UNIT 1 ANALYTIC FUNCTIONS

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1.1 INTRODUCTION

There are equations such as \( x^2 + 3 = 0 \), \( x^2 - 10x + 40 = 0 \), which do not have a root in the real number system \( \mathbb{R} \). There does not exist any real number whose square is a negative real number. If the roots of such equations are to be determined then we need to introduce another number system called complex number system. Complex numbers can be defined as ordered pairs \((x, y)\) of real numbers and represented as points in the complex plane, with rectangular coordinates \( x \) and \( y \).

In Sec. 1.2, we shall review the algebraic and geometric structure of the complex number system. Sec. 1.3 introduces you to the functions of a complex variable. Further, we shall study the concepts of limits, continuity and differentiability of the functions of a complex variable in this section. In Sec. 1.4, you will study about Cauchy-Riemann equations which under certain conditions provide the necessary and sufficient condition for the differentiability of a function of a complex variable at a point. A very important concept of analytic functions which is useful in many application of the complex variable theory is discussed in Sec. 1.5. Lastly, in Sec. 1.6, we shall introduce harmonic functions which play an important role in applied mathematics. The temperature \( T(x, y) \) in thin plates in the \( xy \)-plane and a function \( V(x, y) \) denoting an electrostatic potential in a region free of charges are some of the examples of harmonic function.

Objectives

After studying this unit you should be able to

- define a complex number and represent it as a point in the complex plane;
- define a function of complex variable and also the limit, continuity and differentiability of a function;
- use the Cauchy-Riemann equations to prove the differentiability of a function of a complex variable at a point;
- define the analyticity of a function at a point and discuss it.

1.2 BASIC CONCEPTS

This section introduces you to the concept of complex numbers. It gives you an idea of
how an ordered pair \((x, y)\) of real numbers can be interpreted as a point in the complex plane with rectangular coordinates \(x\) and \(y\).

**Read Secs. (1)-(10) of Chapter 1 of the book from pages 1-32. Carefully go through all the solved examples in each of the ten sections of the chapter and try to do the exercises given at the end of each section.**

This chapter will help you to understand the definition of complex numbers, their algebraic properties of addition and multiplication, geometrical representation of a complex number as a point in the plane, polar and exponential form of a complex numbers and their roots and powers. Carefully go through this chapter as all the results and definitions given there would be used frequently in the chapters to follow. You have to make yourself comfortable with all the notions of Sec. 10. In case you are already familiar with these concepts, a quick glance through this chapter will help you recall these concepts. Chapter 1 of the book is introductory. For the rest of the sections in this unit, we shall confine ourselves to Chapter 2 of the book.

Before proceeding further, we would like to make the following remarks.

**Remark 1:** Every open and convex set (any pair of two points of it can be joined by a line segment lying in the set) is a connected set but every open connected set may not be convex (see Fig. 16 page 31 of the book).

**Remark 2:** Union of two open connected sets may not be an open connected set. For example, see Exercise 5 on page 32 of the book, \(S = \{z : |z| < 1 \text{ or } |z - 2| < 1\}\). Here \(z_1\) cannot be joined to \(z_2\) by a polygonal line without crossing the boundary (see Fig. 1). Thus \(S\) is not connected but is open.

### 1.3 FUNCTIONS OF A COMPLEX VARIABLE

In this section we shall introduce you to functions of a complex variable and develop the concepts of its mappings, limits, continuity and differentiability.

You can start with the following:

**Read Sec. 11, Chapter 2 of the book from pages 33-35. Go through all the four examples and do the exercises 1(c), 1(d), 4 on page 35 of the book.**

### 1.3.1 Mappings

To display information about the function \(w = f(z)\), by indicating pairs of corresponding points \(z = (x, y)\) and \(w = (u, v)\), it is simpler to draw the \(z\) and \(w\) planes separately and this is referred to as a mapping or transformation. Through this we can understand the concept of mapping and get an idea of finding the image in \(w\)-plane of any region in \(z\)-plane under the transformation \(w = f(z)\).

**Read Secs. 12 and 13, Chapter 2 of the book from pages 36-42. Do the exercises 4, 5 on pages 42, 43 of the book.**

We now move on to the concept of limit of a function of a complex variable.
Theorem 1: If \( f(z) \) has a finite limit at \( z_0 \) then \( f(z) \) is a bounded function in some deleted neighbourhood of \( z_0 \).

Since \( \lim_{z \to z_0} f(z) \) exists as a finite complex number, we can take it as some complex number \( w_0 \) and write \( |f(z) - w_0| < \epsilon \) whenever \( 0 < |z - z_0| < \delta \). Hence, for any \( z \) in the neighbourhood of \( z_0 \), we find that

\[
|f(z)| = |f(z) - w_0 + w_0| \leq |f(z) - w_0| + |w_0| < \epsilon + |w_0| = M,
\]

where \( M = \epsilon + |w_0| \) is some finite number. Therefore \( f(z) \) is bounded in some neighbourhood of \( z_0 \).

***

Let us once again consider Theorem 2 on page 47 of the book. Property (9) has been proved using Theorem 1 on page 46. Let us now see how this result can be proved directly by using the definition of the limit of a function of a complex variable.

To prove the result using the definition of limit we write, for given \( \epsilon > 0 \exists \delta_1, \delta_2 > 0 \) such that

\[
|f(z) - w_0| < \epsilon, \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1,
\]
and

\[
|F(z) - W_0| < \epsilon, \quad \text{whenever} \quad 0 < |z - z_0| < \delta_2.
\]

Now,

\[
|f(z)F(z) - w_0 W_0| = |f(z)(F(z) - W_0) + W_0(f(z) - w_0)|
\leq |f(z)(F(z) - W_0) + W_0(f(z) - w_0)|
\leq |f(z)||F(z) - W_0| + |W_0||f(z) - w_0|.
\]

Since \( \lim_{z \to z_0} f(z) \) exists and is finite, \( |f(z)| \leq M \), for a finite constant \( M \) by Theorem 1 above. Choosing \( \delta = \min(\delta_1, \delta_2) \), we get

\[
|f(z)F(z) - w_0 W_0| \leq M \epsilon + |W_0| \epsilon = \epsilon_1, \quad \text{for} \quad 0 < |z - z_0| < \delta.
\]

Thus, \( |f(z)F(z) - w_0 W_0| < \epsilon_1 \)

This establishes Property (9) of Theorem 2 on page 47 of the book.

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Now to get introduced to the limit of a function at a point at infinity and infinite limits you may read the following:

Read Sec. 16, Chapter 2 of the book from Pages 48-51.
Example 1: Show that \( \lim_{z \to 2i} (2x + iy^2) = 4i \)

Solution: For each positive number \( \varepsilon \) we must find a positive number \( \delta \) such that
\[
|2x + iy^2 - 4i| < \varepsilon \text{ whenever } 0 < |z - 2i| < \delta.
\]

To do this we write
\[
|2x + iy^2 - 4i| \leq 2|x| + |y^2 - 4| = 2|x| + |y - 2||y - 2|
\]

Thus, the first of inequality (1) will be satisfied if
\[
2|x| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - 2||y + 2| < \frac{\varepsilon}{2}.
\]

The first of these inequalities is, of course, satisfied if \( x < \varepsilon/4 \). To establish conditions on \( y \) such that the second one holds, we restrict \( y \) so that
\[
|y - 2| < \min\left\{ \frac{\varepsilon}{10}, 1 \right\}.
\]

Then clearly, \( |y - 2| < 1 \) and we find that
\[
|y + 2| = |(y - 2) + 4| \leq |y - 2| + 4 < 5.
\]

Therefore, \( |y - 2||y + 2| < \frac{\varepsilon}{10}.5 = \frac{\varepsilon}{2} \). Hence, from the two conditions,
\[
|x| < \frac{\varepsilon}{4} \quad \text{and} \quad |y - 2| < \min\left\{ \frac{\varepsilon}{10}, 1 \right\}
\]

the appropriate value of \( \delta \) is \( \delta = \min\left\{ \frac{\varepsilon}{10}, 1 \right\} \).

Example 2: Show that \( \lim_{z \to \infty} \frac{1}{z^2} = 0 \)

Solution: For a given real number \( \varepsilon > 0 \), we want to determine a real number \( \delta > 0 \) such that
\[
\left| \frac{1}{z^2} \right| < \varepsilon \quad \text{whenever} \quad |z| > \frac{\delta}{1}
\]

Now,
\[
\left| \frac{1}{z^2} \right| < \varepsilon \Rightarrow |z| > \frac{1}{\sqrt{\varepsilon}}.
\]

Thus we can take \( \delta = \sqrt{\varepsilon} \). With this choice of \( \delta \), we find
\[
\left| \frac{1}{z^2} \right| < \varepsilon \quad \text{whenever} \quad |z| > \frac{1}{\delta}.
\]

Example 3: Show that the following limits do not exist

a) \( \lim_{z \to 0} \frac{z}{|z|} \)

b) \( \lim_{z \to 0} \left[ \frac{\text{Re} z - \text{Im} z}{|z|^2} \right]^2 \)

Solution: We are going to use a simple fact that limit, if exists, is independent of choice of paths or the approaches the variable \( z \) takes to reach the point (at which you intend to calculate the limit).

a) We choose a path such that \( x \to 0 \) is followed by \( y \to 0 \) (see Fig. 2).

We get (when \( x \to 0 \))
\[
\frac{z}{|z|} = \frac{x + iy}{\sqrt{x^2 + y^2}} \to \frac{iy}{|y|}
\]

and when \( y \to 0^+ \), \( \frac{iy}{|y|} \to i \) \[Fig. 2(a)\]

and when \( y \to 0^- \), \( \frac{iy}{|y|} \to -i \) \[Fig. 2(b)\]

Thus limit does not exist.
b) Consider any straight line path \( y = mx \). Taking limit along this path, we have

\[
\lim_{z \to 0} \left| \frac{\Re z - \Im z}{|z|^2} \right|^2 = \lim_{x \to 0} \frac{(x - y)^2}{x^2 + y^2} = \lim_{x \to 0} \frac{(1 - m)^2 x^2}{(1 + m^2)^2 x^2} = \frac{(1 - m)^2}{1 + m^2}
\]

the limit depends on \( m \) and is not unique so, the limit does not exist.

