensure that no partial sum of a_i and b_j are the same. We set up a new problem where

$$\overset{*}{a\,i} = ai + d \qquad i = 1, 2, \&., m$$

$$\overset{*}{b\,j} = bj \qquad j = 1, 2, \&., n-1$$

$$\overset{*}{b\,n} = bn + md \qquad d > 0$$

This modified problem has been constructed in such a manner that no partial sum of $\overset{*}{a\,i}$ is equal to a partial sum of $\overset{*}{b\,j}$. After the problem is solved, we put d = 0 leading to the optimum solution of the original problem.

For illustration, consider the transportation problem presented in Section 5.5 (after the addition of the additional row). Here m=3, n=3. The perturbed problem is given by

Plant	Warehouses			Availability
	W_1	W_2	W_3	
A	25	17	25	300 + d
B	15	10	18	500 + d
Unsatisfied demand	0	0	0	300 + d
Requirement	300	300	500 + 3d	1100 + 3d

The problem can be solved using any of the methods described before. Then we take d = 0 to obtain the solution of the original problem.

Activity 5

Solve the perturbed problem given above.

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5.7 TRANSHIPMENT PROBLEM

In a transportation problem consignments are always transported from an **origin** to a destination. There could be situation where it might be economical to transport items in several stages : First within certain origins and destinations and finally the reform to the ultimate receipt points.

It is not uncommon to maintain dumps for central storage of certain bulk material. Similarly, movement of material involving two different modes of transport - road and railways or between stations connected by broad gauge and metre gauge lines will necessarily require transhipment. Titus for the purpose of transhipment the distinction between an origin and destination is dropped so that from a transportation problem with m origins and n destinations we obtain a transhipment problem with m+n origins and m+n destinations. The formulation and solution of a transhipment problem is illustrated with the help of the following example.

Example 5

Consider a transportation problem where the origins are plants and destinations are depots. The unit transportation costs, capacity at the plants and the requirements at the depots are indicated below :

Table 1

	Depot			
Plant	X	Y	Z	
A	1	3	15	150
B	3	5	25	300
	150	150	150	450

When each plant is also considered a destination and each depot is also considered an origin, there are altogether five origins and five destinations. Some additional cost data are also necessary. There are presented in the following Tables.

Table 2
Unit Transportation Cost from Plant to Plant

From Plant	To Plant A	Plant B
A	0	65
B	1	0

Table 3
Unit Transportation Cost From Depot to Depot

From Depot	Depot X	To Depot Y	Depot Z
X	0	23	1
Y	1	0	3
Z	65	3	0

Table 4
Unit Transportation Cost from Depot to Plant

	Plant	
Depot	A	B
X	3	15
Y	25	3
:	45	55

From Table 1, Table 2, Table 3 and Table 4 we obtain the transportation formulation of the transhipment problem

Table 5
Transhipment Table

	A	B	X	Y	Z	Capacity
A	0	65	1	3	15	150 + 450 = 600
B	1	0	3	5	25	300 + 450 = 750
X	3	15	0	23	1	450
Y	25	3	1	0	3	450
Z	45	55	65	3	0	450
Requirement	450	450	150 +450 =600	150 +450 =600	150 +450 =600	2700

A **bluffer** stock of 450 which is the total capacity and total requirement in the original transportation problem is added to each row and column of the transhipment problem, The resulting transportation problem has m + n = 5 origins and m+n = 5 destinations, On solving the trans^p^ortation problem presented in Table 5

we obtain $x_{11} = 150$ $x_{13} = 300$ $x_{14} = 150$ $x_{21} = 300$
$x_{22} = 450$ $x_{33} = 300$ $x_{35} = 150$ $x_{44} = 450$ $x_{55} = 450.$

1) Transport $x_{21} = 300$ units from plant B to plant A. 'his increase the availability at plant A to 450 units including the 150 units originally available from A.
2) From plant A transport $x_{13} = 300$,to depot X and $X_{14} = 150$ to depot Y.
3) From 300 units available at depot X transport $x_{35} = 150$ units to depot Z.

