

# UNIT 3

## IDENTICAL PARTICLES

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### **3.1 INTRODUCTION**

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In your mechanics courses you have studied many-particle systems. In this unit we discuss the quantum mechanics of a collection of particles, in particular those that are identical in all their intrinsic properties, like the charge / mass / spin etc. In classical physics, even identical particles are distinguishable in some way but that is not true in quantum mechanics. You may have noticed that so far you have applied the techniques of quantum mechanics to study single particle systems. The only exception was the hydrogen atom problem (Unit 9, MPH-004). The hydrogen atom is a two-particle system comprising a proton and an electron. Remember that we got around this problem by transforming to centre of mass and relative coordinates, thereby reducing the Schrodinger equation to two independent single particle problems, which we then went on to solve.

In general, however, an  $N$ -particle system with  $N > 2$ , for example the quantum mechanical problem for any atom heavier than hydrogen, cannot be reduced to  $N$  independent single-particle problems. What is even more surprising is that the quantum mechanical problem of a two-particle system is much more complex when the two particles are identical in all respects. The solution in fact depends on the type of particles. In fact you know that all the

electrons (which are fermions) in the atom do not collect in the lowest energy eigenstate but rather occupy different orbitals. Whereas in a **Bose-Einstein condensate** (a state of matter usually formed when a gas of bosons at very low densities is cooled to temperatures very close to absolute zero) a large fraction of bosons occupy the lowest quantum state.

In Sec. 3.2 we first write down the quantum mechanical operators and commutation relations for a system with  $N$  degrees of freedom. In Sec. 3.3 we study a quantum mechanical system of two distinguishable particles. We see how the Schrödinger equation for the non-interacting two-particle system can be reduced to two independent single-particle Schrödinger equations and there is a similar generalization for a system of  $N$  non-interacting particles. We also study the Schrödinger equation for a system of two interacting distinguishable particles and study the conditions under which the problem can be reduced to two independent single-particle Schrödinger equations. In Sec. 3.4 we study the system of two identical particles. We first learn why the quantum mechanical description is quite different from the classical picture. You study about exchange degeneracy, which is a degeneracy that originates purely from the identical nature of the particles. You learn about the symmetric and antisymmetric states and the permutation operator. In Sec. 3.4 we study the symmetrisation postulate, a new postulate which tells us about the two different kinds of particles that exist in nature, bosons and fermions. You also learn about the Pauli Exclusion Principle and the exchange force.

In the next unit you will learn how to calculate the total angular momentum for a system of particles.

### **Expected Learning Outcomes**

After studying this unit, you should be able to:

- ❖ derive the general solution of the Schrödinger equation for a system of two distinguishable non-interacting particles (or when the system can be described in terms of normal modes) with  $V(x_1, x_2) = V(x_1) + V(x_2)$ ;
- ❖ write the general solution of the Schrödinger equation for a system of  $N$  distinguishable non-interacting particles;
- ❖ derive the general solution of the Schrödinger equation for a system of two distinguishable interacting particles with  $V(x_1, x_2) = V(x_1 - x_2)$  in centre-of-mass and reduced coordinates;
- ❖ derive the symmetric and antisymmetric solutions for a system of two identical particles and explain exchange degeneracy and exchange force;
- ❖ explain the action of the permutation operator on a quantum state of identical particles and derive its properties; and
- ❖ apply the method of canonical quantization.

## **3.2 SYSTEM WITH $N$ DEGREES OF FREEDOM**

Most of the quantum mechanics that you have done so far is for a single particle with one degree of freedom. The position of the classical particle is described by the Cartesian coordinate  $x$  and the

corresponding momentum  $p_x$ . You know that in quantum mechanics we have the corresponding operators  $\hat{x}$  and  $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$ . For the state  $|\psi\rangle$  the wave function in the coordinate representation is  $\psi(x) = \langle x|\psi\rangle$ , where  $|x\rangle$  is the position basis defined by  $\hat{x}|x\rangle = x|x\rangle$  and  $\langle x|\hat{x}|\psi\rangle = x\psi(x)$ .  $\hat{x}$  and  $\hat{p}_x$  satisfy the commutation rule  $[\hat{x}, \hat{p}_x] = i\hbar$  (also called the canonical commutation rule).

Suppose now that we want to describe quantum mechanically a classical system with  $N$  degrees of freedom for which the  $N$  Cartesian coordinates are  $x_1, x_2, \dots, x_N$  and corresponding momenta  $p_1, p_2, \dots, p_N$ . We extend the quantum mechanical description a single particle with one dynamical degree of freedom as follows:

- There is a set  $N$  **mutually commuting** position operators  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$  with

$$[\hat{x}_i, \hat{x}_j] = 0 \text{ for } i = 1, 2, \dots, N \quad (3.1a)$$

- There is a **simultaneous eigenbasis** of these commuting operators, which we can call the coordinate basis for the system:  $|x_1, x_2, \dots, x_N\rangle$ .  $|x_1, x_2, \dots, x_N\rangle$  is a simultaneous eigenket of the position operators  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$ :

$$\hat{x}_i |x_1, x_2, \dots, x_i, \dots, x_N\rangle = x_i |x_1, x_2, \dots, x_i, \dots, x_N\rangle \text{ for } i = 1, 2, \dots, N \quad (3.1b)$$

The eigenbasis is normalized, so

$$\langle x'_1, x'_2, \dots, x'_N | x_1, x_2, \dots, x_N \rangle = \delta(x_1 - x'_1) \delta(x_2 - x'_2) \dots \delta(x_N - x'_N) \quad (3.1c)$$

- The wave function corresponding to the state ket  $|\psi\rangle$  is:

$$\psi(x_1, x_2, \dots, x_N) = \langle x_1, x_2, \dots, x_N | \psi \rangle \quad (3.1d)$$

And

$$\langle x_1, x_2, \dots, x_i, \dots, x_N | \hat{x}_i | \psi \rangle = x_i \psi(x_1, x_2, \dots, x_N) \quad (3.1e)$$

The probability of locating a particle between  $x_1$  and  $x_1 + dx_1$ ,  $x_2 + dx_2, \dots, x_N + dx_N$  is

$$|\psi(x_1, x_2, \dots, x_N)|^2 dx_1 dx_2 \dots dx_N \quad (3.1f)$$

- The momentum operator corresponding to the  $i^{\text{th}}$  degree of freedom is:

$$\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i} \text{ for } i = 1, 2, \dots, N \quad (3.1g)$$

Remember that the specification of Eq. (3.1g) is only when  $x_i$  are Cartesian coordinates.

Any other dynamical variable of the system which depends on  $x_i$  and  $p_i$ , is represented by an operator which is a function of  $\hat{x}_i$  and  $\hat{p}_i$ :

$$\hat{O} = f(\hat{x}_i, \hat{p}_i)$$

- We have the following commutation rules:

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 ; [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (3.1h)$$

- If  $\hat{H}$  is the Hamiltonian of the system, the Heisenberg equation for an operator  $\hat{O}$  is

$$i\hbar \frac{\partial \hat{O}}{\partial t} = [\hat{O}, \hat{H}] \quad (3.1i)$$

In particular when  $\hat{O}$  are the position and momentum operators we get:

$$i\hbar \frac{\partial \hat{x}_i}{\partial t} = [\hat{x}_i, \hat{H}] ; \quad i\hbar \frac{\partial \hat{p}_i}{\partial t} = [\hat{p}_i, \hat{H}] \quad (3.1j)$$

**Note that the coordinates**  $x_1, x_2, \dots, x_N$  could be the  $x, y, z$  coordinates of some  $N/3$  particles or simply the  $x$ -coordinates of  $N$  particles or a mixture of both. All that matters is that they are all independent coordinates.

We now go on to describe a two-particle system.

### 3.3 TWO-PARTICLE SYSTEMS

We first consider a system of two distinguishable particles, which we label as 1 and 2. The masses of the two particles are  $m_1$  and  $m_2$  and they are located at  $\vec{r}_1(x_1, y_1, z_1)$  and  $\vec{r}_2(x_2, y_2, z_2)$  respectively. The state vector for the system is  $|\psi\rangle$ . We generalize our idea of the wave function  $\psi(x, y, z) = \psi(\vec{r})$  of a single particle system with three degrees of freedom, to a two-particle system. We can look upon this two-particle system as a system with 6 degrees of freedom, characterized by the independent Cartesian coordinates  $x_1, y_1, z_1, x_2, y_2, z_2$ , as explained in Sec. 3.2. We write the wave function of this two particle system as  $\psi(\vec{r}_1, \vec{r}_2) = \psi(x_1, y_1, z_1, x_2, y_2, z_2)$ . The eigenket  $|\vec{r}_1, \vec{r}_2\rangle$  in this case is the simultaneous eigenket of the position operators for the two particles which are  $\hat{x}_1, \hat{y}_1, \hat{z}_1$  and  $\hat{x}_2, \hat{y}_2, \hat{z}_2$  respectively. The wavefunction  $\psi(\vec{r}_1, \vec{r}_2, t)$  is defined as follows: the joint probability of finding the particle 1 in the volume  $d^3\vec{r}_1$  and the particle 2 in the volume  $d^3\vec{r}_2$  is:

$$|\psi(\vec{r}_1, \vec{r}_2)|^2 d^3\vec{r}_1 d^3\vec{r}_2 \quad (3.2a)$$

As a result, the wave function is normalized:

$$\int |\psi(\vec{r}_1, \vec{r}_2)|^2 d^3\vec{r}_1 d^3\vec{r}_2 = 1 \quad (3.2b)$$

To simplify the discussion, let us assume that the system is one-dimensional. So the wave function of the two particle system is  $\psi(x_1, x_2) = \langle x_1, x_2 | \psi \rangle$

where:

$$\hat{x}_1 |x_1, x_2\rangle = x_1 |x_1, x_2\rangle ; \quad \hat{x}_2 |x_1, x_2\rangle = x_2 |x_1, x_2\rangle$$

The joint probability of finding the particle 1 in between  $x_1$  and  $x_1 + dx_1$  and the particle 2 between  $x_2$  and  $x_2 + dx_2$  is:

$$|\psi(x_1, x_2)|^2 dx_1 dx_2 \quad (3.3a)$$

With the normalization condition is:

$$\int |\psi(x_1, x_2)|^2 dx_1 dx_2 = 1 \quad (3.3b)$$

The position and momentum operators for the two particles obey the following commutation relations for  $i, j = 1, 2$ :

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \quad (3.4a)$$

$$[\hat{x}_i, \hat{x}_j] = 0 \quad (3.4b)$$

$$[\hat{p}_i, \hat{p}_j] = 0 \quad (3.4c)$$

In the position representation, the position and momentum operators are given by  $\hat{x}_i = x_i$  ;  $\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}$  (3.4d)

Now since the observables corresponding to commuting operators can be measured simultaneously, Eqs. (3.4) tell us that while the position and momentum of any one of the particles (say  $x_1$  and  $p_1$  or  $x_2$  and  $p_2$ ) cannot be measured simultaneously, there is no such restriction on the simultaneous measurement of the position or the momentum of the two different particles ( $x_1$  and  $x_2$  or  $p_1$  and  $p_2$ ) or the position of one and the momentum of the other, say  $x_1$  and  $p_2$ .

The time evolution of the state vector is given by:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \quad (3.5a)$$

Where the Hamiltonian  $\hat{H}$  is

$$\hat{H} = -\left( \frac{\hbar^2}{2m_1} \hat{p}_1^2 + \frac{\hbar^2}{2m_2} \hat{p}_2^2 \right) + V(x_1, x_2, t) \quad (3.5b)$$

When the potential does not depend explicitly on  $t$ ,  $V = V(x_1, x_2)$  and the stationary state solution for Eq. (3.5a) is:

$$|\psi(t)\rangle = \exp\left(-\frac{iEt}{\hbar}\right) |E\rangle \quad (3.6a)$$

where  $|E\rangle$  is the solution of the equation  $\hat{H}|E\rangle = E|E\rangle$ . In the coordinate representation, the time-independent Schrodinger equation for the two particle system in one dimension is:

$$-\left( \frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} \right) \psi_E(x_1, x_2) + V(x_1, x_2) \psi_E(x_1, x_2) = E \psi_E(x_1, x_2) \quad (3.6b)$$

Where  $\psi_E(x_1, x_2) = \langle x_1, x_2 | E \rangle$ . For any given system, the solution to Eq. (3.6b) will depend on the form of  $V(x_1, x_2)$ .

We can generalize the discussion so far to an  $N$ -particle system in one dimension. Let us say that we have a system of  $N$  particles which have the position coordinates  $x_i$  (a system with  $N$  degrees of freedom) and mass  $m_i$  with  $i = 1, 2, 3, \dots, N$ . If the state vector of this  $N$ -particle system is  $|\psi\rangle$ , the wave function is  $\psi(x_1, x_2, \dots, x_N) = \langle x_1, x_2, \dots, x_N | \psi \rangle$  where  $|x_1, x_2, \dots, x_N\rangle$  is the simultaneous position eigenket for the position operators  $\hat{x}_i$  for each of the  $N$  particles. The canonical commutation relations are the same for this  $N$ -particle system as for the two-particle system, so for  $i, j = 1, 2, \dots, N$ :

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}; [\hat{x}_i, \hat{x}_j] = 0; [\hat{p}_i, \hat{p}_j] = 0 \quad (3.7a)$$

where  $\hat{x}_i = x_i$  ;  $\hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}$ . The joint probability of finding the particle 1 in between  $x_1$  and  $x_1 + dx_1$ , the particle 2 between  $x_2$  and  $x_2 + dx_2$ , the particle 3 between  $x_3$  and  $x_3 + dx_3$  and so on is:

$$|\psi(x_1, x_2, \dots, x_N, t)|^2 dx_1 dx_2 \dots dx_N \quad (3.7b)$$

The normalization condition is:

$$\int |\psi(x_1, x_2, \dots, x_N, t)|^2 dx_1 dx_2 \dots dx_N = 1 \quad (3.7c)$$

When the potential function does not depend explicitly on time, the Hamiltonian is

$$\hat{H} = -\left( \frac{\hbar^2}{2m_1} \hat{p}_1^2 + \frac{\hbar^2}{2m_2} \hat{p}_2^2 + \dots + \frac{\hbar^2}{2m_N} \hat{p}_N^2 \right) + V(x_1, x_2, \dots, x_N) \quad (3.7d)$$

The eigenfunction  $\psi_E(x_1, x_2, \dots, x_N)$  is the solution of the time-independent Schrödinger equation:

$$\begin{aligned} & -\left( \frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + \frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\hbar^2}{2m_N} \frac{\partial^2}{\partial x_N^2} \right) \psi_E(x_1, x_2, \dots, x_N) \\ & + V(x_1, x_2, \dots, x_N) \psi_E(x_1, x_2, \dots, x_N) = E \psi_E(x_1, x_2, \dots, x_N) \end{aligned} \quad (3.7e)$$

$$\text{And } \psi_E(x_1, x_2, \dots, x_N, t) = \psi_E(x_1, x_2, \dots, x_N) \exp\left(-\frac{iEt}{\hbar}\right) \quad (3.7f)$$

Let us now go back to the Schrödinger equation for the two-particle system (Eq. 3.6b) and look at possible solutions which depend on the nature of the potential. We consider two possible cases.