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You may now try the following exercises:

Do the exercises 4, 5 on page 53 and exercises 9, 11, 13 on page 54 of the book.

With this background you may now move on to the next section.

1.3.3 Continuity

This section introduces you to the continuity of the function of a complex variable at a point.

Read Sec. 17, Chapter 2 from pages 51-53 of the book.

After you have gone through Sec. 17 of the book we would like you to notice that result given by Eqn.(6) of Theorem 2 on page 53 can be stated as follows:

If \( f(z) \) is continuous in a closed region \( R \), then it is bounded in \( R \), that is,

\[ |f(z)| \leq M \text{ for all } z \text{ in } R \text{ and for some } M > 0. \]

Further, we would like to add the following remark.

Remark: In the definition of continuity given by Eqns.(1)-(3) on page 51 of the book, if both \( f(z_0) \) and \( \lim_{z \to z_0} f(z) = \ell \) (say) exist, but \( f(z_0) \neq \ell \), then point \( z_0 \) is called a point of removable discontinuity. In this case, we can redefine the function \( f(z) \) at \( z_0 \) such that \( f(z_0) = \ell \) and the function can be made continuous at \( z_0 \). This can also be done even if \( f(z_0) \) is not defined there. For instance, consider \( f(z) = \frac{z^2 - 4}{z + 2} \).

Here \( f(z) \) is not defined at \( z = -2 \). If we redefine \( f(-2) \) as

\[ f(-2) = \lim_{z \to -2} f(z) = \lim_{z \to -2} \frac{(z - 2)(z + 2)}{z + 2} = -4, \]

then the function becomes continuous at \( z = -2 \). So that \( z = -2 \) is a removable discontinuity. Thus, function defined as

\[ f(z) = \begin{cases} 
\frac{z^2 - 4}{z + 2}, & z \neq -2 \\
-4, & z = -2
\end{cases} \]

is continuous for all \( z \).

Let us consider the following examples.

Example 4: Show that the functions

a) \( f(z) = e^z \cos y + ie^z \sin y \)

b) \( f(z) = \sin x \cosh y + i \cos x \sinh y \)

c) \( f(z) = x^2 + i(2x - y) \)

are continuous for all \( z \).
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Solution: a) We can write $z = x + iy$. Therefore
\[ f(z) = e^x \cos y + e^x \sin y = u + iv \]
Since the real valued functions $u$ and $v$ of two real variables $x$ and $y$ are continuous at each $(x, y)$ in $\mathbb{R}^2$, the function $f(z)$ is continuous for all $z$.

b) $f(z) = \sin x \cosh y + i \cos x \sinh y = u + iv$.
$f(z)$ is continuous for all $z$ by the same argument as in a) above.

c) Likewise, the function
\[ f(z) = xy^2 + i(2x - y) \]
is continuous everywhere in the complex plane as the component functions are polynomials in $x$ and $y$ are therefore continuous at each point $(x, y) \in \mathbb{R}^2$.

Example 5: Show that the function
\[ f(z) = \begin{cases} \text{Im}(z) & z \neq 0 \\ \frac{y}{z} & z = 0 \end{cases} \]
is not continuous at $z = 0$.

Solution: If $f(z)$ is continuous at $z = 0$ then
\[ \lim_{z \to 0} \frac{\text{Im}(z)}{|z|} = \lim_{(x, y) \to (0, 0)} \frac{y}{\sqrt{x^2 + y^2}} \quad \text{must exist.} \]
Let us consider the path $y = mx$ then
\[ \lim_{(x, y) \to (0, 0)} \frac{y}{\sqrt{x^2 + y^2}} = \lim_{x \to 0} \frac{mx}{\sqrt{x^2 + m^2 x^2}} = \frac{m}{\sqrt{1 + m^2}}. \]
But $m$ is a variable i.e., limiting value depends on the path chosen. Hence $\lim_{z \to 0} f(z)$ does not exist. Therefore the function is not continuous at $z = 0$.

How about trying these exercises now.

E1) Show that the function
a) $f(z) = |z|^2$

b) $f(z) = e^{xy} + i \sin (x^2 - 2xy^3)$

are continuous for all $z$.

E2) Find the value of $f(i)$ so that the function $f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ is continuous at $z = i$.

E3) Find the points in the complex plane at which the following functions are not continuous.

a) $f(z) = \frac{z - 1}{z^2 - 4z + 5}$

b) $f(z) = \frac{z}{(\text{Re}z)^2}$.

Let us now move on to the derivative of a function of a complex variable.

1.3.4 Differentiation

Read Secs. 18-19, Chapter 2 from pages 54-59 of the book.
Secs. 18 and 19 of the book introduce you to the concept of derivative of a function \( f(z) \) at a point in the complex plane. After going through these sections you must have realised:

i) A function which is not continuous at a point \( z = z_0 \) cannot be differentiable at \( z = z_0 \).

ii) A function which is continuous at a point \( z = z_0 \), may or may not be differentiable at \( z = z_0 \).

iii) The rules of differentiation of a function of a real variable \( x \) hold also for a function of a complex variable \( z \).

The following examples may be of help to you for doing the exercises, so go through them before trying problems.

**Example 6:** Show that the function \( f(z) = \overline{z} \) is continuous at the point \( z = 0 \) but not differentiable at \( z = 0 \). Is this function continuous and differentiable elsewhere?

**Solution:** Let \( w = f(z) = u(x, y) + iv(x, y) \).

\[
f(z) = \overline{z} = x - iy \text{ gives } u(x, y) = x, \ v(x, y) = -y.
\]

Now \( \lim_{(x, y) \to (0, 0)} u(x, y) = \lim_{(x, y) \to (0, 0)} x = 0 \)

and \( \lim_{(x, y) \to (0, 0)} v(x, y) = \lim_{(x, y) \to (0, 0)} (-y) = 0 \)

therefore, \( \lim_{z \to 0} f(z) = 0 + 0 = 0 \) and also \( f(0) = 0 \).

Hence, \( \lim_{z \to 0} f(z) = f(0) \Rightarrow f \) is continuous at \( z = 0 \).

In fact, \( f(z) = \overline{z} \) is continuous everywhere. For if \( z = z_0 \) be any point then

\[
|f(z) - f(z_0)| = |\overline{z} - \overline{z}_0| = |z - z_0|.
\]

Thus for given \( \varepsilon > 0 \) such that

\[
|f(z) - f(z_0)| < \varepsilon, \text{ we have } |z - z_0| < \delta = \varepsilon.
\]

To check the differentiability of \( f(z) \)

let \( \Delta w = f(z + \Delta z) - f(z) \).

At \( z = 0 \),

\[
\frac{\Delta w}{\Delta z} = \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \frac{f(\Delta z)}{\Delta z}.
\]

If the limit of \( \frac{\Delta w}{\Delta z} \) exists, it may be found by letting the point \( \Delta z = (\Delta x, \Delta y) \)

approach the origin in the \( \Delta z \)-plane in any manner. In particular, when \( \Delta z \)

approaches the origin horizontally through the point \( (\Delta x, 0) \) on the real axis (see Fig. 3), \( \Delta z = \Delta x + i \Delta y = \Delta z \).

Thus, \( \frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1 \).

When \( \Delta z \) approaches origin vertically through \( \Delta y \)-axis, we get

\[
\frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = \frac{0 + i \Delta y}{\Delta z} = -i \Delta y
\]

and in this case

\[
\frac{\Delta w}{\Delta z} = -1.
\]

Since limits are not unique, \( f(z) = \overline{z} \) is not differentiable at \( z = 0 \).

In fact it is nowhere differentiable. You can check it easily by proceeding as above by considering an arbitrary point \( z_0 \) and observing that
Example 7: Show that the function \( f(z) = z^n \), where \( n \) is a positive integer is differentiable at every point in the finite complex plane.

Solution: Let \( z \) be any point in the finite complex plane.

Then, \[
\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(z + \Delta z)^n - z^n}{\Delta z} = \frac{1}{\Delta z} \left[ \binom{n}{1} z^{n-1} \Delta z + \binom{n}{2} z^{n-2} (\Delta z)^2 + \cdots + \binom{n}{n} (\Delta z)^n \right]
\]

Taking limits on both sides as \( \Delta z \to 0 \), we obtain

\[
\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = nz^{n-1}.
\]

Therefore, \( \frac{d}{dz} (z^n) = nz^{n-1} \), where \( z \) is any point in the finite plane.

Remark 1: Since \( f(z) = z^n \), \( n \) positive integer, is differentiable at every point in the complex plane, any polynomial \( P_n(z) \) in \( z \) is also differentiable at every point in the complex plane.

Remark 2: Suppose that \( f(z) \) is a real-valued function of a complex variable. If \( f(z) \) is differentiable then its derivative is zero otherwise it is not differentiable. For, \( \Delta z = \Delta x \), where \( \Delta x \) is real, we have

\[
\frac{f(z + \Delta z) - f(z)}{\Delta x} = \text{a purely real number (if exists) as } \Delta x \to 0
\]

and for \( \Delta z = i\Delta y \), we have

\[
\frac{f(z + \Delta z) - f(z)}{i\Delta z} = \frac{f(z + i\Delta y) - f(z)}{i\Delta y} = (-i) \left( \frac{f(z + i\Delta y) - f(z)}{\Delta y} \right) = \text{a purely imaginary number.}
\]

Since \( f(z) \) is differentiable \( \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \) exists and from above discussion we conclude that it has to be zero.

Remark 3: Although definition of differentiability of a function of complex variable looks similar as in the case of real functions. But we will see that complex differentiability is stronger than its real counterpart. In fact, we are going to see that if a function is complex differentiable then its derivative is again complex differentiable. This is not true in case of real differentiability.

***

You may now try some exercises to check your understanding of the concept.

