UNIT 2 REVIEW OF PROBABILITY AND STATISTICS

Objectives

After reading this unit, you should be able to:

- Discuss the relevance of probability and statistics in decision making
- Explain the concepts of Random variable and Probability Distribution
- Identify situations where different discrete probability distributions can be applied
- Use suitable continuous distributions to various situations in decision making process
- Summary measures of Probability distributions.

Structure

2.1 INTRODUCTION

Uncertainty is part and parcel of business life. Demand of a product, product quality, stock market prices are but some of the areas, where, prediction about the future with certainty becomes impossible. So, decision making in such areas is facilitated through formal and precise expressions for the uncertainties involved. Formulation the market strategy of a company becomes more easy if the company is able to predict exact demand, product characteristics and competitors strategy for the future with certainty. For example, in order to decide exact time for company's product advertisement on a T.V., the executive needs to know the exact distribution of potential customers according to various socio-economic characteristics, such as income, education, rural/urban living etc.

Probability and statistics provides us with the ways and means to attain the formal precise expressions for uncertainties involved in different situations that come across the decision making process. The objective of this unit is to review the techniques of probability and statistics you have already studied in your earlier course MS-8. Accordingly, the basic concepts: Probability, random variable and Probability distribution are first presented; followed by brief details regarding some specific probability distributions. A brief review of summary statistics for both discrete and continuous probability distribution has also been presented. Finally a few examples relating to business application of these distributions are given along with steps for computation. Usage of these concepts will be facilitated when you will study units relating to inventory management and waiting lines problems in operation research.

Activity 1

Explain three situations relating to your experience with any organisation/firm you know, where you faced uncertainty in taking decisions. Also explain as to how you dealt with the uncertainty in each of the cases.

1) ....

2) ....

3) ....

2.2 RANDOM EXPERIMENT AND PROBABILITY

The experiment whose outcomes are unpredictable before the actual happening is termed as **random experiment** and the outcomes are termed as its **events.** The chance of occurrence associated with each outcome of the random experiment is called its **probability.** Consider following examples for clear understanding of these terms:

1) The demand for a product on any day which is unpredictable and can have three values, viz. high, medium and low demand.

2) Toss of a coin with two outcomes either head or tail.

3) The weight of a student selected at random from a college.

Here, in these examples it is clear that the experiment can be repeated large number of times and each time the set of out cases remains same but occurrence of any is unpredictable. You may recall that we have already given various methods of computation of probability and algebra of probability. It may be quickly pointed out that the **probability** associated with any event of the random experiment is a number **P(A)** such that

i) P(A) is always +ve and less than or equal to one

ii) Sum of P(A) for all the events is always 1.

Thus if P(A) worked out using the assumption that all events of the random experiment are equally likely, it is termed as a **Prior Probability.** If P(A) is worked out as a limit of ratio of favourable trials to total number of trials, then it is called an **empirical-probability.** Here, it is assumed that experiment can be repeated indefinitely under identical conditions. These two definitions are also called objective probabilities. However, if the values to P(A) are assigned on subjective basis satisfying the above conditions. Then the probability is termed as **Subjective Probability.** This subjective probability plays very significant role in business decisions as the experts can assign subjective probabilities to various events on the basis of their expertise and experience. These concepts shall be more clear with following examples:

Example 1

A coin is tossed two times. Describe its sample space along with probabilities associated with the events.

Solution

The sample space consist of following 4 points

S = (HH, HT, TH, TT);

Where H and T denote occurrence of head or tail. These four points are equally likely, mutually exclusive and exhaustive. So using prior probability definition each has probability of occurrence 1/4.

Example 2

Following is record of demand of T.V. sets per day

No. of T.V.s demanded	1	2	3	4	5	6
No. of days	8	12	10	5	3	2

Calculate the probability of demand for more than 3 T.V. sets. Solution

Solution

Here, the no. of experiments (No. of days) = 40

No. of favourable trials = 10 (It is sum of days when demand is 4 to 5 or 6 T.V. sets)

$\therefore$ P (Demand > 3) = 10/40 = $\frac{1}{4}$

Activity 2

List all outcomes of throw of a pair of dice and write all possible outcomes having sum of the values on the face of the dice as 10.

