UNIT 1 2-D and 3-D TRANSFORMATIONS

Structure **Page Nos.**

1.0 INTRODUCTION

In the previous Block, we have presented approaches for the generation of lines and polygonal regions. We know that once the objects are created, the different applications may require variations in these. For example, suppose we have created the scene of a room. As we move along the room we find the object's position comes closer to us, it appears bigger even as its orientation changes. Thus we need to alter or manipulate these objects. Essentially this process is carried out by means of transformations. Transformation is a process of changing the position of the object or maybe any combination of these.

The objects are referenced by their coordinates. Changes in orientation, size and shape are accomplished with **geometric transformations** that allow us to calculate the new coordinates. The basic geometric transformations are translation, rotation, scaling and shearing. The other transformations that are often applied to objects include reflection.

In this Block, we will present transformations to manipulate these geometric 2-D objects through Translation, and Rotation on the screen. We may like to modify their shapes either by magnifying or reducing their sizes by means of Scaling transformation. We can also find similar but new shapes by taking mirror reflection with respect to a chosen axis of references. Finally, we extend the 2-D transformations to 3-D cases.

1.1 OBJECTIVES

After going through this unit, you should be able to:

- describe the basic transformations for 2-D translation, rotation, scaling and shearing;
- discuss the role of composite transformations;

- describe composite transformations for Rotation about a point and reflection about a line;
- define and explain the use of homogeneous coordinate systems for the transformations, and
- extend the 2-D transformations discussed in the unit to 3-D transformations.

1.2 BASIC TRANSFORMATIONS

Consider the xy-coordinate system on a plane. An object (say *Obj*) in a plane can be considered as a set of points. Every object point P has coordinates (x,y), so the object is the sum total of all its coordinate points (see *Figure 1*). Let the object be moved to a new position. All the coordinate points P'(x',y') of a new object *Obj'* can be obtained from the original points P(x,y) by the application of a geometric transformation.

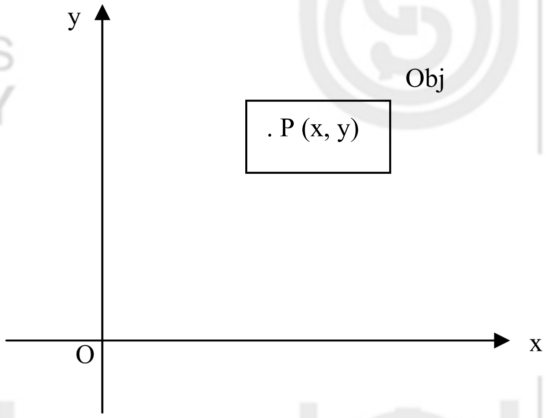


Figure 1

1.2.1 Translation

Translation is the process of changing the position of an object. Let an object point P(x,y)=xI+yJ be moved to P'(x',y') by the given translation vector V= t_xI + t_yJ, where t_x and t_y is the translation factor in x and y directions, such that

P'=P+V. ----------------(1)

In component form, we have

$$Tv = \begin{cases} x' = x + t_x \text{ and} \\ y' = y + t_y \end{cases} \qquad \text{----------------(2)}$$

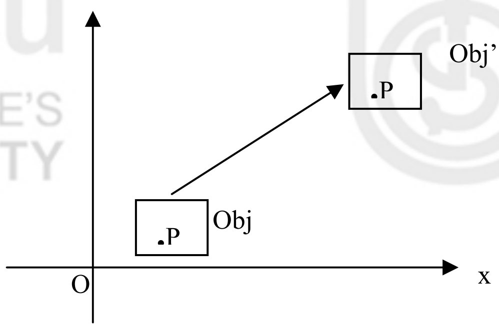


Figure 2

As shown in *Figure 2*, P' is the new location of P, after moving t_x along x-axis and t_y along y-axis. It is not possible to develop a relation of the form.

$P' = P.T_v$ ---------------(3)

Where Tv is the transformation for translation **in matrix form**.

That is, we cannot represent the translation transformation in (2x2) matrix form (2-D Euclidean system).

Any transformation operation can be represented as a (2x2) matrix form, except translation, i.e., translation transformation cannot be expressed as a (2x2) matrix form (2-D Euclidean system). But by using Homogeneous coordinate system (HCS), we can represent translation transformation in matrix form. The HCS and advantages of using HCS is discussed, in detail, in section 1.4.

Relation between 2-D Euclidean (Cartesian) system and HCS

Let P(x,y) be any point in 2-D Euclidean system. In Homogeneous Coordinate system, we add a third coordinate to the point. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; with the condition that (x1,y1,H1)=(x2,y2,H2) ↔ x1/H1 = x2/H2 ; y1/H1 = y2/H2. In two dimensions the value of H is usually kept at 1 for simplicity. (If we take H=0 here, then this represents point at infinity, i.e, generation of horizons).

The following table shows a relationship between 2-D Euclidean (Cartesian coordinate) system and HCS.

2-D Euclidian System	Homogeneous Coordinate System (HCS)
Any point (x,y) ⟶	(x,y,1)
H≠0); (x/H,y/H) ⟵	If (x,y,H) be any point in HCS(such that then (x,y,H)=(x/H,y/H,1), i.e. (x,y,H)

For translation transformation, any point (x,y)→(x+tx,y+ty) in 2-D Euclidian system. Using HCS, this translation transformation can be represented as (x,y,1) → (x+tx,y+ty,1). In two dimensions the value of H is usually kept at 1 for simplicity. Now, we are able to represent this translation transformation in matrix form as:

$$(x',y',1)=(x,y,1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ tx & ty & 1\end{pmatrix}$$

$P'_h = P_h.T_v$ ----------------(4)

Where P'_h and P_h represents object points in Homogeneous Coordinates and Tv is called homogeneous transformation matrix for translation. Thus, P'_h, the new coordinates of a transformed object, can be found by multiplying previous object coordinate matrix, P_h, with the transformation matrix for translation Tv.

The ***advantage*** of introducing the matrix form of translation is that it simplifies the operations on complex objects i.e., we can now build complex transformations by multiplying the basic matrix transformations. This process is called *concatenation of* *matrices* and the resulting matrix is often referred as the *composite transformation* matrix.

We can represent the basic transformations such as rotation, scaling shearing, etc., as 3x3 homogeneous coordinate matrices to make matrix multiplication compatibility with the matrix of translation. This is accomplished by augmenting the 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a third column $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and a third row (0,0,1). That is

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the new coordinates of a transformed object can be found by multiplying previous object coordinate matrix with the required transformation matrix. That is

$$\begin{bmatrix} \text{New Object} \\ \text{Coordinate} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{Previous object} \\ \text{Coordinate} \\ \text{matrix} \end{bmatrix} . \begin{bmatrix} \text{Transformation} \\ \text{matrix} \end{bmatrix}$$

Example1: Translate a square ABCD with the coordinates A(0,0),B(5,0),C(5,5),D(0,5) by 2 units in x-direction and 3 units in y-direction.

Solution: We can represent the given square, in matrix form, using homogeneous coordinates of vertices

as:

$$\begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{pmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \\ x4 & y4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

The translation factors are, tx=2, ty=3

The transformation matrix for translation :

$$T_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

New object point coordinates are:

$$[A'B'C'D'] = [ABCD].T_v$$

$$\begin{matrix} A' \\ B' \\ C' \\ D' \end{matrix} \begin{pmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \\ x'_4 & y'_4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

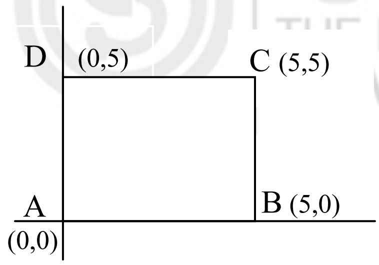
$$= \begin{pmatrix} 2 & 3 & 1 \\ 7 & 3 & 1 \\ 7 & 8 & 1 \\ 2 & 8 & 1 \end{pmatrix}$$

Thus, $A'(x'_1,y'_1)=(2,3)$

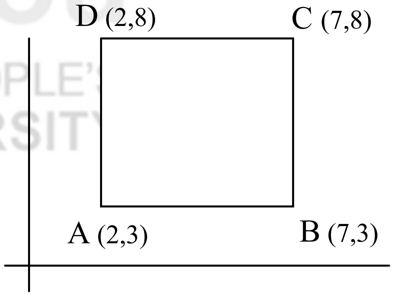
$B'(x'_2,y'_2)=(7,3)$

$C'(x'_3,y'_3)=(7,8)$ and $D'(x'_4,y'_4)=(2,8)$

The graphical representation is given below:



a) Square before Translation



b) Square after Translation

1.2.2 Rotation

In 2-D rotation, an object is rotated by an angle θ with respect to the origin. This angle is assumed to be positive for anticlockwise rotation. There are two cases for 2-D rotation, *case1*- rotation about the origin and *case2* rotation about an arbitrary point. If, the rotation is made about an arbitrary point, a set of basic transformation, i.e., composite transformation is required. For 3-D rotation involving 3-D objects, we need to specify both the angle of rotation and the axis of rotation, about which rotation has to be made. Here, we will consider case1 and in the next section we will consider *case2.*

Before starting case-1 or case-2 you must know the relationship between **polar coordinate system** and **Cartesian system:**

Relation between **polar coordinate system** and **Cartesian system**

A frequently used non-cartesian system is Polar coordinate system. The following *Figure A* shows a polar coordinate reference frame. In polar coordinate system a coordinate position is specified by r and θ, where r is a radial distance from the coordinate origin and θ is an angular displacements from the horizontal (see *Figure 2A*). Positive angular displacements are counter clockwise. Angle θ is measured in degrees. One complete counter-clockwise revolution about the origin is treated as 360^0. A relation between Cartesian and polar coordinate system is shown in *Figure 2B*.

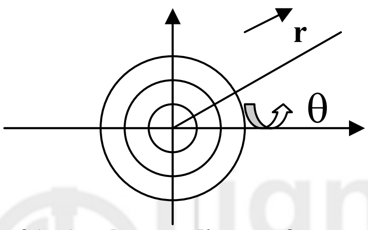


Figure 2A: A polar coordinate reference-frame

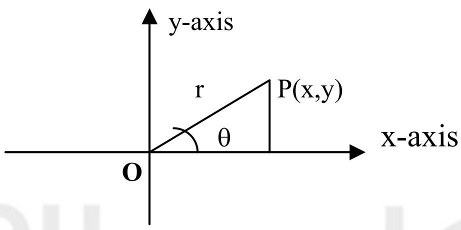


Figure 2B: Relation between Polar and Cartesian coordinates

Consider a right angle triangle in *Figure B*. Using the definition of trigonometric functions, we transform polar coordinates to Cartesian coordinates as:

$x=r.\cos\theta$
$y=r.\sin\theta$

The inverse transformation from Cartesian to Polar coordinates is:

$r=\sqrt{(x^2+y^2)}$ and $\theta=\tan^{-1}(y/x)$

Case 1: Rotation about the origin

Given a 2-D point P(x,y), which we want to rotate, with respect to the origin O. The vector OP has a length 'r' and making a positive (anticlockwise) angle ϕ with respect to x-axis.

Let P'(x'y') be the result of rotation of point P by an angle θ about the origin, which is shown in *Figure 3*.