Do the exercises 7, 8 on page 60 of the book.
You have thus seen that a complex valued function may have derivative at every point, only at a fixed point or at no point. The natural curiosity is then whether there is any formula or method to check differentiability with ease. A partial answer to your curiosity is given by the famous Cauchy-Riemann equations (C-R equations), which are a pair of equations involving first order partial derivatives. Let us see what these equations are.

1.4 CAUCHY-RIEMANN EQUATIONS

Read Secs. 20, 21, 22 Chapter 2 of the book from pages 60-68. Go through Examples 1, 2 (Sec. 20), Examples 1, 2 (Sec. 21) and Examples 1, 2 (Sec. 22).

Let us once again look at the Theorem on Page 62, Sec. 20, of the book. Here we are giving another proof of the theorem for your reference.

Let \( f(z) = u(x, y) + i \, v(x, y) \) and \( f'(z) \) exists at \( z_0 = x_0 + iy_0 \).

Keeping \( y_0 \) fixed, we write

\[
\frac{f(x + iy_0) - f(x_0 + iy_0)}{(x + iy_0) - (x_0 + iy_0)} = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i \, \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}.
\]

Since \( f'(x_0 + iy_0) \) exists therefore,

\[
\lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} \quad \text{and} \quad \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}
\]

both exist. In other words partial derivatives \( u_x(x_0, y_0) \)

and \( v_x(x_0, y_0) \) exist and we have

\[ f'(z_0) = u_x(x_0, y_0) + i \, v_x(x_0, y_0). \] (2)

Repeating the above process by keeping \( x_0 \) fixed we obtain that

\[
\lim_{y \to y_0} \frac{u(x, y) - u(x_0, y_0)}{i(y - y_0)} = u_y(x_0, y_0) \quad \text{and} \quad \lim_{y \to y_0} \frac{i[v(x, y) - v(x_0, y_0)]}{i(y - y_0)} = v_y(x_0, y_0).
\]

Moreover, \( f'(z_0) = \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0) \)

\[
\Rightarrow f'(z_0) = v_y(x_0, y_0) - i \, u_y(x_0, y_0). \] (3)

Eqns.(2) and (3) \( \Rightarrow \) \( u_x(x_0, y_0) = v_y(x_0, y_0) \) and \( u_y(x_0, y_0) = -v_x(x_0, y_0) \)

which are the C-R equations at the point \((x_0, y_0)\).

***

After going through Sec. 20 and Example 1 on Page 63 of the book you must have noticed that if a complex function is differentiable at a point then at that point C-R equations must be satisfied by the real and imaginary parts of the function. However, these equations are merely necessary conditions they do not guarantee the differentiability of a complex function at a point as illustrated in the following example.

Example 8: Let \( f(z) = u(x, y) + i \, v(x, y) \)

where \( u(x, y) = \sqrt{|xy|} \) and \( v(x, y) = 0 \).

Show that C-R equations are satisfied at \( z = 0 \), but \( f \) is not differentiable at \( z = 0 \).

Solution: Obviously, \( \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \) and \( \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \), since \( u(x, y) \) and \( v(x, y) \) are constants (taking value 0) along real and imaginary axes or,
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\[ u_x = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x} = 0 \text{ etc. Thus C-R equations are satisfied at } z = 0. \]

On the other hand,

\[ \frac{f(z) - f(0)}{z - 0} = \sqrt{|xy|}, \]

Putting \( x = r \cos \theta \) and \( y = r \sin \theta \) in the above equation, we find that

\[ \frac{f(z) - f(0)}{z - 0} = \sqrt{|r^2 \cos \theta \sin \theta|} = \sqrt{|\cos \theta \sin \theta|} e^{-i \theta}. \]  

The right hand side of Eqn.(4) is independent of \( r \).

If we consider the line \( x \sin \theta - y \cos \theta = 0 \) and keep \( \theta \) fixed (see Fig. 4),

\[ \frac{f(z) - f(0)}{z - 0} \]

remains constant equal to \( \sqrt{|\cos \theta \sin \theta|} e^{-i \theta} \) for points arbitrarily close to 0. If \( \theta = 0 \) or \( \theta = \pi / 2 \) the constant value is 0 but for \( \theta = \pi / 4 \) this constant takes the value \( \frac{1 - i}{2} \).

Therefore, \( \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} \) does not exist. Hence \( f(z) \) is not differentiable at \( z = 0 \).---x---

Thus the C-R equations are not sufficient for the existence of derivative of a complex function. At this juncture it is clear that we need some additional hypothesis(es). Surprisingly what we require is nothing more than the continuity of the partial derivatives in some neighbourhood of the point at which differentiability is required. This fact has been stated in the theorem on Page 63, Sec. 21. Proof as given on page 64 is straight forward except Eqn. (2) which needs some explanations. In fact, derivation of Eqn. (2) needs an elementary argument of two-variable calculus (the mean-value theorem). We hope the explanation given below will make your task easier.

Consider

\[ \Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \]

\[ = (u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)) + (u(x_0, y_0 + \Delta y) - u(x_0, y_0)) \]

\[ = \Delta x \frac{\partial u}{\partial x}(x_0 + \Delta x, y_0 + \Delta y) + \Delta y \frac{\partial u}{\partial y}(x_0, y_0 + \Delta y), \text{ where } \theta, \phi \in [0, 1] \]

[by the Mean Value Theorem of 2-variable calculus].

Since partial derivatives are continuous we may deduce that

\[ \Delta u = \Delta x \left( \frac{\partial u}{\partial x} + \varepsilon_1 \right) + \Delta y \left( \frac{\partial u}{\partial y} + \varepsilon_2 \right) \]

\( \varepsilon_1, \varepsilon_2 \to 0 \text{ as } (\Delta x, \Delta y) \to (0, 0) \)

or,

\[ \Delta u = \Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y} + (\varepsilon_1 \Delta x + \varepsilon_2 \Delta y) . \]

We can replace \( \varepsilon_1 \) by \( \varepsilon \frac{\Delta x}{\sqrt{\Delta x^2 + \Delta y^2}} \) and \( \varepsilon_2 \) by \( \frac{\varepsilon \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \)

then,

\[ \Delta u = \Delta x \left( \frac{\partial u}{\partial x} + \Delta y \left( \frac{\partial u}{\partial y} \right) + \varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} \right) \]

which is Eqn.(2) on page 64.

---x---

Sec. 22 on page 65-66 of the book introduces you to the polar form of C-R equations in polar coordinates \((r, \theta)\). This change of coordinate sometimes make computational part easier. We shall illustrate it through the following example.
**Example 9:** Show that \( f(z) = \frac{1}{z^4} \) is differentiable at \( z \neq 0 \).

**Solution:** Consider \( z = x + iy \) where \( x = r \cos \theta, y = r \sin \theta \) \((r > 0)\).

Now \( z = re^{i\theta}. \) Thus \( f(z) \) is written as \( f(z) = \frac{1}{r^4} e^{-i4\theta}. \)

Clearly, \( u(r, \theta) = \frac{1}{r^4} \cos 4\theta \) and \( v(r, \theta) = -\frac{1}{r^4} \sin 4\theta. \)

\((u(r, \theta)\) and \(v(r, \theta)\) are defined everywhere, they are continuous and differentiable at every non-zero point \( z = re^{i\theta}. \)

Now we compute partial derivatives:

\[
\begin{align*}
 u_r &= -\frac{4 \cos 4\theta}{r^5}, & u_\theta &= -\frac{4 \sin 4\theta}{r^4} \\
 v_r &= \frac{4 \sin 4\theta}{r^5}, & v_\theta &= -\frac{4 \cos 4\theta}{r^4}
\end{align*}
\]

We have

\[
ru_r = \frac{4 \cos 4\theta}{r^4} = v_\theta \quad \text{and} \quad u_\theta = -\frac{4 \sin 4\theta}{r^4} = -r\left(\frac{4 \sin 4\theta}{r^5}\right)
\]

Therefore C-R equations are satisfied at every non-zero point \( z = re^{i\theta} \) and all partial derivatives with respect to \( r \) and \( \theta \) are continuous. Hence \( f(z) \) is differentiable everywhere when \( z \neq 0 \). The derivative \( f'(z) \) is given by

\[
f'(z) = e^{-i\theta}(u_r + iv_r)
\]

\[
= e^{-i\theta}\left(-\frac{4 \cos 4\theta}{r^5} + i \frac{4 \sin 4\theta}{r^5}\right)
\]

\[
= \frac{-4e^{-i\theta}}{r^5} (\cos 4\theta - i \sin 4\theta) = -\frac{4}{r^5} e^{-i\theta} e^{-i4\theta}
\]

\[
= \frac{-4}{r^5} e^{-i5\theta} = \frac{-4}{r^5} e^{i5\theta} = \frac{-4}{(re^{i\theta})^5} = \frac{-4}{z^5}
\]

\[
\Rightarrow f'(z) = -4z^{-5}.
\]

You may now try the following exercises.

**Do the exercises 1(c), 1(d), 2(a), 2(d), 3(c), 4(c) on page 68 and exercises 5, 6, 10(a), 10(b) on page 69 of the book.**

We would also like you to try the following exercises.

**E4)** Show that the function \( f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y) \) is differentiable only at points that lie on the coordinate axes.

**E5)** Let \( f(z) = (\ln r)^2 - \theta^2 + i 2\theta (\ln r) \) where \( r > 0 \) and \(-\pi < \theta \leq \pi\). Show that \( f \) is differentiable for \( r > 0, -\pi < \theta < \pi \) and find \( f'(z) \).

We now introduce you to a concept that is central to complex analysis.

### 1.5 ANALYTICITY

It is seldom of interest to study functions that are differentiable at only a single point. Complex functions that have a derivative at all points in a neighbourhood of a given
Complex Analysis

point deserve further study. This section is about these functions called analytic functions.

Read Secs. 23 and 24 of Chapter 2 of the book from pages 70-73. Go through the solved Examples 1-3 given in Sec. 24.

After going through these sections you must have noticed that being analytic at a point is much stronger condition then having derivative at that point. A function may have derivative at a single point without being analytic there. Look at \( f(z) = |z|^2 \) (Example 2, Sec.18 on page 55 of the book). It is differentiable only at \( z = 0 \) but it is not analytic at any point.