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Activity 3

Calculate the probability of getting a complete suit in a hand of Bridge. (A complete suit consist of all cards of the same color.)

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Activity 4

Following is record of No. of defective nuts in packets of 150 nuts each

No. of T.V.s demanded	1	2	3	4	5	6
No. of days	8	12	10	5	3	2

Calculate the probability that a packet selected at random will have i) 2 defectives ii) at least 2 defectives iii) atmost 2 defectives.

2.3 RANDOM VARIABLE

When all the outcomes of a random process can be identified through numerical values, then we can redefine the outcomes of the process by a variable called a **random variable.** For example the toss of two coins simultaneously will result into three distinct numerical values, namely two heads, one head and no head. Thus this experiment can be described with the help of random variable X taking values 2,1 and 0 respectively.;

Again, if the random variable has finite possible values, like this example, (tossing of two coins), the variable is called a **discrete random variable (d.r.v.)** and if the variable has infinite possible values it is called **continuous random variable (c.r.v.).** For example, weight of a school student selected at random.

It may be pointed out that the distinction between a continuous and discrete random variable is purely a nomenclature type as a continuous random variable can always be looked upon as a discrete random variable. The idea shall be more clear with this example. Suppose a marketeer regards the demand for sugar to be high, moderate or low according as-the demand is more than 10 quintals, between 10 to 4 quintals and less than 4 quintals respectively.

Though the demand of sugar is a continuous random variable, but from the marketeer point of view it is discrete taking values 1, 2, 3 according as demand is low, moderate or high respectively.

2.4 PROBABILITY DISTRIBUTION

Let X be a discrete random variable taking values X_1, X_2............,X_n. Now, if we can assign probabilities $P(X_1)$, $p(X_2)$,………..., $p(X_n)$ to these values. Then P(X) is called probability distribution function of X, if the following holds statistic

i) $P(X) \geq 0$ for all $X = X_i$, $i = 1, 2, \ldots, n$

ii) $\sum_{i=1}^{n} P(X_i) = P(X_1) + P(X_2) + \ldots\ldots\ldots + P(X_n) = 1$

So, in case of toss of two coins simultaneously, we have

No. of (X) Heads	Probability of X P(X)
0	1/4
1	1/2
2	1/4
Sum	1

From the table it is evident that P(X) defined above satisfy both the conditions, and so it is called Probability distribution of X**.**

Cumulative Probability Distribution Function

The concept and importance of cumulative probability distribution shall be clear from the following example.

Example 3

Following is demand data for last 100 days of number of T.V. sets. Calculate the probability of the following events.

i) The demand on a particular day is 4 T.V. sets

ii) The demand on a particular day is 4 or less T.V. sets

No. of T.V. Demanded	1	2	3	4	5	6	7
No. of days	5	15	20	30	18	9	3

Solution

Table: Probability function and Cumulative Probability function for No. of T.V. demanded per day.

No. of T.V. demanded D	No, of days f	Relative frequency P(D)	Cumulative Frequency (c.f.)	Relative Cumulative frequency = Cumulative Probability C (D)
1	5	0.05	5	0.05
2	15	0.15	20	0.20
3	20	0.20	40	0.40
4	30	0.30	70	0.70
5	18	0.18	88	0.88
6	9	0.09	97	0.97
7	3	0.03	100	1.00

So cumulative probability of the random variableX is sum of the probabilities for values of X less than and equal to a given value. Further, it can be seen that

$C(X) = C(X-1) + P(X)$

Using these expression the last column has been obtained for the table given above.

$P(4) = 0.30$ and $C(4) = 0.70$.

Expectation of Random Variable or Expected value of X denoted by E(X) is an expression for calculating, the value of X in the long run. It is also known as average value of X. Symbolically,

$$\mu = E(X) = \sum XP(X) = X_1P(X_1) + X_2P(X_2) + \ldots\ldots + X_nP(X_n)$$

i.e. expected value of random variable is sum of the product of all possible values of the random variable with corresponding probabilities. So in the previous example average value of demand of T.V. sets/day is

$\mu = E(X) = 1\times0.5+2\times0.15+3\times0.20+4\times0.30+5\times0.18+6\times0.09+7\times0.03$

$= 3.80 \approx 4$ T.V. sets

Thus average or mean demand/day is 4 T.V. sets. It is also known as **measure of central tendency.**

Standard deviation (S.D.)