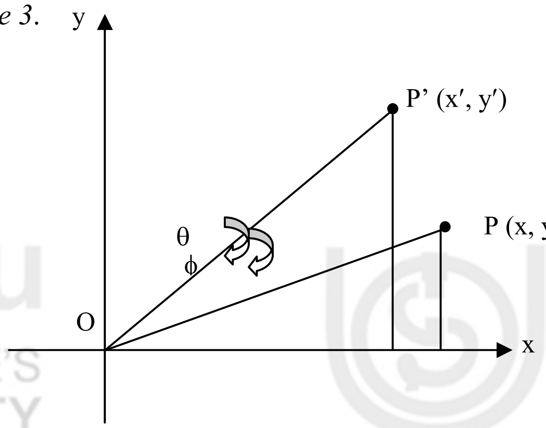


Figure 3

$P(x,y) = P(r.\cos\phi, r.\sin\phi)$
$P'(x',y') = P[r.\cos(\phi+\theta), r\sin(\phi+\theta)]$

The coordinates of P' are:

$x' = r.\cos(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$
$\quad = x.\cos\theta - y.\sin\theta$ (where $x = r\cos\phi$ and $y = r\sin\phi$)

similarly;

$y' = r\sin(\theta+\phi) = r(\sin\theta\cos\phi + \cos\theta.\sin\phi)$
$\quad = x\sin\theta + y\cos\theta$

Thus,

$$R_\theta = \left\{ \begin{array}{l} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{array} \right\} = R_\theta$$

Thus, we have obtained the new coordinate of point P after the rotation. In matrix form, the transformation relation between P' and P is given by:

$$(x'y') = (x,y) \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

that is $P' = P.R_\theta$ ---------(5)

where P'and P represent object points in 2-D Euclidean system and R_θ is transformation matrix for **anti-clockwise** Rotation.

In terms of HCS, equation (5) becomes

$$(x', y', 1) = (x, y, 1) \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(6)}$$

That is $P'_h = P_h.R_\theta$, ---------(7)

Where P'_h and P_h represents object points, after and before required transformation, in Homogeneous Coordinates and R_θ is called homogeneous transformation matrix for **anticlockwise** Rotation. Thus, P'_h, the new coordinates of a transformed object, can be found by multiplying previous object coordinate matrix, P_h, with the transformation matrix for Rotation R_θ.

Note that for **clockwise** rotation we have to put $\theta = -\theta$, thus the rotation matrix R_θ, in HCS, becomes

$$R_{-\theta}=\begin{pmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2: Perform a 45^0 rotation of a triangle A(0,0),B(1,1),C(5,2) about the origin.

Solution: We can represent the given triangle, in matrix form, using homogeneous coordinates of the vertices:

$$[ABC]=\begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{pmatrix}$$

The matrix of rotation is: $R_\theta = R_{45^0} = \begin{pmatrix} \cos45^0 & \sin45^0 & 0 \\ -\sin45^0 & \cos45^0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So the new coordinates A'B'C' of the rotated triangle ABC can be found as:

$$[A'B'C']=[ABC].\,R_{45^\circ}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 3\sqrt{2}/2 & 7\sqrt{2}/2 & 1 \end{bmatrix}$$

Thus A'=(0,0), B'=(0,√2), C'=(3√2/2,7√2/2)

The following *Figure (a)* shows the original, triangle [ABC] and *Figure (b)* shows triangle after the rotation.

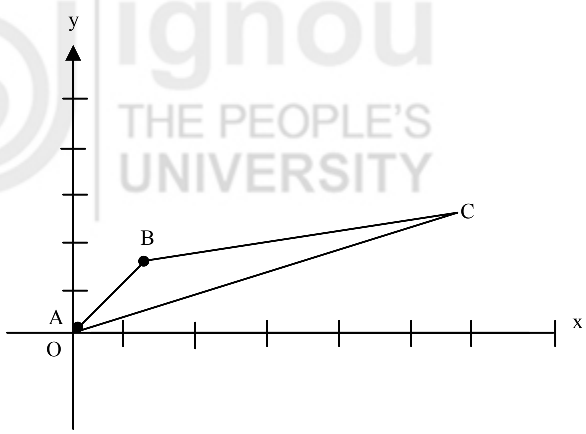


Figure (a)

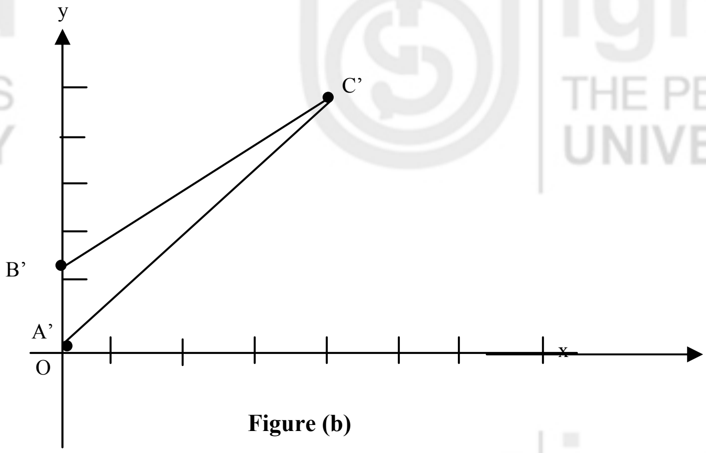


Figure (b)

☞ Check Your Progress 1

1) What are the basic advantages of using Homogeneous coordinates system.

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2) A square consists of vertices A(0,0),B(0,1),C(1,1),D(1,0). After the translation C is found to be at the new location (6,7). Determine the new location of other vertices.

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3) A point P(3,3) makes a rotating of 45^0 about the origin and then translating in the direction of vector v=5I+6J. Find the new location of P.

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4) Find the relationship between the rotations R_θ, $R_{-\theta}$, and R_θ^{-1} .

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1.2.3 Scaling

Scaling is the process of expanding or compressing the dimensions (i.e., size) of an object. An important application of scaling is in the development of viewing transformation, which is a mapping from a window used to clip the scene to a view port for displaying the clipped scene on the screen.

Let P(x,y) be any point of a given object and s_x and s_y be scaling factors in x and y directions respectively, then the coordinate of the scaled object can be obtained as:

$$\left.\begin{array}{l} x' = x.s_x \\ y' = y.s_y \end{array}\right\} \quad \text{--------(8)}$$

If the scale factor is 0<s<1, then it reduces the size of an object and if it is more then 1, it magnifies the size of the object along an axis.

For example, assume $s_x > 1$.

i) Consider (x,y)→(2.x,y) i.e., Magnification in x-direction with scale factor $s_x = 2$.

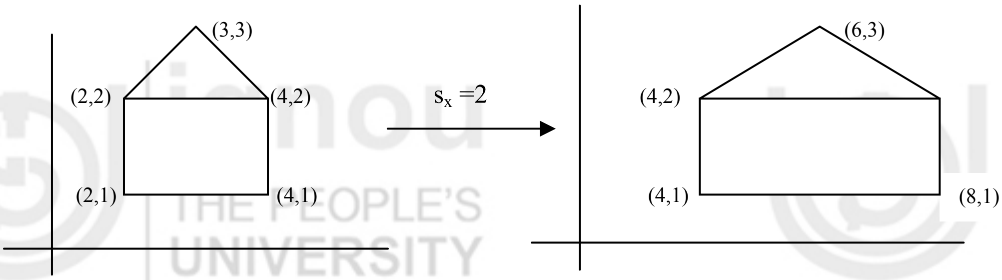

Figure a): Object before Scaling **Figure b): Object after Scaling with s_x =2**

ii) Similarly, assume $s_y > 1$ and consider (x,y)→(x,2.y), i.e., Magnification in y-direction with scale factor $s_y = 2$.

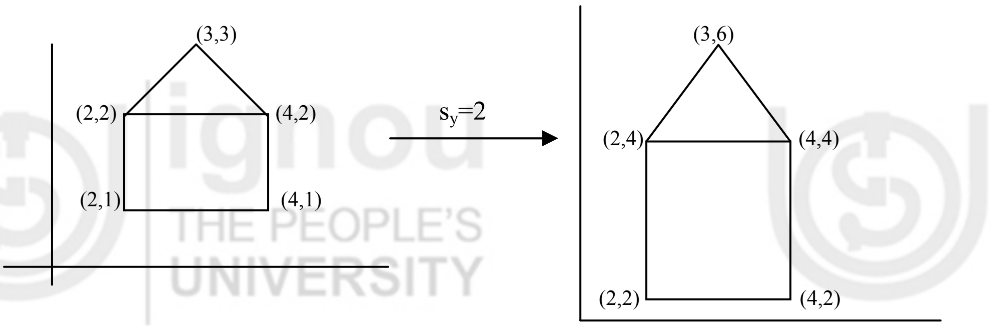

Figure a): Object before Scaling **Figure b): Object after Scaling with Sy=2**

iii) Consider (x,y)→(x.s_x,y) where 0< s_x = y_2 <1 i.e., Compression in x-direction with scale factor s_x=1/2.

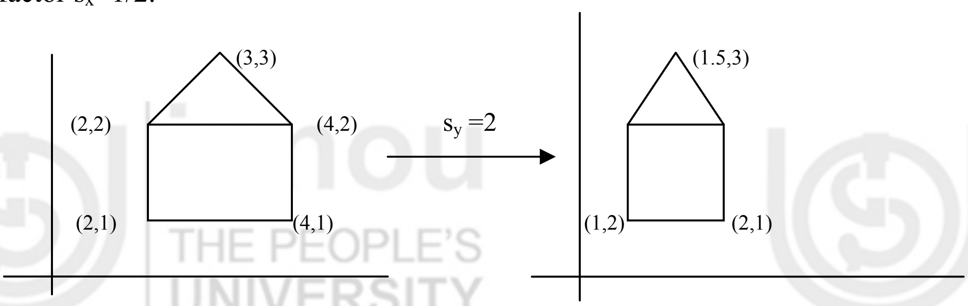

Figure a): Object before Scaling **Figure b): Object after Scaling with Sx=1/2**

Thus, the general scaling is (x,y)→ (x.s_x,y.s_y) i.e., magnifying or compression in both x and y directions depending on Scale factors s_x and s_y. We can represent this in matrix form (2-D Euclidean system) as:

$$(x',y') = (x,y)\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad \text{----(9)}$$

In terms of HCS, equation (9) becomes:

$$(x',y',1)=(x,y,1)\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{----(10)}$$

that is $P'_h=P_h.s_{sx,sy}$ ----(11)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $s_{sx,sy}$ is called transformation matrix for general scaling with scaling factor s_x and s_y .

Thus, we have seen any positive value can be assigned to scale factors s_x and s_y. We have the following three cases for scaling:

Case 1: If the values of s_x and s_y are less then 1, then the size of the object will be reduced.

Case2: If both s_x and s_y are greater then 1, then the size of the object is enlarged.

Case3: If we have the same scaling factor (i.e. s_x=s_x=S), then there will be uniform scaling (either enlargement or compression depending on the value of S_x and Sy) in both x and y directions.

Example3: Find the new coordinates of a triangle A(0,0),B(1,1),C(5,2) after it has been (a) magnified to twice its size and (b) reduced to half its size.