Theorem on page 71 of the book sounds familiar: If the derivative of a function is identically zero in a domain \( D \) then the function is constant throughout \( D \). At this point, we would like to point out the importance of connectedness of \( D \). This theorem will not be true if the domain \( D \) is merely an open set. For example, if the open set \( D \) is the union of two disjoint open discs \( D_1 \) and \( D_2 \) as shown in Fig. 5.

![Fig. 5](image)

then the function \( f(z) \) defined by

\[
f(z) = \begin{cases} 
1 & \text{if } z \in D_1 \\
2 & \text{if } z \in D_2 
\end{cases}
\]

has zero derivative throughout \( D \), but \( f(z) \) is not a constant function in \( D \).

***

To make you comfortable with the contents of Sec.23 we give below some examples. Example 10 is infact, a simple version of Theorem on page 71 of the book.

**Example 10:** If \( f'(z) = 0 \) everywhere in a disk \( D = \{z: |z| \leq R\} \), then \( f(z) \) must be constant throughout \( D \).

**Solution:** If \( f(z) = u(x, y) + i \ v(x, y) \). Then \( f'(z) = 0 \) implies

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0 , \text{ at every point of } D . \quad \text{[using C-R equations]}
\]

Let \( p = a + ib \) and \( q = c + id \) be points in \( D \) (see Fig. 6). Then at least one of \( p_1 = a + id \) and \( q_1 = c + ib \) lies in \( D \) because

\[
p, q \in D \Rightarrow a^2 + b^2 < R^2, c^2 + d^2 < R^2 \Rightarrow a^2 + b^2 + c^2 + d^2 < 2R^2 \Rightarrow \frac{a^2 + b^2}{c^2 + d^2} < R^2 \Rightarrow \text{either } a^2 + d^2 < R^2 \text{ or, } c^2 + b^2 < R^2 \text{ (at least)} \Rightarrow \text{either } p_1 \in D \text{ or, } q_1 \in D .
\]

Suppose \( p_1 \in D \), then \( f(x) = u(x, d) \) and \( g(x) = u(a, y) \) are real functions with zero derivative and so they are constant. Thus \( u(a, d) = u(a, b) = u(c, d) \) and similarly \( v(a, b) = v(c, d) \). Thus \( f(z) \) is constant \( [u(a, b) = u(p), u(c, d) = u(q)] \).
Example 11: The function \( f(z) = x^2 + y^2 + 2i \, xy \) is nowhere analytic.

Solution: We identify the function \( u(x, y) = x^2 + y^2 \) and \( v(x, y) = 2xy \). The equation \( u_x = v_y \) becomes \( 2x = 2x \), which holds everywhere. But the equation \( u_y = -v_x \) becomes \( 2y = -2y \), which holds only when \( y = 0 \). Thus \( f(x) \) is differentiable only at points that lie on the real axis. However, for any point \( z_0 = x_0 + i \, 0 \) on the real axis and any \( \delta \)-neighbourhood of \( z_0 \), the point \( z_1 = x_0 + i \, \delta / 2 \) is a point at which \( f \) is not differentiable (see Fig. 7). Therefore, \( f \) is not differentiable in any neighbourhood of \( z_0 \) and consequently, it is not analytic at \( z_0 \).

Example 12: Suppose that \( f(z) \) is analytic in a domain \( D \). Prove that \( f(z) \) is constant if \( \text{Re} \, f \) is constant.

Solution: Let \( f(z) = u + iv \). Then \( \text{Re} \, f = u \). It is given that \( u \) is constant therefore \( u_x = u_y = 0 \) at every point of \( D \). Since \( f \) is analytic in \( D \) therefore it satisfy C-R equations. Therefore,

\[
\begin{align*}
    u_x &= v_y \quad \text{and} \quad u_y &= -v_x \\
    \Rightarrow \quad v_y &= 0 \quad \text{and} \quad v_x &= 0 \quad \text{at every point of} \quad D
\end{align*}
\]

Now \( f'(z) = u_x + i \, v_x = -i \, u_y + v_y = 0 \quad \forall \, z \in D \).

Then by Theorem on page 71 of the book, \( f \) is constant.

Before we ask you to try some exercises on your own we make the following remark.

Remark: Failure of C-R equations signals non-analyticity. You have already seen that a non-constant real-valued function defined in a domain is non-analytic. As a consequence, many functions derived from an analytic function become non-analytic. For example, none of \( |f|, \, \text{Re} \, f, \, \text{Im} \, f \) is analytic anywhere.

You can work on the following exercises from the book.

Do the exercises 2(b), 3, 4(c), 6, 7(b) on pages 73-74 of the book.

We now move on to harmonic functions in the next section.

### 1.6 HARMONIC FUNCTIONS

This section deals with real-valued functions of two real variables which satisfy the famous Laplace's equation. This equation is of fundamental importance in the mathematical modelling of 2-dimensional physical problems concerning fluid flow, steady heat conduction, electrostatics, and other phenomenon.

Read Sec. 25, Chapter 2 of the book from pages 75-78. Read examples 1-5 carefully. Go through Theorems 1-2 carefully.

Because harmonic functions satisfy Laplace's equations in two dimensions, they occur widely in applied mathematics. Here we shall discuss in brief a few physical situations where these functions occur. In Unit 6 we shall discuss in detail some more applications of harmonic functions.
Harmonic functions are solutions to many physical problems. Applications include 2-dimensional models of heat flow, electrostatics, and fluid flow etc. First we demonstrate how harmonic functions are used to study fluid flows. We assume that an incompressible, frictionless fluid flows over the complex plane and that all cross sections in planes parallel to the complex plane are the same. Situation such as this occur when fluid is flowing in a deep channel. Suppose we have a solution \( \phi = \phi(x, y) \) of Laplace’s equation in two-dimensions and suppose that \( \phi \) is expressible as \( \text{Re} \ w \) for some analytic function \( w \). Therefore \( \phi \) possesses a harmonic conjugate \( \psi = \text{Im} \ w \); which also satisfies the Laplace’s equation.

In the context of 2-dimensional fluid flow, \( \phi \) is the velocity potential and \( \psi \) is the stream function. The function \( w = \phi + i\psi \) is called the complex potential. Its derivative
\[
V(z) = w'(z) = \phi_x(x, y) + iv_x(x, y) = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x}
\]
determines a vector field which models the velocity of fluid motion (see Fig. 8).

![Fig. 8: The vector field \( V(x, y) = \phi_x(x, y) + iv_x(x, y) \), which can be considered as a fluid flow](image)

It can be seen, using elementary vector calculus, that each curve \( \phi = \text{constant} \) is orthogonal to each curve \( \psi = \text{constant} \). The latter curves follow the direction of the fluid flow and are known as \textit{stream lines}. \( \phi = \text{constant} \) are called \textit{Equipotentials}. (see Fig. 9).

![Fig. 9: The families of orthogonal curves \( \phi(x, y) = \text{const} \) and \( \psi(x, y) = \text{const} \)](image)
Let us consider the following example.

**Example 13:** Show that the harmonic function \( \phi(x, y) = x^2 - y^2 \) is the scalar potential function for the fluid flow

\[ V(x, y) = 2x - 2y. \]

**Solution:** The fluid flow can be written as

\[ V(x, y) = \overline{w(z)} = 2x + 2y + \overline{2z} \]

Now \( w(z) = z^2 \) and the real part of \( w \) i.e. \( \text{Re} \ w \) is the desired harmonic function

\[ \phi(x, y) = x^2 - y^2 = \text{Re} \left\{ (x^2 - y^2) + 2xy \right\}. \]

Now observe that the hyperbolas

\[ \phi(x, y) = x^2 - y^2 = c \]

are the equipotentials and the hyperbolas \( \psi(x, y) = 2xy = c \)

are the streamline curves. These curves are orthogonal, as shown in Fig. 10.

---

Let us consider the function \( w(z) = u \left( z + \frac{a^2}{z} \right) (|z| > a) \) where \( u \) is a positive real number. You can easily verify that this function is analytic. Thus, its real and imaginary parts are harmonic. These are given in polar form by

\[ \phi(r, \theta) = u \left( r + \frac{a^2}{r} \right) \cos \theta \]

and

\[ \psi(r, \theta) = u \left( r - \frac{a^2}{r} \right) \sin \theta. \]

We have \( \psi = 0 \) for \( \theta = 0 \) and \( \theta = \pi \), as well as on the circle \( r = a \). For every large \( r \), the streamlines approximate to straight lines parallel to \( x \)-axis (real). This example models uniform 2-dimension fluid flow parallel to the \( x \)-axis, into which a circular cylinder of radius ‘a’ has been placed (see Fig. 11).
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The complex potential has many physical interpretations. Suppose that we have solved a problem related to steady state temperature (see Example 1, page 75, Sec. 25); then a similar problem with the same boundary conditions in electrostatics is obtained by interpreting the isothermals as equipotential curves and the heat flow lines as fluxlines. In fact, this implies that heat flow and electrostatics corresponds directly. Below we give a chart which gives various interpretations of the families of level curves \( \phi(x, y) = \text{const} \) and \( \psi(x, y) = \text{constant} \).