Standard deviation is a tool to describe the variability of the random variable X. Mathematically it is given as

$$\sigma = \sqrt{E[X-E(X)]^2} = \sqrt{\sum (X-\mu)^2 P(X)} = \sqrt{\sum X^2 p(X) - \mu^2}$$

Where $\mu = E(X)$

P(X) is Probability mass function of random variable X.

$\sum$ is sum over all possible values of discrete random variable X.

Activity 5

A flower seller has following distribution of demand for flowers per day:

No. of flowers Demanded per day:	10	11	12	13	14	15
Probability:	0.1	0.2	0.3	0.2	0.1	0.1

He knows from past experience that each flower sold gives him a profit of Rs. 3, where as each unsold flower gives a loss of rupees two. Estimate the number of flowers he would have every day with a view to maximize his earnings on an avera

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2.5 STANDARD DISCRETE PROBABILITY DISTRIBUTIONS

You have already studied standard discrete probability distributions in MS-8 Block Unit 10. These are discussed below in brief alongwith their summary statistics. The usage of these distributions shall be explained with the help of examples.

Binomial Distribution

Range of X: 0, 1, 2, ... n

Frequency Function: $P(X) = {}^nc_x\, p^x\, (1-p)^{n-x}$, where

p: probability of success

n: total No. of Bernoulli Trials

Mean: $\mu = np$

Variance: $\sigma^2 = np(1-p)$

Standard deviation: $\sigma = \sqrt{np(1-p)}$

Poisson Distribution

Range of X: 0, 1, 2, ... ∞

Frequency function: $P(X) = \dfrac{e^{-\lambda}\lambda^x}{X!}$, where

λ: a positive parameter

Mean: $\mu = \lambda$; Variance $= \lambda$; Standard Deviation: $\sigma = \sqrt{\lambda}$

Pascal Distribution (–ve Binomial Distribution)

It is generalised case of geometric distribution

Range of X: k, k+1, k+2, ... ∞

Frequency function: $P(x) = {}^{x-1}C_{k-1}\, p^k q^{x-k}$ where

p: probability of success for any trial

n: No. of trials to get k successes

q: probability of failure for any trial = 1–p

Mean: $\mu = \frac{k}{p}$

Variance: $\sigma^2 = \frac{k(1-p)}{p^2}$

S.D.: $\sigma = \frac{1}{p}\sqrt{k(-p)}$

Uniform Distribution

Range of X: 0, 1, 2, ... n

Frequency function: $P(X) = \frac{1}{1+n}$

Mean : $\mu = n/2$

Variance: $\sigma^2 = \frac{n^2}{12} + \frac{n}{6}$

S.D.: $\sigma = \sqrt{\frac{n(n+2)}{12}}$

Geometric Distribution

Range of X: 0, 1, 2, ... ,∞

Frequency function $P(X) = p^X (1-p)$, where

p: +ve parameter, denotes probability of success for any trial

Mean: $\mu = \frac{p}{1+p}$

Variance: $\sigma^2 = p/(1-p)^2$

S.D.: $\sigma = \frac{\sqrt{p}}{(1-p)}$

Example 4

An owner of a hotel having delux rooms has bought 2 Colour T.V. sets fitted with V.C.R. to rent to occupants of these rooms. He estimates from his past experience that about 1/3 of occupants would be willing to rent sets. Assuming 100% occupancy for all days. Calculate the probability that on a particular evening

i) There are more requests than the T.V. sets

ii) If the owner's cost per set per day is Rs. 100, what rent R must he charge in order to break even in the long run.

Solution

i) Daily request of T.V. sets on rent is a random variable, subject to the Binomial distribution with n = 6 and p = 1/3.

We are interested to find the probability that the random variable takes value more than 2. This probability is

P(A) = 1 - P(0) - P(l) - P(2)

$$= 1 - {}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^6 6 - {}^6C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)5 - {}^6C_2\left(\frac{1}{3}\right)^{2^2}\left(\frac{2}{3}\right)^4$$

$$= 1 - \frac{64}{729} - \frac{6 \times 32}{729} - \frac{15 \times 16}{729} = 1 - \frac{496}{729} = \frac{233}{729}$$

ii) Here, daily revenue from rent is a random variable with the following distribution.