Solution: Magnification and reduction can be achieved by a uniform scaling of s units in both the x and y directions. If, s>1, the scaling produces magnification. If, s<1, the result is a reduction. The transformation can be written as: (x,y,1)→(s.x,s.y,1). In matrix form this becomes

$$(x,y,1).\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = (s.x,s.y,1)$$

We can represent the given triangle, shown in *Figure (a)*, in matrix form, using homogeneous coordinates of the vertices as :

$$\begin{bmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{bmatrix}$$

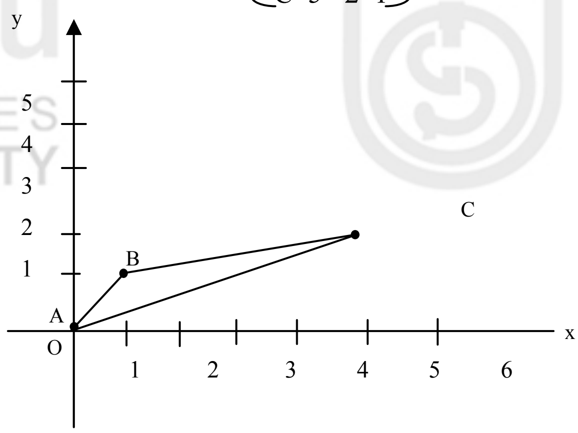

Figure a: Object before scaling

(a) choosing s=2

The matrix of scaling is: $S_{s_x,s_y} = S_{2,2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So the new coordinates A'B'C' of the scaled triangle ABC can be found as:

$$[A'B'C'] = [ABC].\, R_{2,2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 10 & 4 & 1 \end{bmatrix}$$

Thus, A'=(0,0), B'=(2,2), C'= (10, 4)

(b) Similarly, here, s=1/2 and the new coordinates are A''=(0,0), B''=(1/2,1/2), C''=(5/2,1). The following figure (b) shows the effect of scaling with $s_x=s_y=2$ and (c) with $s_x=s_y=s=1/2$.

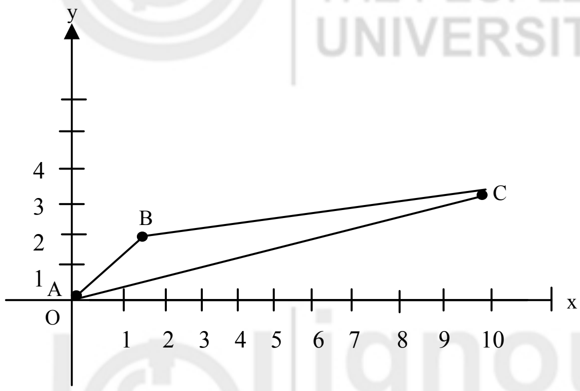

Figure b: Object after scaling with Sx = Sy = 2

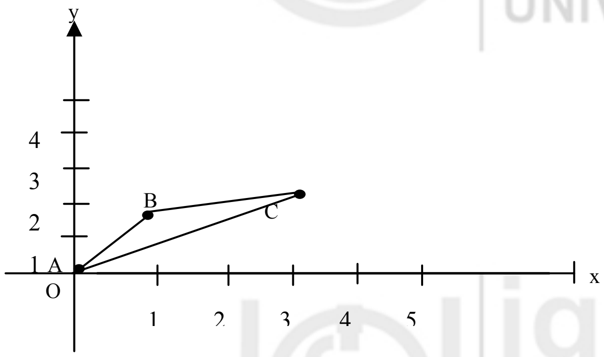

Figure c: Object after scaling with $S_x = S_y = 1/2$

1.2.4 Shearing

Shearing transformations are used for modifying the shapes of 2-D or 3-D objects. The effect of a shear transformation looks like "pushing" a geometric object in a direction that is parallel to a coordinate plane (3D) or a coordinate axis (2D). How far a direction is pushed is determined by its *shearing factor.*

One familiar example of shear is that observed when the top of a book is moved relative to the bottom which is fixed on the table.

In case of 2-D shearing, we have two types namely *x-shear* and *y-shear*.

In *x-shear*, one can push in the *x*-direction, positive or negative, and keep the *y*-direction unchanged, while in *y-shear*, one can push in the *y*-direction and keep the *x*-direction fixed.

x-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') in the x-direction, by the given scale parameter 'a',i.e., P'(x'y') be the result of x-shear of point P(x,y) by scale factor a about the origin, which is shown in *Figure 4.*

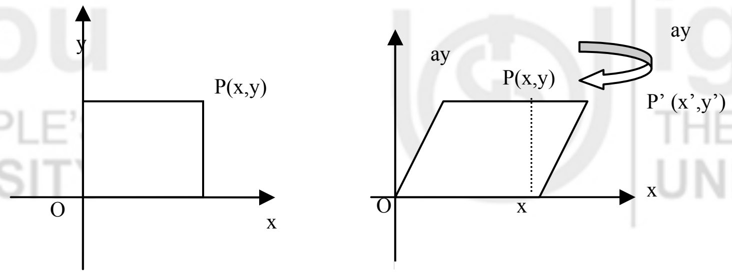

Figure 4

Thus, the points P(x,y) and P'(x',y') have the following relationship:

$$\left.\begin{aligned} x' &= x + ay \\ y' &= y \end{aligned}\right\} = Sh_x(a) \qquad \text{------(11a)}$$

where 'a' is a constant (known as shear parameter) that measures the degree of shearing. If a is negative then the shearing is in the opposite direction.

Note that P(0,H) is taken into P'(aH,H). It follows that the shearing angle A (the angle through which the vertical edge was sheared) is given by:

tan(A) = aH/H = a.

So the parameter a is just the tan of the shearing angle. In matrix form (2-D Euclidean system), we have

$$(x',y')=(x,y)\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad \text{-------(12)}$$

In terms of Homogeneous Coordinates, equation (12) becomes

$$(x',y',1)=(x,y,1).\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{-------(13)}$$

That is, $P'_h = P_h\, Sh_x(a)$ --------(14)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $Sh_x(a)$ is called homogeneous transformation matrix for x-shear with scale parameter 'a' in the x-direction.

y-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') in the x-direction, by the given scale parameter 'b'. i.e., P'(x'y') be the result of y-shear of point P(x,y) by scale factor 'b' about the origin, which is shown in *Figure 5(a)*.

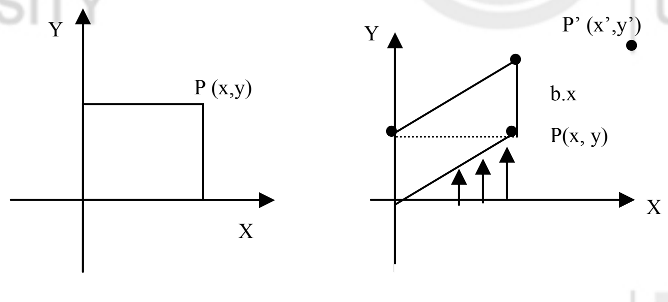

Figure 5 (a)

Thus, the points P(x,y) and P'(x',y') have the following relationship :

$$\left.\begin{array}{l} x' = x \\ y' = y+bx \end{array}\right\} = Sh_y(b) \qquad \text{----------(15)}$$

where 'b' is a constant (known as shear parameter) that measures the degree of shearing. In matrix form, we have

$$(x',y')=(x,y)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \text{----------(16)}$$

In terms of Homogeneous Coordinates, equation (16) becomes

$$(x',y',1)=(x,y,1)\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(17)}$$

That is, $P'_h = P_h.Sh_y(b)$ ---------(18)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and Sh_y (b) is called homogeneous transformation matrix for y-shear with scale factor 'b' in the y-direction.

xy-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') as a result of shear transformation in both *x*- and *y*-directions with shearing factors *a* and *b*, respectively, as shown in *Figure 5(b)*.

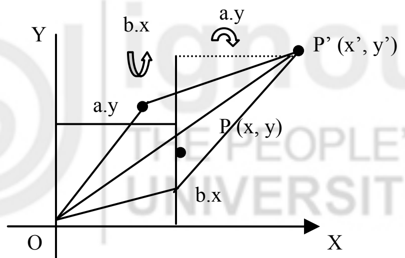

Figure 5 (b)

The points P(x,y) and P'(x',y') have the following relationship :

$$\left.\begin{array}{l} x' = x + ay \\ y' = y+bx \end{array}\right\} = Sh_{xy}(a,b) \qquad \text{----------(19)}$$

where 'ay' and 'bx' are shear factors in x and y directions, respectively. The xy-shear is also called *simultaneous shearing* or *shearing* for short.

In matrix form, we have,

$$(x',y')=(x,y)\begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \qquad \text{---------(20)}$$

In terms of Homogeneous Coordinates, we have

$$(x',y',1)=(x,y,1)\begin{pmatrix} 1 & b & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(21)}$$

That is, $P'_h = P_h.Sh_{xy}(a,b)$ ----------(22)

Where P_h and P'_h represent object points, before and after required transformation, in Homogeneous Coordinates and $Sh_{xy}(a,b)$ is called homogeneous transformation matrix for xy-shear in both *x*- and *y*-directions with shearing factors *a* and *b*, respectively,

Special case: when we put b=0 in equation (21), we have *shearing in x-direction*, and when a=0, we have *Shearing in the y-direction,* respectively**.**

Example 4: A square ABCD is given with vertices A(0,0),B(1,0),C(1,1), and D(0,1). Illustrate the effect of a) x-shear b) y-shear c) xy-shear on the given square, when a=2 and b=3.

Solution: We can represent the given square ABCD, in matrix form, using homogeneous coordinates of vertices as:

$$\begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix}$$

a) The matrix of x-shear is:

$$Sh_x(a) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the new coordinates A'B'C'D' of the x-sheared object ABCD can be found as:
$[A'B'C'D']=[ABCD].\ Sh_x(a)$

$$[A'B'C'D']= \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,0), C'=(3,1) and D'=(2,1).

b) Similarly the effect of shearing in the y direction can be found as:
$[A'B'C'D']=[ABCD].Sh_y(b)$

$$[A'B'C'D']= \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,3), C'=(1,4) and D'=(0,1).

c) Finally the effect of shearing in both directions can be found as:
$[A'B'C'D']=[ABCD].\ Sh_{xy}(a,b)$

$$[A'B'C'D']= \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,3), C'=(3,4) and D'=(2,1).

Figure (a) shows the original square, *figure (b)-(d)* shows shearing in the x, y and both directions respectively.

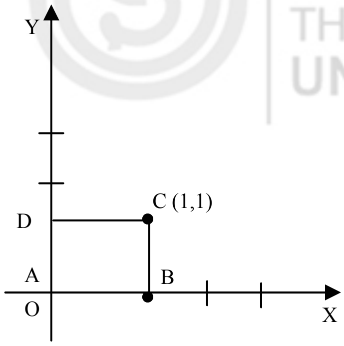

Figure (a)

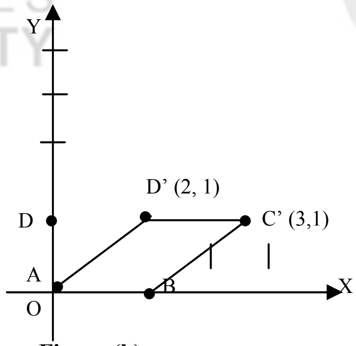

Figure (b)

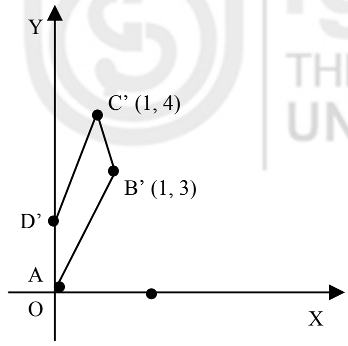

Figure (c)

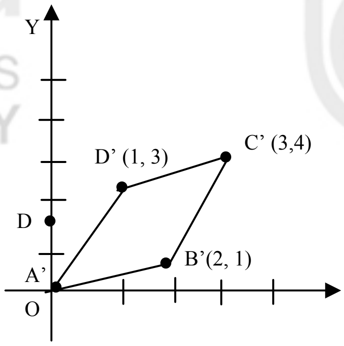

Figure (d)

Example 5: What is the use of Inverse transformation? Give the Inverse transformation for translation, rotation, reflection, scaling, and shearing.