<table>
<thead>
<tr>
<th>Physical Phenomenon</th>
<th>( \phi(x, y) = \text{constant} )</th>
<th>( \psi(x, y) = \text{constant} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat flow</td>
<td>Isothermals</td>
<td>Heat flow lines</td>
</tr>
<tr>
<td>Electrostatic</td>
<td>Equipotential lines</td>
<td>Fluxlines</td>
</tr>
<tr>
<td>Fluid flow</td>
<td>Equipotentials</td>
<td>Streamlines</td>
</tr>
<tr>
<td>Elasticity</td>
<td>Strain function</td>
<td>Stresslines</td>
</tr>
<tr>
<td>Diffusion</td>
<td>Concentration</td>
<td>Lines of flow</td>
</tr>
<tr>
<td>Current flow</td>
<td>Potential</td>
<td>Lines of flow</td>
</tr>
<tr>
<td>Magentism</td>
<td>Potential</td>
<td>Lines of flow</td>
</tr>
</tbody>
</table>

Two-dimensional Electrostatics

A 2-dimensional electrostatic field is produced by a system of charged wires, plates and cylindrical conductors that are perpendicular to the \( z \)-plane. Now this induces an electric field \( E(x, y) \) that can be visualized as force acting on a unit positive charge placed at the point \( (x, y) \). In the study of electrostatics the vector field \( E(x, y) \) is shown to be conservative and derivable from the function \( \phi(x, y) \), called the electrostatic potential, as expressed by the equation:

\[
E(x, y) = -\text{grad } \phi(x, y) = -\phi_x(x, y) - i\phi_y(x, y).
\]

The level curves \( \phi(x, y) = c_1 \) are called the equipotential curves, and the curves \( \psi(x, y) = c_2 \) are called the lines of flux. For example, if a small charge is allowed to move under the influence of \( E(x, y) \), then it will travel along a line of flux. To be more specific let us consider the following situation.

We find the electrical potential \( \phi(x, y) \) in the region between two infinite coaxial cylinder \( r = a \) and \( r = b \), which are kept at the potential \( u_1 \) and \( u_2 \), respectively (see Fig.12).

Fig.12
The function \( w = \log z = \ln |z| + i \arg z \) maps the annular region between the circles \( r = a \) and \( r = b \) onto the infinite strip \( \ln a < u < \ln b \) in the \( w \)-plane as shown in Fig. 12. Now potential \( \phi(u, v) \) in the infinite strip will have the boundary values

\[
\phi(\ln a, v) = u_2 \quad \text{and} \quad \phi(\ln b, v) = u_2 \quad \text{for all} \quad v.
\]

We are seeking a solution \( \phi(u, v) \) that takes on constant values along the vertical lines \( u = \ln a \) and \( u = \ln b \) with \( \phi(u, v) \) being a function of \( u \) alone, that is,

\[
\phi(u, v) = \rho(u) \quad \text{for,} \quad \ln a \leq u \leq \ln b \quad \text{and for all} \quad v
\]

Now Laplace equation, \( \phi_{uu}(u, v) + \phi_{vv}(u, v) = 0 \).

\[
\Rightarrow \rho''(u) = 0 \Rightarrow \rho(u) = mu + c \quad \text{where} \quad m \quad \text{and} \quad c \quad \text{are constants.}
\]

The boundary conditions \( \phi(\ln a, v) = \rho(\ln a) = u_1 \) and \( \phi(\ln b, v) = \rho(\ln b) = u_2 \) lead to the solution

\[
\phi(u, v) = u_1 + \frac{u_2 - u_1}{\ln b - \ln a} (u - \ln a).
\]

Since \( u = \ln |z| \), we conclude that the potential \( \phi(x, y) \) is

\[
\phi(x, y) = u_1 + \frac{u_2 - u_1}{\ln b - \ln a} (\ln |z| - \ln a).
\]

In Fig. 12 you can see that equipotential curves \( \phi(x, y) = \text{const} \) are concentric circles centred at the origin and the line of flux (dotted lines) are portion of rays emanating from the origin.

Do the exercises 1(c), 2, 5, 7, 9 on pages 78-80 of the book.

1.7 SUMMARY

In this unit we have covered the following:

1) Complex numbers can be defined as ordered pairs \((x, y)\) of real numbers and represented as points in the complex plane with rectangular coordinates \(x\) and \(y\).

2) Addition and multiplication properties of complex numbers are the same as for real numbers.

3) The real positive number \(|z| = |x + iy| = \sqrt{x^2 + y^2}\) is called the modulus or the absolute value or the magnitude of the complex number \(z = x + iy\).

4) The distance between two complex numbers \(z_1 = (x_1 + iy_1)\) and \(z_2 = (x_2 + iy_2)\) is given by

\[
d = |z_2 - z_1| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}.
\]

5) For any two complex numbers \(z_1, z_2\), we cannot write \(z_1 < z_2\) or \(z_1 > z_2\). We can however compare the magnitudes of two complex numbers.
6) Complex roots of a polynomial equations $P(x) = 0$ with real coefficients occur in conjugate pairs.

7) For two non-empty sets $S_1$ and $S_2$ of complex numbers, a rule $f$ which assigns a complex number $w$ is $S_2$ for each $z$ in $S_1$, is a complex valued function of a complex variable $z$ and is written as $w = f(z)$.

8) The value of $w$ for a given $z = z_0$, that is $f(z_0)$ is called the image of the point $z = z_0$ under the rule $f$.

9) Mapping or transformation is a way of displaying information about the function $w = f(z), z = (x, y)$ and $w(u, v)$, by drawing the $z$ and $w$ planes separately.

10) Limit of $f(z)$ as $z$ approaches $z_0$ is a number $w_0$, or that $\lim_{z \to z_0} f(z) = w_0$, means that the point $w = f(z)$ can be made arbitrarily close to $w_0$ if we choose the point $z$ close enough to $z_0$ but distinct from it.

11) A function $f$ is continuous at a point $z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

12) The function $f$ is differentiable at $z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is the derivative of $f(z)$ at the point $z = z_0$ and is denoted by $f'(z_0)$.

13) Cauchy-Riemann equations are the pair of equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, which the first-order partial derivatives of the component functions $u$ and $v$ of a function $f(z) = u(x, y) + iv(x, y)$ must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of $f$ exists there.

14) Cauchy-Riemann equations provide necessary conditions for the existence of $f'(z_0)$ but their satisfaction at a point $z_0$ is not sufficient to ensure the existence of the derivatives of a function $f(z)$ at that point.

15) A function $f(z)$ is analytic at a point $z = z_0$ if it is defined, and has a derivative, at every point in some neighbourhood of $z_0$. It is analytic in a domain $D$ if it is analytic at every point in $D$.

16) If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$ then $u(x, y)$ and $v(x, y)$ satisfy Cauchy-Riemann equation at each point in $D$. Conversely, if the first order partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous and satisfy the Cauchy-Riemann equations in a domain $D$, then $f(z) = u(x, y) + iv(x, y)$ is analytic in $D$.

17) A real valued function $H(x, y)$ of real variables $x$ and $y$ is harmonic in a domain $D$ of the $xy$-plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfy the Laplace equation $H_{xx}(x, y) + H_{yy}(x, y) = 0$.

18) The real and imaginary parts of an analytic function are harmonic functions.
Exercises on page 35 of the book.

1. (c) \( f(z) = \frac{z}{z + \bar{z}} \). Clearly domain of definition contains those complex numbers 
   \( z \) for which \( z + \bar{z} \neq 0 \) \( \Rightarrow 2 \Re z \neq 0 \Rightarrow \Re z \neq 0 \).
   Thus domain of definition is \( \{ z : \Re z \neq 0 \} \).

   Geometrically speaking it is whole complex plane without imaginary axis (see Fig.13). This set is open but not connected. You can not join two points say \( z_1, z_2 \), one lying to the left and other to the right of imaginary axis, by a 
   polygonal line without crossing the imaginary axis.

1. (d) \( f(z) = \frac{1}{1 - |z|^2} \). Again it can be seen easily that the domain of definition 
   consists of those complex numbers for which \( |z| \neq 1 \) (i.e. \( 1 - |z|^2 \neq 0 \)). Thus 
   domain of definition is \( \{ z : |z| \neq 1 \} \). Geometrically, it is whole complex plane 
   minus the unit circle centred at the origin (see Fig.14). It is also open but not 
   connected for obvious reasons.

4. \( f(z) = z + \frac{1}{z} \ (z \neq 0) \)

   In polar coordinates: \( (r \neq 0) \)
   
   \( f(re^{i\theta}) = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} \frac{1}{r} e^{-i\theta} \)
   
   \( \Rightarrow f(re^{i\theta}) = r \left( \cos \theta + i \sin \theta \right) + \frac{1}{r} \left( \cos(-\theta) + i \sin(-\theta) \right) \)
   
   \( = r \cos \theta + i r \sin \theta + \frac{1}{r} \cos \theta - i \frac{1}{r} \sin \theta \)
   
   \( = \left( r + \frac{1}{r} \right) \cos \theta + i \left( r - \frac{1}{r} \right) \sin \theta. \)

Exercises on pages 42-43 of the book.

4. \( w = \exp z \)

   \( \Rightarrow u + iv = e^{x+iy} = e^x (\cos y + i \sin y) \)
   
   \( \Rightarrow u = e^x \cos y, \ v = e^x \sin y \)
   
   \( \Rightarrow u^2 + v^2 = e^{2x} \) and \( \tan y = v/u \Rightarrow y = \tan^{-1}(v/u) \)
   
   and \( x = \ln (u^2 + v^2)^{1/2}. \)

   Now \( ay = x(a \neq 0) \) is transformed to a \( \tan^{-1}(v/u) = \ln(u^2 + v^2)^{1/2} \)
   
   \( \Rightarrow (u^2 + v^2)^{1/2} = e^{a \tan^{-1}(v/u)} \) which reduces to \( \rho = e^{x} \) (equation of 
   spiral) in polar form where, \( \rho = (u^2 + v^2)^{1/2}, \ \phi = \tan^{-1}(v/u). \)

5. Go through Example 1, Sec.13 once again.

   Horizontal segment \( AB \) will be mapped onto the portion of ray \( \phi = c \) lying 
   between \( \rho = e^{x}(A') \) and \( \rho = e^{y}(B') \). The images of horizontal segments above 
   \( AB \) and joining the vertical parts of the boundary are portions of ray moving left
of A'B'; eventually, the images of the line segment DC is the portion of ray \( \rho = d \) lying between \( \rho = e^{a}(D') \) and \( \rho = e^{b}(C') \) (see Fig. 15). Thus image of the rectangular region ABCD \((a \leq x \leq b, c \leq x \leq d)\) is the region \( e^{a} \leq \rho \leq e^{b}, c \leq \phi \leq d \) (same as Figure 21, page 41).