T	Probability P(T)	E(T)
0	64/729	0
R	192/729	192R/729
2R	473/729	946R/729
	1.000	1138R/729

Now for Break even in the long run expected total revenue (T) is equal to total cost.

$$\therefore \frac{1138}{729} R = 200 \Rightarrow R = \frac{145800}{1138}$$

Example 5

Assume that cars arrive at a toll booth according to poisson distribution at a mean rate of 4 cars per 5-minute interval. Find the probability that during a random interval of 5 minutes.

i) Exactly 2 cars arrived.

ii) Atmost two cars arrived.

iii) At least three cars arrived.

Solution

Here in this example $\lambda = 4$ and we are to calculate

i) $P(2) = \frac{e^{-4}(4)^2}{2!} = \frac{0.018 \times 16}{2} = 0.144 \; (e^{-4} = 0.018)$

ii) $P(\leq 2) = P(0) + P(1) + P(2) = e^{-4} \left(1 + 4 + \frac{4^2}{2!} \right)$

$= 0.018\,(13) = 0.234$

iii) $P(\geq 3) = 1 - P(\leq 2) = 1 - 0.234 = 0.766$

Example 6

A and B are playing a certain game with odds in favour of A 2 to 3 for each game. Calculate the probability that a will win after loosing first two games. Also calculate mean and variance for No. of failures of A proceeding his first success.

Solution

It is a case of geometric distribution with p=3/5 and n=2

$$\therefore \; p(2) = (p^2)\,(1\text{-}p) = \left(\frac{3}{5}\right)^2 2/5 = \frac{18}{125}$$

$$\text{Mean} = \mu = \frac{\frac{3}{5}}{\frac{8}{5}} = \frac{3}{8}$$

$$\text{Variance} = \frac{\frac{3}{5}}{\left(\frac{2}{5}\right)^2} = \frac{3}{5} \times \frac{5 \times 5}{4} = \frac{15}{4}$$

Activity 6

Incharge of an electronic section of departmental store in New Delhi knows from the past experience that the chance that the customer who is just browsing will buy something is 0.3. Suppose that 10 customers browse in his section every hour during the evening. Calculate the probability of following events during a specified hour in the evening.

a) At least one browsing customer will buy something.

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b) No more than 2 browsing customers will buy something.

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Activity 7

Certain mass produced articles of which 0.5% are defective, are packed in cartons each containing 130 articles. What proportion of cartons are free from defective articles? Also find out the proportion of carton containing 2 or more defectives.

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2.6 CONTINUOUS PROBABILITY DISTRIBUTIONS

Continuous probability distributions are also known as probability, density functions (p.d.f.) for the continuous random variables. Some of the important p.d.f. have already been discussed in course MS=8 Block 3 Unit 11. However, some important continuous density functions are listed below along with their summary statistic for ready reference. Again a few examples are also discussed to explain the usage of these p.d.f. in decision making process.

Rectangular Distribution

Range : 0 to a

Density : $f(x) = \frac{1}{a}$

Mean $: \mu = \frac{a}{2}$

Variance $: \sigma^2 = \frac{a^2}{12}$

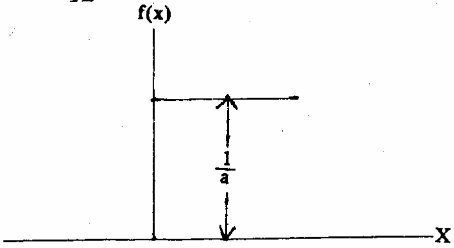

Fig. 2.1: Rectangular distribution

Exponential Distribution

Range : 0 to ∞

Density: $f(x) = ae^{-ax}$

where a is a +ve constant

Cumulative distribution: $F(x) = P(X \leq x) = 1 - e^{-ax}, x \geq 0$

$= 0$, elsewhere

Mean : $\mu = 1/a$

Variance: $\sigma^2 = 1/a^2$

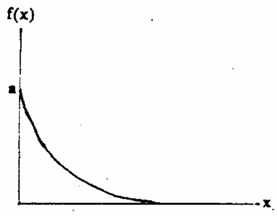

Fig. 2.2: Shape of Exponential Distribution

Normal Distribution

Range: $-\infty$ to ∞

Density: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$ where

μ = Mean of the distribution

σ^2 = variance of the distribution

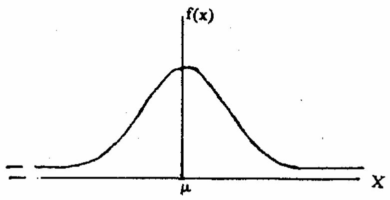

Fig. 2.3: Shape of Normal Distribution.