The total transhipment cost is

$1 \times 300 + 3 \times 150 + 1 \times 300 + 1 \times 150 = 1200$

If, however, the, consignments are transported from plants A, B to depots X, Y, Z only according to the transportation Table 1, the minimum transportation cost schedule is $x_{13} = 150$ $x_{21} = 150$ $x_{22} = 150$ with a minimum cost of 3450. Thus transhipment reduces the cost of cargo movement in this case.

Activity 6

Solve using modified distribution method the transportation problem given in Table 5 (Transhipment problem).

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5.8 MAXIMISATION IN A TRANSPORTATION PROBLEM

There are certain types of transportation problems where the objective function is to be maximised instead of being minimised. These problems can be solved by converting the maximisation problem into a minimisation problem. The formulation and solution of this class of problems ate illustrated with the help of the following example.

Example 6

A firm has three factories located in City A, City B and City C and supplies goods to four dealers spread all over the. country. The production capacities of these factories are 1000, 700 and 900 units per month respectively. The net return per unit product varies for different combinations of dealers and factories which is given in Table 1.

Table 1

Factory	Dealers 1	2	3	4	Factory capacity
City A	6	6	6	4	1000
City B	4	2	4	5	700
City C	5	6	7	8	900
Dealer Requirement	900	800	500	400	2600

Determine a suitable allocation to maximise the total net return. If x_{ij} denotes the number of units to be despatched from the ith city to the jth dealer q be the corresponding return, then the objective function is

$$\text{Maximise } r_{11}x_{11} + r_{12}x_{12} + r_{13}x_{31} + \ldots\ldots\ldots + r_{34}x_{34}$$

The values of decision variables x_{ij} which **maximise** the objective function are also the values where $- r_{11}x_{11} - r_{12}x_{12} - \ldots\ldots - r_{34}x_{34}$ is **minimised.** In order the express the objective function in a more convenient form we observe that the per unit return is maximum corresponding to r_{34} with a value 8. If we add and subtract 8×2600 the **minimisation problem** will **remain unchanged.** Hence the function to be minimised is

$$2600 \times 8 - r_{11}x_{11} - r_{12}x_{12} \ldots\ldots\ldots - r_{34}x_{34} - 8 \times 2600.$$

But $2600 = x_{11} + x_{12} + \ldots\ldots\ldots\ldots + x_{33} + x_{34}$. Hence the function to be minimised is

$$(8 - r_{11})x_{11} + (8 - r_{12})x_{12} + \ldots\ldots\ldots + (8 - r_{33})x_{33} + (8 - r_{34})x_{34} - 2600 \times 8.$$

This is identical to minimise the objective function

$$(8 - r_{11})x_{11} + (8 - r_{12})x_{12} + \ldots\ldots + (8 - r_{33})x_{33} + (8 - r_{34})x_{34}.$$

Hence we have a revised transportation problem given in Table 2.

Table 2

City	Dealers 1	2	3	4	
A	2	2	2	4	$1000 = a_1$
B	4	6	4	3	$700 = a_2$
C	3	2	1	0	$900 = a_3$
	$900 = b_1$	$800 = b_2$	$500 = b_3$	$400 = b_4$	2600

As the partial sum $a_3 = b_3 + b_4$, the problem is degenerate.We consider the corresponding perturbed problem in Table 3.

Table 3

City	Dealers 1	2	3	4	
A	2	2	2	4	1000 + d
B	4	6	4	3	700 + d
C	3	2	1	0	900 + d
	900	800	500	400+3d	2600 + 3d

The initial basic feasible solution by matrix minimum method is $x_{11} = 900$, $x_{12} = 100+d$, $x_{22} = 700-d$, $x_{23} = 2d$, $x_{33} = 500 - 2dx_{34} = 400 + 3d$.