### 3.3.1 Non-Interacting Two-Particle System

When the two particles do not interact among themselves we can write:

$$V(x_1, x_2) = V(x_1) + V(x_2)$$

So while the two particles separately interact with external potentials, they do not interact among themselves. In this case the Hamiltonian is **separable**:

$$\begin{aligned} \hat{H} &= -\left( \frac{\hbar^2}{2m_1} \hat{p}_1^2 + \frac{\hbar^2}{2m_2} \hat{p}_2^2 \right) + V(x_1) + V(x_2) \\ &= \left( -\frac{\hbar^2}{2m_1} \hat{p}_1^2 + V(x_1) \right) + \left( -\frac{\hbar^2}{2m_2} \hat{p}_2^2 + V(x_2) \right) \\ &= \hat{H}_1(x_1) + \hat{H}_2(x_2) \end{aligned} \quad (3.8a)$$

Eq. (3.6b) then reduces to the following:

$$\left(-\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2} + V(x_1)\right) \psi_E(x_1, x_2) + \left(-\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial x_2^2} + V(x_2)\right) \psi_E(x_1, x_2) = E \psi_E(x_1, x_2) \quad (3.8b)$$

This partial differential equation in two variables  $x_1$  and  $x_2$  can be reduced to two ordinary differential equations by the method of separation of variables, as you have studied in your undergraduate courses, by using the following standard ansatz:

$$\psi_E(x_1, x_2) = \psi_1(x_1) \psi_2(x_2) \quad (3.9)$$

The two equations are:

$$\left(-\frac{\hbar^2}{2m_1} \frac{d^2}{dx_1^2} + V(x_1)\right) \psi_1(x_1) = E_1 \psi_1(x_1) \quad (3.10a)$$

$$\text{And } \left(-\frac{\hbar^2}{2m_2} \frac{d^2}{dx_2^2} + V(x_2)\right) \psi_2(x_2) = E_2 \psi_2(x_2) \quad (3.10b)$$

Where  $E = E_1 + E_2$  and  $\psi_1(x_1)$  and  $\psi_2(x_2)$  are the solutions of the one-particle time independent Schrodinger equations  $\hat{H}_1 \psi_1(x_1) = E_1 \psi_1(x_1)$  and  $\hat{H}_2 \psi_2(x_2) = E_2 \psi_2(x_2)$  respectively. So essentially the Schrodinger equation for the two-particle problem is equivalent to two separate single-particle Schrodinger equations, one for each particle.

The wave function for the two particle system:

$$\begin{aligned} \psi_E(x_1, x_2, t) &= \psi_1(x_1) \psi_2(x_2) \exp\left(-\frac{iEt}{\hbar}\right) \\ &= \left[ \psi_1(x_1) \exp\left(-\frac{iE_1 t}{\hbar}\right) \right] \left[ \psi_2(x_2) \exp\left(-\frac{iE_2 t}{\hbar}\right) \right] \end{aligned} \quad (3.11a)$$

which is the product of the solutions to the two single particle problems.

The eigen state  $|E\rangle$  of Eq. (3.6a) corresponds to the particle 1 being in an eigenstate  $|E_1\rangle$  and the particle 2 being in an eigenstate  $|E_2\rangle$  where  $E = E_1 + E_2$ .

In this case we can generalize this result for the  $N$ -particle system. When the particles do not interact among themselves, that is

$V(x_1, x_2, \dots, x_N) = V(x_1) + V(x_2) + \dots + V(x_N)$ , the Hamiltonian of Eq. (3.7c) is separable. And just as we have for the two-particle system, the  $N$ -particle system reduces to  $N$  independent one particle systems, each satisfying a Schrodinger equation of the type:

$$\left(-\frac{\hbar^2}{2m_i} \frac{d^2}{dx_i^2} + V(x_i)\right) \psi_i(x_i) = E_i \psi_i(x_i), \quad i = 1, 2, \dots, N \quad (3.11b)$$

and

In Unit 14 we talk about direct product spaces. You may read this margin remark after you have studied Unit 14. Suppose we have two particles described by  $x_1, p_1$  and  $x_2, p_2$ . In quantum mechanics we can describe particle 1 using a state ket  $|\psi_1\rangle$  in the Hilbert space say  $V_1$ . Using the coordinate basis  $|x_1\rangle$  we can write the wave function  $\psi_1(x_1) = \langle x_1 | \psi_1 \rangle$ . Similarly we have a Hilbert space  $V_2$  for particle 2, where the state ket, coordinate basis ket and wave function are  $|x_2\rangle, |\psi_2\rangle$  and  $\psi_2(x_2) = \langle x_2 | \psi_2 \rangle$  respectively. For the two-particle system we can define the product basis ket  $|x_1\rangle \otimes |x_2\rangle$  in the direct product vector space  $V_1 \otimes V_2$ , which locates particle 1 at  $x_1$  and particle 2 at  $x_2$ . The state ket for the two particle system is then  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ .

$$\psi_E(x_1, x_2, \dots, x_N, t) = \psi_1(x_1)\psi_2(x_2)\dots\psi_N(x_N)\exp\left(-\frac{iEt}{\hbar}\right) \quad (3.11c)$$

$$\text{with } E = E_1 + E_2 + \dots + E_N = \sum_{i=1}^N E_i.$$

This is a simple generalization of the problem from 2 to  $N$ -particles for a non-interacting system of distinguishable particles.

We emphasize again while we have been discussing the problem as an  $N$ -particle system, it is in general a system with  $N$ -degrees of freedom, where the degrees of freedom could correspond to the different spatial dimensions of a single particle or the single spatial dimension of several particles of a mixture of both. For example the three-dimensional isotropic oscillator is a system with three degrees of freedom for a single particle, which is also separable. The hydrogen atom has two particles and three spatial dimensions for each particle, so in all six degrees of freedom.

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### SAQ 1

Derive Eqs 3.10a and 3.10b from Eq. (3.8b) with  $\psi_E(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$ .

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### 3.3.2 Interacting Two-Particle System

When the particles of the system interact with one another

$$V(x_1, x_2) \neq V(x_1) + V(x_2)$$

The Hamiltonian in this case is not separable. This problem cannot in general be reduced to two separate single particle problems, except for the special case **where the only contribution to the potential function is the interaction between the two particles, which further, depends only on the distance between them**, that is:

$$V(x_1, x_2) = V(x_1 - x_2) \quad (3.12a)$$

$$\text{So: } \hat{H} = -\left(\frac{\hbar^2}{2m_1}\hat{p}_1^2 + \frac{\hbar^2}{2m_2}\hat{p}_2^2\right) + V(x_1 - x_2) \quad (3.12b)$$

For the two-particle system which has the potential function defined by Eq. (3.12a) we introduce the centre-of-mass and relative coordinates as follows:

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{m_1x_1 + m_2x_2}{M} \quad (3.13a)$$

$$\text{And } x_r = x_1 - x_2 \quad (3.13b)$$

where  $M = m_1 + m_2$ . You will find that with this transformation the Hamiltonian of Eq. (3.12b) reduces to:



$$\hat{H} = -\left(\frac{\hbar^2}{2M}\hat{p}_{CM}^2 + \frac{\hbar^2}{2\mu}\hat{p}_r^2\right) + V(x_r) \quad (3.14)$$

Where  $p_{CM} = M\dot{x}_{CM} = m_1\dot{x}_1 + m_2\dot{x}_2$ ;  $p_r = \mu\dot{x}_r = \mu(\dot{x}_1 - \dot{x}_2)$  with

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ . From your undergraduate mechanics courses you would

remember that  $\mu$  is the reduced mass of the system.

The transformation to the operators is the following:

$$\hat{x}_{CM} = x_{CM} ; \hat{x}_r = x_r ; \hat{p}_{CM} = -i\hbar \frac{\partial}{\partial x_{CM}} ; \hat{p}_r = -i\hbar \frac{\partial}{\partial x_r}$$

The canonical commutation relations are:

$$[\hat{x}_{CM}, \hat{p}_{CM}] = [\hat{x}_r, \hat{p}_r] = i\hbar \quad (3.15a)$$

$$[\hat{x}_{CM}, \hat{x}_r] = [\hat{x}_{CM}, \hat{p}_r] = [\hat{p}_{CM}, \hat{p}_r] = [\hat{p}_{CM}, \hat{x}_r] = 0 \quad (3.15b)$$

Writing the wave function as  $\psi_E(x_1, x_2) = \psi_E(x_{CM}, x_r)$ , the Schrodinger is:

$$\begin{aligned} & \left(-\frac{\hbar^2}{2M}\hat{p}_{CM}^2\right)\psi_E(x_{CM}, x_r) + \left(-\frac{\hbar^2}{2\mu}\hat{p}_r^2 + V(x_r)\right)\psi_E(x_{CM}, x_r) \\ & \quad = E\psi_E(x_{CM}, x_r) \\ \Rightarrow & \left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x_{CM}^2}\right)\psi_E(x_{CM}, x_r) \\ & \quad + \left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_r^2} + V(x_r)\right)\psi_E(x_{CM}, x_r) = E\psi_E(x_{CM}, x_r) \end{aligned} \quad (3.16)$$

Once again, writing

$$\psi_E(x_{CM}, x_r) = \psi_{CM}(x_{CM})\psi_r(x_r) \quad (3.17a)$$

We see that the Schrodinger equation for the two particle problem is reduced to two independent one-particle Schrodinger equations in the centre of mass and relative coordinates:

$$-\frac{\hbar^2}{2M}\frac{\partial^2\psi_{CM}(x_{CM})}{\partial x_{CM}^2} = E_{CM}\psi_{CM}(x_{CM}) \quad (3.17b)$$

$$\text{And } \left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_r^2} + V(x_r)\right)\psi_r(x_r) = E_r\psi_r(x_r) \quad (3.17c)$$

where  $E = E_{CM} + E_r$ . Eq. (3.17b) is the Schrodinger equation for a particle of mass  $M$  (which is the total mass of the system) located at the centre of mass of the system, drifting as a free particle and  $\psi_{CM}(x_{CM})$  is its energy

eigenfunction. The energy eigenvalue is  $E_{CM} = \frac{p_{CM}^2}{2M}$  and the energy

eigenfunction is a plane wave  $\psi_{CM}(x_{CM}) = \exp\left(-\frac{ip_{CM}x_{CM}}{\hbar}\right)$ .

Eq. (3.17c) is the Schrödinger equation for a particle of mass  $\mu$  (the reduced mass) moving in a potential  $V(x_r)$ .  $\psi_r(x_r)$  is its energy eigenfunction and  $E_r$  is its energy eigenvalue. So now we can write:

$$\begin{aligned} \Psi_E(x_{CM}, x_r, t) \\ = \left[ \exp\left(-\frac{ip_{CM}x_{CM}}{\hbar}\right) \exp\left(-\frac{iE_{CM}t}{\hbar}\right) \right] \left[ \psi_r(x_r) \exp\left(-\frac{iE_r t}{\hbar}\right) \right] \end{aligned} \quad (3.18)$$

where  $E = \frac{p_{CM}^2}{2M} + E_r$ .

We can choose to study this system in the centre of mass coordinates, in which case we can drop Eq. (3.17b) altogether and solve Eq. (3.17c) only. In fact, this is what we do in the Hydrogen atom problem.

Notice that **we cannot generalize this result (Eq. 3.18) to a  $N$ -particle system**. So a quantum mechanical system of  $N$  interacting particles cannot, in general, be treated as  $N$  independent one-particle systems.

### SAQ 2

a) Show that

i) for the Hamiltonian  $\hat{H} = -\left(\frac{\hbar^2}{2m_1}\hat{p}_1^2 + \frac{\hbar^2}{2m_2}\hat{p}_2^2\right) + V(x_1 - x_2)$ , the

Schrodinger equation reduces to

$$\begin{aligned} \left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x_{CM}^2}\right)\Psi_E(x_{CM}, x_r) + \left(-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x_r^2} + V(x_r)\right)\Psi_E(x_{CM}, x_r) \\ = E\Psi_E(x_{CM}, x_r) \end{aligned}$$

where  $x_{CM}$  and  $x_r$  are defined in Eqs. (3.13a and b).

ii)  $[\hat{x}_{CM}, \hat{p}_{CM}] = [\hat{x}_r, \hat{p}_r] = i\hbar$

b) Write down the Schrödinger equations for a

- i) particle in a three-dimensional box
- ii) three-dimensional simple harmonic oscillator

At no point in our discussion so far do we consider that we may not be able to tell the particles apart (the particles are distinguishable). Let us now talk about systems of indistinguishable or identical particles.

## 3.4 IDENTICALITY VERSUS INDISTINGUISHABILITY

Before we discuss identical particles in quantum mechanics let us consider the classical system of  $N$  identical particles.

Two particles are said to be identical, if they are exactly the same in all their intrinsic properties (we could be talking about their mass or charge or spin). The particles are exact replicas of each other, for example like two electrons or two pions etc. So there can be no experiment that detects any difference in

their intrinsic properties (remember that the position or location of a particle is not its intrinsic property). So even when the experiment is not one that measures one of these intrinsic properties, a system of two identical particles cannot be treated the same as a system of two distinguishable particles and vice versa.

Let us first look at the classical picture for a system of identical particles.

### 3.4.1 The Classical Picture

Consider two identical objects in classical mechanics. We label these objects by 1 and 2 respectively and their initial positions by  $\vec{r}_1(t=0) = \vec{R}_0$  and  $\vec{r}_2(t=0) = \vec{R}_0'$  and their initial velocities by  $\vec{v}_1(t=0) = \vec{V}_0$  and  $\vec{v}_2(t=0) = \vec{V}_0'$ , respectively. The labels 1 and 2 do not indicate any intrinsic difference in the two objects, rather it is just our way of keeping track of the fact that they are starting from distinct initial positions. You could also have chosen to color the two objects red and green to distinguish between them so to say. We can study the time evolution of the system of two particles and say that after a time  $t$ , the position of the two particles is  $\vec{r}_1(t) = \vec{R}$  and  $\vec{r}_2(t) = \vec{R}'$  (Fig. 3.1a).

Following the path of the objects from their starting position, you would know that the ball labeled 1 (or colored red) started from  $\vec{R}_0$  and reached  $\vec{R}$  and ball labeled 2 (or colored green) started from  $\vec{R}_0'$  and reached  $\vec{R}'$  (Fig. 3.1a)

Since the two objects are identical the system as a whole does not change if the particles exchanged their places. So, the classical Hamiltonian  $H = H(\vec{r}_1, \vec{r}_2, \vec{v}_1, \vec{v}_2)$  (or the Lagrangian for that matter) of the system would not change if we interchange the labels 1 and 2).

Suppose now we interchange the labels on the particles so that now the initial positions and velocities are (Fig. 3.1b):

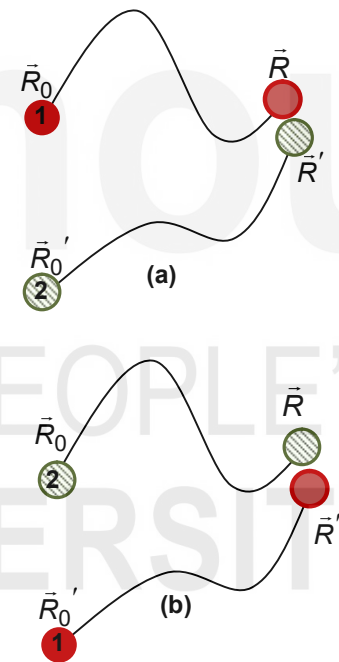
$$\vec{r}_1(t=0) = \vec{R}_0' ; \vec{r}_2(t=0) = \vec{R}_0 ; \vec{v}_1(t=0) = \vec{V}_0' ; \vec{v}_2(t=0) = \vec{V}_0$$

And after a time  $t$ , the position of the two balls is:

$$\vec{r}_1(t) = \vec{R}' ; \vec{r}_2(t) = \vec{R}$$

If you are following the path of the objects from their starting position, you would know that ball 1 (red) in this case started from  $\vec{R}_0'$  and reached  $\vec{R}'$  and ball 2 (green) started from  $\vec{R}_0$  and reached  $\vec{R}$ . You would know the difference in the two cases only because you have labeled the objects and so you can retrace the path of the objects to their respective starting points.