Solution: We have seen the basic matrix transformations for translation, rotation, reflection, scaling and shearing with respect to the origin of the coordinate system. By multiplying these basic matrix transformations, we can build complex transformations, such as rotation about an arbitrary point, mirror reflection about a line etc. This process is called *concatenation of matrices* and the resulting matrix is often referred to as the *composite transformation matrix*. Inverse transformations play an important role when you are dealing with composite transformation. They come to the rescue of basic transformations by making them applicable during the construction of composite transformation. You can observed that the Inverse transformations for translation, rotation, reflection, scaling and shearing have the following relations, and v, θ, a, b, sx, sy, sz are all parameter involved in the transformations.

1) $T_v^{-1} = T_{-v}$
2) $R_\theta^{-1} = R_{-\theta}$
3) (i) $Sh_x^{-1}(a) = Sh_x(-a)$
   (ii) $Sh_y^{-1}(b) = Sh_x(-b)$
   (iii) $Sh_{xy}^{-1}(a,b) = Sh_x(-a,-b)$
4) $S^{-1}_{sx,sy,sz} = S_{1/sx,1/sy,1/sz}$
5) The transformation for mirror reflection about principal axes do not change after inversion.
   (i) $Mx^{-1} = M_{-x} = M_x$
   (ii) $My^{-1} = M_{-y} = M_y$
   (iii) $Mz^{-1} = M_{-z} = M_z$,

6) The transformation for rotations made about x,y,z axes have the following inverse:

(i) $R^{-1}_{x,\theta} = R_{x,-\theta} = R^{T}_{x,\theta}$

(ii) $R^{-1}_{y,\theta} = R_{y,-\theta} = R^{T}_{y,\theta}$

(iii) $R^{-1}_{z,\theta} = R_{z,-\theta} = R^{T}_{z,\theta}$

☞ Check Your Progress 2

1) Differentiate between the Scaling and Shearing transformation.

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2) Show that $S_{a,b} .S_{c,d} = S_{c,d} .S_{a,b} = S_{ac,bd}$

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3) Find the 3x3 homogeneous co-ordinate transformation matrix for each of the following:

a) Shift an image to the right by 3 units.
b) Shift the image up by 2 units and down 1 units.
c) Move the image down 2/3 units and left 4 units.

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4) Find the condition under which we have $Ss_{x,}s_{y} .R_{\theta}= R_{\theta}. Ss_{x,}s_{y}$.

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5) Is a simultaneous shearing the same as the shearing in one direction followed by a shearing in another direction? Why?

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1.3 COMPOSITE TRANSFORMATIONS

We can build complex transformations such as rotation about an arbitrary point, mirror reflection about a line, etc., by multiplying the basic matrix transformations. This process is called *concatenation of matrices* and the resulting matrix is often referred to as the *composite transformation matrix*. In composite transformation, a previous transformation is pre-multiplied with the next one.

In other words we can say that a sequence of the transformation matrices can be concatenated into a single matrix. This is an effective procedure as it reduces because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object. In other words we can say that a sequence of transformation matrix can be concatenated matrix into a single matrix. This is an effective procedure as it reduces computation because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object.

1.3.1 Rotation about a Point

Given a 2-D point P(x,y), which we want to rotate, with respect to an arbitrary point A(h,k). Let P'(x'y') be the result of anticlockwise rotation of point P by angle θ about A, which is shown in *Figure 6*.

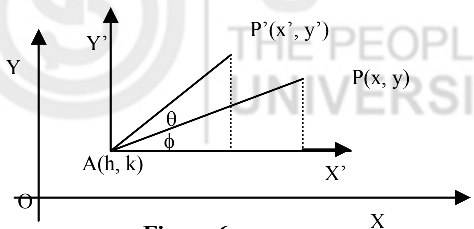


Figure 6

Since, the rotation matrix R_θ is defined only with respect to the origin, we need a set of basic transformations, which constitutes the composite transformation to compute the rotation about a given arbitrary point A, denoted by $R_{\theta,A}$. We can determine the transformation $R_{\theta,A}$ in three steps:

1) Translate the point A(h,k) to the origin O, so that the center of rotation A is at the origin.
2) Perform the required rotation of θ degrees about the origin, and
3) Translate the origin back to the original position A(h,k).

Using **v**=h**I**+k**J** as the translation vector, we have the following sequence of three transformations:

$$R_{\theta,A} = T_{-v}.\ R_\theta.\ T_v$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -k & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ (1-\cos\theta).h+k.\sin\theta & (1-\cos\theta).k-h.\sin\theta & 1 \end{pmatrix} \quad \text{-------(23)}$$

Example 5: Perform a 45^0 rotation of a triangle A (0,0), B (1,1), C (5,2) about an arbitrary point P(–1, –1).

Solution: Given triangle ABC, as show in Figure (a), can be represented in homogeneous coordinates of vertices as:

$$[ABC] = \begin{matrix} A \\ B \\ C \end{matrix}\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix}$$

From equation (23), a rotation matrix R_Q, A about a given arbitrary point A (h, k) is:

$$R_q, A = \begin{pmatrix} Cos\theta & Sin\theta & 0 \\ -Sin\theta & Cos\theta & 0 \\ (1-Cos\theta).h + k.Sin\theta & (1-Cos\theta).k - h.Sin\theta & 1 \end{pmatrix}$$

Thus $R_{45^o,}\ A = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix}$

So the new coordinates $[A'B'C']$ of the rotated triangle [ABC] can be found as:

$$[A'B'C'] = [ABC] \, . \, R_{45^o, A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix} . \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix} =$$

$$\begin{matrix} A' \\ B' \\ C' \end{matrix}\begin{bmatrix} -1 & (\sqrt{2}-1) & 1 \\ -1 & 2\sqrt{2}-1 & 1 \\ \left(\frac{3}{2}\sqrt{2}-1\right) & \left(\frac{9}{2}.\sqrt{2}-1\right) & 1 \end{bmatrix}$$

Thus, $A' = (-1,\ \sqrt{2}-1)$, $B' = (-1,\ 2\sqrt{2}-1)$, and $C' = \left(\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1\right)$. The following *figure (a) and (b)* shows a given triangle, before and after the rotation.

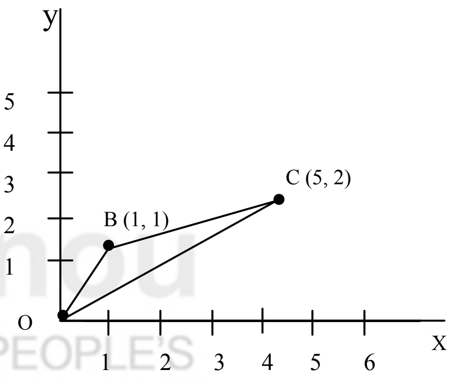


Figure (a)

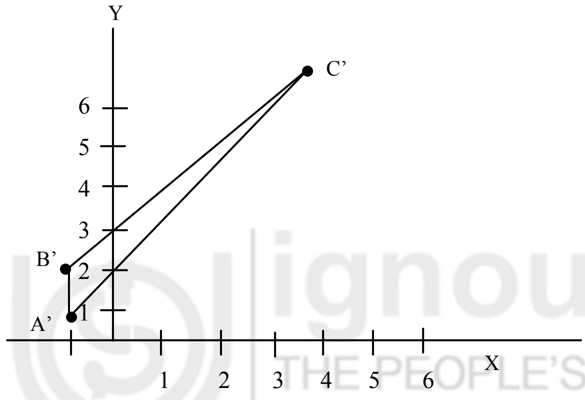


Figure (b)

1.3.2 Reflection about a Line

Reflection is a transformation which generates the mirror image of an object. As discussed in the previous block, the mirror reflection helps in achieving 8-way symmetry for the circle to simplify the scan conversion process. For reflection we need to know the reference axis or reference plane depending on whether the object is 2-D or 3-D.

Let the line L be represented by y=mx+c, where 'm' is the slope with respect to the x axis, and 'c' is the intercept on y-axis, as shown in *Figure 7*. Let P'(x',y') be the mirror reflection about the line L of point P(x,y).

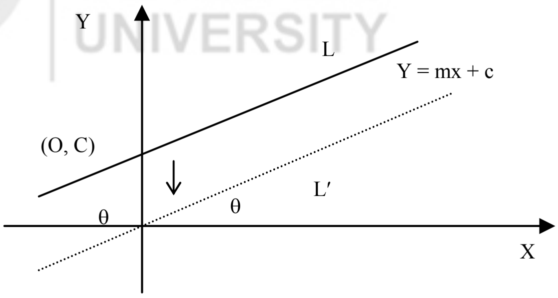


Figure 7

The transformation about mirror reflection about this line L consists of the following basic transformations:

1) Translate the intersection point A(0,c) to the origin, this shifts the line L to L'.
2) Rotate the shifted line L' by –θ degrees so that the line L' aligns with the x-axis.
3) Mirror reflection about x-axis.
4) Rotate the x-axis back by θ degrees
5) Translate the origin back to the intercept point (0,c).

In transformation notation, we have

$$M_L = T_{-v}.R_{-\theta}.M_X.R_{\theta}.T_v \quad , \qquad \text{where } v=0I+cJ$$

$$M_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2.\cos\theta.\sin\theta & 0 \\ 2.\sin\theta.\cos\theta & \sin^2\theta - \cos^2\theta & 0 \\ -2.c.\sin\theta.\cos\theta & -c.(\sin^2\theta - \cos^2\theta)+c & 1 \end{pmatrix} \qquad \text{----------(24)}$$

Let tanθ=m, the standard trigonometry yields $\sin\theta = m/\sqrt{(m^2+1)}$ and $\cos\theta = 1/\sqrt{(m^2+1)}$. Substituting these values for sinθ and cosθ in the equation (24), we have:

$$M_L = \begin{pmatrix} (1-m^2)/(m^2+1) & 2m/(m^2+1) & 0 \\ 2m/(m^2+1) & (m^2-1)/(m^2+1) & 0 \\ -2cm/(m^2+1) & 2c/(m^2+1) & 1 \end{pmatrix} \qquad \text{------------(25)}$$

Special cases

1) If we put c = 0 and m=tanθ=0 in the equation (25) then we have the reflection about the line y = 0 i.e. about x-axis. In matrix form:

$$Mx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(26)}$$

2) If c = 0 and m=tanθ=∞ then we have the reflection about the line x=0 i.e. about y-axis. In matrix form:

$$My = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(27)}$$

4) To get the mirror reflection about the line y = x, we have to put m=1 and c=0. In matrix form:

$$M_{y=x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(28)}$$

5) Similarly, to get the mirror reflection about the line y = – x, we have to put m = –1 and c = 0. In matrix form:

$$M_{y=-x} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(29)}$$

6) The mirror reflection about the Origin (i.e., an axis perpendicular to the xy plane and passing through the origin).

$$M_{org} = \begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \text{----------(30)}$$

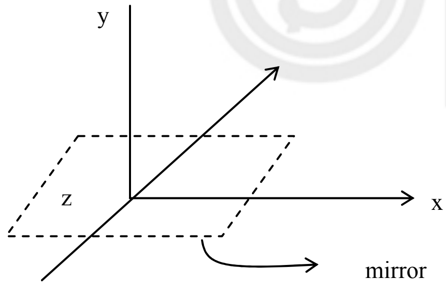


Figure 7(a)

Example 6: Show that two successive reflections about either of the coordinate axes is equivalent to a single rotation about the coordinate origin.