Using modified distribution method we obtain $u_1+v_1 = 2$, $u_1+v_2 = 2$, $u_2+v_2 = 6$ $u_2+v_3 = 4$, $u_3+v_3 = 1$, $u_3+v_4 = 0$.

Taking $u_1 = 0$ arbitrarily we obtain $u_1 = 0$, $u_2 = 4$, $u_3 = 1$, $v_1 = 2$, $v_2 = 3$, $v_3 = 0$.

On verifying the optimality condition, we observe that

$$C_{12} - u_1 - v_2 < 0, \quad C_{32} - u_3 - v_2 < 0.$$

We allocate $x_{12} = 700-d$ and make readjustment in some of the other basic variables. The revised values are $x_{11} = 200+d$, $x_{12} = 800$, $x_{21} = 700-d$, $x_{23} = 2d$, $x_{33} = 500-3d$, $x_{34} = 400+3d$. The dual variables must satisfy the conditions $u_1+v_1 = 2$, $u_1+v_2 = 2$, $u_2+v_1 = 4$, $u_2+v_3 = 4$, $u_3+v_3 = 1$, $u_3+v_4 = 0$.

Taking $u_1 = 0$ arbitrarily

$$u_1 = 0, \ u_2 = 2, \ u_3 = -1, \ v_1 = 2, \ v_2 = 2, \ v_3 = 2, \ v_4 = 1.$$

The optimality conditions are now satisfied. Taking $d = 0$, the optimum solution of the problem is $x_{11} = 200$, $x_{12} = 800$, $x_{21} = 700$, $x_{33} = 500$ $x_{34} = 400$.

The maximum net return is

$$6 \times 200 + 6 \times 800 + 4 \times 700 + 7 \times 500 + 8 \times 400 = 15500.$$

5.9 SUMMARY

Transportation Problem is a special type of linear programming problem. Simplex method is not suitable for the solution of a transportation problem. On the other hand the transportation problem has a special structure which may be utilised to develop efficient computational techniques for its solution.

In the most general form, a transportation problem has a number of **origins** and a number of **destinations.** A certain amount of a particular consignment is **available** in each origin. Likewise, each destination has a certain **requirement.** The transportation problem indicates the amount of consignment to be transported from various origins to different destinations so that the total **transportation cost is minimised** without violating the availability constraints and the requirement constraints. The decision variables x_{ia} of a transportation problem indicate the amount to be transported from the ith origin to the jth destination. Two subscripts are necessary to describe these decision variables.

A number of techniques are available for computing an initial basic feasible solution of a transportation problem. These are the **North West Corner rule, Matrix Minimum method** and **Vogel's Approximation Method** (YAM). Optimum solution of a transportation problem can obtained from **Modified Distribution (MODI) Method** or **Stepping Stone Method.**

Sometimes the total available consignment at the origins is different from the total requirement at the destinations. Such a transportation problem is said to be **unbalanced.** An unbalanced transportation problem can be made **balanced** where the total available consignment at the origins is equal to the total requirement at the destinations by introducing an additional row or column with zero transportation; cost. The basic feasible solutions of a transportation problem with m origins and n destinations should have **m+n - 1 positive basic variables.** However, if one or more basic variables are zero the solution is said to be **degenerate.** A degenerate transportation problem can be modified by **perturbation method** so that the problem can be solved without any difficulty. A problemhaving a structure similar to that of a transportation problem where the objective function is to be **maximised** can also be solved by the techniques developed in this unit with slight modification.

Transportation problem can be generalised into a **transhipment problem** where shipment is possible from origin to origin or destination as well as from destination to origin or destination. This may result in an economy of transportation in some cases. A transhipment problem can be formulated as a transportation problem with an increased number of origin and destinations.

5.10 KEY WORDS

The Origin of a transportation problem is .the' location from which shipments are despatched.

The Destination of, a transportation problem is the location to which shipments are transported.