If you don't label the two objects (or color them for that matter), then starting with two identical objects with initial parameters  $(\vec{R}_0, \vec{V}_0)$  and  $(\vec{R}_0', \vec{V}_0')$ , you would end up with two identical objects at  $(\vec{R}, \vec{R}')$  at a time  $t$ . Essentially by attaching labels to the two objects, **one is able to distinguish between two identical objects by looking at their non-identical trajectories**. So for an observer who watches the whole experiment, the two configurations obtained by interchanging identical objects is not the same and he can distinguish between the two.



**Fig. 3.1: Two identical objects in classical mechanics.**

Off course if a person comes in late and just sees the final positions of the balls and there are no labels on them, he would observe the final configuration of the system to be the same in both cases since the objects are identical.

This immediately makes it clear that **if we were not able to follow the trajectory we would not know the difference**. In quantum mechanics we have no concept of a deterministic particle trajectory (Unit 1, MPH-004), so clearly configurations obtained by exchanging identical particles in quantum mechanics must be equivalent. Let us now describe the quantum mechanical picture.

### 3.4.2 The Quantum Picture

Now let us consider two identical particles in quantum mechanics, for example two electrons. We know that there is no way we can label these electrons (or color them differently) or follow their paths, because in quantum mechanics we cannot define a precise trajectory. An electron as you know is described by a wave function and at most we can talk about the probability of finding an electron at any point in space at a given instant of time. Let us say that we start with two electrons whose wave packets are completely separated (localized at different points) in space. There is no way of saying however that after the time evolution their respective wave packets would still be distinct. In fact their wave packets could overlap in a region of space. And if there is any overlap of the wave functions of the two electrons, then when we detect an electron at any point in space, there is no way of saying which of the electrons we have detected. There could also be any number of paths that could take a system of two well separated identical particles to the same final state. So **in quantum mechanics, identical particles are truly indistinguishable**.

Let us look at a particular example to understand this idea. Consider the collision of two identical particles (Fig. 3.2a) which are moving towards each other from opposite directions. In our minds let us call these as particle 1 and particle 2. Before the collision let say that the wave packets ( $\psi_1$  and  $\psi_2$ ) of the two particles are clearly separated in space, with the individual probability densities ( $|\psi_1|^2$  and  $|\psi_2|^2$ ) for particles 1 and 2 respectively. This is the initial state of the system.

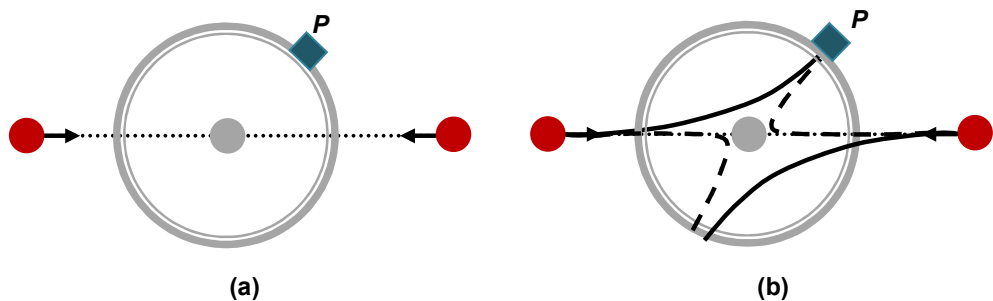
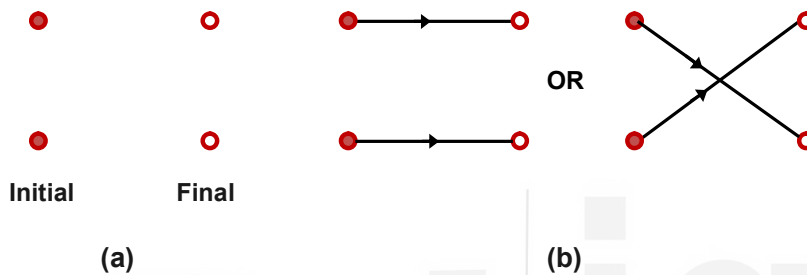


Fig. 3.2: Two identical objects in quantum mechanics.

Now the particles collide, say after a time  $t$  and during the process of collision there is an overlap between the wave packets of particles one and two. After the collision the particles are scattered. So let us say that we place a particle detector at a point  $P$ . To calculate the probability density of finding a particle at

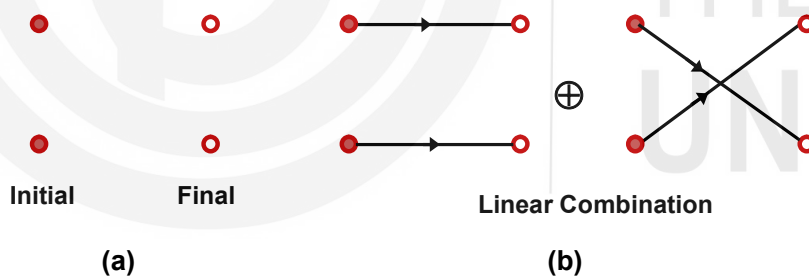
$P$  you need the wave function of the system. If you now detect a particle at  $P$  would you know whether the particle was associated with the wave packet  $\psi_1$  or  $\psi_2$  before the collision? Not really, because there are two different ways in which you could end up with an electron at  $P$ , as you can see in Fig. 3.2b (solid and dashed lines). Now both these situations are distinct but they correspond to the same physical state of the system because you cannot distinguish between these situations in an experiment.

So in classical mechanics given an initial and final configuration for a pair of identical particles (Fig. 3.3a), we would know which of the two sets of trajectories shown in Fig. (3.3b) actually happened.



**Fig. 3.3: Identical Particles in Classical Physics.**

In quantum mechanics, as we know, we cannot trace trajectories. All we know is that there were two incoming (identical) particles and two outgoing ones (as in Fig. 3.2). We cannot identify with certainty, as to which outgoing state corresponds to which incoming state. Hence we need to consider a linear combination of both possibilities (Fig. 3.4 b):



**Fig. 3.4: Identical Particles in Quantum Mechanics**

### 3.4.3 Exchange Degeneracy

Consider two particles, labeled 1 and 2. We assume that we have states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  where  $P$  is the index that characterizes a complete set of observables and the states are distinct and orthogonal ( $P \neq P'$ ). Let us say we carry out a measurement of the system and determine that particle 1 is in the state  $|\psi_P\rangle$  and particle 2 is in the state  $|\psi_{P'}\rangle$ . For example, the index  $P$  could be the position of the particle, in which case the measurement would tell you that particle 1 is at  $P$  and particle 2 is at  $P'$ . So the two particle state just after the measurement would be  $|\psi_P\rangle_1 |\psi_{P'}\rangle_2$ . If on the other hand, the

measurement shows that particle 1 is at  $P'$  and particle 2 at  $P$ , then the two-particle state for the system would be  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$ .

The two states are different if the particles are distinguishable. So if you make a measurement, you would know whether the final state is  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$  or  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$ .

However if the particles are identical then  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$  and  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$  are not distinguishable although they are different. Suppose you make a measurement on the two particle system, the result of the measurement may give you  $P$  and  $P'$ . But you would not be able to say whether the final state of the system was  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$  or  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$ . It would therefore be correct to say that the final state is a linear combination of the two states, that is the final state is  $C_1|\psi_P\rangle_1|\psi_{P'}\rangle_2 + C_2|\psi_{P'}\rangle_1|\psi_P\rangle_2$ .

So if you were to make a measurement of an observable, on the state  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$  or  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$  or a state of the form  $C_1|\psi_P\rangle_1|\psi_{P'}\rangle_2 + C_2|\psi_{P'}\rangle_1|\psi_P\rangle_2$  you would get the same eigenvalues of the observables.

So these states  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$ ,  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$  are degenerate states of the system of two identical particles. Here you see a degeneracy that can happen only in a system of identical particles and not in a single-particle system (remember that you have degenerate eigenstates even in single particle systems). Here the degeneracy is only because in a quantum mechanical system of identical particles, the particles may be exchanged without changing the eigenvalues associated with the quantum mechanical state of the system. And specifying the eigenvalue does not completely specify the state of the system. This kind of a degeneracy is called an **exchange degeneracy**.

We next talk about a special property of the final state of the system of two identical particles.

### 3.4.4 Symmetric and Antisymmetric States

We have already said that in a system of two identical particles, the states of the system obtained by the exchange of particles are indistinguishable though they give us the same eigenvalue. Before we go into a formal description of exchange symmetry using permutation operators, let us take up a simple example.

Let us say that the two particle wavefunction for a system of two identical particles in one-dimension is  $\psi(x_1, x_2, t)$  (Sec. 3.2). From our discussion in Sec. (3.4.3) we know that a state with one particle, say particle 1 at  $x_1$  and particle 2 at  $x_2$  ( $\psi(x_1, x_2, t)$ ) is quite indistinguishable from the reverse ( $\psi(x_2, x_1, t)$ ), so we can say that

$$|\psi(x_1, x_2, t)|^2 dx_1 dx_2 = |\psi(x_2, x_1, t)|^2 dx_1 dx_2 \Rightarrow |\psi(x_1, x_2, t)|^2 = |\psi(x_2, x_1, t)|^2 \quad (3.19)$$

For Eq. (3.19) to hold, we must have:

$$\psi(x_1, x_2, t) = e^{i\theta} \psi(x_2, x_1, t) \quad (3.20)$$

Since the two particles are identical, we can go ahead and interchange the labels 1 and 2 in Eq. (3.20) to get

$$\psi(x_2, x_1, t) = e^{i\theta} \psi(x_1, x_2, t) \quad (3.21)$$

Comparing Eqs. (3.20) and (3.21) we get:

$$e^{i2\theta} = 1 \Rightarrow \theta = 0 \text{ or } \pi \quad (3.22)$$

Thus under the exchange of particles, the wave function of the two particle system should satisfy either:

$$\psi(x_2, x_1, t) = +\psi(x_1, x_2, t) : \text{Symmetric State} \quad (3.23a)$$

$$\text{Or } \psi(x_2, x_1, t) = -\psi(x_1, x_2, t) : \text{Antisymmetric State} \quad (3.23b)$$

So the wave function of the composite system is either symmetric under exchange of particles (Eq. (3.23a)) or antisymmetric (Eq. 3.23b). It also turns out that these are the only two possibilities under exchange of particles.

The exchange of identical particles in a system is formally executed through an operator called the **Permutation** operator in quantum mechanics.

### 3.4.5 The Permutation Operator

We start with the simple picture of two particles 1 and 2 and two states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  where the index  $P$  and  $P'$  represent any complete set of observables (not necessarily the position). We define a permutation operator  $\hat{P}_{12}$  as follows:

$$\hat{P}_{12} |\psi_P\rangle_1 |\psi_{P'}\rangle_2 = |\psi_{P'}\rangle_1 |\psi_P\rangle_2 \quad (3.24)$$

Under the effect of the permutation operator the particle 1 which was in the state  $|\psi_P\rangle$  goes to the state  $|\psi_{P'}\rangle$  and particle 2 which was in the state  $|\psi_{P'}\rangle$  goes to the state  $|\psi_P\rangle$ . So the effect of the permutation operator is to exchange the particle labels 1 and 2. Clearly the permutation operator  $\hat{P}_{21}$  would also have the same effect, that is to exchange the particle labels 2 and 1. What would be the effect of the operator  $\hat{P}_{12}^2$ . Now:

$$\hat{P}_{12}^2 |\psi_P\rangle_1 |\psi_{P'}\rangle_2 = \hat{P}_{12} |\psi_{P'}\rangle_1 |\psi_P\rangle_2 = |\psi_P\rangle_1 |\psi_{P'}\rangle_2 \Rightarrow \hat{P}_{12}^2 = \hat{I} \quad (3.25)$$

So  $\hat{P}_{12}^2$  is the identity operator and  $\hat{P}_{12} = \hat{P}_{12}^{-1}$ . Let us now say that the states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  are the eigenkets of a single observable  $O$  for the particles, where  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  are orthogonal and normalized. So if  $\hat{O}_1$  is the operator corresponding to the observable  $O$  for particle 1 and  $\hat{O}_2$  is the operator corresponding to  $O$  for particle 2, then let us say:

$$\hat{O}_1 |\psi_P\rangle_1 |\psi_{P'}\rangle_2 = \omega_P |\psi_P\rangle_1 |\psi_{P'}\rangle_2 \quad (3.26a)$$

$$\text{And } \hat{O}_2 |\psi_P\rangle_1 |\psi_{P'}\rangle_2 = \omega_{P'} |\psi_P\rangle_1 |\psi_{P'}\rangle_2 \quad (3.26b)$$

where  $\omega_P$  and  $\omega_{P'}$  are the eigen values for  $\hat{O}_1$  and  $\hat{O}_2$  in the states  $|\psi_P\rangle_1$  and  $|\psi_{P'}\rangle_2$  respectively. Now Eq. (3.26a) can be written as:

$$\hat{P}_{12}\hat{O}_1|\psi_P\rangle_1|\psi_{P'}\rangle_2 = \omega_P\hat{P}_{12}|\psi_P\rangle_1|\psi_{P'}\rangle_2 \quad (3.27a)$$

And inserting  $\hat{P}_{12}^{-1}\hat{P}_{12} = \hat{I}$  we get

$$\hat{P}_{12}\hat{O}_1\hat{P}_{12}^{-1}\hat{P}_{12}|\psi_P\rangle_1|\psi_{P'}\rangle_2 = \omega_P\hat{P}_{12}|\psi_P\rangle_1|\psi_{P'}\rangle_2 \quad (3.27b)$$

Using Eq. (3.24), Eq. (3.27b) reduces to:

$$\hat{P}_{12}\hat{O}_1\hat{P}_{12}^{-1}|\psi_{P'}\rangle_1|\psi_P\rangle_2 = \omega_P|\psi_{P'}\rangle_1|\psi_P\rangle_2 \quad (3.27c)$$

However from Eq. (3.26b) we also know that:

$$\hat{O}_2|\psi_{P'}\rangle_1|\psi_P\rangle_2 = \omega_P|\psi_{P'}\rangle_1|\psi_P\rangle_2 \quad (3.27d)$$

Since this is true for any arbitrary  $|\psi_P\rangle$ ,  $|\psi_{P'}\rangle$ , the Eqs. (3.27c) and (3.27d) can be consistent with the definition of Eqs. (3.26a and b) only if:

$$\hat{P}_{12}\hat{O}_1\hat{P}_{12}^{-1} = \hat{O}_2 \quad (3.28)$$

So the permutation operator  $\hat{P}_{12}$  exchanges the particle labels on the observables. Now let us consider the Hamiltonian for a system of two identical particles. The Hamiltonian is symmetric in the position and momentum operators of the two particles,  $V(x_1, x_2) = V(x_2, x_1)$ , so we should have:

$$\hat{H}(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2) = -\left(\frac{\hbar^2}{2m}\hat{p}_1^2 + \frac{\hbar^2}{2m}\hat{p}_2^2\right) + V(x_1) + V(x_2) + V(|x_1 - x_2|) \quad (3.29)$$

From Eq. (3.28), we know that

$$\hat{P}_{12}\hat{x}_1\hat{P}_{12}^{-1} = \hat{x}_2; \hat{P}_{12}\hat{x}_2\hat{P}_{12}^{-1} = \hat{x}_1; \hat{P}_{12}\hat{p}_1\hat{P}_{12}^{-1} = \hat{p}_2; \hat{P}_{12}\hat{p}_2\hat{P}_{12}^{-1} = \hat{p}_1 \quad (3.30)$$

Substituting from Eq. (3.30) into Eq. (3.29), we get that the permutation operator commutes with the Hamiltonian for a system of two identical particles:

$$\hat{P}_{12}\hat{H}\hat{P}_{12}^{-1} = \hat{H} \Rightarrow [\hat{P}_{12}, \hat{H}] = 0 \quad (3.31)$$

Since  $\hat{P}_{12}$  commutes with  $\hat{H}$ , **the eigenkets of the Hamiltonian operator are also eigenkets of the permutation operator**. The eigenkets of the Hamiltonian, therefore, must also be symmetric or antisymmetric under the exchange of a pair of particles.