![Diagram showing images of segments and regions](image)

**Fig. 15**

**Exercises on pages 53-54 of the book.**

4. We observe that for \( n = 1 \)

\[
\lim_{z \to z_0} z = z_0.
\]  

(5)

Therefore it holds for \( n = 1 \). Suppose it holds for \( n = k \) i.e.

\[
\lim_{z \to z_0} z^k = z_0^k.
\]  

(6)

Now, \( \lim_{z \to z_0} z^{k+1} = \lim_{z \to z_0} (z^k \cdot z) \)

\[= \left( \lim_{z \to z_0} z^k \right) \left( \lim_{z \to z_0} z \right) \]

(using Property (9) Sec. 15)

\[= z_0^k \cdot z_0 \] (using Eqns. (5) and (6))

\[= z_0^{k+1}.\]

Thus it holds for \( n = k + 1 \). Hence by Mathematical induction \( \lim_{z \to z_0} z^n = z_0^n \) holds for all positive integer \( n \).

5. Go back to Example 2, Sec. 14 for one more reading. We know that if \( \lim_{z \to 0} \left( \frac{z}{Z} \right)^2 \)

exists, it could be found by letting the point \( z = (x, y) \) approach the origin in any manner (along any path).

When \( z = (x, 0), \ (x \neq 0) \) then

\[\left( \frac{z}{Z} \right)^2 = \left( \frac{x}{x} \right)^2 = 1\]

and when \( z = (x, x), \ (x \neq 0) \) then

\[\left( \frac{z}{Z} \right)^2 = \left( \frac{x + ix}{x - ix} \right)^2 = \left( \frac{1 + i}{1 - i} \right)^2 = -1\]

Thus, by letting \( z \) approach the origin along real axis, we would find that the desired limit is 1. An approach along line \( y = x \), on the other hand, gives the
limit as \(-1\). Since a limit has to be unique by definition, therefore limit \(\lim_{z \to 0} \left( \frac{z}{z} \right)^2\) does not exist. If we consider, letting \(z\) approaches origin along imaginary axis then \(\left( \frac{z}{z} \right)^2 = \left( \frac{iy}{-iy} \right)^2 = 1\), it is the same limit as we get along the real axis. This was the reason for considering another path namely \(y = x\).

9. We are given that \(\lim f(z) = 0\). By the definition of limit, for a given \(\varepsilon > 0\) there exists \(\delta_1 > 0\) such that \(|f(z)| = |f(z) - 0| < \varepsilon\), whenever \(0 < |z - z_0| < \delta_1\).

If \(|g(z)| \leq M\) (\(M\) a positive number) for all \(z\) in some neighborhood of \(z_0\), it means there is \(\delta_2 > 0\) such that \(|g(z)| \leq M\) \(\forall z\) such that \(|z - z_0| < \delta_2\).

Now choosing \(\delta = \min\{\delta_1, \delta_2\}\), we have

\[|f(z)g(z) - 0| = |f(z)g(z)| = |f(z)||g(z)| < \varepsilon M = \varepsilon_1,\] whenever \(0 < |z - z_0| < \delta\).

(given \(\varepsilon > 0\), \(\exists \varepsilon_1 > 0\) such that \(\varepsilon M = \varepsilon_1\) and conversely consequently).

Hence \(|f(z)g(z)| < \varepsilon_1\) whenever \(0 < |z - z_0| < \delta\).

\[\Rightarrow \lim_{z \to z_0} f(z)g(z) = 0.\]

11. a) \(\lim_{z \to 0} T(z) = \infty\)

if \(\lim_{z \to 0} \frac{1}{T(1/z)} = 0\)

if \(\lim_{z \to 0} \frac{a + bz}{c + dz} = 0\)

if \(\lim_{z \to 0} \frac{c + dz}{a + bz} = 0\)

if \(c = 0\)

b) \(\lim_{z \to \infty} T(z) = a/c\) and \(\lim_{z \to -\infty} T(z) = \infty\)

if \(\lim_{z \to 0} T(1/z) = a/c\) and \(\lim_{z \to -\infty} \frac{1}{T(z)} = 0\)

if \(\lim_{z \to 0} \frac{a + bz}{c + dz} = a/c\) and \(\lim_{z \to -\infty} \frac{cz + d}{az + b} = 0\)

if \(c \neq 0\)

13. Suppose \(S\) is unbounded. We have to show that every neighborhood of the point at infinity contains at least one point of \(S\).

Let there be a neighborhood of the point at infinity containing no point of \(S\). That is, \(\exists\) a small \(\varepsilon > 0\) such that \(|z| > 1/\varepsilon\) contains no point of \(S\). It means \(S \subset \{z : |z| \leq 1/\varepsilon\} \Rightarrow S\) is contained in \(|z| \leq 1/\varepsilon\) i.e. \(S\) is bounded by the circle \(|z| = 1/\varepsilon\). \(S\) is bounded. It is a contradiction. Therefore every neighborhood of the point at infinity contains at least one point of \(S\).

Conversely, suppose every neighborhood of the point at infinity contains at least one point of \(S\). Let \(S\) be bounded. Then by definition of boundedness \(\exists\) a positive number \(R\) such that \(S\) lies inside the circle \(|z| = R\). Now choosing
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\( \varepsilon = 1/R \), we see that \(|z| > 1/\varepsilon \) contains no point of \( S \). It is again a contradiction. Therefore \( S \) is unbounded.

Exercises given in the unit on page 10.

E1) a) \( f(z) = |z|^2 = x^2 + y^2 \), \( u(x, y) = x^2 + y^2 \), \( v(x, y) = 0 \)

\( u(x, y) \) and \( v(x, y) \) are continuous for all \( x, y \in \mathbb{R}^2 \). Therefore \( f(z) \) is continuous for all \( z \) in the complex plane.

b) Do it yourself.

E2) \( f(i) = 4(i + 1) \).

E3) a) \( z = 4 + i, z = 4 - 1 \).

b) On imaginary axis.

Exercises on page 60 of the book.

7. We write \( m = -n \). Then \( f(z) = z^n = z^{-m} = \frac{1}{z^m} \) (where \( m > 0 \) and \( z \neq 0 \)). Now using the formula for the derivative of a quotient of two functions and

\[
\frac{d}{dz} z^n = n z^{n-1} \quad \text{(where \( n \) is positive integer), we have}
\]

\[
\frac{d}{dz} \left( \frac{1}{z^m} \right) = \frac{\frac{d}{dz} (1) \cdot z^m - 1 \cdot \frac{d}{dz} (z^m)}{(z^m)^2}
\]

\[
= \frac{0 - mz^{m-1}}{z^m} = -m z^{-m-1} = n z^{n-1}
\]

\( \Rightarrow \frac{d}{dz} z^n = n z^{n-1} \).

8. a) Here \( \frac{\Delta w}{\Delta z} = \frac{(z + \Delta z) - \bar{z}}{\Delta z} = \frac{\bar{z} + \Delta z - \bar{z}}{\Delta z} = \frac{\Delta z}{\Delta z} \).

When \( \Delta z \) approaches the origin horizontally through the point \((\Delta x, 0)\) on the real axis:

\( \bar{\Delta z} = \Delta x + i0 = \Delta x = \Delta z \)

In this case, \( \frac{\Delta w}{\Delta z} = \frac{\Delta z}{\Delta z} = 1 \). Hence if the limit of \( \frac{\Delta w}{\Delta z} \) exists, its value must be 1. However, when \( \Delta z \) approaches the origin vertically through the point \((0, \Delta y)\) on the imaginary axis, we find \( \bar{\Delta z} = \Delta x + i\Delta y = 0 + i\Delta y = -i\Delta y = -\Delta z \)

and \( \frac{\Delta w}{\Delta z} = \frac{-\Delta z}{\Delta z} = -1 \). Thus limit is not unique and so \( \frac{dw}{dz} \) does not exist.

b) Do it yourself.

c) Here \( \frac{\Delta w}{\Delta z} = \frac{\text{Im}(z + \Delta z) - \text{Im} z}{\Delta z} = \frac{\text{Im} z + \text{Im} \Delta z - \text{Im} z}{\Delta z} = \frac{\text{Im} (\Delta z)}{\text{Im} \Delta z} \).

If \( \Delta z \) approaches the origin horizontally through \((\Delta x, 0)\), we get

\( \text{Im} \Delta z = 0 \). In this case \( \frac{\Delta w}{\Delta z} = 0 \). Hence if the limit exists, its value must be
0. However, when \( \Delta z \) approaches the origin vertically through \( (0, \Delta y) \) then
\[
\text{Im} \Delta z = \Delta y \quad \text{and} \quad \frac{\Delta w}{\Delta z} = \frac{\Delta y}{\Delta y} = 1. \quad \text{Thus} \quad \frac{dw}{dz} = f'(z) \text{ does not exist.}
\]

Exercises on page 68 of the book.

1. c) \( f(z) = 2x + ixy^2 \).

Here \( u(x, y) = 2x, \ v(x, y) = xy^2 \).

Then \( u_x = 2, \ u_y = 0, \ v_x = y^2, \ v_y = 2xy \).

If \( f(z) \) is differentiable at any point \( (x, y) \) then by Theorem page 62 of the book.

\[
\begin{align*}
\Rightarrow & \quad 2 = 2xy \quad \text{and} \quad 0 = -y^2 \\
\Rightarrow & \quad xy = 1 \quad \text{and} \quad y = 0 \\
\end{align*}
\]

which is not possible for any \( (x, y) \). Therefore \( f'(z) \) does not exist at any point.

d) \( f(z) = e^x e^{-iy} = e^x (\cos y - i \sin y) \).

We have, \( u = e^x \cos y \) and \( v = -e^x \sin y \).

Now \( u_x = e^x \cos y, \ u_y = -e^x \sin y, \ v_x = -e^x \sin y \) and \( v_y = -e^x \cos y \).

If \( f(z) \) is analytic at a point \( (x, y) \) then by C-R equations

\[
\begin{align*}
\Rightarrow & \quad e^x \cos y = -e^x \cos y \quad \text{and} \quad -e^x \sin y = e^x \sin y \\
\Rightarrow & \quad e^x \cos y = 0 \quad \text{and} \quad e^x \sin y = 0 \forall (x, y) \\
\Rightarrow & \quad \cos y = 0 \quad \text{and} \quad \sin y = 0 \forall (x, y) \\
\end{align*}
\]

which is not possible for any \( (x, y) \). Therefore \( f'(z) \) does not exist at any point.