The Unit Transportation Cost is the cost of transporting one unit of the consignment from an origin to a destination.

The North West Corner Rule is a method of computing a basic feasible solution of a transportation problem where basic variables are selected from the North West Corner, i.e. top left corner.

The Matric Minimum Method is a method of computing a basic feasible solution of a transportation problem where the basic variables are chosen according to the unit cost of transportation.

The Vogel's Approximation Method (VAM) is an iterative procedure of computing a basic feasible solution of the transportation problem.

The Modified Distribution Method (MODI) is a method of computing optimum solution of a transportation problem.

The Stepping Stone Method is a method of computing optimum solution of a transportation problem.

An Unbalanced Transportation Problem is a transportation problem where the total availability at the origins is different from the total requirement at the destinations.

A **Degenerate Transportation Problem** with in origins and n destinations has a basic feasible solution with fewer than m+n - 1 positive basic variables.

The Perturbation Technique is a method of modifying a degenerate transportation problem so that the degeneracy can be resolved.

The Transhipment Problem is a transportation problem where shipment is possible from an origin to an origin or a destination as well as from a destination to an origin or a destination.

5.11 SELF-ASSESSMENT EXERCISES

1) Consider the following transportation problem with the following unit transportation costs, availability and requirement.

	Destination			
Origin	1	2	3	Available
1	8	9	10	42
2	9	11	11	30
3	10	12	9	28
Requirement	35	40	25	100

Find a basic feasible solution of problem by (i) North West Corner Rule. (ii) Matrix Minimum Method (iii) VAM. Compute the corresponding transportation costs.

2) Find the optimum solution of the transportation problem given in exercise 1 by (i) Modified Distribution Method (ii) Stepping Stone Method.

3) High Speed Transport Corporation has the following requirement and availability of goods from I, II and III to A, B and C with the following unit cost matrix.

	A	B	C	Available
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Required	4	2	2	8

Find the minimum cost transportation schedule.

4) A company has three plants at locations A, B and C which supply to warehouses located at D, E, F, G and Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400, and 800 units respectively. The unit transportation costs are given below :

	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	6
C	8	4	6	6	3

Determine an optimum distribution for the company. in order to minimise the total transportation cost.

5) A transportation problem has two origins and three destinations.. The unit costs of transportation, availability at the origins and the requirement at the destinations are given below.

Origin	Destination			Available
	D_1	D_2	D_3	
S_1	9	8	1	30
S_2	1	7	8	30
Requirement	20	20	20	60

Further suppose that the unit transportation costs from S_1, S_2 to S_1, S_2 and from D_1, D_2, D_2 to D_1, D_2, D_3 are as follows :

	S_1	S_2		D_1	D_2	D_3
S_1	0	1	D_1	0	2	1
S_2	2	0	D_2	2	0	9
			D_3	1	9	0

Find a minimum cost transhipment solution of the problem. Compare this cost with the corresponding minimum cost transportation solution.

6) A Department Store wishes to purchase the following quantities of dresses :

Dress Type	A	B	C	D	E
Quantity	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities indicated below :

Manufacturer	W	X	Y	Z
Total Dress Quantity	300	250	150	200

Tenders are submitted by four different manufacturers who undertake to supply not more than the quantities indicated below :

Manufacturer	W	X	Y	Z
Total Dress Quantity	300	250	15	200

The store estimates that its profit (in suitable units) per dress will vary according to the manufacture as shown in the following Table. .

	A	B	C	D	E
W	2.75	3.50	4.25	2.25	1.50
X	3.00	3.25	4.50	1.75	1.00
Y	2.50	3.50	4.75	2.00	1.25
Z	3.25	2.75	4.00	2.50	1.75

How should the orders for dresses be placed so as to maximise the profit?

5.12 ANSWERS

Activity 1

i) Minimise
ii) Equations
iii) mn, m+n.