Let us now look at possible eigenstates of the permutation operator. If  $|\psi\rangle$  is an eigenstate  $\hat{P}_{12}$  with the eigenvalue  $\lambda$ , then  $\hat{P}_{12}|\psi\rangle = \lambda|\psi\rangle$ . Since  $\hat{P}_{12}^2 = \hat{I}$ ,

$$\hat{P}_{12}^2|\psi\rangle = |\psi\rangle \Rightarrow \lambda^2 = 1 \text{ or } \lambda = 1, -1 \quad (3.32)$$



So now we look for states  $|\psi\rangle$  that satisfy the condition of Eq. (3.32). An eigenstate of  $\hat{P}_{12}$  with  $\lambda = 1$  is called a symmetric state and an eigenstate of  $\hat{P}_{12}$  with  $\lambda = -1$  is called an antisymmetric state.

Of all the possible linear combinations, we look at the following two (unnormalized) linear combinations of the degenerate two-particle states  $|\psi_P\rangle_1|\psi_{P'}\rangle_2$  and  $|\psi_{P'}\rangle_1|\psi_P\rangle_2$ , which are :

$$|\psi_{PP'}\rangle_+ = N_+ (|\psi_P\rangle_1|\psi_{P'}\rangle_2 + |\psi_{P'}\rangle_1|\psi_P\rangle_2) : \text{Symmetric State} \quad (3.33a)$$

$$|\psi_{PP'}\rangle_- = N_- (|\psi_P\rangle_1|\psi_{P'}\rangle_2 - |\psi_{P'}\rangle_1|\psi_P\rangle_2) : \text{Antisymmetric State} \quad (3.33b)$$

You can show that (SAQ 3)

$$\hat{P}_{12}|\psi_{PP'}\rangle_+ = |\psi_{PP'}\rangle_+ ; \hat{P}_{12}|\psi_{PP'}\rangle_- = -|\psi_{PP'}\rangle_- \quad (3.33c)$$

The eigenvalue of  $\hat{P}_{12}$  is +1 for  $|\psi\rangle_+$  and -1 for  $|\psi\rangle_-$ .

### SAQ 3

a) Show that  $\hat{P}_{12}\hat{H}\hat{P}_{12}^{-1} = \hat{H}$  for the Hamiltonian of Eq. (3.29).

b) Show that (i)  $\hat{P}_{12}|\psi_{PP'}\rangle_+ = |\psi_{PP'}\rangle_+$  (ii)  $\hat{P}_{12}|\psi_{PP'}\rangle_- = -|\psi_{PP'}\rangle_-$

c) Show that  $N_+ = N_- = \frac{1}{\sqrt{2}}$ .

These are the **only two possible eigenstates for  $\hat{P}_{12}$**  for a system of identical particles. The quantum state of the two particle system can either be symmetric ( $|\psi_{PP'}\rangle_+$ ) or antisymmetric ( $|\psi_{PP'}\rangle_-$ ) under particle exchange.

Since  $\hat{P}_{12}$  commutes with  $\hat{H}$ , it is a constant of motion. This also means that if initially the two particle system is in a symmetric state, the system stays in a symmetric state for all times to come. Similarly if initially the two particle system is in an antisymmetric state, the system stays in an antisymmetric state for all times.

Given that the particles are identical, we now drop the subscripts 1 and 2 on the states and say that for two single particle states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$ , the two particle state is either:

$$|\psi_{PP'}\rangle_+ = \frac{1}{\sqrt{2}} (|\psi_P\rangle|\psi_{P'}\rangle + |\psi_{P'}\rangle|\psi_P\rangle) \quad (3.34)$$

Or: 
$$|\psi_{PP'}\rangle_- = \frac{1}{\sqrt{2}} (|\psi_P\rangle|\psi_{P'}\rangle - |\psi_{P'}\rangle|\psi_P\rangle) \quad (3.35)$$

We can write the symmetric and antisymmetric states in the coordinate representation as follows:

$$\begin{aligned}\Psi_+(x_1, x_2) &= \langle x_1, x_2 | \Psi_{PP'} \rangle_+ = \frac{1}{\sqrt{2}} (\langle x_1 | \Psi_P \rangle \langle x_2 | \Psi_{P'} \rangle + \langle x_1 | \Psi_{P'} \rangle \langle x_2 | \Psi_P \rangle) \\ &= \frac{1}{\sqrt{2}} [\Psi_P(x_1) \Psi_{P'}(x_2) + \Psi_{P'}(x_1) \Psi_P(x_2)]\end{aligned}\quad (3.36a)$$

$$\begin{aligned}\Psi_-(x_1, x_2) &= \langle x_1, x_2 | \Psi_{PP'} \rangle_- = \frac{1}{\sqrt{2}} (\langle x_1 | \Psi_P \rangle \langle x_2 | \Psi_{P'} \rangle - \langle x_1 | \Psi_{P'} \rangle \langle x_2 | \Psi_P \rangle) \\ &= \frac{1}{\sqrt{2}} [\Psi_P(x_1) \Psi_{P'}(x_2) - \Psi_{P'}(x_1) \Psi_P(x_2)]\end{aligned}\quad (3.36b)$$

Notice that the antisymmetric wave function can be written as a determinant:

$$\Psi_-(x_1, x_2) = \begin{vmatrix} \Psi_P(x_1) & \Psi_{P'}(x_1) \\ \Psi_P(x_2) & \Psi_{P'}(x_2) \end{vmatrix}\quad (3.36c)$$

How does this result generalize for  $N > 2$ ? Let us consider the particular case of three identical particles labelled 1, 2 and 3, and three single particle states  $|\Psi_P\rangle$ ,  $|\Psi_{P'}\rangle$ ,  $|\Psi_{P''}\rangle$ . Now we have a total of six (3!) possible three-particle states which are all degenerate:

$$\begin{aligned}|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3, |\Psi_P\rangle_1 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_3, |\Psi_P\rangle_2 |\Psi_{P'}\rangle_3 |\Psi_{P''}\rangle_1, \\ |\Psi_P\rangle_2 |\Psi_{P''}\rangle_3 |\Psi_{P'}\rangle_1, |\Psi_P\rangle_3 |\Psi_{P'}\rangle_1 |\Psi_{P''}\rangle_2, |\Psi_P\rangle_3 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_1\end{aligned}$$

As before, under the exchange of any two of the particle labels the state vector can change only by a factor of +1 or -1, hence the only two possible states are:

$$\begin{aligned}|\Psi_{PP'P''}\rangle_+ &= \frac{1}{\sqrt{6}} [|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3 + |\Psi_P\rangle_1 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_3 + |\Psi_P\rangle_2 |\Psi_{P'}\rangle_3 |\Psi_{P''}\rangle_1 \\ &\quad + |\Psi_P\rangle_2 |\Psi_{P''}\rangle_3 |\Psi_{P'}\rangle_1 + |\Psi_P\rangle_3 |\Psi_{P'}\rangle_1 |\Psi_{P''}\rangle_2 + |\Psi_P\rangle_3 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_1]\end{aligned}\quad (3.37a)$$

where every interchange of a pair of particle labels is accompanied by a factor of +1. And

$$\begin{aligned}|\Psi_{PP'P''}\rangle_- &= \frac{1}{\sqrt{6}} [|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3 - |\Psi_P\rangle_1 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_3 + |\Psi_P\rangle_2 |\Psi_{P'}\rangle_3 |\Psi_{P''}\rangle_1 \\ &\quad - |\Psi_P\rangle_2 |\Psi_{P''}\rangle_3 |\Psi_{P'}\rangle_1 + |\Psi_P\rangle_3 |\Psi_{P'}\rangle_1 |\Psi_{P''}\rangle_2 \\ &\quad - |\Psi_P\rangle_3 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_1]\end{aligned}\quad (3.37b)$$

where every exchange of a pair of particle labels is accompanied by a factor of -1. Notice that all terms that are obtained from the term  $|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3$  by exchanging an odd number of pairs of particle labels are negative: for example

$$|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3 \xrightarrow{2 \leftrightarrow 3 (\text{one exchange})} -|\Psi_P\rangle_1 |\Psi_{P''}\rangle_2 |\Psi_{P'}\rangle_3$$

whereas all terms that are obtained from the term  $|\Psi_P\rangle_1 |\Psi_{P'}\rangle_2 |\Psi_{P''}\rangle_3$  by exchanging an even number of pairs of particle labels are positive: for example

$$|\psi_P\rangle_1|\psi_{P'}\rangle_2|\psi_{P''}\rangle_3 \xrightarrow{1\leftrightarrow 2} -|\psi_P\rangle_2|\psi_{P'}\rangle_1|\psi_{P''}\rangle_3 \xrightarrow{1\leftrightarrow 3} |\psi_P\rangle_2|\psi_{P'}\rangle_3|\psi_{P''}\rangle_1$$

The state vector of Eq. (3.37a) is called the totally symmetric state of the three-particle system and the state vector of Eq. (3.37b) is called the totally antisymmetric state because:

$$\hat{P}_{ij}|\psi_{PP'P''}\rangle_+ = |\psi_{PP'P''}\rangle_+ \quad (3.38a)$$

$$\text{And } \hat{P}_{ij}|\psi_{PP'P''}\rangle_- = -|\psi_{PP'P''}\rangle_- \quad (3.38b)$$

Where  $\hat{P}_{ij}$  is the permutation operator for the exchange of the particle labels  $i$  and  $j$ , where  $i, j = 1, 2, 3$  with  $i \neq j$ .

We can once again write the symmetric and antisymmetric wave functions for the three particle state in the coordinate representation (SAQ 4).

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### SAQ 4

Write the symmetric and antisymmetric wave function for system of three identical particles.

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Notice that for the totally symmetric state every exchange of particle labels ( $1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 1$ ) picks up a factor of  $+1$  and for the totally antisymmetric state every exchange of particle labels ( $1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 1$ ) picks up a factor of  $-1$ . A mixture of these properties is not possible. That is you cannot construct a state in which the factors are a mixture of  $+1$  or  $-1$ , say an exchange of  $1 \leftrightarrow 2$  gives you  $+1$  and an exchange  $2 \leftrightarrow 3$  gives you a factor of  $-1$ . You will see later that this is an inherent property of the nature of the identical particles. The totally symmetric and antisymmetric states can be constructed for any  $N$ -particle state in the same fashion.

It also turns out that it is the **spin** of the particles that decides whether they exist in symmetrical or antisymmetrical states. It is not possible to explain the connection between the spin of the particles and the symmetry or antisymmetry of the states within the realm of quantum mechanics that we have studied so far. For that one has to study quantum field theory.

Let us now study an important postulate of quantum mechanics which deals with a systems of identical particles.

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## 3.5 THE SYMMETRIZATION POSTULATE

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What we have established so far is that the two-particle state can be either symmetric or antisymmetric under an exchange of particles. However it turns out that all the allowed states for a system of any number of identical particles of a particular kind must always be either be symmetric or anti symmetric under exchange of a pair of particles. This is a new law of quantum mechanics, and is accepted as an axiom like the other postulates.

The symmetrization postulate stipulates that the available states of many particles of the same type have to be either exclusively symmetric or exclusively antisymmetric. So essentially there are only **two kinds** of identical particles:

In two-dimensions it is to have objects called **anyons**, which have statistical properties ranging between fermions and bosons.

- those which are always found in symmetric states (called **bosons**)
- those which are always found in antisymmetric states (called **fermions**)

Therefore all particles are either bosons or fermions, but we still do not know how to categorize a particle as either a boson or a fermion. This is decided by the spin of the particle which tells us the statistics followed by it. The **bosons are particles with integer spin** and obey the Bose-Einstein statistics, like the pion and the photon. The **fermions are particles with half integer spins** and they obey the Fermi-Dirac statistics, like the electron and the photon.

You know that the spin of a particle is a purely quantum mechanical concept and the magnitude of the spin of a particle, which is one of its invariant properties, can have only the following values:  $\frac{\hbar}{2}, \hbar, \frac{3\hbar}{2}, 2\hbar, \dots$ . Quantum field theory says that particles whose spins are even multiples of  $\frac{\hbar}{2}$  follow Bose-Einstein statistics and particles whose spins are odd multiples of  $\frac{\hbar}{2}$  follow Fermi-Dirac statistics.

With this postulate we can now establish an important rule for fermions, which you have studied in your undergraduate courses and you will be using in other courses in this programme.

### 3.5.1 The Pauli Exclusion Principle

Let us consider a system of two identical fermions which can occupy only two states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$ . Now from the symmetrisation postulate we know that the state of the two-fermion system can only be antisymmetric. So we can say that the only possible state for the fermions is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_P\rangle|\psi_{P'}\rangle - |\psi_{P'}\rangle|\psi_P\rangle) \quad (3.39)$$

If we now assert that the two single fermion states are identical, i.e.:  $P = P'$ , the two fermion state is:

$$|\Psi_{PP}\rangle = \frac{1}{\sqrt{2}} (|\psi_P\rangle|\psi_P\rangle - |\psi_P\rangle|\psi_P\rangle) = 0 \quad (3.40)$$

This is the famous Pauli Exclusion principle which says that no two identical fermions can exist in the same quantum state.

Now suppose we have two bosons which can occupy only two states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$ . Then the three possible states are:

$$|\Psi\rangle = |\psi_P\rangle|\psi_{P'}\rangle, |\psi_{P'}\rangle|\psi_P\rangle, \frac{1}{\sqrt{2}} (|\psi_P\rangle|\psi_{P'}\rangle + |\psi_{P'}\rangle|\psi_P\rangle)$$

If the two single boson states are identical, i.e.:  $P = P'$ , the two possible states are  $|\psi_P\rangle|\psi_P\rangle, |\psi_{P'}\rangle|\psi_{P'}\rangle$ . So while fermions cannot exist in the same state, bosons do. This tendency of bosons shows up in particular at very low temperatures where a collection of bosons tends to exist in the ground state, a phenomenon called Bose-Einstein condensation.

---

### SAQ 5

Calculate the normalization constant for two bosons in the same single particle state.

The symmetrisation postulate is also applied to multiparticle states of bosons and fermions with  $N > 2$  with respect to the interchange of any pair of particles (say  $i$  and  $j$ ):

$$\hat{P}_{ij} |\Psi_N \text{ Bosons}\rangle = + |\Psi_N \text{ Bosons}\rangle \quad (3.41a)$$

$$\hat{P}_{ij} |\Psi_N \text{ Fermions}\rangle = - |\Psi_N \text{ Fermions}\rangle \quad (3.41b)$$

$\hat{P}_{ij}$  is the permutation operator for the exchange of the  $i^{\text{th}}$  and  $j^{\text{th}}$  particles.