Solution: Let (x, y) be any object point, as shown in *Figure (a)*. Two successive reflection of P, either of the coordinate axes, i.e., Reflection about x-axis followed by reflection about y-axis or *vice-versa* can be reprosecuted as:

$$(x, y) \xrightarrow[M_x]{} (x, -y) \xrightarrow[M_y]{} (-x, -y) \qquad \text{----(i)}$$

$$(x, y) \xrightarrow[M_y]{} (x, -y) \xrightarrow[M_x]{} (-x, -y) \qquad \text{----(ii)}$$

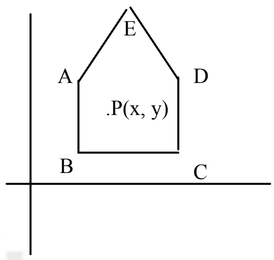
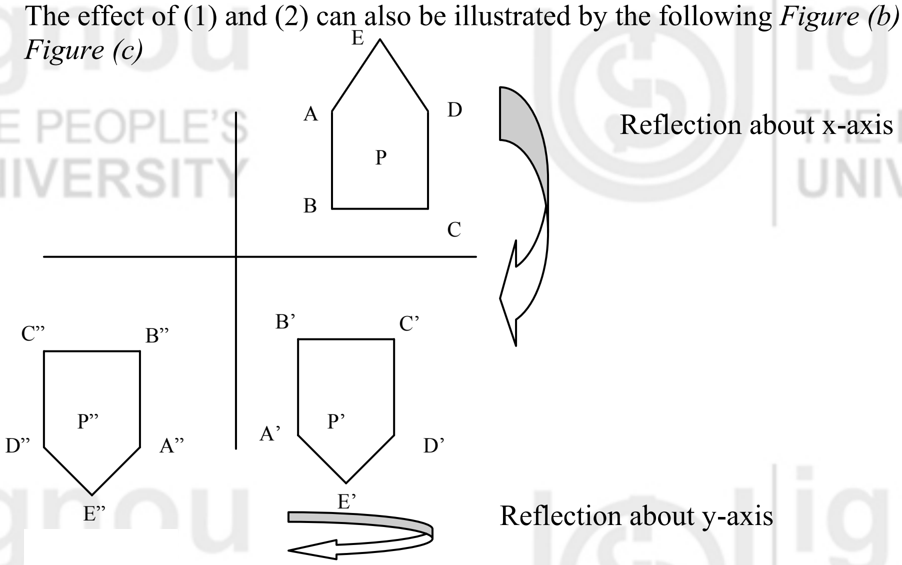


Figure (a)

The effect of (1) and (2) can also be illustrated by the following *Figure (b) and Figure (c)*



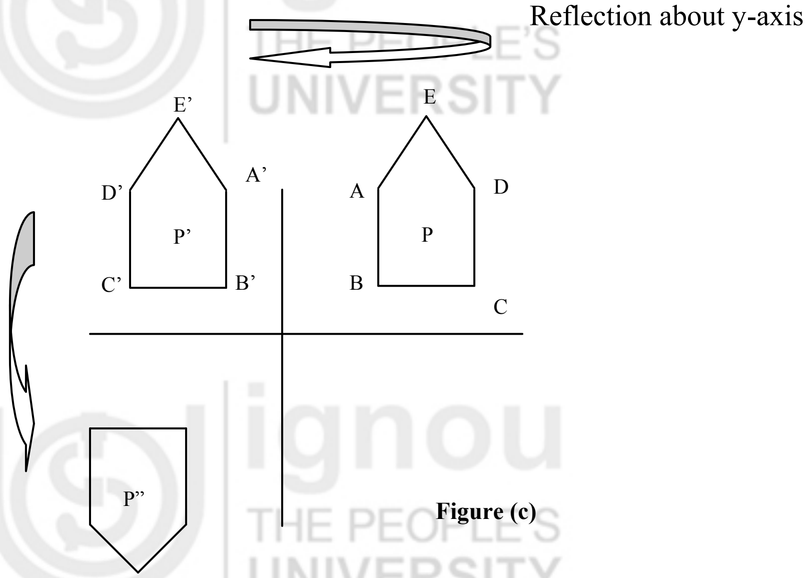


Figure (c)

From equation (i) and (ii), we can write:

$$(x, y) \longrightarrow (-x, -y) \;=\; (x, y)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(iii)}$$

Equation (3) is the required reflection about the origin. Hence, two successive reflections about either of the coordinate axes is just equivalent to a single rotation about the coordinate origin.

Example 7: Find the transforation matrix for the reflection about the line y = x.

Solution: The transformation for mirror reflection about the line y = x, consists of the following three basic transformations.

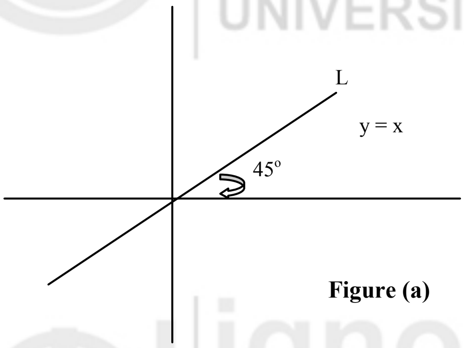


Figure (a)

1) Rotate the line L through 45° in clockwise rotation,
2) Perform the required Reflection about the x-axis.
3) Rotate back the line L by – 45°

i.e.,

$M_L = R_{45^o} \,.\, M_x \,.\, R_{-45^o}$

$$= \begin{bmatrix} \text{Cos}\,45^o & -\text{Sin}\,45^o & 0 \\ \text{Sin}\,45^o & \text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} \;.\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} Cos\,45^o & +Sin\,45^o & 0 \\ -Sin\,45^o & Cos\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^o & \sin 45^o & 0 \\ \sin 45^o & -\cos 45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \cos 45^o & \sin 45^o & 0 \\ -\sin 45^o & \cos 45^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 90 & \sin 90 & 0 \\ \sin 90 & -\cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_y = x$$

Example 8 : Reflect the diamond-shaped polygon whose vertices are A(–1,0), B(0, –2),C(1,0) and D(0,2) about (a) the horizontal line y=2, (b) the vertical line x=2, and (c) the line y=x+2.

Solution: We can represent the given polygon by the homogeneous coordinate matrix as

$$V=[ABCD] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

a) The horizontal line y=2 has an intercept (0,2) on y axis and makes an angle of 0 degree with the x axis. So m=0 and c=2. Thus, the reflection matrix

$M_L = T_{-v}.R_{-\theta}.M_x.R_{\theta}.T_{-v}$, where $\mathbf{v}=0\mathbf{I}+2\mathbf{J}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

So the new coordinates A'B'C'D' of the reflected polygon ABCD can be found as:

[A'B'C'D']=[ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 6 & 1 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A'=(-1,4), B'=(0,6), C'=(1,4) and D'=(0,2).

b) The vertical line x=2 has no intercept on y-axis and makes an angle of 90 degree with the x-axis. So m=tan90^0=∞ and c=0. Thus, the reflection matrix

$M_L = T_{-v}.R_{-\theta}.M_y.R_{\theta}.T_{-v}$, where $\mathbf{v}=2\mathbf{I}$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

So the new coordinates A'B'C'D' of the reflected polygon ABCD can be found as:

[A'B'C'D']=[ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 1 \\ 4 & -2 & 1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

Thus, A'=(5,0), B'=(4,-2), C'=(3,0) and D'=(4,2)

c) The line y=x+2 has an intercept (0,2) on y-axis and makes an angle of 45^0 with the x-axis. So m=tan45^0=1 and c=2. Thus, the reflection matrix

$$M_L = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

The required coordinates A',B', C', and D' can be found as:
[A'B'C'D']=[ABCD]. M_L

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}. \quad = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 1 \\ -2 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A'=(–2,1), B'=(–4,2), C'=(–2,3) and D'=(0,2)

The effect of the reflected polygon, which is shown in *Figure (a)*, about the line y=2, x=2, and y=x+2 is shown in *Figure (b) - (d)*, respectively.

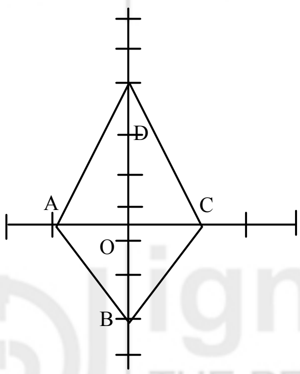


Figure (a)

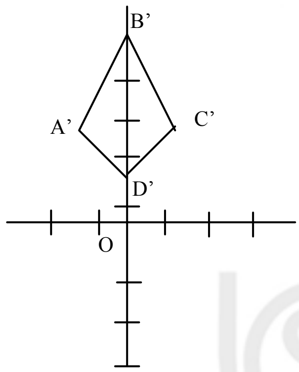


Figure (b)

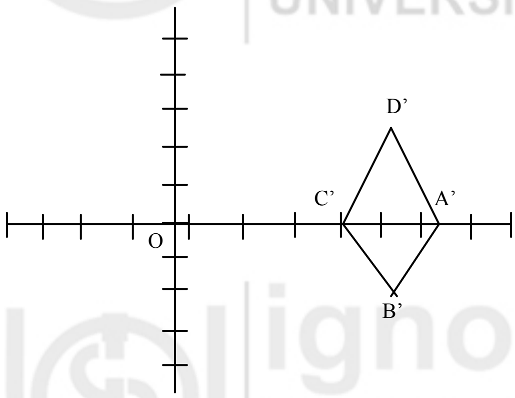


Figure (c)

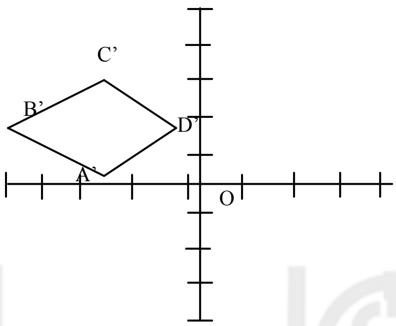


Figure (d)

1.4 HOMOGENEOUS COORDINATE SYSTEMS

Let P(x,y) be any point in 2-D Euclidean (Cartesian) system.

In Homogeneous Coordinate system, we add a third coordinate to a point. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; with the condition that (x1,y1,H1)=(x2,y2,H2) ↔ x1/H1 = x2/H2 ; y1/H1 = y2/H2.

(Here, if we take H=0, then we have point at infinity, i.e., generation of horizons).

Thus, (2,3,6) and (4,6,12) are the same points are represented by different coordinate triples, i.e., each point has many different Homogeneous Coordinate representation.