Activity 2

North West Corner Rule $x_{11} = 15$, $x_{21} = 10$, $x_{22} = 10$, $x_{32} = 16$, $x_{33} = 14$, $x_{43} = 35$. Minimum cost 668.

Matrix Minimum Method $x_{13} = 15$, $x_{22} = 20$, $x_{33} = 30$, $x_{41} = 25$, $x_{42} = 6$, $x_{43} = 4$. Minimum Cost 538.

Vogel Approximation Method $x_{13} = 15$, $x_{22} = 20$, $x_{33} = 30$, $x_{41} = 25$, $x_{42} = 6$, $x_{43} = 4$. Minimum Cost 538.

Activity 3

$u_1+v_1 = 1$, $u_1+v_2 = 9$, $u_2+v_2 = 12$, $u_2+v_5 = 1$, $u_3+v_1 = 14$, $u_3+v_3 = 1$. $u_3+v_4 = 23$.

$u_1 = -13$, $u_2 = -10$, $u_3 = 0$, $v_1 = 14$, $v_2 = 22$, $v_3 = 1$, $v_4 = 25$, $v_5 = 11$.

All $c_{ij} - u_i - v_j \geq 0$. The solution is optimum.

Activity 4

$x_{11} = 10$, $x_{21} = 3$, $x_{22} = 22$, $x_{31} = 2$, $x_{33} = 18$

or

$x_{11} = 10$, $x_{22} = 22$, $x_{23} = 3$, $x_{31} = 5$, $x_{33} = 15$

Minimum Transportation Cost 173.

Activity 5

$x_{12} = 300$, $x_{21} = 300$, $x_{23} = 200$, $x_{33} = 300$

Minimum Transportation Cost 13200

Self-assessment Exercises

1) i) North West Corner Rule $x_{11} = 35$, $x_{12} = 7$, $x_{22} = 30$, $x_{32} = 3$, $x_{33} = 25$. Minimum transportation cost 934.

   ii) Matrix Minimum Method $x_{11} = 35$, $x_{12} = 7$, $x_{22} = 30$, $x_{32} = 12$, $x_{33} = 25$. Minimum Transportation Cost 934.

   iii) VAM $x_{11} = 2$, $x_{21} = 30$, $x_{12} = 40$, $x_{31} = 3$, $x_{33} = 25$. Minimum transportation cost 901.

2) $x_{11} = 2$, $x_{12} = 40$, $x_{21} = 30$, $x_{31} = 3$, $x_{33} = 25$. Minimum transportation cost 901.

3) $x_{11} = 1$, $x_{21} = 3$, $x_{32} = 2$, $x_{33} = 2$. Minimum Transportation Cost 820.

4) $x_{15} = 800$, $x_{21} = 400$, $x_{24} = 100$, $x_{32} = 400$, $x_{33} = 200$, $x_{34} = 300$,

x_{43} (short supply) = 300.
Minimum Transportation Cost 9200.

5) Transhipment problem $x_{11} = 60$, $x_{12} = 10$, $x_{15} = 20$, $x_{22} = 50$, $x_{23} = 40$.
$x_{33} = 40$, $x_{34} = 20$, $x_{44} = 60$, $x_{55} = 60$.
Minimum Transhipment Cost = 100.

Transportation problem S_1 to $D_2 = 10$, S_1 to $D_3 = 20$, S_2 to $D_1 = 20$,
S_2 to $D_2 = 10$.
Minimum Transportation Cost = 100.

6) 150 dresses of A and 50 of E by W, 250 of D by X, 150 of E by Y, 100 of B and 75 of C by Z.
Maximum Profit 1687.50.

5.13 FURTHER READINGS

Hadley, G. 1969. *Linear Programming,* Addison Wesley, Reading, Massachusetts USA.

Mittal, K.V. 1976. *Optimisation Methods in Operations Research and Systems Analysis,* Wiley Eastern Limited : New Delhi.

Mustafi, C.K. 1988. *Operations Research Methods and Practice,* Wiley Eastern Limited : New Delhi.