Gamma Distribution

Range: 0 to ∞

Density: $f(x) = \frac{b^{a+1}}{\Gamma(a+1)} x^a e^{-bx}$, where

b is a +ve parameter,

a is a parameter greater than -1;

Γa denote gamma function with argument a

$$\Gamma a = \int_0^\infty e^{-x} x^{a-1} dx$$

where $\int$ is a sign for definite integral from 0 to ∞ with respect to x.

Mean: $\mu = \frac{a+1}{b}$

Variance: $\alpha = \frac{a+1}{b^2}$

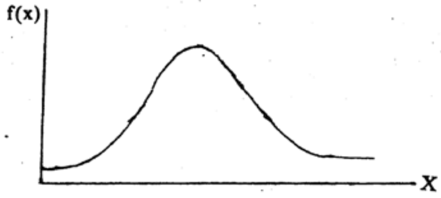

Fig. 2.4: Shape of Gamma Distribution.

Beta Distribution

Range: 0 to 1

Density: $f(x) = \frac{(a+b+1)!}{a!b!} x^a (1-x)^b$, where

a and b are both parameters having value greater than –1.

Mean: $\mu = \frac{a+1}{a+b+2}$

Variance: $\sigma^2 = \frac{a+2}{a+b+3} \quad \frac{a+1}{a+b+2}$

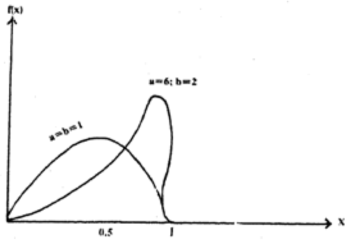

Fig. 2.5: Shape of Beta Distribution.

Example 7

It is known from past experience that average life of a bulb (assumed to be a continuous random variable following exponential distribution) is 120 hours. Calculate the probability that the bulb will work for atmost 30 hours.

Solution

Here Mean $\mu = 120 = 1/a$

$\therefore a = \frac{1}{120}$

Required to find $F(30) = P(x \leq 30) = 1 - 3^{-30/120} = 1 - 3^{-1/4}$

$= 1 - e^{-0.25} = 1 - 0.7796$

$= 0.2204$

(the value of $e^{-0.25} = 0.7796$ from standard tables).

Example 8

A company manufacturing plastic rope knows from the past experience that average load bearing capacity of the rope is 200 lbs with a S.D. of 20 lbs. Calculate the probability of the following events.

i) The roap fails to bear a load of 170 lbs.

ii) The roap bears a load of 240 N.

Solution

To calculate these probabilities following sequence of steps will be performed.

1) Identify mean and S.D. of the normal distribution. In this case these are

   μ = 200 lbs.

   σ = 20 lbs.

2) Sketch the graph to depict the desired area to be calculated as probability of the events.

   For (i) and (ii) they are given as Fig. 2.6.

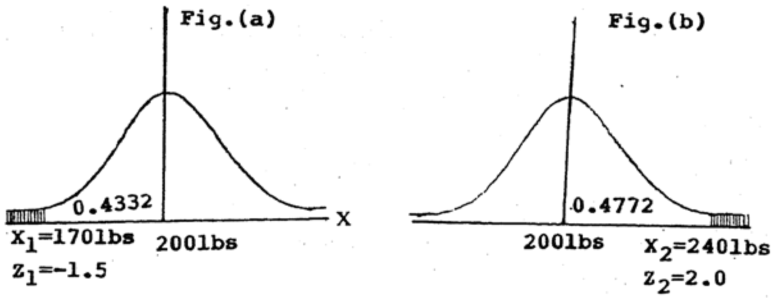

Fig. 2.6: Required Area for (i) and (ii) under the normal curve.