2. a) \( f(z) = i(z + 2) \).

This function is defined everywhere. We express \( f(z) \) as
\[
\begin{align*}
f(z) &= (2 - y) + i x \\
\end{align*}
\]

Here, \( u(x, y) = 2 - y \) and \( v(x, y) = x \). We observe that \( u(x, y) \) and \( v(x, y) \) are polynomials in \( x \) and \( y \) therefore partial derivatives exists everywhere, they are \( u_x = 0, \ u_y = -1, \ v_x = 1, \ v_y = 0 \). Clearly, being constant functions they are continuous everywhere.

Moreover, C-R equations are satisfied everywhere i.e.
\[
\begin{align*}
u_x &= 0 = u_y \quad \text{and} \quad u_y = -1 = -v_x. \\
\end{align*}
\]

Thus every condition of the Theorem on page 63 is satisfied. Therefore \( f'(z) \) exists at every point and is given by
\[
f'(z) = u_x + iv_x = 0 + i \cdot 1 = i.
\]

We have shown that \( f'(z) \) exists everywhere and it is given by \( f'(z) = i \).

Thus derivative is a constant function. Put \( f'(z) = g(z) \).

Now \( g(z) = i = u(x, y) + i v(x, y) \)
\[
\Rightarrow \quad u(x, y) = 0, \quad v(x, y) = 1.
\]

Clearly being constant functions \( u_x, u_y, v_x, v_y \) exist everywhere and
\[
\begin{align*}
u_x &= 0, \ u_y = 0, \ v_x = 0, \ v_y = 0 \quad \text{and} \quad u_x = v_y \quad \text{and} \quad u_y = -v_x. \quad \text{Thus all the}
\end{align*}
\]
conditions of Theorem on page 63 are satisfied. Therefore \( g(z) \) is differentiable i.e. \( g'(z) \) exists everywhere i.e. \( f'(z) = g'(z) \) exists everywhere and is given by \( f'(z) = g'(z) = u_x + i v_x = 0 \).

More generally, if \( f(z) = az + b \), a linear function, then \( f'(z) = a \) and \( f'(z) = 0 \).

d) \( f(z) = \cos x \cosh y - i \sin x \sinh y \).
We have \( u(x, y) = \cos x \cosh y \), \( v(x, y) = -\sin x \sinh y \).
It is defined everywhere. We observe that \( u(x, y) \) and \( v(x, y) \) are simply a product of trigonometric and hyperbolic functions, therefore differentiable at every point. Thus partial derivatives exist everywhere i.e.
\[ u_x = -\sin x \cosh y, \quad u_y = \cos x \sinh y, \quad v_x = -\cos x \sinh y, \quad v_y = -\sin x \cosh y. \]

For every \((x, y)\), \( u_x = v_y \) and \( u_y = -v_x \). Thus every hypothesis of Theorem on page 63 is satisfied. Hence \( f(z) \) is differentiable everywhere and \( f'(z) \) exists everywhere. Therefore \( f'(z) = u_x + iv_x \).
\[ \Rightarrow f'(z) = -\sin x \cosh y - i \cos x \sinh y. \]
By the similar arguments we see that \( f'(z) \) is also differentiable everywhere with derivative
\[ f'(z) = -\cos x \cosh y + i \sin x \sinh y = -f(z). \]

3. c) \( f(z) = z \Im z. \)
We express \( f(z) \) as \( f(z) = (x + iy)y = xy + iy^2 \).
Thus, \( u(x, y) = xy \) and \( v(x, y) = y^2 \).
First we try to find out those points at which C-R equations are satisfied i.e., \( u_x = v_y \) and \( u_y = -v_x \).
We get \( y = 2y \) and \( x = 0 \) \( \Rightarrow \) \( y = 0 \) and \( x = 0 \).
Thus C-R equations are satisfied only at \((0, 0)\), i.e. at \( z = 0 \). We see that being combinations of simple functions (elementary) \( u \) and \( v \) are differentiable everywhere, i.e., there is a neighbourhood of \( z = 0 \) where partial derivatives exist and are continuous. Thus according to the Theorems of Secs.20 and 21, \( f(z) \) is differentiable only at \( z = 0 \) with \( f'(0) = u_x(0) + iv_x(0) = 0 + i0 = 0. \)

4. c) \( f(z) = e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r) \) \( (r > 0, 0 < \theta < 2\pi) \)
We have \( u(r, \theta) = e^{-\theta} \cos(\ln r) \) and \( v(r, \theta) = e^{-\theta} \sin(\ln r) \). It is obvious that in the given domain \( u(r, \theta) \) and \( v(r, \theta) \) are defined and their partial derivative exists and we get \( u_r = -e^{-\theta} \sin(\ln r) \frac{1}{r} \), \( u_\theta = -e^{-\theta} \cos(\ln r) \) and
\[ v_\theta = -e^{-\theta} \sin(\ln r), \quad v_r = e^{-\theta} \cos(\ln r) \frac{1}{r}. \]
Now \( ru_r = r\left(-e^{-\theta} \sin(\ln r) \frac{1}{r}\right) = -e^{-\theta} \sin(\ln r) \)
and \( rv_r = r\left(-e^{-\theta} \cos(\ln r) \frac{1}{r}\right) = -e^{-\theta} \cos(\ln r). \)
Clearly these partial derivatives are continuous in the given domain and C-R equations are satisfied. Therefore, Theorem on page 66, Sec.22 says that
function is differentiable in the indicated domain of definition and its
derivative is given by \( f'(z) = e^{-i\theta}(u_r + iv_r) \)
\[
\Rightarrow f'(z) = e^{-i\theta} \left( -e^{-\theta} \sin(\ln r) \frac{1}{r} + i e^{-\theta} \cos(\ln r) \frac{1}{r} \right) \\
= \frac{i}{r} e^{i\theta} \left( e^{-\theta} \cos(\ln r) + i e^{-\theta} \sin(\ln r) \right) = \frac{if(z)}{z}.
\]

Exercises on page 69 of the book.

5. \( f(z) = x^3 + i(1 - y)^3 \).
   
   Here \( u(x, y) = x^3 \), \( v(x, y) = (1 - y)^3 \).

   Now C-R equations are satisfied only if \( 3x^2 = -3(1 - y)^2 \) and \( 0 = 0 \) which imply \( x = 0 \) and \( y = 1 \). Thus, C-R equations are satisfied only at \((0,1)\), i.e., \( z = i \). Also observe that in every neighbourhood of \( z = i \), partial derivatives exist and are continuous. Therefore \( f(z) \) is differentiable only at \( z = i \). It is therefore legitimate to write \( f'(z) = u_x + iv_x = 3x^2 + i \cdot 0 = 3x^2 \).

6. \( f(z) = \begin{cases} \frac{z^2}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases} \)

   We can write \( \frac{z^2}{z} = \frac{\bar{z}^2}{z} \).

   Hence, \( u(x, y) = \begin{cases} \\
\end{cases} \)

   \( x^3 - 3xy^2 \), \( (x, y) \neq (0,0) \)

   \( 0 \), \( (x, y) = (0,0) \).

   and \( v(x, y) = \begin{cases} \\
\end{cases} \)

   \( y^3 - 3x^2y \), \( (x, y) \neq (0,0) \)

   \( 0 \), \( (x, y) = (0,0) \).

   Now \( u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{h^3 / h^2}{h} = 1 \)

   likewise, \( u_y(0,0) = 0 \), \( v_x(0,0) = 0 \) and \( v_y(0,0) = 1 \).

   Thus C-R equations are satisfied at \((0,0)\) or at \( z = 0 \).

10. a) We express \( F(x, y) \) as follows:
   
   \( F = F[x(z, \bar{z}), y(z, \bar{z})] \) \( x \) and \( y \) are treated as functions of \( z \) and \( \bar{z} \).

   Now differentiating \( F \) w.r.t \( \bar{z} \) and using chain rule we get,

   \[
   \frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}} \\
   = \frac{1}{2} \frac{\partial F}{\partial x} - i \frac{\partial F}{\partial y} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right). 
   \]

   b) For a function \( f(z) = u(x, y) + iv(x, y) \) we are given that \( u_x = v_y \) and \( u_y = -v_x \).

   Now \( \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \) \( \) [by part (a)]
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\[ \frac{1}{2} (u_x + i v_x + i (u_y + i v_y)) \]
\[ = \frac{1}{2} [(u_x - v_y) + i (v_x + u_y)] \]
\[ = \frac{1}{2} (0 + i, 0) = 0 \]

[complex form of C-R equations]

Thus, \( \frac{\partial f}{\partial z} = 0 \).

Exercises on page 15 of the unit.

E4. \( f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2 y) \).

Here \( u(x, y) = x^3 + 3xy^2 \) and \( v(x, y) = y^3 + 3x^2 y \).

Now \( u_x = 3x^2 + 3y^2, \ u_y = 6xy, \ v_x = 6xy \) and \( v_y = 3y^2 + 3x^2 \).

If \( f(z) \) is differentiable at \( (x, y) \) then it must satisfy C-R equations:

\[ u_x = v_y \text{ and } u_y = -v_x \]

\[ \Rightarrow 3(x^2 + y^2) = 3(x^2 + y^2) \] [which is identically satisfied] and \( 6xy = -6xy \)

\[ \Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0. \]

So, if \( f(z) \) has to be differentiable then the points of differentiability are only those points which lie along coordinate axes. Since partial derivatives are continuous everywhere, therefore \( f(z) \) is differentiable along coordinate axes, i.e., \( f(z) \) is differentiable at points of the form \((x_0, 0)\) or \((0, y_0)\) only.

E5. We have \( u(r, \theta) = (\ln r)^2 - \theta^2, \ v(r, \theta) = 2\theta (\ln r) \).