So essentially the rules for a system of 2 identical bosons or fermions also extends to a system of  $N$  bosons or  $N$  fermions. The  $N$ -particle state for a system of  $N$  bosons is always symmetric and the system of  $N$  fermions is always antisymmetric under particle exchange.

### 3.5.2 The Exchange Force

Let us consider the wave functions be  $\psi_P(x)$  and  $\psi_{P'}(x)$  for two single particle states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  which are **orthogonal and normalized**. If we now have a system of two particles, 1 and 2, which are distinguishable, with one in the state  $\psi_P(x)$  and the other in the state  $\psi_{P'}(x)$ , the two particle wavefunction is :

$$\psi(x_1, x_2) = \psi_P(x_1)\psi_{P'}(x_2) \quad (3.42a)$$

If the two particles are identical bosons, the wavefunction is:

$$\psi_B(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_P(x_1)\psi_{P'}(x_2) + \psi_{P'}(x_1)\psi_P(x_2)] \quad (3.42b)$$

where the subscript  $B$  denotes bosons. And if they are identical fermions the wave function is

$$\psi_F(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_P(x_1)\psi_{P'}(x_2) - \psi_{P'}(x_1)\psi_P(x_2)] \quad (3.42c)$$

where the subscript  $F$  denotes fermions. Let us now calculate the expectation value of the separation between the two particles which is:

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle = \langle (\hat{x}_1)^2 \rangle + \langle (\hat{x}_2)^2 \rangle - 2\langle (\hat{x}_1\hat{x}_2) \rangle \quad (3.42d)$$

- For distinguishable particles:

$$\begin{aligned} \langle (\hat{x}_1)^2 \rangle &= \int dx_1 dx_2 \psi_P^*(x_1)\psi_{P'}^*(x_2) x_1^2 \psi_P(x_1)\psi_{P'}(x_2) \\ &= \left[ \int dx_1 \psi_P^*(x_1) x_1^2 \psi_P(x_1) \right] \left[ \int dx_2 \psi_{P'}^*(x_2)\psi_{P'}(x_2) \right] \\ &= \int dx \psi_P^*(x) x^2 \psi_P(x) = \langle \hat{x}^2 \rangle_P \end{aligned} \quad (3.43a)$$

which is the expectation value of the square of position in the state  $\psi_P(x)$ .

Notice that in deriving this result we have used the properties of orthogonality and normalization of the wavefunctions:

$$\int dx \psi_{P'}^*(x) \psi_{P'}(x) = \int dx \psi_P^*(x) \psi_P(x) = 1;$$

$$\int dx \psi_P^*(x) \psi_{P'}(x) = 0 = \int dx \psi_{P'}^*(x) \psi_P(x)$$

Similarly

$$\langle (\hat{x}_2)^2 \rangle = \int dx \psi_{P'}^*(x) x^2 \psi_{P'}(x) = \langle \hat{x}^2 \rangle_{P'} \quad (3.43b)$$

And

$$\begin{aligned} \langle (\hat{x}_1 \hat{x}_2) \rangle &= \int dx_1 dx_2 \psi_P^*(x_1) \psi_{P'}^*(x_2) x_1 x_2 \psi_P(x_1) \psi_{P'}(x_2) \\ &= \left[ \int dx_1 \psi_P^*(x_1) x_1 \psi_P(x_1) \right] \left[ \int dx_2 \psi_{P'}^*(x_2) x_1 \psi_{P'}(x_2) \right] \\ &= \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} \end{aligned} \quad (3.43c)$$

Hence the expectation value  $\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle$  defined in Eq. (3.42d) can be written as follows for a pair of distinguishable particles:

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_D = \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} - 2 \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} \quad (3.44)$$

where the subscript  $D$  is for distinguishable particles.

- **For identical bosons**, we use Eq. (3.42b) for the wave function. So

$$\begin{aligned} \langle (\hat{x}_1)^2 \rangle &= \int dx_1 dx_2 \psi_B^*(x_1, x_2) x_1^2 \psi_B(x_1, x_2) \\ &= \frac{1}{2} \int dx_1 dx_2 \psi_P^*(x_1) \psi_{P'}^*(x_2) x_1^2 \psi_P(x_1) \psi_{P'}(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_{P'}^*(x_1) \psi_P^*(x_2) x_1^2 \psi_{P'}(x_1) \psi_P(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_P^*(x_1) \psi_P^*(x_2) x_1^2 \psi_P(x_1) \psi_{P'}(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_P^*(x_1) \psi_{P'}^*(x_2) x_1^2 \psi_{P'}(x_1) \psi_P(x_2) \\ &= \frac{1}{2} \left[ \int dx_1 \psi_P^*(x_1) x_1^2 \psi_P(x_1) \right] \left[ \int dx_2 \psi_{P'}^*(x_2) \psi_{P'}(x_2) \right] \\ &\quad + \frac{1}{2} \left[ \int dx_1 \psi_{P'}^*(x_1) x_1^2 \psi_{P'}(x_1) \right] \left[ \int dx_2 \psi_P^*(x_2) \psi_P(x_2) \right] \\ &\quad + \frac{1}{2} \left[ \int dx_1 \psi_P^*(x_1) x_1^2 \psi_P(x_1) \right] \left[ \int dx_2 \psi_P^*(x_2) \psi_{P'}(x_2) \right] \\ &\quad + \frac{1}{2} \left[ \int dx_1 \psi_P^*(x_1) x_1^2 \psi_{P'}(x_1) \right] \left[ \int dx_2 \psi_{P'}^*(x_2) \psi_P(x_2) \right] \\ &= \frac{1}{2} \langle \hat{x}^2 \rangle_P + \frac{1}{2} \langle \hat{x}^2 \rangle_{P'} + 0 + 0 = \frac{1}{2} \left[ \langle \hat{x}^2 \rangle_P + \langle \hat{x}^2 \rangle_{P'} \right] \end{aligned} \quad (3.45a)$$

$$\text{Similarly } \langle (\hat{x}_2)^2 \rangle = \frac{1}{2} \left[ \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} \right] \quad (3.45b)$$

And

$$\begin{aligned} \langle (\hat{x}_1 \hat{x}_2) \rangle &= \int dx_1 dx_2 \psi_B^*(x_1, x_2) x_1 x_2 \psi_B(x_1, x_2) \\ &= \frac{1}{2} \int dx_1 dx_2 \psi_P^*(x_1) \psi_{P'}^*(x_2) x_1 x_2 \psi_P(x_1) \psi_{P'}(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_{P'}^*(x_1) \psi_P^*(x_2) x_1 x_2 \psi_{P'}(x_1) \psi_P(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_P^*(x_1) \psi_P^*(x_2) x_1 x_2 \psi_P(x_1) \psi_{P'}(x_2) \\ &\quad + \frac{1}{2} \int dx_1 dx_2 \psi_{P'}^*(x_1) \psi_{P'}^*(x_2) x_1 x_2 \psi_{P'}(x_1) \psi_P(x_2) \\ &= \frac{1}{2} \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} + \frac{1}{2} \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} + \frac{1}{2} \langle \hat{x} \rangle_{PP'} \langle \hat{x} \rangle_{P'P} + \frac{1}{2} \langle \hat{x} \rangle_{P'P} \langle \hat{x} \rangle_{PP'} \\ &= \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} + |\langle \hat{x} \rangle_{PP'}|^2 \end{aligned} \quad (3.45c)$$

Hence the expectation value of  $\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle$  for a pair of bosons is:

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_B = \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} - 2 \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} - 2 |\langle \hat{x} \rangle_{PP'}|^2 \quad (3.46)$$

where the subscript  $B$  is for a pair of bosons.

- **For identical Fermions**, we use Eq. (3.42c) for the wave function and the results is (SAQ 6) :

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_F = \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} - 2 \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} + 2 |\langle \hat{x} \rangle_{PP'}|^2 \quad (3.47)$$

where the subscript  $F$  is for a pair of fermions.

### SAQ 6

For a system of two identical fermions show that

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_F = \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} - 2 \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} + 2 |\langle \hat{x} \rangle_{PP'}|^2$$

Comparing the three results we can easily note the difference between the system of two distinguishable particles, bosons and fermions

#### **Distinguishable Particles**

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_D = \langle (\hat{x})^2 \rangle_P + \langle (\hat{x})^2 \rangle_{P'} - 2 \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} \quad (3.48a)$$

#### **Identical Bosons**

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_B = \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_D - 2 |\langle \hat{x} \rangle_{PP'}|^2 \quad (3.48b)$$

#### **Identical Fermions**

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_F = \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_D + 2 |\langle \hat{x} \rangle_{PP'}|^2 \quad (3.48c)$$

Notice that the difference between Eqs. (3.48a), (3.48b) and 3.48c) is the presence of the overlap integral between the single particle states in Eqs. (3.48b) and 3.48c):

$$\langle \hat{x} \rangle_{PP'} = \frac{1}{2} \int dx \psi_P^*(x) x \psi_{P'}(x)$$

This tells us the following:

- i) Unless and until the overlap integral is non-zero, that is the wavefunction  $\psi_P(x)$  and  $\psi_{P'}(x)$  are both finite at  $x$ , it would not matter whether we take a symmetric/antisymmetric wavefunction as in Eqs. (3.41b and c) or just the wave function for distinguishable particles. So for example if you had one proton in Delhi and one proton in Kolkata, and the single particle states are Gaussian wave functions centred around Delhi ( $\psi_D(x)$ ) and Kolkata ( $\psi_K(x)$ ) respectively, then given that  $\psi_K(x)$  is negligible where  $\psi_D(x)$  is finite and vice versa, you need not antisymmetrize the wavefunction.
- ii) If the overlap integral is non-zero, then a pair of bosons tends to be closer together than a pair of fermions. So it as if there is a **force that tends to draw two bosons together and push two fermions apart.**

This force between a pair of bosons or a pair of fermions is called the **exchange force**. It is attractive for bosons and repulsive for fermions. Remember it is not an actual force but a consequence of the symmetrisation property of wave function of identical bosons or fermions. This is therefore a purely quantum mechanical effect and has no classical counterpart.

We have so far discussed the properties of identical bosons and fermions without any reference to the spin. At this point we just point out that as you have studied for an electron the wave function is actually a spinor which includes information about the spin state of the electron. So when one refers to the symmetrisation/ antisymmetrization of the wavefunction, we are considering the complete wavefunction which includes the spin part of the wave function as well.

So for a collection of  $N$  electrons for example, the wavefunction is  $\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \chi(s_1, s_2, \dots, s_N)$  where  $s_j$  indicates the spin state of each electron. And the wave function is antisymmetric with exchange, means exchange of both position and spin labels.

We will discuss this in a little more detail after we discuss addition of angular momentum which we do in the next Unit.

### 3.6 CANONICAL QUANTIZATION

Let us now try to relate the postulates of quantum mechanics that you studied in Sec. 3.2 to what you have studied in classical mechanics.

You have learnt about canonical coordinates in classical mechanics. A classical system with  $N$  degrees of freedom has the canonical coordinates  $x_1, x_2, \dots, x_N$  and the corresponding conjugate canonical momenta:  $p_1, p_2, \dots, p_N$ . The Poisson bracket (denoted by  $\{A, B\}_{PB}$ ) for any two dynamical variables, say  $A$  and  $B$  is:



$$\{A, B\}_{PB} = \sum_{j=1}^N \left[ \frac{\partial A}{\partial x_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial x_j} \right] \quad (3.49a)$$

Further, as you know:

$$\{x_i, x_j\}_{PB} = \{p_i, p_j\}_{PB} = 0 ; \{x_i, p_j\}_{PB} = \delta_{ij} \quad (3.49b)$$

And the time variation of a dynamical variable  $O$  is:

$$\frac{\partial O}{\partial t} = \{O, H\}_{PB} \quad (3.49c)$$

For the canonical coordinates and momenta:

$$\frac{\partial x_i}{\partial t} = \{x_i, H\}_{PB} ; \frac{\partial p_i}{\partial t} = \{p_i, H\}_{PB} \quad (3.49d)$$

Comparing the equations for the classical dynamical variables and the corresponding quantum operators: it **appears** that any classical Hamiltonian system with  $N$  canonical coordinates  $x_1, x_2, \dots, x_N$  and the corresponding canonical momenta:  $p_1, p_2, \dots, p_N$  can be quantized with the prescription:

$$x_i \rightarrow \hat{x}_i ; p_i \rightarrow \hat{p}_i ; \{ , \}_{PB} \rightarrow \frac{1}{i\hbar} [ , ] \quad (3.50)$$

And dynamical variable of the system which depends on the position  $x_i$  and momenta  $p_i$ , say  $O = f(x_i, p_i)$  can be represented by the corresponding operator expression obtained by replacing  $x_i$  by  $\hat{x}_i$  and  $p_i$  by  $\hat{p}_i$ :  $\hat{O} = f(\hat{x}_i, \hat{p}_i)$ .

We have then a plausible procedure for determining a quantum system that reduces to a particular classical system in the classical limit: a method of deriving a quantum version of a classical system. This process is called **canonical quantization** because it uses the Hamiltonian or canonical formulation of classical mechanics.

There is however no unique way of arriving at the quantum version of a classical system. Because you can always add terms to the quantum equations of motion that would vanish in the classical limit. So this process just gives you the simplest quantum system that has the correct classical limit, something like a first guess.

You must also remember that the quantum and classical equations of motion refer to essentially different quantities, because operators in quantum mechanics, unlike dynamical variables in classical mechanics, are not observable. Observables are the corresponding expectation values of the operators. It is only the form of the equations that are similar in classical and quantum mechanics. So essentially the method of canonical quantizations gives us a first formulation of quantum mechanics, however only for those systems which have a classical Hamiltonian formulation.

There are operators like spin that do not have any classical analogue and the quantum mechanics of a spin-half particle cannot be arrived at from a classical Hamiltonian

### 3.7 SUMMARY

- A classical system with  $N$  degrees of freedom for which the  $N$  Cartesian coordinates and momenta are  $x_1, x_2, \dots, x_N$  and  $p_1, p_2, \dots, p_N$  respectively has the following quantum mechanical description

- There is a set  $N$  **mutually commuting** position operators  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$

The **simultaneous eigenbasis** of these commuting operators is the coordinate basis for the system:  $|x_1, x_2, \dots, x_N\rangle$ .  $|x_1, x_2, \dots, x_N\rangle$  is a simultaneous eigenket of the position operators  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$ .

- The wave function corresponding to the state ket  $|\psi\rangle$  is:

$$\psi(x_1, x_2, \dots, x_N) = \langle x_1, x_2, \dots, x_N | \psi \rangle$$

- The commutation relations are

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 ; [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \text{ for } i, j = 1, 2, \dots, N$$

- For a system of two distinguishable particles in one-dimension labelled 1 and 2, characterized by the independent Cartesian coordinates  $x_1, x_2$  the wave function is  $\psi(x_1, x_2)$ . For

- **non-interacting particles:**  $V(x_1, x_2) = V(x_1) + V(x_2)$

A stationary state with total energy  $E$  is  $\psi_E(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$  where

$$\left( -\frac{\hbar^2}{2m_1} \frac{d^2}{dx_1^2} + V(x_1) \right) \psi_1(x_1) = E_1 \psi_1(x_1);$$

$$\left( -\frac{\hbar^2}{2m_2} \frac{d^2}{dx_2^2} + V(x_2) \right) \psi_2(x_2) = E_2 \psi_2(x_2);$$

with  $E = E_1 + E_2$

- interacting particles with  $V(x_1, x_2) = V(x_1 - x_2)$

The problem is solved in centre of mass coordinates with the following two Schrodinger equations:

$$-\frac{\hbar^2}{2M} \frac{\partial^2 \psi_{CM}(x_{CM})}{\partial x_{CM}^2} = E_{CM} \psi_{CM}(x_{CM});$$

$$\left( -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial x_r^2} + V(x_r) \right) \psi_r(x_r) = E_r \psi_r(x_r)$$

with  $\psi_E(x_{CM}, x_r) = \psi_{CM}(x_{CM})\psi_r(x_r)$ ,  $M$  is the total mass,  $\mu$  is the reduced mass,  $x_{CM}$  is the centre of mass coordinate and  $x_r$  is the relative coordinate.