Step (iii) Convert the X values to standard normal score (z)

values using $z = \frac{X-\mu}{\sigma}$

Here $X_1 = 170$ $X_2 = 240$

$$\therefore Z_1 = \frac{170 - 200}{20} = -\frac{30}{20} = -1.5$$

$$Z_2 = \frac{240 - 200}{200} = \frac{40}{20} = 2$$

Step (iv): Read area from 0 to Z under the standard normal curve using statistical table area under the normal curve.

Here (i) Area from 0 to - 1.5is 0.4332 which is same as area from 0 to 1.5 (ii) Area from 0 to 2 is 0.4772

Step (v) Calculate required probabilities using the fact that area to the left of the mean is 0.5.

$\therefore$ (i) Probability that rope will fail to bear a load of 170

lbs = 0.5 - 0.4332 = 0.0668

(ii) Probability that rope will bear to a load of 240 lbs = 0.5 - 0.4772 = 0.0228

Activity 8

In an industrial complex, the average no. of total accidents per month is one-half. What is the probability that 4 months will pass without a fatal accident? Assume No. of accidents per month follows Poisson distribution. Further mean of exponential distribution is 1/A, where A is mean of Poisson distribution.

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Activity 9

Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1,000 would you expect to have height more than six feet? Assume that height of soldiers follows normal distribution.

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2.7 SUMMARY

Probability in simple terms means the chance of occurrence of an event. The need to develop a formal and precise expression for uncertainty in decision making, has lead to different approaches to probability measurement namely, priori, relative frequency and subjective. However, these approaches follows the same axioms. The concept of random variable enables one to assign numerical values to all possible events of the random experiment. If the events are finite, then the random variable associated with it is called discrete random variable and corresponding probability function as probability mass function. Again, if the events are infinite; then the random variable is called continuous and probability function as the density function of this random variable. Binomial, Poisson, geometric and negative Binomial or Pascal are a few important discrete probability functions. Normal; uniform, exponential, Gamma and Beta probability density function are a few examples in case of continuous random variable. These probability functions enable us to calculate precise expression for uncertainties attached with day-to-day business decisions.

2.8 KEY WORDS

Random Experiment is an experiment whose outcomes are unpredictable before the actual happening.

Probability is a precise expression for chance of occurrence of an event if the experiment is performed large number of times.

Random Variable is a numerical valued function defined .on outcomes of the random experiment.

Discrete Random Variable is a random variable which takes only finite values.

Continuous Random Variable is a random variable which takes infinite possible values.

Probability function is an expression to describe probability of occurrence for all possible values of the random variable.

Probability mass function is a probability function defined for discrete random variable.

Probability density function is a probability function defined for continuous random variable:

Cumulative probability function is an expression for probability of the random variable taking a value equal to or less than a given value.

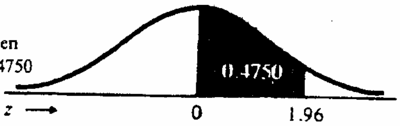
Expectation is the average value of the random variable in the long run.

Variance is an expression to measure the variability of random variable about its expectation.

Standard deviation is an expression to measure the variability of random variable about its mean which has the same unit of measurement as that of random variable.

Appendix 1

Example:
If $z = 1.96$, then
$P(0 \text{ to } z) = 0.4750$



AREAS UNDER THE NORMAL CURVE

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.3015
1.3	0.3032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4389
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Appendix 2

VALUES OF e^{-j}

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1,0000	0.9048	0.8187	0.7408	0.6703	0.6055	0.5488	0.4966	0.4493	0.4066
1	0.3679	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827	0.1653	0.1496
2	0.1353	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672	0.0608	0.0550
3	0.0498	0.0450	0.0408	0.0369	0.0334	0.0302	0.0273	0.0247	0.0224	0.0202
4	0.0183	0.0166	0.0150	0.0136	0.0123	0.0111	0.0101	0.0091	0.0082	0.0074
5	0.0067	0.0061	0.0055	0.0050	0.0045	0.0041	0.0037	0.0033	0.0030	0.0027
6	0.0025	0.0022	0.0020	0.0018	0.0017	0.0015	0.0014	0.0012	0.0011	0.0010