Clearly, \( u \) and \( v \) are defined in the indicated domain. We compute partial derivatives

\[ u_r = \frac{2}{r} (\ln r), \ u_\theta = -2\theta, \ v_r = \frac{2}{r} \theta, \ v_\theta = 2(\ln r) \], which exist in the domain \( \{r, \theta): r > 0, -\pi < \theta < \pi \} \) and are also continuous here in.

Also, \( r u_r = r \cdot \frac{2}{r} (\ln r) = 2(\ln r) = v_\theta \)

and \( u_\theta = -2\theta = -r \left( \frac{2\theta}{r} \right) = -r v_r \). C-R equations are satisfied.

Hence, \( f'(z) \) exist for all \( z \) in the indicated domain and

\[ f'(z) = e^{-i\theta} (u_r + iv_r) = e^{-i\theta} \left( \frac{2}{r} (\ln r) + i \frac{2}{r} \theta \right) = \frac{2}{r} e^{-i\theta} (\ln r + i \theta). \]

Exercises on pages 73-74 of the book.

2. b) \( f(z) = 2xy + i (x^2 - y^2) \).

We have, \( u(x, y) = 2xy, \ v(x, y) = x^2 - y^2 \).

Therefore, \( u_x = 2y, \ u_y = 2x, \ v_x = 2x, \ v_y = -2y \) and so C-R equations are satisfied only at \((0, 0)\). This \( f(z) \) is nowhere analytic. Note that \( f'(0) \) exists.

3. Let \( f(z) \) and \( g(z) \) be any two entire functions. Then by definitions \( \frac{df}{dz} \) and \( \frac{dg}{dz} \) exists everywhere in the complex plane. Clearly, \( fog \) and \( gof \) are defined in the plane and \( (fog)(z) = f(g(z)) \ \forall z \) and \( (gof)(z) = g(f(z)) \ \forall z \).
Then by chain rule:

\[
\frac{d}{dz} (gof)(z) = \frac{d}{dz} g(f(z)) = \frac{d}{dz} g(u) \quad \text{[where } f(z) = u]\]

\[
= \frac{d}{du} g(u) \cdot \frac{du}{dz}
\]

\[
= g'(f(z)) \frac{d}{dz} (f(z))
\]

\[
= g'(f(z)) \cdot f'(z) \forall z
\]

\[\Rightarrow \text{gof is analytic everywhere. Complete the remaining part yourself.}\]

4. c) \( f(z) = \frac{z^2 + 1}{(z + 2) (z^2 + 2z + 2)} \).

Singular points are the solution of the equation:

\[(z + 2) (z^2 + 2z + 2) = 0\]

\[\Rightarrow z = -2, \quad z = -1 \pm i.\]

Since \( f(z) = \frac{P(z)}{Q(z)} \) where \( P(z) \) and \( Q(z) \) are polynomials we know from quotient rule of differentiations that \( f'(z) \) exists everywhere except at \( z = -2, -1 \pm i \) because function is not defined at these points.

6. We have \( g(z) = \ln r + i\theta \quad (r > 0, \ 0 < \theta < 2\pi) \) (\( z \neq 0 \)). Component functions are:

\[u(r, \theta) = \ln r \quad \text{and} \quad v(r, \theta) = \theta.\]

We have,

\[u_r = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1.\]

Observe that partial derivatives exists in the indicated domain and are continuous there. C-R equations are also satisfied as

\[r u_r = 1 = v_\theta \quad \text{and} \quad u_\theta = 0 = -r v_r.\]

Hence, \( g(z) \) is analytic in the indicated domain.

Again,

\[g'(z) = e^{-i \theta} (u_r + i v_r) = e^{-i \theta} \left( \frac{1}{r} + i 0 \right)\]

\[= e^{-i \theta} \frac{1}{r} = \frac{1}{\rho e^{i \theta}} = \frac{1}{z}.\]

Applying chain rule and \( g'(z) = \frac{1}{z} \), we get

\[G'(z) = g'(z^2 + 1). \quad 2z\]

\[= \frac{1}{z^2 + 1}. \quad 2z = \frac{2z}{z^2 + 1}.\]

7. b) Suppose \( |f(z)| = c \ \forall z \in \mathbb{D} \).

When \( c = 0 \), then \( |f(z)| = 0 \Rightarrow f(z) = 0 \ \forall z \in \mathbb{D} \Rightarrow f(z) \) is constant.

When \( c \neq 0 \) then \( |f(z)|^2 = c^2\)

\[\Rightarrow u^2 + v^2 = c^2 \quad \text{[} f(z) = u + iv \text{]}\]

Therefore,

\[u u_x + v v_x = 0, \quad u u_y + v v_y = 0 \quad \text{[} u_x = \frac{\partial u}{\partial x} \text{ etc.}\]

Hence, by the C-R equations.

\[u u_x - v u_y = 0, \quad u u_y + v u_x = 0\]
Elimination of $u_y$ gives $(u^2 + v^2)u_x = 0 = c^2 u_x$

$\Rightarrow u_x = 0$, everywhere in $D$.

Similarly, $u_y, v_x, v_y$ are zero everywhere in $D$. We deduce that $f$ is a constant.

**Exercises on Page 78 of the book.**

1. c) Given function is $u(x, y) = \sinh x \sin y$.

   We observe that

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sinh x \sin y = \sinh x \sin y = 0 \quad \forall x \text{ and } y,$$

   i.e. $u(x, y)$ is harmonic throughout the entire complex plane. If $v(x, y)$ is conjugate to $u(x, y)$ then by C-R equations:

   $$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

   Now $v_y = \frac{\partial u}{\partial x} = \cosh x \sin y$, therefore

   $$v(x, y) = -\cosh x \cos y + \phi(x)$$

   and its differentiation partially w.r.t. $x$ gives

   $$v_x = -\sinh x \cos y + \phi'(x)$$

   $\Rightarrow -u_y = -\sinh x \cos y + \phi'(x)$

   $\Rightarrow -\sinh x \cos y = -\sinh x \cos y + \phi'(x)$

   $\Rightarrow \phi'(x) = 0$

   $\Rightarrow \phi(x) = C$ where $C$ is an arbitrary real number

   and we get,

   $$v(x, y) = -\cosh x \cos y + C.$$ 

   For $C = 0$, we get a particular harmonic conjugate $v(x, y) = -\cosh x \cos y$.

2. Since $v$ and $V$ if exist are harmonic conjugate of $u$ in a domain $D$, $f(z) = u + iv$ and $f'(z) = u + iV$ are analytic functions in $D$. Now by C-R equations:

   $$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_x = V_y, \quad u_y = -V_x$$

   $\Rightarrow v_y = V_y \quad \text{and} \quad v_x = V_x$

   $\Rightarrow (v - V)_y = 0 \quad \text{and} \quad (v - V)_x = 0$

   $\Rightarrow v - V = \text{const.}$

   i.e. $v = V + C$ where $C$ is a constant.

5. We have C-R equations in Polar-form as:

   $$r u_r = v_\theta \quad \text{(i)}$$

   $$u_\theta = -r v_r \quad \text{(ii)}$$

   Differentiating (i) w.r.t. $r$, we get

   $$u_r + r u_{rr} = v_{\theta r} \quad \text{(iii)}$$

   Differentiating (ii) w.r.t. $\theta$, we get

   $$u_{\theta \theta} = -r v_\theta \quad \text{(iv)}$$

   Multiplying (iii) by $r$ and adding to (iv) we get

   $$r u_r + r^2 u_{rr} + u_{\theta \theta} = 0.$$ 

7. As given in suggestion:

   slope of tangent at $z_0 = (x_0, y_0)$ to curve $u(x, y) = c_1$ is given as

   $$m_1 = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}$$
and slope of tangent to curve \( v(x, y) = c_2 \) is given by

\[
    m_2 = -\frac{\partial v / \partial x}{\partial v / \partial y}.
\]

\[
    \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} = -1 \quad (u_x = v_y \quad \text{and} \quad u_y = -v_x)
\]

\[\Rightarrow u(x, y) = c_1 \quad \text{and} \quad v(x, y) = c_2 \] are orthogonal at \( z_0 \).

9. We are given \( f(z) = \frac{1}{z} \). It is an analytic function throughout the plane except at point \((0, 0)\) i.e. at origin.

We can express \( f(z) = \frac{1}{z} \) as:

\[
f(z) = \frac{x - iy}{x^2 + y^2} \Rightarrow f(z) = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}.
\]

Here \( u(x, y) = \frac{x}{x^2 + y^2} \) and \( v(x, y) = -\frac{y}{x^2 + y^2} \).

Now families of level curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \) where \( c_1 \) and \( c_2 \) are arbitrary real constants. \( (f(z) \neq 0 \text{ for any } z \in \Omega) \).

Now, \( \frac{x}{x^2 + y^2} = c_1 \) and \( \frac{-y}{x^2 + y^2} = c_2 \) or \( \frac{y}{x^2 + y^2} = -c_2 \).

If \( c_1 \neq 0, \frac{x}{c_1} = x^2 + y^2 \Rightarrow x^2 + y^2 - C_1 x = 0 \Rightarrow \left( x - \frac{C_1}{2} \right)^2 + y^2 = \frac{C_1^2}{4} \quad \left[ c_1 = \frac{1}{C_1} \right] \)

\[\Rightarrow \left( x - \frac{C_1}{2} \right)^2 + y^2 = \left( \frac{C_1}{2} \right)^2,\]

and if \( c_2 \neq 0, \frac{y}{c_2} = -c_2 \Rightarrow x^2 + \left( y - \frac{C_2}{2} \right)^2 = \left( \frac{C_2}{2} \right)^2 \quad \left[ c_2 = \frac{1}{C_2} \right] \)

Families of level curves are shown in Fig.16.

Remember \( v(x, y) = c_2 \) and \( u(x, y) = c_1 \) are circles centred at \( y \)-axis and \( x \)-axis, respectively and passing through the points \((0, 0)\) but this point is
removed in all cases. For $c_1 = 0$ we get $y$-axis (minus origin) and for $c_2 = 0$, we get $x$-axis (minus origin).

Check the functions $u(x, y) = c_1$ and $v(x, y) = c_2$ for orthogonality yourself.