- In a system of identical particles the quantum mechanical state of the system has an exchange degeneracy only because the particles may be exchanged without changing the eigenvalues associated with the quantum mechanical state of the system.

- Under the exchange of particles, the wave function of a system of two identical particles  $\psi(x_1, x_2, t)$  can be

$$\text{Symmetric} : \psi(x_2, x_1, t) = +\psi(x_1, x_2, t)$$

$$\text{or Antisymmetric: } \psi(x_2, x_1, t) = -\psi(x_1, x_2, t)$$

- For two particles 1 and 2 and two states  $|\psi_P\rangle$  and  $|\psi_{P'}\rangle$  where  $P$  and  $P'$  represent any set of observables we define a permutation operator  $\hat{P}_{12}$  as follows:  $\hat{P}_{12}|\psi_P\rangle_1|\psi_{P'}\rangle_2 = |\psi_{P'}\rangle_1|\psi_P\rangle_2$ .

The effect of the permutation operator is to exchange the particle labels 1 and 2.

The **only two possible eigenstates** for  $\hat{P}_{12}$  for a system of identical particles are

$$|\psi_{PP'}\rangle_+ = N_+ (|\psi_P\rangle_1|\psi_{P'}\rangle_2 + |\psi_{P'}\rangle_1|\psi_P\rangle_2) : \text{Symmetric State}$$

$$|\psi_{PP'}\rangle_- = N_- (|\psi_P\rangle_1|\psi_{P'}\rangle_2 - |\psi_{P'}\rangle_1|\psi_P\rangle_2) : \text{Antisymmetric State}$$

$$\text{where } \hat{P}_{12}|\psi_{PP'}\rangle_+ = |\psi_{PP'}\rangle_+ ; \hat{P}_{12}|\psi_{PP'}\rangle_- = -|\psi_{PP'}\rangle_-$$

- The symmetrization postulate stipulates that the available states of many particles of the same type have to be either exclusively symmetric or exclusively antisymmetric. So essentially there are only **two kinds** of identical particles:
  - those which are always found in symmetric states (called **bosons**)
  - those which are always found in antisymmetric states (called **fermions**)
- The principle of **canonical quantization** states that any classical Hamiltonian system with  $N$  canonical coordinates  $x_1, x_2, \dots, x_N$  and the corresponding canonical momenta:  $p_1, p_2, \dots, p_N$  can be quantized with the prescription:

$$x_i \rightarrow \hat{x}_i ; p_i \rightarrow \hat{p}_i ; \{ , \}_{PB} \rightarrow \frac{1}{i\hbar} [ , ]$$

### 3.8 TERMINAL QUESTIONS

- Solve the Schrodinger equations for a particle in a three dimensional box and obtain the energy eigenvalues and eigenfunctions.
- Solve the Schrodinger equations for a three simple harmonic oscillator and obtain the energy eigenvalues and eigenfunctions for the ground and first excited state. State the parity of each of the energy eigen function.
- Consider two non-interacting particles confined to a one dimensional box. Calculate the eigen energy and eigen functions of the ground state and first excited state if the two particles are
  - Distinguishable
  - Identical Bosons
  - Identical Fermions

4. Write the Schrodinger equations for the coupled spring mass system (Unit 7, MPH-004) and obtain the solutions.
5. For two non-interacting distinguishable particles in an one-dimensional box of length  $L$ , calculate  $\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle$  if one particle is in a state  $\psi_m(x)$  and the other is in a state  $\psi_n(x)$ .
6. For two non-interacting particles in an one-dimensional box of length  $L$ , calculate  $\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle$  if one particle is in a state  $\psi_m(x)$  and the other is in a state  $\psi_n(x)$  if the particles are a) identical bosons; b) identical fermions.
7. Construct the wave function  $\psi(x_1, x_2, x_3)$  for i) a system of 3 fermions in the states  $|\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$ , respectively; ii) a system of 3 bosons in the states  $|\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$ , respectively; and iii) 3 bosons in the states  $|\psi_2\rangle, |\psi_2\rangle, |\psi_3\rangle$ , respectively, of an one-dimensional box of size  $L$

### 3.9 SOLUTIONS AND ANSWERS

#### Self-Assessment Questions

1. With  $\psi_E(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$  we get

$$\frac{\partial^2 \psi_E}{\partial x_1^2} = \frac{d^2 \psi_1}{dx_1^2} \psi_2 \quad (i)$$

$$\text{And : } \frac{\partial^2 \psi_E}{\partial x_2^2} = \frac{d^2 \psi_2}{dx_2^2} \psi_1 \quad (ii)$$

Using Eqs. (i) and (ii), Eq. (3.8b) reduces to

$$\begin{aligned} & -\frac{\hbar^2}{2m_1} \frac{d^2 \psi_1}{dx_1^2} \psi_2 + V(x_1) \psi_1 \psi_2 \\ & -\frac{\hbar^2}{2m_2} \frac{d^2 \psi_2}{dx_2^2} \psi_1 + V(x_2) \psi_1 \psi_2 = E \psi_1 \psi_2 \end{aligned} \quad (iii)$$

Dividing Eq. (iii) by  $\psi_1 \psi_2$  we get

$$\begin{aligned} & \left[ -\frac{\hbar^2}{2m_1} \frac{1}{\psi_1} \frac{d^2 \psi_1}{dx_1^2} + V(x_1) \right] + \left[ -\frac{\hbar^2}{2m_2} \frac{1}{\psi_2} \frac{d^2 \psi_2}{dx_2^2} + V(x_2) \right] = E \\ \Rightarrow & -\frac{\hbar^2}{2m_1} \frac{1}{\psi_1} \frac{d^2 \psi_1}{dx_1^2} + V(x_1) = E - \left[ -\frac{\hbar^2}{2m_2} \frac{1}{\psi_2} \frac{d^2 \psi_2}{dx_2^2} + V(x_2) \right] = E_1 \quad (\text{say}) \end{aligned}$$

Hence

$$-\frac{\hbar^2}{2m_1} \frac{d^2 \psi_1}{dx_1^2} + V(x_1) \psi_1 = E_1 \psi_1$$

$$\text{and } -\frac{\hbar^2}{2m_2} \frac{d^2 \psi_2}{dx_2^2} + V(x_2) \psi_2 = (E - E_1) \psi_2 = E_2 \psi_2 \quad (\text{say})$$

where  $E - E_1 = E_2$  or  $E = E_1 + E_2$

$$2. a) i) \quad x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1}{M} x_1 + \frac{m_2}{M} x_2$$

$$x_r = x_1 - x_2$$

changing variables from  $(x_1, x_2) \rightarrow (x_{CM}, x_r)$

$$\frac{\partial}{\partial x_1} = \frac{\partial x_{CM}}{\partial x_1} \frac{\partial}{\partial x_{CM}} + \frac{\partial x_r}{\partial x_1} \frac{\partial}{\partial x_r} = \frac{m_1}{M} \frac{\partial}{\partial x_{CM}} + \frac{\partial}{\partial x_r}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial x_{CM}}{\partial x_2} \frac{\partial}{\partial x_{CM}} + \frac{\partial x_r}{\partial x_2} \frac{\partial}{\partial x_r} = \frac{m_2}{M} \frac{\partial}{\partial x_{CM}} - \frac{\partial}{\partial x_r}$$

$$\begin{aligned} \frac{\partial^2}{\partial x_1^2} &= \frac{\partial}{\partial x_{CM}} \left( \frac{m_1}{M} \frac{\partial}{\partial x_{CM}} + \frac{\partial}{\partial x_r} \right) \cdot \frac{m_1}{M} \\ &\quad + \frac{\partial}{\partial x_r} \left( \frac{m_1}{M} \frac{\partial}{\partial x_{CM}} + \frac{\partial}{\partial x_r} \right) \end{aligned}$$

$$= \left( \frac{m_1}{M} \right)^2 \frac{\partial^2}{\partial x_{CM}^2} + \frac{2m_1}{M} \frac{\partial^2}{\partial x_{CM} \partial x_r} + \frac{\partial^2}{\partial x_r^2}$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{m_2}{M} \frac{\partial}{\partial x_{CM}} \left( \frac{m_2}{M} \frac{\partial}{\partial x_{CM}} - \frac{\partial}{\partial x_r} \right) - \frac{\partial}{\partial x_r} \left( \frac{m_2}{M} \frac{\partial}{\partial x_{CM}} - \frac{\partial}{\partial x_r} \right)$$

$$= \left( \frac{m_2}{M} \right)^2 \frac{\partial^2}{\partial x_{CM}^2} - \frac{2m_2}{M} \frac{\partial^2}{\partial x_{CM} \partial x_r} + \frac{\partial^2}{\partial x_r^2}$$

$$\therefore -\frac{\hbar^2}{2m_1} \frac{\partial^2 \psi_E}{\partial x_1^2} - \frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_E}{\partial x_2^2} + V(x_1 - x_2) \psi_E = E \psi_E$$

$$\Rightarrow -\frac{\hbar^2}{2} \frac{m_1}{M^2} \frac{\partial^2 \psi_E}{\partial x_{CM}^2} - \frac{\hbar^2}{M} \frac{\partial^2}{\partial x_{CM} \partial x_r} - \frac{\hbar^2}{2m_1} \frac{\partial^2 \psi_E}{\partial x_r^2}$$

$$- \frac{\hbar^2}{2} \frac{m_2}{M^2} \frac{\partial^2 \psi_E}{\partial x_{CM}^2} + \frac{\hbar^2}{M} \frac{\partial^2 \psi_E}{\partial x_{CM} \partial x_r} - \frac{\hbar^2}{2m_2} \frac{\partial^2 \psi_E}{\partial x_r^2} + V(x_r) \psi_r = E \psi_E$$

$$\Rightarrow -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_E}{\partial x_{CM}^2} - \frac{\hbar^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \frac{\partial^2 \psi_E}{\partial x_r^2} + V(x_r) \psi_E = E \psi_E$$

$$\Rightarrow -\frac{\hbar^2}{2M} \frac{\partial^2 \psi_E}{\partial x_{CM}^2} - \frac{\hbar^2}{2\mu} \frac{\partial^2 \psi_E}{\partial x_r^2} + V(x_r) \psi_E = E \psi_E$$

$$\text{where } \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}$$

$$ii) \quad \dot{x}_{CM} = \frac{m_1}{M} \dot{x}_1 + \frac{m_2}{M} \dot{x}_2$$

$$\Rightarrow M \dot{x}_{cm} = m_1 \dot{x}_1 + m_2 \dot{x}_2$$

$$\Rightarrow p_{CM} = p_1 + p_2$$

$$\Rightarrow \hat{p}_{CM} = \hat{p}_1 + \hat{p}_2$$

$$\begin{aligned} [\hat{x}_{CM}, \hat{p}_{CM}] &= \left[ \frac{m_1}{M} \hat{x}_1 + \frac{m_2}{M} \hat{x}_2, \hat{p}_1 + \hat{p}_2 \right] \\ &= \frac{m_1}{M} [\hat{x}_1, \hat{p}_1] + \frac{m_2}{M} [\hat{x}_2, \hat{p}_2] \quad (\because [\hat{x}_1, \hat{p}_2] = [\hat{x}_2, \hat{p}_1] = 0) \\ &= i\hbar \frac{m_1}{M} + i\hbar \frac{m_2}{M} = i\hbar \end{aligned}$$

$$x_r = x_1 - x_2 \Rightarrow \dot{x}_r = \dot{x}_1 - \dot{x}_2$$

$$\Rightarrow \mu \dot{x}_r = \mu \dot{x}_1 - \mu \dot{x}_2 = \frac{m_2}{M} p_1 - \frac{m_1}{M} p_2$$

$$\Rightarrow p_r = \frac{m_2}{M} p_1 - \frac{m_1}{M} p_2$$

$$\therefore \hat{p}_r = \frac{m_2}{M} \hat{p}_1 - \frac{m_1}{M} \hat{p}_2$$

$$\begin{aligned} \therefore [\hat{x}_r, \hat{p}_r] &= \left[ \hat{x}_1 - \hat{x}_2, \frac{m_2}{M} \hat{p}_1 - \frac{m_1}{M} \hat{p}_2 \right] \\ &= [\hat{x}_1, \hat{p}_1] \frac{m_2}{M} + \frac{m_1}{M} [\hat{x}_2, \hat{p}_2] = i\hbar \end{aligned}$$

b) i) The Schrödinger equation for the particle is (since  $V = 0$  inside the box):

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial z^2} = E\psi$$

With  $\psi = \psi_x(x)\psi_y(y)\psi_z(z)$  we get

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_x}{\partial x^2} = E_x \psi_x \quad 0 \leq x \leq L \quad (i)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_y}{\partial y^2} = E_y \psi_y \quad 0 \leq y \leq L \quad (ii)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_z}{\partial z^2} = E_z \psi_z \quad 0 \leq z \leq L \quad (iii)$$

where  $E = E_x + E_y + E_z$ .

ii) The potential function for an isotropic 3d oscillator is

$$V(x, y, z) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$$

$\therefore$  The Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) \psi = E\psi$$

With  $\psi = \psi_x(x) \psi_y(y) \psi_z(z)$  we get:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_x}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi_x = E_x \psi_x \quad (i)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_y}{\partial y^2} + \frac{1}{2} m \omega^2 y^2 \psi_y = E_y \psi_y \quad (ii)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_z}{\partial z^2} + \frac{1}{2} m \omega^2 z^2 \psi_z = E_z \psi_z \quad (iii)$$

where  $E = E_x + E_y + E_z$

$$\begin{aligned} 3. a) \quad \hat{P}_{12} \hat{H} \hat{P}_{12}^{-1} &= \hat{P}_{12} \left\{ -\frac{\hbar^2}{2m} \hat{p}_1^2 - \frac{\hbar^2}{2m} \hat{p}_2^2 \right\} \hat{P}_{12}^{-1} \\ &+ \hat{P}_{12} V(x_1) \hat{P}_{12}^{-1} + \hat{P}_{12} V(x_2) \hat{P}_{12}^{-1} + \hat{P}_{12} V(x_1 - x_2) \hat{P}_{12}^{-1} \\ \hat{P}_{12} \hat{p}_1^2 \hat{P}_{12}^{-1} &= \hat{P}_{12} \hat{p}_1 \hat{P}_{12}^{-1} \hat{P}_{12} \hat{p}_1 \hat{P}_{12}^{-1} = \hat{p}_2^2 \end{aligned}$$

Similarly  $\hat{P}_{12} \hat{p}_2^2 \hat{P}_{12}^{-1} = \hat{p}_1^2$ . Since  $V(x_1)$  is a function of  $x_1$  only:

$$\hat{P}_{12} V(x_1) \hat{P}_{12}^{-1} = V(x_2) \quad \text{and} \quad \hat{P}_{12} V(x_2) \hat{P}_{12}^{-1} = V(x_1)$$