2-D Euclidian System	Homogeneous Coordinate System
Any point (x,y) ⟶	(x,y,1)
	If (x,y,H) be any point in HCS(such that H≠0); Then (x,y,H)=(x/H,y/H,1)
(x/H,y/H) ⟵	(x,y,H)

Now, we are in the position to construct the matrix form for the translation with the use of homogeneous coordinates.

For translation transformation (x,y)→(x+tx,y+ty) in Euclidian system, where tx and ty are the translation factor in x and y direction, respectively. Unfortunately, this way of describing translation does not use a matrix, so it cannot be combined with other transformations by simple matrix multiplication. Such a combination would be desirable; for example, we have seen that rotation about an arbitrary point can be done by a translation, a rotation, and another translation. We would like to be able to combine these three transformations into a single transformation for the sake of efficiency and elegance. One way of doing this is to use homogeneous coordinates. In homogeneous coordinates we use 3x3 matrices instead of 2x2, introducing an additional dummy coordinate H. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; In two dimensions the value of H is usually kept at 1 for simplicity.

Thus, in HCS (x,y,1) → (x+tx,y+ty,1), now, we can express this in matrix form as:

$$(x',y',1)=(x,y,1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ t_x & t_y & 1\end{pmatrix}$$

The ***advantage*** of introducing the matrix form of translation is that it simplifies the operations on complex objects, i.e., we can now build complex transformations by multiplying the basic matrix transformations.

In other words, we can say, that a sequence of transformation matrices can be concatenated into a single matrix. This is an effective procedure as it reduces the computation because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object. Matrix representation is standard method of implementing transformations in computer graphics.

Thus, from the point of view of matrix multiplication, with the matrix of translation, the other basic transformations such as scaling, rotation, reflection, etc., can also be expressed as 3x3 *homogeneous coordinate matrices*. This can be accomplished by augmenting the 2x2 matrices with a third row (0,0,x) and a third column. That is

$$\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} \qquad \begin{pmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix}$$

Example 9: Show that the order in which transformations are performed is important by applying the transformation of the triangle ABC by:

(i) Rotating by 45^o about the origin and then translating in the direction of the vector (1,0), and

(ii) Translating first in the direction of the vector (1,0), and then rotating by 45^o about the origin, where A = (1, 0) B = (0 ,1) and C = (1, 1).

Solution: We can represent the given triangle, as shown in *Figure (a)*, in terms of Homogeneous coordinates as:

$$[ABC] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

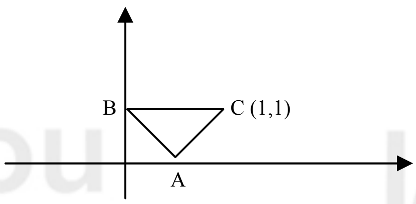


Figure (a)

Suppose the rotation is made in the counter clockwise direction. Then, the transformation matrix for rotation, R_{45^o}, in terms of homogeneous coordinate system is given by:

$$R_{45^o} = \begin{bmatrix} \text{Cos}\,45^o & \text{Sin}\,45^o & 0 \\ -\text{Sin}\,45^o & \text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Translation matrix, T_v, where V = 1I + 0J is:

$$T_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where t_x and t_y is the translation factors in the x and y directions respectively.

i) Now the rotation followed by translation can be computed as:

$$R_{45^o} \,.\, T_v = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

So the new coordinates $A'B'C'$ of a given triangle ABC can be found as:

$$[A'B'C'] = [A\,B\,C] \,.\, R_{45^o} \,.\, T_v$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1/\sqrt{2}+1) & 1/\sqrt{2} & 1 \\ (-1/\sqrt{2}+1) & 1/\sqrt{2} & 1 \\ 1 & \sqrt{2} & 1 \end{bmatrix} \qquad \text{(I)}$$

implies that the given triangle A(1,0), B (0, 1) C (1, 1) be transformed into $A'\left(\frac{1}{\sqrt{2}}+1, \frac{1}{\sqrt{2}}\right)$, $B'\left(\frac{-1}{\sqrt{2}}+1, \frac{1}{\sqrt{2}}\right)$ and $C'\left(1, \sqrt{2}\right)$, respectively, as shown in *Figure (b)*.

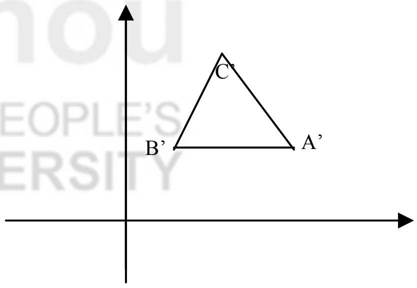


Figure (b)

Similarly, we can obtain the translation followed by rotation transformation as:

$$T_v \cdot R_{45^o} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

And hence, the new coordinates $A'B'C'$ can be computed as:

$$[A'B'C'] = [A\,B\,C].T_v \;\; R_{45^o}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 1 \\ 0 & 2/\sqrt{2} & 1 \\ 1/\sqrt{2} & 3/\sqrt{2} & 1 \end{bmatrix} \qquad \text{(II)}$$

Thus, in this case, the given triangle A(1,0), B(0, 1) and C(1,1) are transformed into $A''\left(2/\sqrt{2}, 2/\sqrt{2}\right)$, $B''\left(0, 2/\sqrt{2}\right)$ *and* $C''\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, respectively, as shown in *Figure (c)*.

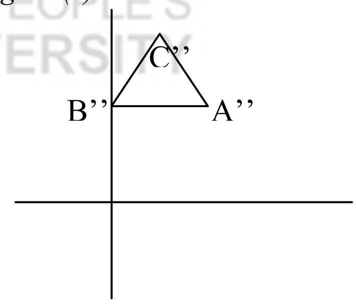


Figure (c)

By (I) and (II), we see that the two transformations do not commute.

☞ Check Your Progress 3

1) Show that transformation matrix (28), for the reflection about the line y=x, is equivalent to the reflection relative to the x-axis followed by a counterclockwise rotation of 90^0.

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2) Give a single 3x3 homogeneous coordinate transformation matrix, which will have the same effect as each of the following transformation sequences.
   a) Scale the image to be twice as large and then translate it 1 unit to the left.
   b) Scale the x direction to be one-half as large and then rotate counterclockwise by 90^0 about the origin.
   c) Rotate counterclockwise about the origin by 90^0 and then scale the x direction to be one-half as large.
   d) Translate down ½ unit, right ½ unit, and then rotate counterclockwise by 45^0.

3) Obtain the transformation matrix for mirror reflection with respect to the line y=ax, where 'a' is a constant.

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4) Obtain the mirror reflection of the triangle formed by the vertices A(0,3),B(2,0) and C(3,2) about the line passing through the points (1,3) and (–1, –1).

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1.5 3-D TRANSFORMATIONS

The ability to represent or display a three-dimensional object is fundamental to the understanding of the shape of that object. Furthermore, the ability to rotate, translate, and project views of that object is also, in many cases, fundamental to the understanding of its shape. Manipulation, viewing, and construction of three-dimensional graphic images require the use of three-dimensional *geometric* and *coordinate transformations*. In *geometric transformation*, the coordinate system is fixed, and the desired transformation of the object is done with respect to the coordinate system. In *coordinate* transformation, the object is fixed and the desired transformation of the object is done on the coordinate system itself. These transformations are formed by composing the basic transformations of translation, scaling, and rotation. Each of these transformations can be represented as a matrix transformation. This permits more complex transformations to be built up by use of matrix multiplication or concatenation. We can construct the complex objects/pictures, by instant transformations. In order to represent all these transformations, we need to use homogeneous coordinates.

Hence, if P(x,y,z) be any point in 3-D space, then in HCS, we add a fourth-coordinate to a point. That is instead of (x,y,z), each point can be represented by a Quadruple (x,y,z,H) such that H≠0; with the condition that x1/H1=x2/H2; y1/H1=y2/H2; z1/H1=z2/H2. For two points $(x_1, y_1, z_1, H_1) = (x_2, y_2, z_2, H_2)$ where $H_1 \neq 0$, $H_2 \neq 0$. Thus any point (x,y,z) in Cartesian system can be represented by a four-dimensional vector as (x,y,z,1) in HCS. Similarly, if (x,y,z,H) be any point in HCS then (x/H,y/H,z/H) be the corresponding point in Cartesian system. Thus, a point in three-dimensional space (x,y,z) can be represented by a four-dimensional point as: (x',y',z',1)=(x,y,z,1).[T], where [T] is some transformation matrix and (x',y'z',1) is a new coordinate of a given point (x,y,z,1), after the transformation.

The generalized 4x4 transformation matrix for three-dimensional homogeneous coordinates is:

$$[T]=\begin{pmatrix} a & b & c & w \\ d & e & f & x \\ g & h & I & y \\ l & m & n & z \end{pmatrix} = \left(\begin{array}{c|c} (3x3) & (3x1) \\ \hline (1x3) & (1x1) \end{array}\right) \quad \text{--------(31)}$$

The upper left (3x3) sub matrix produces *scaling, shearing, rotation* and *reflection* transformation. The lower left (1x3) sub matrix produces *translation*, and the upper right (3x1) sub matrix produces a *perspective* transformation, which we will study in the next unit. The final lower right-hand (1x1) sub matrix produces overall *scaling*.

1.5.1 Transformation for 3-D Translation

Let P be the point object with the coordinate (x,y,z). We wish to translate this object point to the new position say, P'(x',y',z') by the translation Vector $V=t_x.\mathbf{I}+t_y.\mathbf{J}+t_z.\mathbf{K}$, where t_x , t_y and t_z are the translation factor in the x, y, and z directions respectively, as shown in *Figure 8*. That is, a point (x,y,z) is moved to $(x+ t_x,y+ t_y,z+ t_z)$. Thus the new coordinates of a point can be written as:

$$\left.\begin{array}{l} x'=x+t_x \\ y'=y+t_y \\ z'=z+t_z \end{array}\right\} =T_v \quad \text{---------(32)}$$

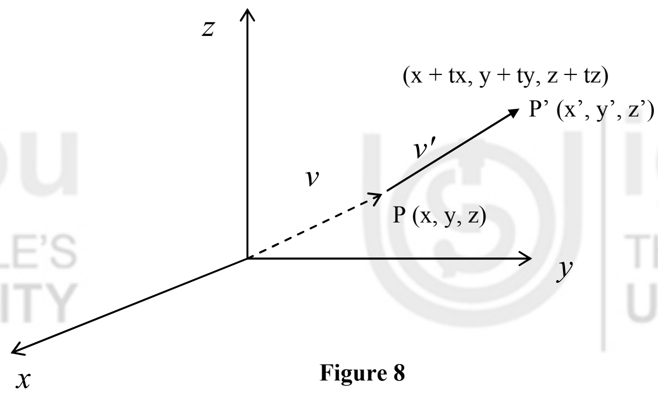


Figure 8

In terms of homogeneous coordinates, equation (32) can be written as

$$(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{pmatrix} \quad \text{--------(33)}$$

i.e., $P'_h = P_h.T_v$ ----------(34)

1.5.2 Transformation for 3-D Rotation

Rotation in three dimensions is considerably more complex then rotation in two dimensions. In 2-D, a rotation is prescribed by an angle of rotation θ and a centre of rotation, say P.