$V(|x_1 - x_2|)$  depends on the difference between  $|x_1 - x_2|$  and hence is not affected by the permutation operator so

$$\hat{P}_{12} V(|x_1 - x_2|) \hat{P}_{12}^{-1} = V(|x_1 - x_2|).$$

$$\therefore \hat{P}_{12} \hat{H} \hat{P}_{12}^{-1} = \hat{H}$$

$$\begin{aligned} b) i) \quad \hat{P}_{12} |\psi_{PP'}\rangle_+ &= N_+ \hat{P}_{12} [|\psi_P\rangle_1 |\psi_{P'}\rangle_2 + |\psi_{P'}\rangle_1 |\psi_P\rangle_2] \\ &= N_+ [|\psi_P\rangle_2 |\psi_{P'}\rangle_1 + |\psi_{P'}\rangle_2 |\psi_P\rangle_1] \\ &= |\psi_{PP'}\rangle_+ \end{aligned}$$

$$\begin{aligned} ii) \quad \hat{P}_{12} |\psi_{PP'}\rangle_- &= N_- \hat{P}_{12} [|\psi_P\rangle_1 |\psi_{P'}\rangle_2 - |\psi_{P'}\rangle_1 |\psi_P\rangle_2] \\ &= N_- [|\psi_P\rangle_2 |\psi_{P'}\rangle_1 - |\psi_{P'}\rangle_2 |\psi_P\rangle_1] \\ &= -N_- [|\psi_P\rangle_1 |\psi_{P'}\rangle_2 - |\psi_{P'}\rangle_1 |\psi_P\rangle_2] = -|\psi_{PP'}\rangle_- \end{aligned}$$

$$c) \quad |\psi_{PP'}\rangle_+ = N_+ [|\psi_P\rangle_1 |\psi_{P'}\rangle_2 + |\psi_{P'}\rangle_1 |\psi_P\rangle_2]$$

$$\Rightarrow \psi_+(x_1, x_2) = N_+ [\psi_P(x_1) \psi_{P'}(x_2) + \psi_{P'}(x_1) \psi_P(x_2)]$$

$$\text{And } \psi_+^*(x_1, x_2) = N_+^* [\psi_P^*(x_1) \psi_{P'}^*(x_2) + \psi_{P'}^*(x_1) \psi_P^*(x_2)]$$

$$\int \psi_+^* \psi_+ dx_1 dx_2 = 1$$

$$\begin{aligned} \Rightarrow |N_+|^2 &\left[ \int \psi_P^*(x_1) \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_{P'}(x_2) dx_2 \right. \\ &\left. + \int \psi_{P'}^*(x_1) \psi_{P'}(x_1) dx_1 \int \psi_P^*(x_2) \psi_P(x_2) dx_2 \right] \end{aligned}$$

$$\begin{aligned}
& + \int \psi_{P'}^*(x_1) \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_P(x_2) dx_2 \\
& + \int \psi_{P'}^*(x_1) \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_P(x_2) dx_2 = 1
\end{aligned}$$

$$\Rightarrow |N_+|^2 [1 + 0 + 0 + 1] = 1 \Rightarrow N_+ = \frac{1}{\sqrt{2}}$$

Similarly

$$\psi_-(x_1, x_2) = N_- [\psi_P(x_1) \psi_{P'}(x_2) - \psi_{P'}(x_1) \psi_P(x_2)]$$

$$\psi_-^*(x_1, x_2) = N_- [\psi_{P'}^*(x_1) \psi_P^*(x_2) - \psi_P^*(x_1) \psi_{P'}^*(x_2)]$$

$$\int \psi_-^*(x_1, x_2) \psi_-(x_1, x_2) dx_1 dx_2 = 1$$

$$\begin{aligned}
\Rightarrow |N_-|^2 & \left[ \int \psi_P^*(x_1) \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_{P'}(x_2) dx_2 \right. \\
& - \int \psi_{P'}^*(x_1) \psi_{P'}(x_1) dx_1 \int \psi_P^*(x_2) \psi_P(x_2) dx_2 \\
& - \int \psi_P^*(x_1) \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_{P'}(x_2) dx_2 \\
& \left. + \int \psi_{P'}^*(x_1) \psi_{P'}(x_1) dx_1 \int \psi_P^*(x_2) \psi_P(x_2) dx_2 \right] = 1
\end{aligned}$$

$$\Rightarrow |N_-|^2 [1 - 0 - 0 + 1] = 1 \Rightarrow N_- = \frac{1}{\sqrt{2}}$$

$$4. \quad \psi_+(x_1, x_2, x_3) = \langle x_1, x_2, x_3 | \psi_{PP'P''} \rangle_+$$

$$\begin{aligned}
& = \frac{1}{\sqrt{6}} [\psi_P(x_1) \psi_{P'}(x_2) \psi_{P''}(x_3) + \psi_P(x_1) \psi_{P'}(x_3) \psi_{P''}(x_2) \\
& + \psi_P(x_2) \psi_{P'}(x_3) \psi_{P''}(x_1) + \psi_P(x_2) \psi_{P'}(x_1) \psi_{P''}(x_3) \\
& + \psi_P(x_3) \psi_{P'}(x_2) \psi_{P''}(x_1) + \psi_P(x_3) \psi_{P'}(x_1) \psi_{P''}(x_2)]
\end{aligned}$$

$$\psi_-(x_1, x_2, x_3) = \langle x_1, x_2, x_3 | \psi_{PP'P''} \rangle_-$$

$$\begin{aligned}
& = \frac{1}{\sqrt{6}} [\psi_P(x_1) \psi_{P'}(x_2) \psi_{P''}(x_3) - \psi_P(x_1) \psi_{P'}(x_3) \psi_{P''}(x_2) \\
& + \psi_P(x_2) \psi_{P'}(x_3) \psi_{P''}(x_1) - \psi_P(x_2) \psi_{P'}(x_1) \psi_{P''}(x_3) \\
& - \psi_P(x_3) \psi_{P'}(x_2) \psi_{P''}(x_1) + \psi_P(x_3) \psi_{P'}(x_1) \psi_{P''}(x_2)]
\end{aligned}$$

$$= \frac{1}{\sqrt{6}} \begin{vmatrix} \psi_P(x_1) & \psi_{P'}(x_1) & \psi_{P''}(x_1) \\ \psi_P(x_2) & \psi_{P'}(x_2) & \psi_{P''}(x_2) \\ \psi_P(x_3) & \psi_{P'}(x_3) & \psi_{P''}(x_3) \end{vmatrix}$$



5. The wave function for the two boson state is

$$\psi_+(x_1, x_2) = N_+ [2\psi_P(x_1)\psi_P(x_2)]$$

using Eq. (3.33a) with  $P = P'$

$$\therefore \int \psi_+^* \psi_+ dx_1 dx_2 = |N_+|^2 4 \left[ \int \psi_P^*(x_1)\psi_P(x_1) dx_1 \right] \left[ \int \psi_P^*(x_2)\psi_P(x_2) dx_2 \right]$$

$$\Rightarrow |N_+|^2 = \frac{1}{4} \quad \text{or} \quad N_+ = \frac{1}{2}$$

$$6. \quad \psi_F(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_P(x_1)\psi_{P'}(x_2) - \psi_{P'}(x_1)\psi_P(x_2)]$$

$$\psi_F^*(x_1, x_2) = \frac{1}{\sqrt{2}} [\psi_P^*(x_1)\psi_{P'}^*(x_2) - \psi_{P'}^*(x_1)\psi_P^*(x_2)]$$

$$\langle \hat{x}_1^2 \rangle = \frac{1}{2} \int \psi_P^*(x_1) x_1^2 \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) \psi_{P'}(x_2) dx_2$$

$$+ \frac{1}{2} \int \psi_{P'}^*(x_1) x_1^2 \psi_{P'}(x_1) dx_1 \int \psi_P^*(x_2) \psi_P(x_2) dx_2$$

$$= \frac{1}{2} [\langle \hat{x}_1^2 \rangle_P + \langle \hat{x}_1^2 \rangle_{P'}]$$

$$\text{Similarly} \quad \langle \hat{x}_2^2 \rangle = \frac{1}{2} [\langle \hat{x}_2^2 \rangle_P + \langle \hat{x}_2^2 \rangle_{P'}]$$

$$\therefore \langle x_1^2 \rangle + \langle x_2^2 \rangle = \langle \hat{x}^2 \rangle_P + \langle \hat{x}^2 \rangle_{P'}$$

$$\langle \hat{x}_1 \hat{x}_2 \rangle = \frac{1}{2} \int \psi_P^*(x_1) x_1 \psi_P(x_1) dx_1 \int \psi_{P'}^*(x_2) x_2 \psi_{P'}(x_2) dx_2$$

$$- \frac{1}{2} \int \psi_P^*(x_1) x_1 \psi_{P'}(x_1) dx_1 \int \psi_{P'}^*(x_2) x_2 \psi_P(x_2) dx_2$$

$$- \frac{1}{2} \int \psi_{P'}^*(x_1) x_1 \psi_P(x_1) dx_1 \int \psi_P^*(x_2) x_2 \psi_{P'}(x_2) dx_2$$

$$+ \frac{1}{2} \int \psi_{P'}^*(x_1) \psi_{P'}(x_1) x_1 dx_1 \int \psi_P^*(x_2) \psi_P(x_2) x_2 dx_2$$

$$= \frac{1}{2} \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} - \frac{1}{2} \langle \hat{x} \rangle_{PP'} \langle \hat{x} \rangle_{P'P} - \frac{1}{2} \langle \hat{x} \rangle_{P'P} \langle \hat{x} \rangle_{PP'} + \frac{1}{2} \langle \hat{x} \rangle_{P'} \langle \hat{x} \rangle_P$$

$$= \langle \hat{x} \rangle_P \langle \hat{x} \rangle_{P'} - \left| \langle \hat{x} \rangle_{PP'} \right|^2$$

Therefore

$$\left\langle (\hat{x}_1 - \hat{x}_2)^2 \right\rangle_F = \langle \hat{x}_1^2 \rangle + \langle \hat{x}_2^2 \rangle - 2 \langle \hat{x}_1 \hat{x}_2 \rangle$$

$$= \left\langle \hat{x}^2 \right\rangle_P + \left\langle x^2 \right\rangle_{P'} - 2 \left\langle x \right\rangle_P \left\langle x \right\rangle_{P'} + 2 \left| \left\langle x \right\rangle_{PP'} \right|^2$$

### Terminal Questions

1. We solve Eqs. (i) to (iii) of the solution of SAQ 2b(i). Each of these equations is the equation of a particle of mass  $m$  in a one-dimensional box of length  $L$ .

Eq. (i) can be written as:

$$\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + E_x \psi_x = 0 \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE_x}{\hbar^2} \psi_x = 0$$

which has the solutions:

$$\psi_x = A \cos kx + B \sin kx, \text{ where } k = \sqrt{\frac{2mE_x}{\hbar^2}} \quad (\text{i})$$

We get on applying the boundary conditions  $\psi_x(x=0) = \psi_x(x=L) = 0$  for Eq. (i):

$$\psi_x = B \sin kx; \quad k = \frac{n\pi}{L}, n = 1, 2, 3, \dots$$

The normalization condition is  $\int_0^L \psi_x^* \psi_x dx = 1$  which gives us  $B = \sqrt{\frac{2}{L}}$

$$\text{and hence } \psi_x = \sqrt{\frac{2}{L}} \sin \frac{n_x \pi x}{L}, n_x = 1, 2, 3 \quad (\text{i})$$

You may refer to Unit 6 of MPH-004 to verify these steps and obtain the solution of Eq. (i) SAQ 2b(i).

Similarly the solution for Eq. (ii) of SAQ 2b(i) with  $\psi_y(0) = \psi_y(L) = 0$  is

$$\psi_y = \sqrt{\frac{2}{L}} \sin \frac{n_y \pi y}{L}, n_y = 1, 2, 3 \quad (\text{ii})$$

and for Eq. (iii) of SAQ 2b(i) with  $\psi_z(0) = \psi_z(L) = 0$

$$\psi_z = \sqrt{\frac{2}{L}} \sin \frac{n_z \pi z}{L}, n_z = 1, 2, 3 \quad (\text{iii})$$

Hence the energy eigenfunctions are

$$\psi_{n_x, n_y, n_z}(x, y, z) = \left( \frac{2}{L} \right)^{3/2} \sin \frac{n_x \pi x}{L} \sin \frac{n_y \pi y}{L} \sin \frac{n_z \pi z}{L}$$

With  $n_x, n_y, n_z = 1, 2, 3, \dots$  and are labelled by  $n_x, n_y, n_z$ .

$$\text{The energy eigenvalues are: } E_{n_x, n_y, n_z} = \frac{\hbar^2 \pi^2}{2m} [n_x^2 + n_y^2 + n_z^2] \quad (\text{iv})$$

Let us write down the energy eigenfunction for the ground state and first excited state.

The lowest energy eigenvalue is obtained when  $n_x, n_y, n_z$  all have their lowest value, which is 1, so the ground state energy eigenfunction is

$$\psi_{111} = \left(\frac{2}{L}\right)^{3/2} \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \sin \frac{\pi z}{L}$$

The corresponding energy eigenvalue is  $E_{111} = \frac{\hbar^2 \pi^2}{2m} (1^2 + 1^2 + 1^2) = \frac{3\hbar^2 \pi^2}{2m}$

The first excited state corresponds to the next highest energy eigenvalue and from Eq. (iv) we can see that we can get this by using one of following sets of values of  $(n_x, n_y, n_z)$ :

$$n_x = 2, n_y = n_z = 1; n_x = n_z = 1, n_y = 2 \quad \text{or} \quad n_x = n_y = 1; n_z = 2$$

So we have the following three energy eigenfunctions:

$$\psi_{211} = \left(\frac{2}{L}\right)^{3/2} \sin \frac{2\pi x}{L} \sin \frac{\pi y}{L} \sin \frac{\pi z}{L} \quad (\text{v})$$

$$\psi_{121} = \left(\frac{2}{L}\right)^{3/2} \sin \frac{\pi x}{L} \sin \frac{2\pi y}{L} \sin \frac{\pi z}{L} \quad (\text{vi})$$

$$\psi_{112} = \left(\frac{2}{L}\right)^{3/2} \sin \frac{\pi x}{L} \sin \frac{\pi y}{L} \sin \frac{2\pi z}{L} \quad (\text{vii})$$

All three eigenfunctions correspond to the same energy eigenvalue which is:

$$E_{211} = \frac{\hbar^2 \pi^2}{2m} [2^2 + 1^2 + 1^2] = \frac{3\hbar^2 \pi^2}{2m} = E_{121} = E_{112}$$

Hence the first excited state has a three-fold degeneracy, corresponding to the three eigen functions of Eqs. (v) to (vii)

2. Solving Eqs. (i), (ii) and (iii) of SAQ 2b(ii) we can write

$$\psi(x, y, z) = \psi_{n_x, n_y, n_z}(x, y, z) = \psi_{n_x}(x) \psi_{n_y}(y) \psi_{n_z}(z)$$

where  $\psi_{n_x}(x), \psi_{n_y}(y), \psi_{n_z}(z)$  are the energy eigenfunctions of the corresponding one-dimensional SHO Schrodinger equations, such that

$$E_x = \left(n_x + \frac{1}{2}\right) \hbar \omega; E_y = \left(n_y + \frac{1}{2}\right) \hbar \omega; E_z = \left(n_z + \frac{1}{2}\right) \hbar \omega$$

and  $E = E_{n_x n_y n_z} = \left(n_x + n_y + n_z + \frac{3}{2}\right) \hbar \omega$  where  $n_x, n_y, n_z = 0, 1, 2, 3, \dots$

The ground state (lowest energy state) is obtained for  $n_x = n_y = n_z = 0$ , so the eigenfunction and energy eigenvalue are:

$$\psi_{000} = \psi_0(x) \psi_0(y) \psi_0(z); E_{000} = \left(0 + 0 + 0 + \frac{3}{2}\right) \hbar \omega = \frac{3}{2} \hbar \omega$$

The first excited state corresponds to the next higher energy eigenvalue, which is given by  $n_x + n_y + n_z = 1$  and the possible values of  $n_x, n_y, n_z$  are

$$n_x = 1, n_y = n_z = 0; \quad n_x = 0, n_y = 1, n_z = 0; \quad n_x = n_y = 0; n_z = 1$$

$\therefore E_{n_x n_y n_z} = E_{100} = E_{010} = E_{001} = \frac{5}{2} \hbar \omega$  and the degenerate eigen

functions for the first excited states are:

$$\psi_{100} = \psi_1(x) \psi_0(y) \psi_0(z)$$

$$\psi_{010} = \psi_0(x) \psi_1(y) \psi_0(z)$$

$$\psi_{001} = \psi_0(x) \psi_0(y) \psi_1(z)$$

The parity of the SHO eigenfunction  $\psi_n(x) = (-1)^n$

∴ The parity of the state  $\psi_{n_x n_y n_z}(x, y, z)$  is hence

$$(-1)^{n_x} (-1)^{n_y} (-1)^{n_z} = (-1)^{n_x + n_y + n_z}$$

For the ground state  $n_x + n_y + n_z = 0$  hence  $\psi_{000}(-x) = \psi_{000}(x)$

∴ It is a state of even parity. The first excited states have

$n_x + n_y + n_z = 1$  so they are states of odd parity:

$$\psi_{100}(-x) = -\psi_{100}(x); \psi_{010}(-x) = -\psi_{010}(x); \psi_{001}(-x) = -\psi_{001}(x).$$

3. The eigenfunctions for a particle in a one-dimensional box are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots$$

i) For two distinguishable particles, which we label 1 and 2, the wave function is written using Eq. (3.9) as

$$\psi(x_1, x_2) = \psi_n(x_1) \psi_m(x_2)$$

where particle 1 is in the state  $\psi_n(x_1)$  and particle 2 is in the state

$\psi_m(x_2)$  with  $E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$ ;  $E_m = \frac{\hbar^2 m^2 \pi^2}{2mL^2}$ . The total energy is

$$E = E_n + E_m = \frac{\hbar^2 \pi^2}{2mL^2} (n^2 + m^2).$$

The lowest energy state is the one in which both particles occupy the lowest energy state which corresponds to  $n = m = 1$ , so

$$\psi_0(x_1, x_2) = \psi_1(x_1) \psi_1(x_2) = \frac{2}{L} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right)$$

The energy of the ground state is  $E = \frac{\hbar^2 \pi^2}{2mL^2} (1^2 + 1^2) = \frac{\hbar^2 \pi^2}{mL^2}$

The first excited state is one in which one particle is in the state and the other is in the first excited state with  $n = 1, m = 2$  or  $n = 2, m = 1$ .

There are two possible states:

$$\psi_{1a}(x_1, x_2) = \psi_1(x_1) \psi_2(x_2) (n = 1, m = 2)$$

$$\psi_{1b}(x_1, x_2) = \psi_2(x_1) \psi_1(x_2) (n = 2, m = 1)$$

So:  $\psi_{1a} = \frac{2}{L} \sin \frac{\pi x_1}{L} \sin \frac{2\pi x_2}{L}$ ;  $\psi_{1b} = \frac{2}{L} \sin \frac{2\pi x_1}{L} \sin \frac{\pi x_2}{L}$

The energy of the states are

$$E_{1a} = \frac{\hbar^2 \pi^2}{2mL^2} (1^2 + 2^2) = \frac{5\hbar^2 \pi^2}{2mL^2}; \quad E_{1b} = \frac{\hbar^2 \pi^2}{2mL^2} (2^2 + 1^2) = \frac{5\hbar^2 \pi^2}{2mL^2}$$

The energy of the eigenfunctions  $\psi_{1a}$  and  $\psi_{1b}$  are equal. Hence the first excited state is doubly degenerate.

ii) **For two identical bosons**

We know that identical bosons can occupy the same state, so the ground state will have both bosons occupying the lowest energy state. So, once again

$$\psi_{0B}(x_1, x_2) = \psi_1(x_1)\psi_1(x_2) = \frac{2}{L} \sin \frac{\pi x_1}{L} \sin \frac{\pi x_2}{L}$$

$$\text{The energy of the ground state is } E = \frac{\hbar^2 \pi^2}{2mL^2} (1^2 + 1^2) = \frac{\hbar^2 \pi^2}{mL^2}$$

For the first excited, the wave function will be a symmetric combination corresponding to one boson in the lowest energy state  $\psi_1$  and one in the first excited state (Eq. 3.14b):

$$\begin{aligned} \psi_{1B}(x_1, x_2) &= \frac{1}{\sqrt{2}} [\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)] \\ &= \frac{1}{\sqrt{2}} \left[ \frac{2}{L} \sin \frac{\pi x_1}{L} \sin \frac{2\pi x_2}{L} + \frac{2}{L} \sin \frac{2\pi x_1}{L} \sin \frac{\pi x_2}{L} \right] \\ &= \frac{\sqrt{2}}{L} \left[ \sin \frac{\pi x_1}{L} \sin \frac{2\pi x_2}{L} + \sin \frac{2\pi x_1}{L} \sin \frac{\pi x_2}{L} \right] \end{aligned}$$

The first excited state is non-degenerate in this case.

$$\text{The energy of the first excited state is } \frac{5\hbar^2 \pi^2}{2mL^2}.$$

Two identical fermions cannot exist in same state, so in this case the lowest energy (ground state) eigenfunction is given by the following antisymmetric wavefunction corresponding to one fermion in the lowest energy state  $\psi_1$  and one in the first excited state  $\psi_2$  (Eq. 3.14c).

$$\begin{aligned} \psi_{0F}(x_1, x_2) &= \frac{1}{\sqrt{2}} [\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)] \\ &= \frac{\sqrt{2}}{L} \left[ \sin \frac{\pi x_1}{L} \sin \frac{2\pi x_2}{L} - \sin \frac{2\pi x_1}{L} \sin \frac{\pi x_2}{L} \right] \end{aligned}$$

$$\text{The energy of the ground state is } \frac{5\hbar^2 \pi^2}{2mL^2}.$$

The first excited state eigenfunction is given by the antisymmetric wavefunction corresponding to one fermion in the state  $\psi_1$  and one in the state  $\psi_3$ :

$$\begin{aligned} \psi_{1F}(x_1, x_2) &= \frac{1}{\sqrt{2}} [\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2)] \\ &= \frac{\sqrt{2}}{L} \left[ \sin \frac{\pi x_1}{L} \sin \frac{3\pi x_2}{L} - \sin \frac{3\pi x_1}{L} \sin \frac{\pi x_2}{L} \right] \end{aligned}$$

$$\text{The energy of the first excited state is } \frac{5\hbar^2 \pi^2}{mL^2}.$$

4. The Schrödinger equation (using Eq. 7.47, unit 7, MPH-004) is:

$$\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_2^2} + \frac{1}{2} m \omega^2 x_1^2 + \frac{1}{2} m (\sqrt{3} \omega)^2 x_2^2 \right) \psi(x_1, x_2) = E \psi(x_1, x_2)$$

$$\text{where } x_1 = \frac{1}{\sqrt{2}}(x_A + x_B); x_2 = \frac{1}{\sqrt{2}}(x_A - x_B)$$

The solution is:  $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$

Since  $V = V(x_1) + V(x_2)$ , we get the following two equations:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1}{\partial x_1^2} + \frac{1}{2} m \omega^2 x_1^2 \psi_1 = E_1 \psi_1 \quad (\text{i})$$

And

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_2}{\partial x} + \frac{1}{2} m (\sqrt{3} \omega)^2 x_2^2 \psi_2 = E_2 \psi_2 \quad (\text{ii})$$

with  $E = E_1 + E_2$ .

Eqs. (i) and (ii) are the Schrödinger equations for two one-dimensional simple harmonic oscillators with frequencies  $\omega, \sqrt{3}\omega$  and wave functions  $\psi_{n_1}(x_1), \psi_{n_2}(x_2)$  respectively. So the wavefunction

$$\psi(x_1, x_2) = \psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1) \psi_{n_2}(x_2)$$

$$E_1 = E_{n_1} = \left( n_1 + \frac{1}{2} \right) \hbar \omega; E_2 = E_{n_2} = \left( n_2 + \frac{1}{2} \right) \hbar (\sqrt{3} \omega)$$

$$\text{The energy is } E = E_{n_1 n_2} = \left( n_1 + \frac{1}{2} \right) \hbar \omega + \left( n_2 + \frac{1}{2} \right) \hbar (\sqrt{3} \omega)$$

5. For two distinguishable particles we use Eq. (3.44). With

$$\psi_m(x) = \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L}; \psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

We can calculate the following:

$$\langle x \rangle_m = \frac{2}{L} \int_0^L x \sin^2 \frac{m\pi x}{L} dx = \frac{1}{L} \left[ \frac{L^2}{2} \right] = \frac{L}{2} \quad (\text{i})$$

$$\langle x \rangle_n = \frac{L}{2} \quad (\text{ii})$$

$$\langle x^2 \rangle_m = \frac{2}{L} \int_0^L x^2 \sin^2 \frac{m\pi x}{L} dx = L^2 \left( \frac{1}{3} - \frac{1}{2\pi^2 m^2} \right) \quad (\text{iii})$$

$$\langle x^2 \rangle_n = L^2 \left( \frac{1}{3} - \frac{1}{2n^2 \pi^2} \right) \quad (\text{iv})$$

Using the results of Eqs. (i-iv) in Eq. (3.44) we get:

$$\therefore \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle = L^2 \left( \frac{1}{3} - \frac{1}{2m^2 \pi^2} \right) + L^2 \left( \frac{1}{3} - \frac{1}{2n^2 \pi^2} \right) - 2 \cdot \frac{L}{2} \cdot \frac{L}{2}$$

$$= L^2 \left[ \frac{1}{6} - \frac{1}{2\pi^2} \left( \frac{1}{n^2} + \frac{1}{m^2} \right) \right]$$

6. a) For identical bosons using Eq. (3.45) we can write

$$\begin{aligned} \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle &= \langle \hat{x} \rangle_m + \langle x^2 \rangle_n - 2 \langle x \rangle_m \langle \hat{x} \rangle_n - 2 |\langle \hat{x} \rangle_{mn}|^2 \\ &= \langle (\hat{x}_1 - \hat{x}_2)^2 \rangle_D - 2 |\langle x \rangle_{mn}|^2 \end{aligned}$$

$$\begin{aligned} \langle x \rangle_{mn} &= \frac{2}{L} \int_0^L x \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\ &= \frac{L}{\pi^2} \left[ \frac{1}{(m-n)^2} \{(-1)^{m-n} - 1\} - \frac{1}{(m+n)^2} \{(-1)^{m+n} - 1\} \right] \end{aligned}$$

If  $m, n$  are both even or both odd the  $(m-n)$  and  $(m+n)$  is even so  $(-1)^{m-n} = (-1)^{m+n} = 1$

$$\therefore \langle x_{mn} \rangle = 0$$

If  $m, n$  are not both even or both odd, then,  $(-1)^{m+n} = (-1)^{m-n} = -1$

$$\therefore \langle x \rangle_{mn} = \frac{2L}{\pi^2} \left[ \frac{1}{(m+n)^2} - \frac{1}{(m-n)^2} \right] = -\frac{8Lmn}{(m^2 - n^2)^2 \pi^2}$$

$$\therefore \langle (x_1 - x_2)^2 \rangle = L^2 \left[ \frac{1}{6} - \frac{1}{2\pi^2} \left( \frac{1}{m^2} + \frac{1}{n^2} \right) \right] - \frac{128L^2 m^2}{(m^2 - n^2)^4 \pi^4}$$

when  $\psi_m, \psi_n$  have opposite parity.

$$= L^2 \left[ \frac{1}{6} - \frac{1}{2\pi^2} \left( \frac{1}{m^2} + \frac{1}{n^2} \right) \right]$$

when  $\psi_m, \psi_n$  have the same parity.

b) For identical fermions

$$\langle (x_1 - x_2)^2 \rangle = L^2 \left[ \frac{1}{6} - \frac{1}{2\pi^2} \left( \frac{1}{m^2} + \frac{1}{n^2} \right) \right] + \frac{128L^2 m^2 n^2}{\pi^4 (m^2 - n^2)^4 \pi^4}$$

when  $\psi_m, \psi_n$  have opposite parity

$$= L^2 \left[ \frac{1}{6} - \frac{1}{2\pi^2} \left( \frac{1}{m^2} + \frac{1}{n^2} \right) \right]$$

when  $\psi_m, \psi_n$  have same parity.

7. i) We write the states determinant for the completely antisymmetric wave function as

$$\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}} \left( \frac{2}{L} \right)^{3/2} \begin{vmatrix} \sin \frac{2\pi x_1}{L} & \sin \frac{3\pi x_1}{L} & \sin \frac{4\pi x_1}{L} \\ \sin \frac{2\pi x_2}{L} & \sin \frac{3\pi x_2}{L} & \sin \frac{4\pi x_2}{L} \\ \sin \frac{2\pi x_3}{L} & \sin \frac{3\pi x_3}{L} & \sin \frac{4\pi x_3}{L} \end{vmatrix}$$

ii) Using the results of SAQ4, we can write

$$\begin{aligned} \psi(x_1, x_2, x_3) = & \frac{1}{\sqrt{6}} \left(\frac{2}{L}\right)^{3/2} \left[ \sin \frac{2\pi x_1}{L} \sin \frac{3\pi x_2}{L} \sin \frac{4\pi x_3}{L} \right. \\ & + \sin \frac{2\pi x_1}{L} \sin \frac{4\pi x_2}{L} \sin \frac{3\pi x_3}{L} + \sin \frac{3\pi x_1}{L} \sin \frac{4\pi x_2}{L} \sin \frac{2\pi x_3}{L} \\ & + \sin \frac{3\pi x_1}{L} \sin \frac{2\pi x_2}{L} \sin \frac{4\pi x_3}{L} + \sin \frac{4\pi x_1}{L} \sin \frac{2\pi x_2}{L} \sin \frac{3\pi x_3}{L} \\ & \left. + \sin \frac{4\pi x_1}{L} \sin \frac{3\pi x_2}{L} \sin \frac{2\pi x_3}{L} \right] \end{aligned}$$

iii) If two bosons are in the state  $\psi_2$  and one is in the state  $\psi_3$ .

$$\begin{aligned} = & \sqrt{\frac{2}{3}} \left(\frac{2}{L}\right)^{3/2} \left[ \sin \frac{2\pi x_1}{L} \sin \frac{2\pi x_2}{L} \sin \frac{3\pi x_3}{L} \right. \\ & + \sin \frac{2\pi x_1}{L} \sin \frac{3\pi x_2}{L} \sin \frac{2\pi x_3}{L} \\ & \left. + \sin \frac{3\pi x_1}{L} \sin \frac{2\pi x_2}{L} \sin \frac{2\pi x_3}{L} \right] \end{aligned}$$



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