However, in 3-D rotations, we need to mention the angle of rotation and the axis of rotation. Since, we have now three axes, so the rotation can take place about any one of these axes. Thus, we have rotation about x-axis, y-axis, and z-axis respectively.

Rotation about z-axis

Rotation about z-axis is defined by the xy-plane. Let a 3-D point P(x,y,z) be rotated to P'(x',y',z') with angle of rotation θ see *Figure 9*. Since both P and P' lies on xy-plane i.e., z=0 plane their z components remains the same, that is z=z'=0.

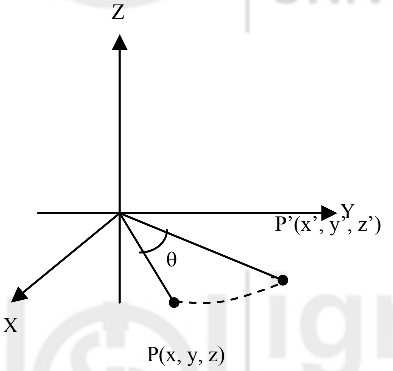


Figure 9

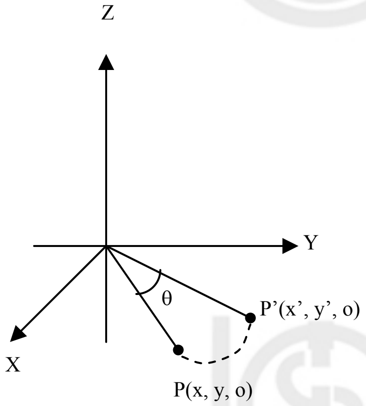


Figure 10

Thus, P'(x'y',0) be the result of rotation of point P(x,y,0) making a positive (anticlockwise) angle ϕ with respect to z=0 plane, as shown in *Figure 10*.

From *figure (10)*,

$P(x,y,0) = P(r.\cos\phi, r.\sin\phi, 0)$
$P'(x',y',0) = P[r.\cos(\phi+\theta), r\sin(\phi+\theta), 0]$

The coordinates of P' are:

$x' = r.\cos(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$
$\quad = x.\cos\theta - y.\sin\theta \quad$ (where $x = r\cos\phi$ and $y = r\sin\phi$)

similarly;

$y' = r\sin(\theta+\phi) = r(\sin\theta\cos\phi + \cos\theta.\sin\phi)$
$\quad = x\sin\theta + y\cos\theta$

Thus,

$$[Rz]_\theta = \begin{Bmatrix} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{Bmatrix} \qquad \text{----------(35)}$$

In matrix form,

$$(x'y',z') = (x,y,z)\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(36)}$$

In terms of HCS, equation (36) becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(37)}$$

That is, $P'_h = P_h.[Rz]_\theta$ ---------(38)

Rotations about x-axis and y-axis

Rotation about the x-axis can be obtained by cyclic interchange of x→y→z→x in equation (35) of the z-axis rotation i.e.,

$$[Rz]_\theta = \begin{cases} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

↓ After cyclic interchange of x→y→z→x

$$[Rx]_\theta = \begin{cases} y' = y.\cos\theta - z.\sin\theta \\ z' = y.\sin\theta + z.\cos\theta \\ x' = x \end{cases} \qquad \text{--------(39)}$$

So, the corresponding transformation matrix in homogeneous coordinates becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P'_h = P_h.[Rx]_\theta$ -------(40)

Similarly, the rotation about y-axis can be obtained by cyclic interchange of x→y→z→x in equation (39) of the x-axis rotation $[Rx]_\theta$ i.e.,

$$[Rx]_\theta = \begin{cases} y' = y.\cos\theta - z.\sin\theta \\ z' = y.\sin\theta + z.\cos\theta \\ x' = x \end{cases}$$

↓ After cyclic interchange of x→y→z→x

$$[R_y]_\theta = \begin{cases} z' = z.\cos\theta - x.\sin\theta \\ x' = z.\sin\theta + x.\cos\theta \\ y' = y \end{cases} \qquad \text{-------(41)}$$

So, the corresponding transformation matrix in homogeneous coordinates becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.\,[R_y]_\theta$ -------(42)

1.5.3 Transformation for 3-D Scaling

As we have seen earlier, the scaling process is mainly used to change the size of an object. The scale factors determine whether the scaling is a magnification, s>1, or a reduction, s<1. Two-dimensional scaling, as in equation (8), can be easily extended to scaling in 3-D case by including the z-dimension.

For any point (x,y,z), we move into $(x.s_x, y.s_y, z.s_z)$, where s_x, s_y, and s_z are the scaling factors in the x,y, and z-directions respectively.

Thus, scaling with respect to origin is given by:

$$S_{s_x,s_y,s_z} = \begin{cases} x' = x.s_x \\ y' = y.s_y \\ z' = z.s_z \end{cases} \quad \text{---------(43)}$$

In matrix form,

$$(x'y',z') = (x,y,z) \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \quad \text{---------(44)}$$

In terms of HCS, equation (44) becomes

$$(x'y',z',1) = (x,y,z,1) \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.\ S_{s_x,s_y,s_z}$ --------(45)

1.5.4 Transformation for 3-D Shearing

Two-dimensional xy- shearing transformation, as defined in equation (19), can also be easily extended to 3-D case. Each coordinate is translated as a function of displacements of the other two coordinates. That is,

$$Sh_{xyz} = \begin{cases} x' = x + a.y + b.z \\ y' = y + c.x + d.z \\ z' = z + e.x + f.y \end{cases} \quad \text{--------(46)}$$

where a,b,c,d,e and f are the shearing factors in the respective directions.

In terms of HCS, equation (46) becomes

$$(x'y',z',1) = (x,y,z,1) \begin{pmatrix} 1 & c & e & 0 \\ a & 1 & f & 0 \\ b & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P'_h = P_h.Sh_{xyz}$ -------(47)

Note that the off-diagonal terms in the upper left 3x3 sub matrix of the generalized 4x4 transformation matrix in equation (31) produce shear in three dimensions.

1.5.5 Transformation for 3-D Reflection

For 3-D reflections, we need to know the reference plane, i.e., a plane about which the reflection is to be taken. Note that for each reference plane, the points lying on the plane will remain the same after the reflection.

Mirror reflection about xy-plane

Let P(x,y,z) be the object point, whose mirror reflection is to be obtained about xy-plane(or z=0 plane). For the mirror reflection of P about xy-plane, only there is a change in the sign of z-coordinate, as shown in *Figure (11)*. That is,

$$M_{xy} = \begin{cases} x' = x \\ y' = y \\ z' = -z \end{cases} \qquad \text{-------(48)}$$

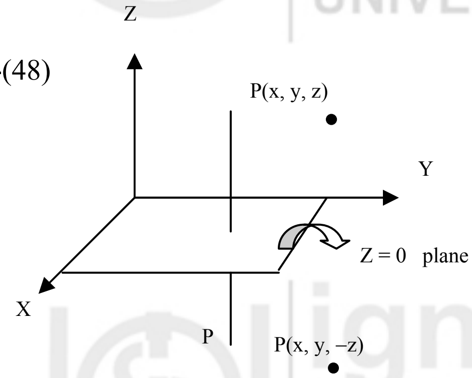


Figure 11

In matrix form,

$$(x'\,y',z') = (x,y,z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{----(49)}$$

In terms of HCS (Homogenous coordinate systems), equation (49) becomes

$$(x'\,y',z',1) = (x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.M_{xy}$ -----(50)

Similarly, the mirror reflection about yz plane shown in *Figure 12* can be represented as:

$$M_{yz} = \begin{cases} x' = -x \\ y' = y \\ z' = z \end{cases} \qquad \text{-----(51)}$$

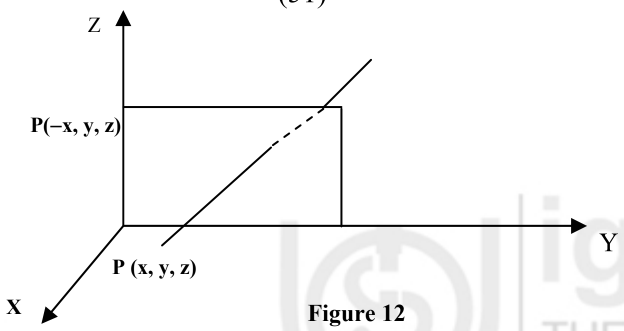


Figure 12

In matrix form,

$$(x'\,y',z') = (x,y,z)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{------(52)}$$

In terms of HCS, equation (52) becomes

$$(x'\,y',z',1) = (x,y,z,1)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P \cdot M_{yz}$; ------(53)

and similarly, the reflection about xz plane, shown in *Figure 13*, can be presented as:

$$M_{xz} = \begin{cases} x' = x \\ y' = -y \\ z' = z \end{cases} \quad \text{-------(54)}$$

In matrix form,

$$(x'y',z') = (x,y,z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{-------(55)}$$

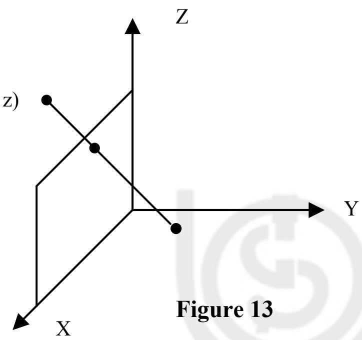


Figure 13

In terms of HCS, equation (55) becomes

$$(x'y',z',1) = (x,y,z,1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P \cdot M_{xz}$ --------(56)

1.6 SUMMARY

In this unit, the following things have been discussed in detail:

- Various geometric transformations such as translation, rotation, reflection, scaling and shearing.
- Translation, Rotation and Reflection transformations are used to manipulate the given object, whereas Scaling and Shearing transformation changes their sizes.
- Translation is the process of changing the position (not the shape/size) of an object w.r.t. the origin of the coordinate axes.
- In 2-D rotation, an object is rotated by an angle θ. There are two cases of 2-D rotation: *case1*- rotation about the origin and *case2*- rotation about an arbitrary point. So, in 2-D, a rotation is prescribed by an angle of rotation θ and a centre of rotation, say P. However, in 3-D rotations, we need to mention the angle of rotation and the axis of rotation.
- Scaling process is mainly used to change the shape/size of an object. The scale factors determine whether the scaling is a magnification, $s>1$, or a reduction, $s<1$.
- Shearing transformation is a special case of translation. The effect of this transformation looks like "pushing" a geometric object in a direction that is parallel to a coordinate plane (3D) or a coordinate axis (2D). How far a direction is pushed is determined by its *shearing factor.*
- Reflection is a transformation which generates the mirror image of an object. For reflection we need to know the reference axis or reference plane depending on whether the object is 2-D or 3-D.
- Composite transformation involves more than one transformation concatenated into a single matrix. This process is also called *concatenation of matrices.* Any transformation made about an arbitrary point makes use of composite transformation such as Rotation about an arbitrary point, reflection about an arbitrary line, etc.
- The use of homogeneous coordinate system to represent the translation transformation in matrix form, extends our N-coordinate system with (N+1) coordinate system.

- The transformations such as translation, rotation, reflection, scaling and shearing can be extended to 3D cases.

1.7 SOLUTIONS/ANSWERS

Check Your Progress 1

1) Matrix representation are standard method of implementing transformations in computer graphics. But unfortunately, we are not able to represent all the transformations in a (2 x 2) matrix form; such as translation. By using Homogeneous coordinates system (HCS), we can represent all the transformations in matrix form. For translation of point (x, y) $\rightarrow$ (x $+t_x$, y $+ t_y$), it is not possible to represent this transformation in matrix form. But, now in HCS;

$$(x', y', 1) = (x, y, 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

The advantage of introducing the matrix form for translation is that we can now build a complex transformation by multiplying the basic matrix transformation. This is an effective procedure as it reduces the computations.

2) The translation factor, t_x and t_y can be obtained from new old coordinates of vertex C.

$t_x = 6 \; -1 \; = \; 5$

$t_y = 7 \; -1 \; = 6$

The new coordinates [A′ B′ C′ D′] = [A B C D] . T_v

$$\begin{matrix} A' \\ B' \\ C' \\ D' \end{matrix}\begin{bmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \\ x'_4 & y'_4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 5 & 7 & 1 \\ 6 & 7 & 1 \\ 6 & 6 & 1 \end{bmatrix}$$

Thus A′ = (5, 6), B′ = (5, 7), C′ = (6, 7) and D′ = (6, 6)

3) The new coordinate P′ of a point P, after the Rotation of 45° is:

$P' = P.R_{45^o}$

$$(x', y', 1) = (x, y, 1)\begin{bmatrix} \text{Cos}45^o & \text{Sin}45^o & 0 \\ -\text{Sin}45^o & \text{Cos}45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = (x, y, 1)\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\frac{1}{\sqrt{2}}(x - y), \; \frac{1}{\sqrt{2}}(x + y, 1)\right] = \left(0, \; 6/\sqrt{2}, 1\right)$$

Now, this point P′is again translated by $t_x = 5$ and $t_y = 6$. So the final coordinate P″ of a given point P, can be obtained as:

$$(x'', y'', 1) = (x', y', 1) \; . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix}$$

$$= \left(0, 6/\sqrt{2}, 1\right) . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix} = \left(5, \frac{6}{\sqrt{6}} + 6, 1\right)$$

Thus P″ (x″, y″) = $(5, \frac{6}{\sqrt{2}} + 6)$

4) $R_{\theta} = \begin{pmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{pmatrix}$ $R_{-\theta} = \begin{pmatrix} \text{Cos}(-\theta) & \text{Sin}(-\theta) \\ -\text{Sin}(-\theta) & \text{Cos}(\theta) \end{pmatrix} = \begin{pmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{pmatrix}$

$$\because \quad R_{\theta} . R_{-\theta} = \begin{pmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{pmatrix} \begin{pmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{Identity matrix}$$

Therefore, we can say that $R_{\theta} . R_{-\theta}$ are inverse because $R_{\theta} . R_{-\theta} = I$. So $R_{-\theta} = R_{\theta}^{-1}$ i.e., inverse of a rotation by θ degree is a rotation in the opposite direction.

Check Your Progress 2

1) Scaling transformation is mainly used to change the size of an object. The scale factors determines whether the scaling is a compression, S < 1 or a enlargement, S > 1, whereas the effect of shearing is "pushing" a geometric object is a direction parallel to the coordinate axes. Shearing factor determines, how far a direction is pushed.

2) $S_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $S_{c,d} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ and $S_{ac,bd} = \begin{pmatrix} a.c & 0 \\ 0 & b.d \end{pmatrix}$

since

$$S_{a,b}. S_{c,d} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} . \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a.c & 0 \\ 0 & b.d \end{pmatrix} \quad -(1)$$

and $S_{c,d}. S_{a,b} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} . \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} c.a & 0 \\ 0 & d.b \end{pmatrix} \quad -(2)$

from (1) and (2) we can say:

$$S_{a,b}. S_{c,d} = S_{c,d}. S_{a,b} = S_{ac,\, bd}$$

3)

a) Shift an image to the right by 3 units

$$\therefore S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

b) Shift the image up by 2 units and down by 1 units i.e. $S_x = S_x + 2$ and $S_y = S_y - 1$

$$\therefore\ S = \begin{pmatrix} (S_x+2) & 0 & 0 \\ 0 & (S_y-1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \therefore\ S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

c) Move the image down 2/3 units and left 4 units

$$\therefore S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -2/3 & 1 \end{pmatrix}$$

4) $S_{Sx,\ Sy} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix}$ and $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

we have to find out condition under which $S_{Sx,\ Sy}.\ R_\theta = R_{\theta.}\ S_s,\ S_y$

so $S_{Sx,\ Sy}.\ R_\theta = \begin{pmatrix} S_x & o \\ o & S_y \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} S_x.\cos\theta & S_x.\sin\theta \\ -S_y.\sin\theta & S_y.\cos\theta \end{pmatrix}$ — (1)

and $R_\theta.\ S_{Sx,Sy} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \begin{pmatrix} S_x & o \\ o & S_y \end{pmatrix} = \begin{pmatrix} \cos\theta.S_x & Sin\theta.S_y \\ -S_x.Sin\theta & \cos\theta.S_y \end{pmatrix}$ — (2)

In order to satisfy $S_{Sx,\ Sy.}\ R\theta = R\theta.\ S_{Sx,Sy}$

We have Sy. $\sin\theta = \sin\theta.S_x \Rightarrow$ either $\sin\theta = 0$ or $\theta = n\pi$, where n is an integer.

$\sin\theta\ (Sy - S_x) = 0$ or $S_x = S_y$ i.e. scaling transform is uniform.

5) No, since $Sh_x\ (a).\ Sh_y(b) = \begin{pmatrix} 1 & o \\ a & 1 \end{pmatrix} . \begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & ab+1 \end{pmatrix}$ — (1)

$Sh_y\ (b).Sh_x\ (a) = \begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} \begin{pmatrix} 1 & o \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1+ba & b \\ a & 1 \end{pmatrix}$ — (2)

and $Sh_{xy}\ (a, b) = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$

from (1) , (2) and (3), we can say that

$Sh_{xy}\ (a, b) \neq Sh_x\ (a)\ .\ Sh_y\ (b) \neq Sh_y\ (b).\ Sh_x\ (a)$

Check Your Progress 3

1) $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and

Counter clockwise Rotation of 90°; $R_{90^o} = \begin{bmatrix} \text{Cos}\,90^o & \text{Sin}\,90^o \\ -\text{Sin}\,90^o & \text{Cos}\,90^o \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

We have to show that

$M_{y=x} = M_x \, . \, R_{90^o}$

Since $M_x \, . \, R_{90^o} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_{y=x}$

Hence, a reflection about the line y = x, is equivalent to a reflection relative to the x-axis followed by a counter clockwise rotation of 90°.

2) The required single (3 x 3) homogeneous transformation matrix can be obtained as follows:

a) $T = S_{2,2} \, . \, T_{tx-1,\, ty} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) $T = S_{s_x + \frac{3}{2}, s_y} . R_{90^o} = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} Cos 90^o & Sin 90^o & 0 \\ -Sin 90^o & Cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3/2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $T = R_{90^o} \, . \, S_{S_x + \frac{3}{2}, S_y} = \begin{bmatrix} Cos 90^o & Sin 90^o & 0 \\ -Sin 90^o & Cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

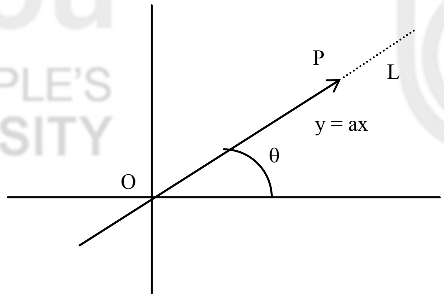
$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 3/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = T_{tx \frac{-1}{2}, t_y + \frac{1}{2}} \, . \, R_{45^o} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1\sqrt{2} & 0 & 1 \end{bmatrix}$$

3) Let OP be given line L, which makes an angle θ with respect to



The transformation matrix for reflection about an arbitrary line y = mx + c is (see equation 25).

$$M_L = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & 0 \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & 0 \\ \frac{-2cm}{m^2+1} & \frac{-2C}{m^2+1} & 1 \end{bmatrix} \text{ where } m = \tan\theta$$

For line y = ax; m = tanθ = a and intercept on y-axis is 0 i.e. c = 0. Thus, transformation matrix for reflection about a line y = ax is:

$$M_L = M_{y=ax} = \begin{bmatrix} \frac{1-a^2}{a^2+1} & \frac{2a}{a^2+1} & 0 \\ \frac{2.a}{a^2+1} & \frac{a^2-1}{a^2+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ where } a = \tan\theta = m$$

4) The equation of the line passing through the points (1,3) and (–1, –1) is obtained as:

y = 2x + 1 (1)

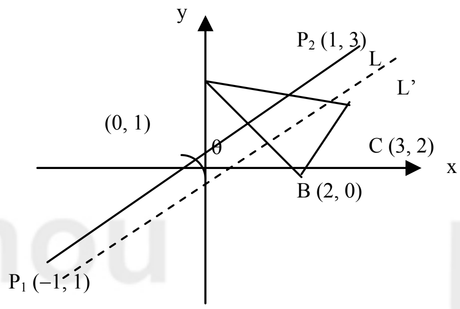


Figure (a)

If θ is the angle made by the line (1) with the positive x-axis, then

$\tan\theta = 2 \Rightarrow \text{Cos}\theta = \frac{1}{\sqrt{2}}$ and $\text{Sin}\theta \ \frac{2}{\sqrt{5}}$

To obtain the reflection about the line (1), the following sequence of transformations can be performed:

1) Translate the intersection point (0, 1) to the origin, this shift the line L to L′
2) Rotate the shifted line L′ by $-\theta^o$ (i.e. clockwise), so that the L′aligns with the x-axis.

3) Perform the reflection about x-axis.
4) Apply the inverse of the transformation of step (2).
5) Apply the inverse of the transformation of step (1).

By performing step 1 – step 5, we get

$$M_L = T_V \,.\, R_\theta \; M_X \,.\, R_\theta^{-1} \,.\, T_V^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/\sqrt{5} & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ -4/5 & 2/5 & 1 \end{bmatrix}$$

So the new coordinates A′B′C′ of the reflected triangle ABC can be found as:
[A′ B′ C′] = [ABC] . M_L

$$= \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} . \begin{bmatrix} -3/\sqrt{5} & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ -4/5 & 2/5 & 1 \end{bmatrix} = \begin{bmatrix} 8/5 & 11/5 & 1 \\ -2 & 2 & 1 \\ -1 & 4 & 1 \end{bmatrix}$$

Thus, $A' = \left(8/5, \dfrac{11}{5}\right)$, $B' = (-2, 2)$ and $C' = (-1, 4)$, which is shown in *Figure (b)*.

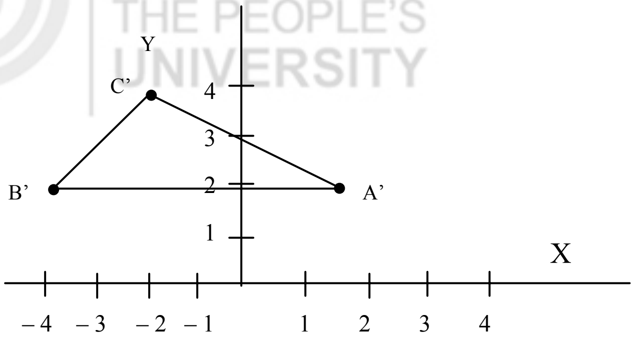


Figure (b)